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Spectral-like duality for distributive Hilbert algebras with infimum

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Abstract. Distributive Hilbert algebras with infimum, or $DH^{\wedge}$-algebras for short, are algebras with implication and conjunction, in which the implication and the conjunction do not necessarily satisfy the residuation law. These algebras do not fall under the scope of the usual duality theory for lattice expansions, precisely because they lack residuation. We propose a new approach, that consists of regarding the conjunction as the additional operation on the underlying implicative structure. In this paper, we introduce a class of spaces, based on compactly-based sober topological spaces. We prove that the category of these spaces and certain relations is dually equivalent to the category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms. We show that the restriction of this duality to a wide subcategory of spaces gives us a duality for the category of $DH^{\wedge}$-algebras and algebraic homomorphisms. This last duality generalizes the one given by the author in 2003 for implicative semilattices. Moreover, we use the duality to give a dual characterization of the main classes of filters for $DH^{\wedge}$-algebras, namely, (irreducible) meet filters, (irreducible) implicative filters and absorbent filters.

1. Introduction

The classical Stone representation theory for distributive lattices leans on the fact that any distributive lattice is isomorphic to the lattice of compact and open subsets of a spectral space, that is, a sober space with a base of compact open sets closed under finite intersections [26]. Further generalizations of this approach lead to dualities for distributive meet-semilattices [6, 8, 20], implicative semilattices [4], Hilbert algebras [7] and Hilbert algebras with supremum [9]. What they all have in common is that they provide representations in terms of compactly-based sober spaces. Other interesting results on generalizations of Stone duality are the papers [15], [14], and [11]. We refer to this class of dualities as *spectral-like dualities*.

A different approach, initiated by Priestley [25], leads to a representation in terms of ordered Hausdorff topological spaces. Although both approaches have been followed to generalize the pioneering work on representation of

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Boolean algebras with operators [23], the latter can be considered to be advantageous [19], especially in view of recent developments of the theory of canonical extensions (see [18] and its references). The key point of this theory, called *extended Priestley duality*, is to represent n-ary (dual) quasioperators by means of $n+1$-ary relations.

One of the strengths of Stone/Priestley duality is mainly that it allows us to use topological tools in the study of logic. Duality theory in logic has been proven to be a fruitful field of study from which, among other results, completeness with respect to Kripke-style semantics of a wide range of non-classical logics has been proven.

Recall that Hilbert algebras represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. It is well known that every poset $\langle X, \leq \rangle$ with greatest element 1 induces a structure of Hilbert algebra defining an implication $\to$ on X as follows: $a \to b = 1$ when $a \leq b$, and $a \to b = b$ when $a \nleq b$. This example allows us to define Hilbert algebras on semilattices or lattices which are not implicative semilattices or Heyting algebras. For instance, the Boolean lattice with four elements and the implication given by the order is a Hilbert algebra with bounded distributive lattice reduct that is not a Heyting algebra. These examples motivate the study of Hilbert algebras with lattice operations. We note that these classes of enriched Hilbert algebras are subclasses of BCK-algebras with lattice operations considered by P. M. Idziak in [22] (see also [3]). In this paper, we will consider mainly Hilbert algebras where the induced order is a distributive meet-semilattice. In this class of algebras, the conjunction and the implication define the same order, but these operations need not be related by the residuation law. We call these algebras distributive Hilbert algebras with infimum (or $DH^{\wedge}$-algebras for short). The lack of residuation forces us to search for a completely different route for a topological representation of this class of algebras.

Our results are supported by already existing dualities for distributive meet-semilattices and Hilbert algebras. These dualities have the characteristic that the duals are pairs of the form $\langle X, \kappa \rangle$, where κ is a base for a topology τ_κ on X satisfying certain additional properties. This strategy is also used in other works. For example, in [11] (see also [14]), a generalization of the classical Stone duality was established proving that there exists a dual equivalence between the category of ideal-distributive posets with the so-called $\wedge$-stable ideal-continuous maps and the category of pairs $\langle X, B \rangle$, where B is a fixed base of a sober topology on X, and B is meet-dense in the collection of all compact open sets. The morphisms in this category preserve the distinguished bases under inverse images. Another interesting example is the equivalence between the category of T_1-spaces with a distinguished base and a certain category of conditionally up-complete, algebraic and maximized posets proved in [15].

In this paper, we provide a spectral-like duality for two categories based on $DH^{\wedge}$-algebras. A parallel study involving a Priestley-style duality for the same categories is being developed in a forthcoming paper.

The strategy consists of looking at the meet operation as an additional operation on the underlying Hilbert algebra, instead of what is customary, namely looking at the implication as an additional operation on the (semi)-lattice structure. Accordingly, the meet is represented by a subset satisfying certain conditions, instead of being represented by a relation.

The organization of the paper goes as follows. In Section 2, we present the preliminaries and we establish the basic notational conventions. Particularly, we recall the spectral-like duality for distributive meet-semilattices given in [6] and [20] (see also [13]), the spectral-like duality for Hilbert algebras developed in [7] and [9], and we introduce the class of $DH^{\wedge}$-algebras. In Section 3, we examine different notions of filters associated with a $DH^{\wedge}$-algebra, and the relations between them, that yield the keypoint of our representation strategy. In Section 4, we present the duality for objects, where duals of $DH^{\wedge}$-algebras are certain spectral-like spaces augmented with a subset that satisfies some conditions. In Section 5, we extend the duality to morphisms between $DH^{\wedge}$-algebras. Following [7], we deal with two different notions of morphism, namely, the usual algebraic notion of homomorphism and a weaker notion related to that of semi-homomorphism between Hilbert algebras. Then two categories are defined and the dual equivalences of these categories are proved. As was pointed out to us by the referee, the principal new approach in paper [6] was the consideration of relations as morphisms between DS-spaces instead of functions, but such approaches were investigated already before this paper appeared, one typical instance being the paper by Hofmann and Watson [21]. Finally, in the last section, a topological characterization of the main classes of filters is given.

2. Preliminaries

2.1. Basic notation and terminology. We denote by ω the set of natural numbers and by $\emptyset$ the empty set. For X a set and $Y \subseteq X$, we denote by Y^c the complement of Y, namely $\{x \in X : x \notin Y\}$. For a binary relation $R \subseteq X_1 \times X_2$ between sets X_1 and X_2, and for any $x_1 \in X_1$, we denote by $R(x_1)$ the set $\{x_2 \in X_2 : (x_1, x_2) \in R\}$, and for any $Y \subseteq X_2$, we denote by $R^{-1}(Y)$ the set $\{x_1 \in X_1 : \exists y \in Y((x_1, y) \in R)\}$. For sets X_1, X_2, X_3, functions $f\colon X_1 \longrightarrow X_2$, $g\colon X_2 \longrightarrow X_3$ and relations $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, the composition is denoted by $g \circ f$ and $S \circ R$, respectively.

Let $\langle X, \leq\rangle$ be a poset. A subset $Y \subseteq X$ is an *up-set* when for every $y \in Y$ and every $x \in X$, if $y \leq x$, then $x \in Y$. *Down-sets* are defined order-dually. By $\mathcal{P}^{\uparrow}(X)$ we denote the collection all up-sets of $\langle X, \leq\rangle$. For any $Y \subseteq X$, we denote by $\uparrow Y$ (resp. $\downarrow Y$) the up-set (resp. down-set) generated by Y, i.e., $\{x \in X : \exists y \in Y(y \leq x)\}$ (resp. $\{x \in X : \exists y \in Y(x \leq y)\}$). If Y is a singleton $\{x\}$, then we write $\uparrow x$ and $\downarrow x$ instead of $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively.

Let $X = \langle X, \tau\rangle$ be a topological space. As usual, we shall refer to it by X. We denote by $\mathcal{O}(X)$ (resp. $\mathcal{C}(X)$) the collection of open (resp. closed) subsets of

X and by $\mathcal{KO}(X)$ the collection of open and compact subsets of X. For $Y \subseteq X$, we denote by $\mathrm{cl}(Y)$ the closure of Y, i.e., the least closed set that contains Y. Recall that a subset Y of X is *saturated* provided it is an intersection of open sets. The saturation of a subset Y of X is the least saturated set that contains Y, and we denote it by $\mathrm{sat}(Y)$. If Y is a singleton $\{x\}$, then we write $\mathrm{cl}(x)$ and $\mathrm{sat}(x)$ instead of $\mathrm{cl}(\{x\})$ and $\mathrm{sat}(\{x\})$, respectively. We also recall that the *specialization pre-order* of $\langle X, \tau \rangle$ is given by $x \preceq_X y$ if and only if $x \in \mathrm{cl}(y)$. When $\langle X, \tau \rangle$ is T_0, the pre-order $\preceq$ is an order. A nonempty subset $Y \subseteq X$ is *irreducible* provided for any $Y_1, Y_2 \in \mathcal{C}(X)$, if $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$. The space X is *sober* when each closed irreducible subset is the closure of a unique point.

Distributive meet-semilattices. A *meet-semilattice with top element* is an algebra $\mathbf{A} = \langle A, \wedge, 1 \rangle$ of type $(2, 0)$ such that the operation $\wedge$ is idempotent, commutative, associative, and $a \wedge 1 = a$ for each $a \in A$. As usual, the binary relation $\leq$ defined by $a \leq b$ if and only if $a \wedge b = a$ is a partial order. In what follows we shall use *semilattice*, instead of meet-semilattice with top element.

An *order ideal* of a semilattice $\mathbf{A}$ is a non-empty up-directed down-set of A, i.e., a down-set I with $\emptyset \neq I \subseteq A$ such that whenever $a, b \in I$, there exists $c \in I$ such that $a, b \leq c$. We denote by $\mathrm{Id}(\mathbf{A})$ the collection of all order ideals of $\mathbf{A}$. Notice that all principal down-sets are order ideals.

A *meet filter* of a semilattice $\mathbf{A}$ is a non-empty up-set closed under the meet operation, i.e., an up-set $F \subseteq A$ such that $1 \in F$ and $a \wedge b \in F$ whenever $a, b \in F$. Notice that all principal up-sets are meet filters. A meet filter F is *proper* when $F \neq A$. We denote by $\mathrm{Fi}(\mathbf{A})$ the collection all meet filters of $\mathbf{A}$. The set $\mathrm{Fi}(\mathbf{A})$ is closed under arbitrary intersections, so for each $B \subseteq A$, we denote by $[\![B\rangle\!\rangle$ the least meet filter that contains B. We call $[\![B\rangle\!\rangle$ the *meet filter generated by* B. It is well known that

$$[\![B\rangle\!\rangle = \Big\{ a \in A : \bigwedge F \leq a, \text{ for some finite subset } F \subseteq B \Big\}.$$

Notice that for each $a \in A$, $[\![a\rangle\!\rangle = {\uparrow} a$. We consider the bounded lattice $\mathbf{Fi}(\mathbf{A}) := \langle \mathrm{Fi}(\mathbf{A}), \cap, \vee, A, \{1\} \rangle$, in which the meet operation is given by forming intersection and the join operation is given by the meet filter generated by the union. We say that a meet filter F is $\wedge$*-irreducible* when it is a meet irreducible element of the lattice $\mathbf{Fi}(\mathbf{A})$. We denote by $\hat{X}(\mathbf{A})$ the collection of $\wedge$-irreducible meet filters of $\mathbf{A}$.

Definition 2.1. A semilattice $\mathbf{A}$ is *distributive* if for each $a, b, c \in A$ with $a \wedge b \leq c$, there exist $a', b' \in A$ such that $a \leq a'$, $b \leq b'$ and $c = a' \wedge b'$.

A representation theorem for distributive semilattices may be obtained from [20], where Grätzer defines distributive semilattices as a general framework to discuss topological representations of distributive lattices. Elementary properties of distributive semilattices are studied in [20] and [8], one being that a semilattice $\mathbf{A}$ is distributive if and only if the lattice of meet filters $\mathbf{Fi}(\mathbf{A})$

is distributive. We recall that a filter F of $\mathbf{A}$ is a $\wedge$-irreducible meet filter iff $F^c \in \mathrm{Id}(\mathbf{A})$.

The following lemma is an analogue of Birkhoff's Prime Filter Lemma. We note that this Lemma and the next Lemma 2.5 are special instances of a general Separation Lemma due to Banaschewski and Erné in [1].

Lemma 2.2. ($\wedge$*-irreducible Meet Filter Lemma*) *Let* $\mathbf{A}$ *be a distributive semilattice. Let* $F \in \mathrm{Fi}(\mathbf{A})$ *and* $I \in \mathrm{Id}(\mathbf{A})$ *be such that* $F \cap I = \emptyset$. *Then there is* $G \in \hat{X}(\mathbf{A})$ *such that* $F \subseteq G$ *and* $G \cap I = \emptyset$.

Let $\mathbf{A}$ be a semilattice. Let $a \in A$ with $a \neq 1$. The element a is *meet irreducible* when for all $b, c \in A$, if $a = b \wedge c$, then $a = b$ or $a = c$, and a is *meet prime* when for all $b, c \in L$, if $b \wedge c \leq a$, then $b \leq c$ or $c \leq a$. It is well known that prime and irreducible elements coincide for any distributive semilattice.

A categorical duality for distributive semilattices and homomorphisms preserving top was studied in [6], where dual objects are called DS-spaces. Recall that a *DS-space* [6, Definition 14] is a topological space $X = \langle X, \tau \rangle$ such that:

- (DS1) The collection $\mathcal{KO}(X)$ of open and compact subsets of X forms a basis for the topology τ,
- (DS2) $\langle X, \tau \rangle$ is sober.

Remark 2.3. As a corollary of the duality for distributive semilattices and homomorphisms preserving top, there exists a duality between the following categories. On the one hand, the category of distributive join-semilattices with least element whose morphism are prime-ideal continuous maps. On the other hand, the category of algebraic distributive lattices whose morphism are frame homomorphisms preserving compactness (see [13, Section 5] for the details).

Let X be a DS-space. Consider the family $D(X) := \{U : U^c \in \mathcal{KO}(X)\}$, which is closed under finite intersection. In [20], it is proven that $\mathbf{D}(X) := \langle D(X), \cap, X \rangle$ is a distributive semilattice, called the *dual distributive semilattice* of X.

Let $\mathbf{A} = \langle A, \wedge, 1 \rangle$ a distributive semilattice. Recall that $\hat{X}(\mathbf{A})$ is the set of all $\wedge$-irreducible meet filters of $\mathbf{A}$. Consider the map $\sigma_{\mathbf{A}} : A \longrightarrow \mathcal{P}^{\uparrow}(\hat{X}(\mathbf{A}))$ defined by $\sigma_{\mathbf{A}}(a) = \{P \in \hat{X}(\mathbf{A}) : a \in P\}$. In [20] (see also [6]), it is proven that $\{\sigma_{\mathbf{A}}(a)^c = \hat{X}(\mathbf{A}) - \sigma_{\mathbf{A}}(a) : a \in A\}$ is a base for a topology $\tau_{\mathbf{A}}$ on $\hat{X}(\mathbf{A})$. Moreover, $\langle \hat{X}(\mathbf{A}), \tau_{\mathbf{A}} \rangle$ is shown to be a DS-space, called the *dual DS-space* of $\mathbf{A}$.

If X is a DS-space, then it is homeomorphic to $\langle \hat{X}(\mathbf{D}(X)), \tau_{\mathbf{D}(X)} \rangle$ by means of the map $\hat{\varepsilon} : X \to \hat{X}(\mathbf{D}(X))$, given by $\hat{\varepsilon}(x) = \{U \in D(X) : x \in U\}$. If $\mathbf{A}$ is a distributive semilattice, then it is isomorphic to $\mathbf{D}(\hat{X}(\mathbf{A}))$ by means of the map $\sigma_{\mathbf{A}}$.

Hilbert algebras. In this subsection, we recall the representation theory for Hilbert algebras.

Definition 2.4. A *Hilbert algebra* is an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $(2, 0)$ in which

(1) $a \to (b \to a) = 1$,
(2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$,
(3) $a \to b = 1 = b \to a$ implies $a = b$.

In [12], Diego proves that the class of Hilbert algebras is a variety. It is easy to check that the binary relation $\leq_{\mathbf{A}}$ defined on a Hilbert algebra $\mathbf{A}$ by $a \leq_{\mathbf{A}} b$ if and only if $a \to b = 1$ is a partial order on A with top element 1. This order is called the *natural order* on $\mathbf{A}$. When the context is clear, we omit the subscript of $\leq_{\mathbf{A}}$. Let us recall that the class of Hilbert algebras is a subclass of BCK-algebras (see [10, page 165]).

Let $\mathbf{A}$ be a Hilbert algebra. An *implicative filter* (or *deductive system*) of $\mathbf{A}$ is a subset $P \subseteq A$ such that $1 \in P$ and if $a, a \to b \in P$, then $b \in P$. Notice that implicative filters are up-sets, and all principal up-sets are implicative filters. We denote by $\mathrm{Fi}_{\to}(\mathbf{A})$ the collection of implicative filters of $\mathbf{A}$. The set $\mathrm{Fi}_{\to}(\mathbf{A})$ is closed under arbitrary intersections, so for any $B \subseteq A$, there exists the least implicative filter that contains B. We call this implicative filter the *implicative filter generated by* B and we denote it by $\langle B\rangle$. Notice that for all $a \in A$, $\langle a\rangle = {\uparrow} a$. The algebra $\mathbf{Fi}_{\to}(\mathbf{A}) := \langle \mathrm{Fi}_{\to}(\mathbf{A}), \cap, \vee, A, \emptyset\rangle$, in which $\vee$ is given by the implicative filter generated by the union, is a bounded distributive lattice (see [12] or [24] for more details on implicative filters in Hilbert algebras). We say that an implicative filter P of $\mathbf{A}$ is $\to$*-irreducible* when it is a meet irreducible element of the lattice $\mathbf{Fi}_{\to}(\mathbf{A})$. Since the lattice $\mathbf{Fi}_{\to}(\mathbf{A})$ is distributive, meet irreducible and meet prime elements of $\mathbf{Fi}_{\to}(\mathbf{A})$ coincide. Thus, an implicative filter P of $\mathbf{A}$ is irreducible iff $P^c \in \mathrm{Id}(\mathbf{A})$. We denote by $X(\mathbf{A})$ the collection of $\to$-irreducible implicative filters of $\mathbf{A}$.

We note that Lemma 2.5 is a special case of a general Separation Lemma due to Banaschewski and Erné in [1] (for a proof for Hilbert algebras, see [4]).

Lemma 2.5. (*Irreducible Implicative Filter Lemma*) *Let* $\mathbf{A}$ *be a Hilbert algebra. Let* $P \in \mathrm{Fi}_{\to}(\mathbf{A})$ *and* $I \in \mathrm{Id}(\mathbf{A})$ *be such that* $P \cap I = \emptyset$. *Then there is* $Q \in X(\mathbf{A})$ *such that* $P \subseteq Q$ *and* $Q \cap I = \emptyset$.

Let X be a set and let $\kappa \subseteq \mathcal{P}(X)$. A topological space with a *fixed* base κ is denoted by $\langle X, \tau_\kappa\rangle$ or directly by $\langle X, \kappa\rangle$. We note that the topology τ_κ is completely determinate for the base κ. Recall that an H*-space* is a pair $\langle X, \kappa\rangle$ where κ is a base of compact open sets for a sober topology τ_κ on X satisfying the condition:

(H) for every $U, V \in \kappa$, $\mathrm{sat}(U \cap V^c) \in \kappa$.

For an H-space $\langle X, \kappa\rangle$, we consider the family $D_\kappa(X) := \{U : U^c \in \kappa\}$, and we define a binary operation $\Rightarrow$ on it given by $U \Rightarrow V = (\mathrm{sat}(U \cap V^c))^c$. By Condition (H), this operation is well defined, and in [9] (see also [7]), it is proven that $\mathbf{D}_\kappa(X) := \langle D_\kappa(X), \Rightarrow, X\rangle$ is a Hilbert algebra, called the *dual Hilbert algebra* of X. If $\langle X, \kappa\rangle$ is an H-space, then the topology τ_κ is T_0, and so its specialization pre-order is a partial order. We deal with its dual order, which we denote by $\leq_X$, or by $\leq$ when the context is clear. Then we have that for all $x \in X$, $\mathrm{cl}(x) = {\uparrow}x$, and for all $U \subseteq X$, $\mathrm{sat}(U) = {\downarrow}U$, with respect to the order $\leq$.

Let $\mathbf{A}$ be a Hilbert algebra. We consider the map $\varphi_{\mathbf{A}} \colon A \longrightarrow \mathcal{P}^{\uparrow}(X(\mathbf{A}))$ defined by

$$\varphi_{\mathbf{A}}(a) = \{P \in X(\mathbf{A}) : a \in P\}.$$

For convenience, we omit the subscript of $\varphi_{\mathbf{A}}$, when no confusion is possible. In [7], it is proven that the family $\kappa_{\mathbf{A}} := \{\varphi(a)^c : a \in A\}$ is a base for a topology $\tau_{\kappa_{\mathbf{A}}}$ on $X(\mathbf{A})$. Moreover, $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$ is an H-space, called the *dual H-space* of $\mathbf{A}$. The dual of the specialization order of this space is the inclusion relation. We note that for each $x \in X(\mathbf{A})$, $\mathrm{cl}(x) = {\uparrow}x$, and every closed subset Y of $X(\mathbf{A})$ is an up-set. We note that $\uparrow$ and 'up-set' refer to inclusion, instead of the the specialization order.

If $\langle X, \kappa\rangle$ is an H-space, then it is homeomorphic to $\langle X(\mathbf{D}_{\kappa}(X)), \kappa_{\mathbf{D}(X)}\rangle$ by means of the map $\varepsilon_X : X \to X(\mathbf{D}_{\kappa}(X))$, given by

$$\varepsilon_X\left(x\right) = \left\{U \in D_{\kappa}(X) : x \in U\right\}.$$

If $\mathbf{A}$ is a Hilbert algebra, then $\varphi_{\mathbf{A}}$ establishes an isomorphism between $\mathbf{A}$ and $\mathbf{D}_{\kappa_{\mathbf{A}}}(X(\mathbf{A}))$.

In [7], two different categories with Hilbert algebras as objects were considered: On the one hand, the usual algebraic category, with algebraic homomorphisms as morphisms, and on the other hand, a category with a weaker notion of morphism, namely maps $h \colon \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ that preserve the top element and such that $h(a \to b) \leq h(a) \to h(b)$ for all $a, b \in A$. The latter are called *$\to$-semi-homomorphisms*, and in [7], it is proven that they dually correspond to binary relations $R \subseteq X_1 \times X_2$ between two H-spaces $\langle X_1, \kappa_1\rangle$ and $\langle X_2, \kappa_2\rangle$, satisfying the following conditions:

(HR1) $R^{-1}(U) \in \kappa_1$, for every $U \in \kappa_2$,
(HR2) $R(x)$ is a closed subset of $\langle X_2, \kappa_2\rangle$, for all $x \in X_1$.

Such relations are called *H-relations* [7, Definition 3.2]. If $h \colon \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ is a semi-homomorphism between Hilbert algebras, then $R_h \subseteq X(\mathbf{A}_2) \times X(\mathbf{A}_1)$, given by: $(P, Q) \in R_h$ if and only if $h^{-1}[P] \subseteq Q$, is an H-relation between the corresponding dual H-spaces. Moreover, for a given H-relation between H-spaces, $R \subseteq X_1 \times X_2$, the map $\Box_R \colon D_{\kappa_2}(X_2) \longrightarrow D_{\kappa_1}(X_1)$, defined by

$$\Box_R(U) := \{x \in X_1 : R(x) \subseteq U\},$$

is a $\to$-semi-homomorphism between the corresponding dual Hilbert algebras. Recall that the dual of homomorphisms between Hilbert algebras are H-relations $R \subseteq X_1 \times X_2$ satisfying the condition:

(HF) If $(x, y) \in R$, then there exists $z \in X_1$ such that $x \leq z$ and $R(z) = \mathrm{cl}(y)$.

Such relations are called *functional H-relations* in [7], where it is proven that the correspondence between $\to$-semi-homomorphisms and H-relations restricts to homomorphisms and functional H-relations, respectively.

Hilbert algebras with infimum. Now we define the class of Hilbert algebras, where the order given by the implication defines the structure of a meet semilattice.

Definition 2.6. An algebra $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ of type $(2, 2, 0)$ is a *Hilbert algebra with infimum* or *$H^\wedge$-algebra* if

(1) $\langle A, \to, 1\rangle$ is a Hilbert algebra,
(2) $\langle A, \wedge, 1\rangle$ is a meet semilattice with top element 1,
(3) for all $a, b \in A$, $a \to b = 1$ iff $a \wedge b = a$.

Notice that by Condition (3) in the previous definition, we have that the natural order given by the implication and the order given by the semilattice coincide. In [17], it is proven that the class of $H^\wedge$-algebras is a variety. We note that this result also follows from results given by P. M. Idziak in [22] for BCK-algebras with lattice operations.

Example 2.7. In any semilattice $\langle A, \wedge, 1\rangle$, it is possible to define the structure of Hilbert algebra with infimum if we take the implication $\to$ given by

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

We call such an operation *the implication defined by the order* on A.

Example 2.8. Implicative semilattices (see [10]), also called Hertz algebras or Brouwerian semilattices, are Hilbert algebras with infimum in which the implication is the right residuum of the meet operation, or equivalently, in which the following equation holds:

(PA) $a \to (b \to (a \wedge b)) = 1$.

The next example shows that the class of implicative semilattices is strictly included in the class of $H^\wedge$-algebras.

Example 2.9. Let $A = \{0, a, b, 1\}$ be the four-element Boolean lattice considered as a distributive meet-semilattice $\langle A, \wedge, 1\rangle$ with top element 1. Consider on A the implication defined by the order $\leq$. Then we have that $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ is an $H^\wedge$-algebra. We can see that $\to$ does not preserve meets in the second coordinate, since

$$0 = a \to (a \wedge b) \neq (a \to a) \wedge (a \to b) = b,$$

and thus it is not an implicative semilattice.

Definition 2.10. We say that an $H^\wedge$-algebra $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ is *distributive* (or a *$DH^\wedge$-algebra*) when the underlying semilattice $\langle A, \wedge, 1\rangle$ is distributive.

Notice that any non-distributive semilattice augmented with the implication defined by the order is a Hilbert algebra with infimum that is not distributive. Therefore, the class of $DH^{\wedge}$-algebras is strictly included in the class of $H^{\wedge}$-algebras. It is well known that the same way the lattice reduct of a Heyting algebra is distributive, the semilattice reduct of an implicative semilattice is distributive, so the class of implicative semilattices is included in the class of $DH^{\wedge}$-algebras. Notice that it follows from Example 2.9 that this inclusion is strict.

Lemma 2.11. *Let* $\mathbf{A}$ *be an* $H^{\wedge}$*-algebra. Then for all* $a, b, c \in A$*, we have* $a \to (b \to c) \leq (a \wedge b) \to c$*.*

Proof. Let $a, b, c \in A$. From $a \wedge b \leq a$ we get $a \to (b \to c) \leq (a \wedge b) \to (b \to c)$. From $a \wedge b \leq b$, we get $b \to c \leq (a \wedge b) \to c$, and so

$$(a \wedge b) \to (b \to c) \leq (a \wedge b) \to ((a \wedge b) \to c) = (a \wedge b) \to c,$$

and we are done. □

3. Filters in $H^{\wedge}$-algebras

In an $H^{\wedge}$-algebra $\mathbf{A}$, we distinguish two classes of *filters*. On the one hand, we have the collection of implicative filters $\mathrm{Fi}_{\to}(\mathbf{A})$ associated with the $(\to, 1)$-reduct of $\mathbf{A}$. On the other hand, we have the collection of meet filters $\mathrm{Fi}(\mathbf{A})$ associated with the $(\wedge, 1)$-reduct of $\mathbf{A}$. Both classes of filters play an important role in the representation of $H^{\wedge}$-algebras. In the present section, we study the relations between these classes; in addition, one more notion of filter is considered.

Let $\mathbf{A}$ be an $H^{\wedge}$-algebra. It is easy to prove that all meet filters of $\mathbf{A}$ are also implicative filters. Indeed, let $F \in \mathrm{Fi}(\mathbf{A})$ and $a, a \to b \in F$. Then $a \wedge b = a \wedge (a \to b) \in F$, and thus $a \wedge b \leq b \in F$. Clearly, $1 \in F$ since it is a non-empty up-set. Therefore, we have $\mathrm{Fi}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$. Moreover, the following relation between $\to$-irreducible implicative filters and $\wedge$-irreducible meet filters holds for any $H^{\wedge}$-algebra.

Proposition 3.1. *Let* $\mathbf{A}$ *be an* $H^{\wedge}$*-algebra. Then* $X(\mathbf{A}) \cap \mathrm{Fi}(\mathbf{A}) \subseteq \hat{X}(\mathbf{A})$*.*

Proof. This is immediate. □

The next proposition gives a characterization of distributive $H^{\wedge}$-algebras by means of the relation between $\to$-irreducible implicative filters and $\wedge$-irreducible meet filters.

Proposition 3.2. *An* $H^{\wedge}$*-algebra* $\mathbf{A}$ *is distributive iff* $\hat{X}(\mathbf{A}) \subseteq X(\mathbf{A})$*.*

Proof. Assume that $\mathbf{A}$ is distributive and let $P \in \hat{X}(\mathbf{A})$. On the one hand, we have that $P \in \mathrm{Fi}_{\to}(\mathbf{A})$. Since P is a $\wedge$-irreducible meet filter and P^c is an order ideal, we conclude that $P \in X(\mathbf{A})$.

Let now $\mathbf{A}$ be an $H^{\wedge}$-algebra such that $\hat{X}(\mathbf{A}) \subseteq X(\mathbf{A})$. Since the $(\rightarrow, 1)$-reduct of $\mathbf{A}$ is a Hilbert algebra, we obtain that P^c is an order ideal for all $P \in \hat{X}(\mathbf{A})$. Thus, by [5, Theorem 10], the $(\wedge, 1)$-reduct of $\mathbf{A}$ is a distributive semilattice, so $\mathbf{A}$ is a $DH^{\wedge}$-algebra, as required. □

Corollary 3.3. *Let* $\mathbf{A}$ *be a* $DH^{\wedge}$*-algebra. Then*

(1) $X(\mathbf{A}) \cap \mathrm{Fi}(\mathbf{A}) = \hat{X}(\mathbf{A})$.

(2) *For each* $P \in X(\mathbf{A})$*, there exists* $Q \in \hat{X}(\mathbf{A})$ *such that* $Q \subseteq P$*, i.e.,* $X(\mathbf{A}) = \uparrow\hat{X}(\mathbf{A})$.

Proof. (1): This follows from Proposition 3.1 and Proposition 3.2.

(2): Let $P \in X(\mathbf{A})$. As P is not empty, there exists $a \in P$. So, $\uparrow a \cap P^c = \emptyset$, and as P^c is an order-ideal, by Lemma 2.2, there exists $Q \in \hat{X}(\mathbf{A})$ such that $a \in Q$ and $Q \subseteq P$. □

We note that the inclusion $\hat{X}(\mathbf{A}) \subseteq X(\mathbf{A})$ in Proposition 3.2 may be strict, as the following example shows.

Example 3.4. Consider the $DH^{\wedge}$-algebra given in Example 2.9. Let us denote by F_{ab} the implicative filter $\uparrow(\{a, b\}) = \{a, b, 1\}$. It is easy to see that $\mathrm{Fi}(\mathbf{A})$ is the collection of principal up-sets. Moreover, $\mathrm{Fi}_{\rightarrow}(\mathbf{A})$ is $\mathrm{Fi}(\mathbf{A})$ together with F_{ab}. It is not difficult to check that $F_{ab} \in X(\mathbf{A})$, but since it is not closed under meet, $F_{ab} \notin \hat{X}(\mathbf{A})$. Hence, we have: $\hat{X}(\mathbf{A}) \subsetneq X(\mathbf{A})$.

Finally, we mention one more notion of filter for $H^{\wedge}$-algebras, that was introduced in [17]. This notion corresponds to the notion of *logical filter* for the logic $\mathcal{H}^{\wedge}$ defined in [17]. Although these filters do not play any role in the representation of $DH^{\wedge}$-algebras, we will obtain a dual characterization of them in the last section of the paper.

Definition 3.5. Let $\mathbf{A}$ be an $H^{\wedge}$-algebra. An implicative filter H of $\mathbf{A}$ is *absorbent* if for all $a \in A$ and $b \in H$, $a \rightarrow (a \wedge b) \in H$.

We denote by $\mathrm{Ab}(\mathbf{A})$ the collection of all absorbent filters of $\mathbf{A}$. It is easy to prove that $\mathrm{Ab}(\mathbf{A}) \subseteq \mathrm{Fi}(\mathbf{A})$. Indeed, let $a, b \in H \in \mathrm{Ab}(\mathbf{A})$. Clearly, P is an up-set and moreover $a \rightarrow (a \wedge b) \in H$. Since H is an implicative filter, we have $a \wedge b \in H$.

Notice that $\mathrm{Ab}(\mathbf{A})$ is closed under arbitrary intersections, so for $B \subseteq A$, we may consider the least absorbent filter that contains B. Unlike the case of meet filters or implicative filters, we do not have an alternative characterization of the absorbent filter generated by a set. But we have the following proposition, that will be used later on.

Proposition 3.6. *For all* $F \in \mathrm{Fi}(\mathbf{A})$*,* $F \in \mathrm{Ab}(\mathbf{A})$ *if and only if for all* $a \in A$*,* $\langle F \cup \uparrow a\rangle$ *is a meet filter.*

Proof. Let $F \in \mathrm{Ab}(\mathbf{A})$ and let $a \in A$. If $a \in F$, there is nothing to prove, so suppose $a \notin F$. We claim that

$$\langle F \cup \uparrow a\rangle \in \mathrm{Fi}(\mathbf{A}).$$

For this, we need only show $\langle F\cup\uparrow a\rangle$ is closed under meets, so let $b, c \in \langle F\cup\uparrow a\rangle$. As $F \neq \emptyset$, we may assume that there are $b_0, \dots, b_n, c_0, \dots, c_m \in F \cup \uparrow a$ such that $b_0 \to (\cdots(b_n \to b)\cdots) = 1$ and $c_0 \to (\cdots(c_m \to c)\cdots) = 1$. By Lemma 2.11, this implies $(b_0 \wedge \cdots \wedge b_n) \to b = 1$ and $(c_0 \wedge \cdots \wedge c_m) \to c = 1$. Then we have $b_0 \wedge \cdots \wedge b_n \wedge c_0 \wedge \cdots \wedge c_m \leq b \wedge c$. Since $b_0, \dots, b_n, c_0, \dots, c_m \in F \cup \uparrow a$ and F and $\uparrow a$ are both closed under meets, we have $d_1 \in F$ and $d_2 \in \uparrow a$ such that $b_0 \wedge \cdots \wedge b_n \wedge c_0 \wedge \cdots \wedge c_m = d_1 \wedge d_2 \leq b \wedge c$. Moreover, by definition of an absorbent filter, $d_2 \to (d_1 \wedge d_2) \in F \subseteq \langle F \cup \uparrow a\rangle$. Since $d_2 \in \uparrow a \subseteq \langle F \cup \uparrow a\rangle$, by definition of an implicative filter, we obtain $d_1 \wedge d_2 \in \langle F \cup \uparrow a\rangle$, as required.

For the converse, let $F \in \mathrm{Fi}(\mathbf{A})$ be such that for all $a \in A$, $\langle F \cup \uparrow a\rangle$ is a meet filter. We show that F is absorbent. Let $b \in F$ and $a \in A$. We prove that $a \to (a \wedge b) \in F$. Notice first that $\langle F \cup \uparrow a\rangle = \langle F \cup \{a\}\rangle$. As $a \in \uparrow a$ and $b \in F$, we have by hypothesis that $a \wedge b \in \langle F \cup \uparrow a\rangle$. Now we use the definition of a generated implicative filter, and we get that there are $c_0, \dots, c_n \in F$, for some $n \in \omega$, such that $c_0 \to (c_1 \to (\cdots(c_n \to (a \to (a \wedge b)))\cdots)) = 1$. But this implies that $a \to (a \wedge b) \in F$, as required. □

4. Representation theorem for $DH^{\wedge}$-algebras

In this section, we shall define spectral-like dual objects of $DH^{\wedge}$-algebras, called $DH^{\wedge}$-spaces, and we shall prove that any $DH^{\wedge}$-algebra can be represented by means of a $DH^{\wedge}$-space. Recall that if $\langle X, \tau\rangle$ is a topological space and Y is a subset of X, then the family $\{U \cap Y : U \in T\}$ of subsets of Y is a topology for Y called the relative topology inherited from $\langle X, \tau\rangle$, or the subspace topology on Y. If Y is equipped with the subspace topology, then it is a topological space in its own right, and is called a subspace of $\langle X, \tau\rangle$. Subsets of topological spaces are usually assumed to be equipped with the subspace topology unless otherwise stated.

Definition 4.1. A $DH^{\wedge}$*-space* is a triple $\langle X, \kappa, \hat{X}\rangle$ such that $\hat{X}$ is a subset of X, and

- $(DH^{\wedge}1)$ $\langle X, \kappa\rangle$ is an H-space,
- $(DH^{\wedge}2)$ $\hat{X}$ is a DS-space under the subspace topology inherited from the topology τ_κ of the H-space $\langle X, \kappa\rangle$,
- $(DH^{\wedge}3)$ $\kappa = \left\{(\uparrow V)^c : V \in D(\hat{X})\right\}$.

Remark 4.2. Let $\langle X, \kappa, \hat{X}\rangle$ be a $DH^{\wedge}$-space. We need to be careful when dealing with complements, since we are working with two spaces at the same time. From now on we establish the following convention: complements V^c always refer to the set X. Therefore, the complement of $V \subseteq \hat{X}$ with respect to $\hat{X}$ is $V^c \cap \hat{X}$.

On the other hand, since $\hat{X} \in D(\hat{X})$, by condition $(DH^{\wedge}3)$, we have $(\uparrow\hat{X})^c = \uparrow(X \cap \hat{X})^c = \emptyset$. Thus, $X = \uparrow(X \cap \hat{X}) = \uparrow\hat{X}$.

Now we are left to define an operation on $D_\kappa(X) = \{U : U^c \in \kappa\}$ that aims to represent the meet operation. The following proposition will be useful for this purpose. Notice that by the definition of generated subspace, the family $\{U \cap \hat{X} : U \in \kappa\}$ is a base for the subspace $\hat{X}$.

Proposition 4.3. *Let $\langle X, \kappa, \hat{X}\rangle$ be a $DH^\wedge$-space.*

(1) $U^c = \uparrow(U^c \cap \hat{X})$*, for each* $U \in \kappa$.
(2) $(\uparrow(U_1^c \cap \cdots \cap U_n^c \cap \hat{X}))^c \in \kappa$*, for every finite subset* $\{U_1, \ldots, U_n\}$ *of* κ.
(3) $\mathcal{KO}(\hat{X}) = c$.

Proof. (1): Let $U \in \kappa$. Then by condition $(DH^\wedge 3)$, there exists $V \in D(\hat{X})$ such that $U = (\uparrow V)^c$. So, $\uparrow(U^c \cap \hat{X}) = \uparrow((\uparrow V) \cap \hat{X}) = \uparrow(V \cap \hat{X}) = \uparrow V = U^c$.

(2): Let $U_1, \ldots, U_n \in \kappa$. Then there exist $V_1, \ldots, V_n \in D(\hat{X})$ such that $U_1 = (\uparrow V_1)^c, \ldots, U_n = (\uparrow V_n)^c$. So,

$$(\uparrow(U_1^c \cap \cdots \cap U_n^c \cap \hat{X}))^c = (\uparrow((\uparrow V_1) \cap \cdots \cap (\uparrow V_1) \cap \hat{X}))^c$$
$$= (\uparrow(V_1 \cap \cdots \cap V_1 \cap \hat{X}))^c = (\uparrow(V_1 \cap \cdots \cap V_1))^c \in \kappa.$$

(3): Note that $\{U \cap \hat{X} : U \in \kappa\} = \left\{\hat{X} \cap (\uparrow V)^c : V^c \in \mathcal{KO}(\hat{X})\right\} = \mathcal{KO}(\hat{X})$, because $V = \hat{X} \cap (\uparrow V)^c$ for each $V^c \in \mathcal{KO}(\hat{X})$. □

For any $DH^\wedge$-space $\langle X, \kappa, \hat{X}\rangle$, the structure $\langle D(\hat{X}), \cap, \hat{X}\rangle$ is a distributive semilattice, where by Proposition 4.3, $D(\hat{X}) = \{U \cap \hat{X} : U \in D_\kappa(X)\}$. Item (2) of Proposition 4.3 guarantees that we can lift to $D_\kappa(X)$ the meet operation on $D(\hat{X})$ given by intersection, and come up with a binary operation $\sqcap$ on $D_\kappa(X)$, given by

$$U \sqcap V = \uparrow(U \cap V \cap \hat{X}).$$

It is not difficult to see that $\langle D_\kappa(X), \sqcap, X\rangle$ is isomorphic to $\langle D(\hat{X}), \cap, \hat{X}\rangle$ by means of the map $\gamma\colon D_\kappa(X) \longrightarrow D(\hat{X})$, given by $\gamma(U) = U \cap \hat{X}$. Clearly, γ is a surjective map such that $\gamma(X) = \hat{X}$, and from Proposition 4.3, it follows that it is injective. Moreover, from $U, V \in D_\kappa(X)$ being up-sets and by item (2) of Proposition 4.3, we get

$$\gamma(U \sqcap V) = (\uparrow(U \cap V \cap \hat{X})) \cap \hat{X} = U \cap V \cap \hat{X} = \gamma(U) \cap \gamma(V).$$

Proposition 4.4. *Let $\langle X, \kappa, \hat{X}\rangle$ be a $DH^\wedge$-space. Then for all $U, V \in D_\kappa(X)$, $U \Rightarrow V = X$ if and only if $U \sqcap V = U$.*

Proof. Let $U, V \in D_\kappa(X)$. By definition of $\Rightarrow$, we have that $U \Rightarrow V = X$ if and only if $U \subseteq V$. Then we show that $U \subseteq V$ if and only if $U \sqcap V = U$. By item (1) of Proposition 4.3, if $U \subseteq V$, then $U \sqcap V = \uparrow(U \cap V \cap \hat{X}) = \uparrow(U \cap \hat{X}) = U$. The converse is immediate because $U \sqcap V = \uparrow(U \cap V \cap \hat{X}) \subseteq \uparrow(V \cap \hat{X}) = V$. □

Corollary 4.5. *Let $\langle X, \kappa, \hat{X}\rangle$ be a $DH^\wedge$-space. Then $\langle D_\kappa(X), \Rightarrow, \sqcap, X\rangle$ is a $DH^\wedge$-algebra.*

Given a $DH^{\wedge}$-space $\langle X, \kappa, \hat{X}\rangle$, the $DH^{\wedge}$-algebra $\langle D_{\kappa}(X), \Rightarrow, \sqcap, X\rangle$ will be called the *dual* $DH^{\wedge}$*-algebra* of $\mathbf{X}$.

Now we provide a construction that shows that any $DH^{\wedge}$-algebra $\mathbf{A}$ is (isomorphic to) the dual $DH^{\wedge}$-algebra of some $DH^{\wedge}$-space.

Let $\mathbf{A}$ be a $DH^{\wedge}$-algebra. Let $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$ be the dual H-space of $\langle A, \rightarrow, 1\rangle$. As $\kappa_{\mathbf{A}}$ is a base for a topology $\tau_{\kappa_{\mathbf{A}}}$on $X(\mathbf{A})$, we have that the family,

$$\{U \cap \hat{X}(\mathbf{A}) : U \in \kappa_{\mathbf{A}}\} = \{\varphi(a)^c \cap \hat{X}(\mathbf{A}) : a \in A\},$$

is a base for the induced topology $\tau_{\hat{X}(\mathbf{A})}$ on $\hat{X}(\mathbf{A})$. But as

$$\varphi(a)^c \cap \hat{X}(\mathbf{A}) = \{G \in \hat{X}(\mathbf{A}) : a \notin G\} = \sigma(a)^c,$$

for each $a \in A$, we have that $\tau_{\hat{X}(\mathbf{A})} = \tau_{\mathbf{A}}$, i.e., $\langle \hat{X}(\mathbf{A}), \tau_{\hat{X}(\mathbf{A})}\rangle = \left\langle \hat{X}(\mathbf{A}), \tau_{\mathbf{A}}\right\rangle$ is the dual DS-space of $\langle A, \wedge, 1\rangle$.

Proposition 4.6. *Let* $\mathbf{A}$ *be a* $DH^{\wedge}$*-algebra. Then* $\varphi(a) = \uparrow\sigma(a)$*, for each* $a \in A$.

Proof. Let $P \in X(\mathbf{A})$. Assume that $a \in P$. As P^c is an order ideal such that $a \notin P^c$, we have by Lemma 2.2, there exists $Q \in \hat{X}(\mathbf{A})$ such that $a \in Q \subseteq P$. So, $Q \in \sigma(a)$ and $Q \subseteq P$. Hence, $P \in \uparrow\sigma(a)$. As $\sigma(a) \subseteq \varphi(a)$ and $\varphi(a)$ is an up-set, $\uparrow\sigma(a) \subseteq \varphi(a)$. □

Theorem 4.7. *Let* $\mathbf{A}$ *be a* $DH^{\wedge}$*-algebra. Then*

$$\left\langle X(\mathbf{A}), \kappa_{\mathbf{A}}, \hat{X}(\mathbf{A})\right\rangle$$

is a $DH^{\wedge}$*-space and the map* $\varphi\colon A \longrightarrow \mathcal{P}^{\uparrow}(X(\mathbf{A}))$ *is an isomorphism between the* $DH^{\wedge}$*-algebras* $\mathbf{A}$ *and* $< D_{\kappa}(X(\mathbf{A})), \Rightarrow, \sqcap, X(\mathbf{A}) >$.

Proof. That $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}, \hat{X}(\mathbf{A})\rangle$ is a $DH^{\wedge}$-space follows from the previous proposition and the spectral-like duality for Hilbert algebras and distributive semilattices, as was already remarked. It also follows that φ is an isomorphism of Hilbert algebras $\langle A, \rightarrow, 1\rangle$ and $\langle D_{\kappa_{\mathbf{A}}}(X(\mathbf{A})), \Rightarrow, X(\mathbf{A})\rangle$. Moreover, it follows from the definition and item (2) of Proposition 4.6, that

$$\varphi(a) \sqcap \varphi(c) = \uparrow(\varphi(a) \cap \varphi(c) \cap \hat{X}(\mathbf{A})) = \varphi(a \wedge c).$$

Thus, φ is an isomorphism of meet semilattices. □

Given a $DH^{\wedge}$-algebra $\mathbf{A}$, the $DH^{\wedge}$-space $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}, \hat{X}(\mathbf{A})\rangle$ will be called the *dual* $DH^{\wedge}$*-space* of $\mathbf{A}$.

Recall that given a $DH^{\wedge}$-space $\langle X, \kappa, \hat{X}\rangle$, by the results on duality for Hilbert algebras, the map

$$\varepsilon_X : X \longrightarrow X(D_{\kappa}(X)), \text{ given by } \varepsilon_X(x) = \{U \in D_{\kappa}(X) : x \in U\},$$

is a homeomorphism between the H-spaces $\langle X, \kappa\rangle$ and $\langle X(\mathbf{D}_{\kappa}(X)), \kappa_{\mathbf{D}_{\kappa}(X)}\rangle$. Moreover, by the duality for distributive semilattices, we get that the map $\hat{\varepsilon}_{\hat{X}} : \hat{X} \longrightarrow \hat{X}(\mathbf{D}(\hat{X}))$, given by

$$\hat{\varepsilon}_{\hat{X}}(x) = \{U \in D(\hat{X}) : x \in U\} = \{V \cap \hat{X} : x \in V \in D_{\kappa}(X)\} = \gamma[\varepsilon_X(x)],$$

is a homeomorphism between the DS-spaces $\hat{X}$ and $\hat{X}(\mathbf{D}(\hat{X}))$.

Theorem 4.8. *Let $\langle X, \kappa, \hat{X}\rangle$ be a $DH^{\wedge}$-space. Then $\varepsilon_X[\hat{X}] = \hat{X}(\mathbf{D}(\hat{X}))$.*

Proof. Notice that $\gamma[\varepsilon_X[\hat{X}]] = \hat{\varepsilon}_{\hat{X}}[\hat{X}] = \hat{X}(\mathbf{D}(\hat{X})) = \gamma[\hat{X}(\mathbf{D}(\hat{X}))]$. Since γ is an isomorphism between $\langle D_\kappa(X), \sqcap, X\rangle$ and $\mathbf{D}(\hat{X})$, we conclude that $\varepsilon_X[\hat{X}] = \hat{X}(\hat{\mathbf{D}}(X))$. □

5. Categorical duality

We now extend the topological representation studied in the previous section to a dual equivalence of categories. Following the same approach as in [7], we consider two different categories with $DH^{\wedge}$-algebras as objects. The morphisms we consider are of algebraic homomorphisms, and a weaker notion that naturally extends the notion of *$\rightarrow$-semi-homomorphism* between Hilbert algebras introduced in [5].

Definition 5.1. A *semi-homomorphism* between two $DH^{\wedge}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ is a map $h\colon A_1 \longrightarrow A_2$ such that for all $a, b \in A_1$,

(1) $h(1_1) = 1_2$,
(2) $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$,
(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$.

If moreover h satisfies $h(a) \rightarrow_2 h(b) \leq h(a \rightarrow_1 b)$, then it is called a *homomorphism.*

Recall that we call *$\rightarrow$-semi-homomorphism* those maps between Hilbert algebras that satisfy conditions (1) and (2) in previous definition. Thus, a semi-homomorphism is a $\rightarrow$-semi-homomorphism and it is a homomorphism with respect to the meet.

Definition 5.2. A relation $R \subseteq X_1 \times X_2$ is a *$DH^{\wedge}$-morphism* between the $DH^{\wedge}$-spaces $\langle X_1, \tau_{\kappa_1}, \hat{X}_1\rangle$ and $\langle X_2, \tau_{\kappa_2}, \hat{X}_2\rangle$ if R is an H-relation between the H-spaces $\langle X_1, \tau_{\kappa_1}\rangle$ and $\langle X_2, \tau_{\kappa_2}\rangle$, and ($DH^{\wedge}$M) $R(x) = \uparrow(R(x) \cap \hat{X}_2)$, for every $x \in \hat{X}_1$.

By the spectral-like duality for Hilbert algebras, for any $DH^{\wedge}$-morphism $R \subseteq X_1 \times X_2$ between the $DH^{\wedge}$-spaces $\langle X_1, \kappa_1, \hat{X}_1\rangle$ and $\langle X_2, \kappa_2, \hat{X}_2\rangle$, the function $\Box_R\colon D_{\kappa_2}(X_2) \longrightarrow D_{\kappa_1}(X_1)$, given by

$$\Box_R(U) = \{x \in X_1 : R(x) \subseteq U\},$$

is a $\rightarrow$-semi-homomorphism of Hilbert algebras. We also get (see [7, Example 3.1] that for a $DH^{\wedge}$-space $\langle X, \kappa, \hat{X}\rangle$, the order $\leq$ on X, given by the dual of the specialization order, is a functional H-relation. Notice that for all $x \in \hat{X}$, $\uparrow x = \uparrow(\uparrow x \cap \hat{X})$. Therefore, the relation $\leq$ also satisfies the condition ($DH^{\wedge}$M), and hence it is a $DH^{\wedge}$-morphism. Furthermore, it follows easily that $\Box_{\leq} = \mathrm{id}_{D_\kappa(X)}$. It is also easy to see that for all $DH^{\wedge}$-spaces $\langle X_i, \tau_{\kappa_i}, \hat{X}_i\rangle$

with $1 \leq i \leq 3$, and for all $DH^{\wedge}$-morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, we have that $\Box_{S \circ R} = \Box_R \circ \Box_S$.

Proposition 5.3. *Suppose $R \subseteq X_1 \times X_2$ is a $DH^{\wedge}$-morphism between the $DH^{\wedge}$-spaces $\langle X_1, \tau_{\kappa_1}, \hat{X}_1 \rangle$ and $\langle X_2, \tau_{\kappa_2}, \hat{X}_2 \rangle$. Then for all $U, V \in D_{\kappa_2}(X_2)$,*

$$\Box_R(\uparrow(U \cap V \cap \hat{X}_2)) = \uparrow(\Box_R(U) \cap \Box_R(V) \cap \hat{X}_1).$$

Proof. Let $U, V \in D_{\kappa_2}(X_2)$. First we show the inclusion from left to right. Let $x \in \Box_R(\uparrow(U \cap V \cap \hat{X}_2))$. By item (1) of Proposition 4.3, we know that $\Box_R(\uparrow(U \cap V \cap \hat{X}_2)) = \uparrow(\Box_R(\uparrow(U \cap V \cap \hat{X}_2)) \cap \hat{X}_1)$. Then there is $y \in \hat{X}_1$ such that $y \in \Box_R(\uparrow(U \cap V \cap \hat{X}_2))$ and $y \leq x$. By definition, we have that $R(y) \subseteq \uparrow(U \cap V \cap \hat{X}_2)$. From U, V being up-sets, it follows that $R(y) \subseteq U \cap V$, i.e., $y \in \Box_R(U) \cap \Box_R(V)$. From the assumption $x \geq y \in \hat{X}_1$, it follows that $x \in \uparrow(\Box_R(U) \cap \Box_R(V) \cap \hat{X}_1)$, as required.

Let us show now the reverse inclusion. Since $\Box_R(\uparrow(U \cap V \cap \hat{X}_2))$ is an up-set, it is enough to show that $\Box_R(U) \cap \Box_R(V) \cap \hat{X}_1 \subseteq \Box_R(\uparrow(U \cap V \cap \hat{X}_2))$. Let $x \in \Box_R(U) \cap \Box_R(V) \cap \hat{X}_1$, i.e., $R(x) \subseteq U \cap V$ and $x \in \hat{X}_1$. In order to show that $R(x) \subseteq \uparrow(U \cap V \cap \hat{X}_2)$, let $y \in R(x)$. By condition ($DH^{\wedge}$M), we know that $R(x) = \uparrow(R(x) \cap \hat{X}_2)$. Thus, there is $y' \in R(x) \cap \hat{X}_2$ such that $y' \leq y$. By assumption, $y' \in U \cap V$, so, $y' \in U \cap V \cap \hat{X}_2$, and therefore $y \in \uparrow(U \cap V \cap \hat{X}_2)$. Hence, we have proved that $R(x) \subseteq \uparrow(U \cap V \cap \hat{X}_2)$, i.e., $x \in \Box_R(\uparrow(U \cap V \cap \hat{X}_2))$, as required. $\Box$

Corollary 5.4. *Suppose that $R \subseteq X_1 \times X_2$ is a $DH^{\wedge}$-morphism between the $DH^{\wedge}$-spaces $\langle X_1, \tau_{\kappa_1}, \hat{X}_1 \rangle$ and $\langle X_2, \tau_{\kappa_2}, \hat{X}_2 \rangle$. Then $\Box_R$ is a semi-homomorphism between the $DH^{\wedge}$-algebras $D_{\kappa_2}(X_2)$ and $D_{\kappa_1}(X_1)$.*

Corollary 5.4 provides the *dual semi-homomorphism* of a $DH^{\wedge}$-morphism. Before showing how to define a $DH^{\wedge}$-morphism from a semi-homomorphism, we consider a different class of relations between $DH^{\wedge}$-spaces. Following [7], we call a $DH^{\wedge}$-morphism *functional* when it satisfies condition (HF). The next corollary follows straightforwardly from the above results and the duality for Hilbert algebras.

Corollary 5.5. *Let $R \subseteq X_1 \times X_2$ be a $DH^{\wedge}$-functional morphism between the $DH^{\wedge}$-spaces $\langle X_1, \tau_{\kappa_1}, \hat{X}_1 \rangle$ and $\langle X_2, \tau_{\kappa_2}, \hat{X}_2 \rangle$. Then $\Box_R$ is a homomorphism between the dual $DH^{\wedge}$-algebras of $D_{\kappa_2}(X_2)$ and $D_{\kappa_1}(X_1)$.*

Recall that by spectral-like duality for Hilbert algebras, we have that for any semi-homomorphism $h\colon A_1 \longrightarrow A_2$ between $DH^{\wedge}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, the relation $R_h \subseteq X(\mathbf{A}_1) \times X(\mathbf{A}_2)$, given by $(P, Q) \in R_h$ iff $h^{-1}[P] \subseteq Q$, is an H-relation between the dual H-spaces of $\langle A_2, \rightarrow_2, 1_2 \rangle$ and $\langle A_1, \rightarrow_1, 1_1 \rangle$. It also follows that for the identity morphism $\mathrm{id}_{\mathbf{A}} : A \longrightarrow A$ for an $DH^{\wedge}$-algebra $\mathbf{A}$, we have $R_{\mathrm{id}_{\mathbf{A}}} = \subseteq$, and for $DH^{\wedge}$-algebras $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ and semi-homomorphisms $h\colon A_1 \longrightarrow A_2$ and $g\colon A_2 \longrightarrow A_3$, we have $R_{g \circ h} = R_g \circ R_h$.

Lemma 5.6. *Suppose $h\colon A_1 \longrightarrow A_2$ is a semi-homomorphism between the $DH^{\wedge}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$. Then for any $P \in \mathrm{Fi}(\mathbf{A}_2)$, $h^{-1}[P] \in \mathrm{Fi}(\mathbf{A}_1)$.*

Proof. Let us show that $h^{-1}[P]$ is a meet filter. Let $a \in h^{-1}[P]$ and $b \in A_1$ such that $a \leq_1 b$. Then $a \to_1 b = 1_1$, so $1_2 = h(1_1) = h(a \to_1 b) \leq h(a) \to_2 h(b)$. Therefore, $h(a) \leq_2 h(b)$, and by assumption, $h(a) \in P$. As P is an up-set, so $h(b) \in P$, and so $b \in h^{-1}[P]$. This shows that $h^{-1}[P]$ is an up-set. Let $a, b \in h^{-1}[P]$. Then $h(a), h(b) \in P$, and since P is a meet filter and $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$, we have $h(a \wedge_1 b) \in P$. So, $a \wedge_1 b \in h^{-1}[P]$. This shows that $h^{-1}[P]$ is closed under meets. □

Proposition 5.7. *Suppose $h\colon A_1 \longrightarrow A_2$ is a semi-homomorphism between $DH^{\wedge}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$; then $R_h \subseteq X(\mathbf{A}_2) \times X(\mathbf{A}_1)$ is a $DH^{\wedge}$-morphism.*

Proof. Let $P \in \hat{X}(\mathbf{A}_2)$. We prove that $R_h(P) = \uparrow(R_h(P) \cap \hat{X}(\mathbf{A}_1))$. Since R_h is an H-relation, we know that $R_h(P)$ is a closed subset of the H-space $\langle X(A), \kappa_{\mathbf{A}}\rangle$. So, $R_h(P)$ is an up-set. Thus, $\uparrow(R_h(P) \cap \hat{X}(\mathbf{A}_1)) \subseteq R_h(P)$. Let us show the reverse inclusion. Let $Q \in R_h(P)$, i.e., $h^{-1}[P] \subseteq Q$. By Lemma 5.6, $h^{-1}[P] \in \mathrm{Fi}(\mathbf{A}_1)$. So $h^{-1}[P] \cap Q^c = \emptyset$. As Q^c is an order ideal and $h^{-1}[P]$ a meet filter, by Lemma 2.2 we have that there is $Q' \in \hat{X}(\mathbf{A}_1)$ such that $h^{-1}[P] \subseteq Q'$ and $Q' \cap Q^c = \emptyset$. Then Q' is the required element such that $Q' \in R_h(P) \cap \hat{X}(\mathbf{A}_1)$ and $Q' \subseteq Q$, i.e., $Q \in \uparrow(R_h(P) \cap \hat{X}(\mathbf{A}_1))$. □

Corollary 5.8. *Suppose $h\colon A_1 \longrightarrow A_2$ is a semi-homomorphism between the $DH^{\wedge}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$. Then R_h is a $DH^{\wedge}$-morphism between the dual $DH^{\wedge}$-spaces of $\mathbf{A}_2$ and $\mathbf{A}_1$. Moreover, if h is a homomorphism, then R_h is functional.*

5.1. Dual equivalences of categories. We show first that$DH^{\wedge}$-spaces are taken as objects and $DH^{\wedge}$-morphisms as morphisms, we obtain indeed a category. As a corollary, we get that $DH^{\wedge}$-spaces and $DH^{\wedge}$-functional morphisms form a subcategory of the former.

Theorem 5.9. *Suppose that $\langle X_1, \tau_{\kappa_1}, \hat{X}_1\rangle$, $\langle X_2, \tau_{\kappa_2}, \hat{X}_2\rangle$ and $\langle X_3, \tau_{\kappa_3}, \hat{X}_3\rangle$ are $DH^{\wedge}$-spaces; let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be two $DH^{\wedge}$-morphisms.*

(1) *The $DH^{\wedge}$-morphism $\leq_2 \subseteq X_2 \times X_2$ satisfies the conditions $\leq_2 \circ R = R$, and $S \circ \leq_2 = S$.*
(2) *$S \circ R \subseteq X_1 \times X_3$ is a $DH^{\wedge}$-morphism.*
(3) *If R, S are functional, then so is $S \circ R$.*

Proof. (1): This has been proven in [7, Theorem 3.1] for H-relations, so it holds particularly for $DH^{\wedge}$-morphisms.

(2): By [7, Theorem 3.1], we get that $S \circ R$ is an H-relation. We just have to show that $S \circ R$ satisfies condition ($DH^{\wedge}$M), i.e., to show that for all $x \in \hat{X}_1$, $(S \circ R)(x) = \uparrow(S \circ R)(x) \cap \hat{X}_3)$. Let $x \in \hat{X}_1$. First we prove that $(S \circ R)(x)$ is an up-set. Let $z \in (S \circ R)(x)$ and let $z \leq_3 w$ for some $w \in X_3$. By definition, there is $y \in X_2$ such that $y \in R(x)$ and $z \in S(y)$. By condition ($DH^{\wedge}$M) on R, we have $R(x) = \uparrow(R(x) \cap \hat{X}_2)$. Then there is $y' \in R(x) \cap \hat{X}_2$ such that $y' \leq_2 y$. Now since $S \circ \leq_2 = S$, we get $z \in S(y')$. As $y' \in \hat{X}_2$, by condition

$(DH^{\wedge}\mathrm{M})$ on S, $S(y') = {\uparrow}(S(y') \cap \hat{X}_3)$. Then there is $z' \in S(y') \cap \hat{X}_3$ such that $z' \leq_3 z \leq_3 w$. Therefore, we have $w \in S(y')$, and then from $y' \in R(x)$, we get $w \in (S \circ R)(x)$.

From $(S \circ R)(x)$ being an up-set, we have that ${\uparrow}(S{\circ}R)(x){\cap}\hat{X}_3 \subseteq (S \circ R)(x)$. For the other inclusion, let $z \in (S \circ R)(x)$. By a similar argument as before, we conclude that there is $z' \in (S \circ R)(x) \cap \hat{X}_3$ such that $z' \leq z$. Therefore, $z \in {\uparrow}(S \circ R(x) \cap \hat{X}_3)$.

(3) This follows from item (2) and results in [7, Section 5.2]. □

Corollary 5.10. *$DH^{\wedge}$-spaces and $DH^{\wedge}$-morphisms form a category.*

Proof. For a $DH^{\wedge}$-space $\langle X, \kappa, \hat{X}\rangle$, we already pointed out that the order $\leq$ on X given by the dual of the specialization order, is a $DH^{\wedge}$-morphism. Then by item (1) in Theorem 5.9, it is the identity morphism on $\langle X, \kappa, \hat{X}\rangle$. By item (2) in Theorem 5.9, the relational composition works as composition between $DH^{\wedge}$-morphisms. □

Corollary 5.11. *$DH^{\wedge}$-spaces and $DH^{\wedge}$-functional morphisms form a category.*

Proof. This follows from the previous corollary and item (3) in Theorem 5.9. □

Let us consider the following categories:

$\mathsf{DH}^{\wedge}_{S}$	$DH^{\wedge}$-algebras and semi-homomorphisms,
$\mathsf{DH}^{\wedge}_{H}$	$DH^{\wedge}$-algebras and homomorphisms,
$\mathsf{Sp}^{DH^{\wedge}}_{M}$	$DH^{\wedge}$-spaces and $DH^{\wedge}$-morphisms,
$\mathsf{Sp}^{DH^{\wedge}}_{F}$	$DH^{\wedge}$-spaces and $DH^{\wedge}$-functional morphisms.

We now complete the dualities by exhibiting the contravariant functors and the natural isomorphisms involved in them.

By previous results, we define a contravariant functor $(\,)_*\colon \mathsf{DH}^{\wedge}_{S} \to \mathsf{Sp}^{DH^{\wedge}}_{M}$ as follows. For any $DH^{\wedge}$-algebras $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2$ and any semi-homomorphism $h\colon A_1 \longrightarrow A_2$,

$$\mathbf{A}_* = \langle X(\mathbf{A}), \kappa_{\mathbf{A}}, \hat{X}(\mathbf{A})\rangle \quad \text{and} \quad h_* = R_h.$$

Similarly, we shall define the contravariant functor $(\,)^*\colon \mathsf{Sp}^{DH^{\wedge}}_{M} \to \mathsf{DH}^{\wedge}_{S}$ as follows. For any $DH^{\wedge}$-spaces $\langle X, \kappa, \hat{X}\rangle$, $\langle X_1, \kappa_1, \hat{X}_1\rangle$, and $\langle X_2, \kappa_2, \hat{X}_2\rangle$ and any $DH^{\wedge}$-morphism $R \subseteq X_1 \times X_2$,

$$(\langle X, \kappa, \hat{X}\rangle)^* = \langle D_\kappa(X), \Rightarrow, \sqcap, X\rangle \quad \text{and} \quad R^* = \Box_R.$$

Consider now the following family of morphisms in $\mathsf{DH}^{\wedge}_{S}$:

$$\varphi = \left(\varphi_{\mathbf{A}}\colon A \longrightarrow D_{\kappa_{\mathbf{A}}}(X(\mathbf{A}))\right)_{\mathbf{A}\in\mathsf{DH}^{\wedge}_{S}}.$$

Lemma 5.12. *φ is a natural isomorphism between the identity functor in $\mathsf{DH}^{\wedge}_{S}$ and $((\,)_*)^*$.*

Proof. Suppose that $\mathbf{A}_1, \mathbf{A}_2$ are two $DH^\wedge$-algebras, and let $h\colon A_1 \longrightarrow A_2$ be a semi-homomorphism between them. Then by [7, Lemma 3.5], we get that $\square_{R_h} \circ \varphi_{\mathbf{A}_1} = \varphi_{\mathbf{A}_2} \circ h$, and then by Theorem 4.7 we get that for all $\mathbf{A} \in \mathsf{DH}^\wedge_S$, $\varphi_{\mathbf{A}}$ is an isomorphism. □

For any $DH^\wedge$-space $\langle X, \kappa, \hat{X}\rangle$, we defined the map $\varepsilon_X\colon X \longrightarrow X(\mathbf{D}_\kappa(X))$. Associated with this map, we define now the relation $\psi_X \subseteq X \times X(\mathbf{D}_\kappa(X)$ as follows:

$$(x, P) \in \psi_X \ \text{ iff } \ \varepsilon_X(x) \subseteq P.$$

Lemma 5.13. *ψ_X is a $DH^\wedge$-functional morphism.*

Proof. By the results of [7], we know that ψ_X is a functional H-relation, so we just have to check that condition ($DH^\wedge$M) of Definition 5.2 is satisfied. Let $x \in \hat{X}$. It is immediate that $\uparrow(\psi_X(x) \cap \hat{X}(\mathbf{D}(\hat{X}))) \subseteq \psi_X(x)$. Let $P \in \psi_X(x)$, i.e., $\varepsilon_X(x) \subseteq P$. By Theorem 4.8, we know that $\varepsilon_X(x) \in \hat{X}(\mathbf{D}(\hat{X}))$, and clearly $\varepsilon_X(x) \in \psi_X(x)$. Therefore, $P \in \uparrow(\psi_X(x) \cap \hat{X}(\mathbf{D}(\hat{X}))$, as required. □

Consider now the following family of morphisms in $\mathsf{Sp}_M^{DH^\wedge}$:

$$\psi = \big(\psi_X \subseteq X \times X(\mathbf{D}_\kappa(X))\big)_{\langle X,\kappa,\hat{X}\rangle \in \mathsf{Sp}_M^{DH^\wedge}}.$$

Lemma 5.14. *ψ is a natural isomorphism between then identity functor in $\mathsf{Sp}_M^{DH^\wedge}$ and $((\,)^*)_*$.*

Proof. Suppose that $\langle X_1, \tau_{\kappa_1}, \hat{X}_1\rangle$ and $\langle X_2, \tau_{\kappa_2}, \hat{X}_2\rangle$ are two $DH^\wedge$-spaces, and let $R \subseteq X_1 \times X_2$ be a $DH^\wedge$-morphism between them. By [7, Lemma 3.4], we get that $(x, y) \in R$ iff $(\varepsilon_{X_1}(x), \varepsilon_{X_2}(y)) \in R_{\square_R}$, and from this it follows that $R_{\square_R} \circ \psi_{X_1} = \psi_{X_2} \circ R$. Moreover, by Theorem 4.8, we have that ε_X is an homeomorphism such that $\varepsilon_X[\hat{X}] = \hat{X}(\mathbf{D}(\hat{X}))$. This implies, together with results from [7, Theorem 3.2], that E_X is an isomorphism in $\mathsf{Sp}_M^{DH^\wedge}$, as required. □

Theorem 5.15. *The categories $\mathsf{Sp}_M^{DH^\wedge}$ and $\mathsf{DH}^\wedge_S$ are dually equivalent by means of the contravariant functors $(\,)_*$ and $(\,)^*$ and the natural equivalences φ and ψ. Similarly, the categories $\mathsf{Sp}_F^{DH^\wedge}$ and $\mathsf{DH}^\wedge_H$ are dually equivalent.*

Remark 5.16. A subclass of $DH^\wedge$-algebras already mentioned are the implicative semilattices (or IS-algebras). A duality for IS-algebras was studied in [5], where *IS-spaces* are defined as those DS-spaces $\langle X, \tau\rangle$ that satisfy the following condition:

(IS) for all $U, V \in \mathcal{KO}(X)$, $\mathrm{sat}(U \cap V^c) \in \mathcal{KO}(X)$.

Notice that condition (IS) is similar to condition (H) of definition of H-space. It is clear that the duality given in [5] is a particular case of the one presented here when $X = \hat{X}$ and $\kappa = \mathcal{KO}(X)$.

6. Topological characterization of filters

In [7, Section 5], the authors give a topological characterization of implicative filters of an H-algebra. Here we give a topological characterization of implicative filters, meet filters and absorbent filters of a $DH^{\wedge}$-algebra.

Let $\mathbf{A}$ be a $DH^{\wedge}$-algebra. Let $\mathcal{C}(X(\mathbf{A}))$ be the family of all closed subsets of the H-space $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$. We consider the following maps:

$$C_{(-)}\colon \mathrm{Fi}_{\rightarrow}(\mathbf{A}) \longrightarrow \mathcal{C}(X(\mathbf{A})) \quad \text{given by} \quad F \longmapsto C_F = \bigcap\{\varphi(a) : a \in F\},$$
$$F_{(-)}\colon \mathcal{C}(X(\mathbf{A})) \longrightarrow \mathrm{Fi}_{\rightarrow}(\mathbf{A}) \quad \text{given by} \quad C \longmapsto F_C = \{a \in A : C \subseteq \varphi(a)\}.$$

By [7, Proposition 5.1], we get that these maps are well defined, and moreover, they are inverses of each other. So, the ordered sets $\langle \mathrm{Fi}_{\rightarrow}(\mathbf{A}), \subseteq\rangle$ and $\langle \mathcal{C}(X(\mathbf{A})), \supseteq\rangle$ are order isomorphic. Let C be an irreducible closed subset $X(\mathbf{A})$. We prove that the implicative filter $F_C = \{a \in A : C \subseteq \varphi(a)\}$ is irreducible. If F_1,F_2 are implicative filters of $\mathbf{A}$ such that $F_1 \cap F_2 \subseteq F_C$, then $C \subseteq C_{F_1} \cup C_{F_2}$. As C is irreducible, $C \subseteq C_{F_1}$ or C_{F_2}. So, $F_1 \subseteq F_C$ or $F_2 \subseteq F_C$, i.e., F_C is irreducible. Thus, $\rightarrow$-irreducible implicative filters of $\mathbf{A}$ correspond to irreducible closed subsets of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$. We now identify which closed subsets of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$ correspond to meet filters of $\mathbf{A}$.

Proposition 6.1. *For any $DH^{\wedge}$-algebra $\mathbf{A}$, the following hold:*

(1) *If $F \in \mathrm{Fi}(\mathbf{A})$, then $C_F = {\uparrow}(C_F \cap \hat{X}(\mathbf{A}))$.*
(2) *If $C \in \mathcal{C}(X(\mathbf{A}))$ is such that $C = {\uparrow}(C \cap \hat{X}(\mathbf{A}))$, then $F_C \in \mathrm{Fi}(\mathbf{A})$.*

Proof. (1): For $F \in \mathrm{Fi}(\mathbf{A})$, as C_F is closed subset of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$, we have that it is an up-set. Thus, ${\uparrow}(C_F \cap \hat{X}(\mathbf{A})) \subseteq C_F$. Let $P \in X(\mathbf{A})$ be such that $P \in C_F$. Then $F \subseteq P$. Consider the order ideal P^c. We have $F \cap P^c = \emptyset$, so by Lemma 2.2, there is $F' \in \hat{X}(\mathbf{A})$ such that $F \subseteq F'$ and $F' \cap P^c = \emptyset$, i.e., $F' \subseteq P$. Thus, $F' \in C_F \cap \hat{X}(\mathbf{A})$, and therefore $P \in {\uparrow}(C_F \cap \hat{X}(\mathbf{A}))$.

(2): Let $C \in \mathcal{C}(X(\mathbf{A}))$ be a closed subset such that $C = {\uparrow}(C \cap \hat{X}(\mathbf{A}))$. Let $a, b \in F_C$. We show $a \wedge b \in F_C$. By assumption, $C \subseteq \varphi(a), \varphi(b)$. Then $C \cap \hat{X}(\mathbf{A}) \subseteq \varphi(a) \cap \varphi(b) \cap \hat{X}(\mathbf{A})$, and so

$$C = {\uparrow}(C \cap \hat{X}(\mathbf{A})) \subseteq {\uparrow}(\varphi(a) \cap \varphi(b) \cap \hat{X}(\mathbf{A})) = \varphi(a \wedge b).$$

Therefore, $a \wedge b \in F_C$. □

Recall that we have already considered the DS- space $\hat{X}(\mathbf{A}) = \langle \hat{X}(\mathbf{A}), \tau_{\mathbf{A}}\rangle$ where the topology $\tau_{\mathbf{A}}$ is generated by the base $\{\sigma(a) : a \in A\}$. Since $\hat{X}(\mathbf{A})$ is a subspace of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$, we have that C is a closed subset of $\hat{X}(\mathbf{A})$ if and only if $C = C' \cap \hat{X}(\mathbf{A})$ for some closed subset C' of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$. Therefore, by means of the same maps as before, we obtain that meet filters of $\mathbf{A}$ correspond to closed sets of $\langle \hat{X}(\mathbf{A}), \tau_{\mathbf{A}}\rangle$, and vice versa. As above, we can see that $\wedge$-irreducible meet filters correspond to irreducible closed subsets of the DS-space $\langle \hat{X}(\mathbf{A}), \tau_{\mathbf{A}}\rangle$.

Finally, we identify which closed sets of the H-space $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$ correspond to absorbent filters of $\mathbf{A}$.

Proposition 6.2. *For any $DH^{\wedge}$-algebra $\mathbf{A}$,*

(1) *If $F \in \mathrm{Ab}(\mathbf{A})$, then for all $a \in A$, $C_F \cap \varphi(a) = \uparrow(C_F \cap \varphi(a) \cap \hat{X}(\mathbf{A}))$.*
(2) *If C is a closed subset of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$ with $C \cap \varphi(a) = \uparrow(C \cap \varphi(a) \cap \hat{X}(\mathbf{A}))$ for all $a \in A$, then $F_C \in \mathrm{Ab}(\mathbf{A})$.*

Proof. (1): For $F \in \mathrm{Ab}(\mathbf{A})$ and $a \in A$, if $a \in F$, then $C_F \cap \varphi(a) = C_F$, and since F is a meet filter, Proposition 6.1 implies

$$C_F \cap \varphi(a) = C_F = \uparrow(C_F \cap \hat{X}(\mathbf{A})) = \uparrow(C_F \cap \varphi(a) \cap \hat{X}(\mathbf{A})).$$

Assume that $a \notin F$. Then by Proposition 3.6, we get that $\langle F \cup \uparrow a\rangle$ is a meet filter. Let $P \in C_F \cap \varphi(a)$, i.e., $\{a\} \cup F \subseteq P$. We show $P \in \uparrow(C_F \cap \varphi(a) \cap \hat{X}(\mathbf{A}))$. As $\langle F \cup \uparrow a\rangle$ is a meet filter and P^c is an order ideal such that $\langle F \cup \uparrow a\rangle \cap P^c = \emptyset$, we have by Lemma 2.2, that there is $F' \in \hat{X}(\mathbf{A})$ such that $F' \cap P^c = \emptyset$ and $\langle F \cup \uparrow a\rangle \subseteq F'$, i.e., $\{a\} \cup F \subseteq F' \subseteq P$. Therefore, $F' \in C_F \cap \varphi(a) \cap \hat{X}(\mathbf{A})$, and consequently $P \in \uparrow(C_F \cap \varphi(a) \cap \hat{X}(\mathbf{A}))$, as required.

(2): For C a closed subset of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$ with $C \cap \varphi(a) = \uparrow(C \cap \varphi(a) \cap \hat{X}(\mathbf{A}))$ for all $a \in A$, we show that F_C is an absorbent filter, i.e., we show that for any $b \in F_C$ and $c \in A$, $c \to (b \wedge c) \in F_C$. By definition, we have to show that we have $C \subseteq \varphi(c \to (b \wedge c)) = \varphi(c) \Rightarrow (\varphi(b) \sqcap \varphi(c))$. By assumption, we have that $C \cap \varphi(c) \subseteq \uparrow(C \cap \varphi(c) \cap \hat{X}(\mathbf{A}))$ and $C \subseteq \varphi(b)$. Then we have that $C \cap \varphi(c) \subseteq \uparrow(\varphi(b) \cap \varphi(c) \cap \hat{X}(\mathbf{A})) = \varphi(b) \sqcap \varphi(c)$, so $C \cap \varphi(c) \cap (\varphi(b) \sqcap \varphi(c))^c = \emptyset$. Since C is a closed subset of $\langle X(\mathbf{A}), \kappa_{\mathbf{A}}\rangle$, it is an up-set, so this implies that $C \cap \mathrm{sat}(\varphi(c) \cap (\varphi(b) \sqcap \varphi(c))^c) = \emptyset$, i.e.,

$$C \subseteq (\mathrm{sat}(\varphi(c) \cap (\varphi(b) \sqcap \varphi(c))^c))^c = \varphi(a \to (b \wedge a)),$$

as required. □

Finally, by means of the same map as before, we obtain that absorbent filters of $\mathbf{A}$ correspond to closed sets C of the DS-space $\langle \hat{X}(\mathbf{A}), \tau_{\mathbf{A}}\rangle$ with the property that for all $a \in A$, $C \cap \varphi(a) = \uparrow(C \cap \varphi(a) \cap \hat{X}(\mathbf{A}))$.

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