Algebra Univers. 77 (2017) 237–249
DOI 10.1007/s00012-017-0423-6
Published online January 17, 2017

Algebra Universalis

Weak complemented and weak invertible elements in C-lattices

C. Jayaram

Abstract. In this paper, we prove that an indecomposable M-lattice is either a principal element domain or a special principal element lattice. Next, we introduce weak complemented elements and characterize reduced M-lattices in terms of weak complemented elements. We also study weak invertible elements and locally weak invertible elements in C-lattices and characterize reduced Prüfer lattices, *WI*-lattices, reduced almost principal element lattices, and reduced principal element lattices in terms of locally weak invertible elements.

1. Introduction

By a *multiplicative lattice*, we mean a complete lattice L, with least element 0 and compact greatest element 1, on which there is defined a commutative, associative, completely join distributive product for which 1 is a multiplicative identity. By a *C-lattice* we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset C of compact elements. Throughout this paper L denotes a principally generated (i.e., every element of L is a join of principal elements of L) C-lattice and L_* denotes the set of all compact elements of L. R denotes a commutative ring with identity and $L(R)$ denotes the lattice of all ideals of R.

Obviously, C-lattices arise as abstractions of ideal systems, in particular when considering rings with identity. There the principal ideals form a generating set of compact elements whereas the finitely generated ideals form the set of all compact elements. Like the ideal lattice of a ring, any C-lattice can be localized. If S is a multiplicative closed subset of L_*, then for any $a \in L$, $a_S = \bigvee\{x \in L_* \mid xs \leq a \text{ for some } s \in S\}$ and $L_S = \{x_S \mid x \in L\}$. L_S is again a multiplicative lattice under the same order as L with the product $a_S \circ b_S = (ab)_S$. We denote the meet and join operations in L_S by $\wedge$ and $\vee$. The meet and join operations for L_S are given by $a_S \wedge b_S = (a \wedge b)_S$ and $a_S \vee b_S = (a \vee b)_S$, for any $a_S, b_S \in L_S$. For $a, b \in L$, we denote $\bigvee\{x \in L \mid xb \leq a\}$ by $(a{:}b)$. If p is a prime element of L and $S = \{x \in L_* \mid x \not\leq p\}$, then L_S is denoted by L_p. For more details on C-lattices and

Presented by W. McGovern.
Received June 26, 2015; accepted in final form August 22, 2016.
2010 *Mathematics Subject Classification*: Primary: 06F10; Secondary: 06F05, 13A15.
Key words and phrases: principal element lattice, M-lattice, Prüfer lattice, multiplication ring, general *ZPI*-ring, arithmetical ring, *WI*-ring, σ-element, weak complemented element, weak invertible element and locally weak invertible element.

Birkhäuser

their localization theory, the reader is referred to [21] and [26]. If $L = L(R)$ is the lattice of ideals of a ring and if p is a prime ideal of R, then the lattice L_p of ideals of R_p is naturally isomorphic to the localization $L(R)_p$ of the lattice $L(R)$.

We note that in a C-lattice, product is order preserving, a finite product of compact elements is again compact, and for any $a, b \in L$, $a = b$ if and only if $a_m = b_m$ for all maximal elements m of L. Since in C-lattices, every element is a join of compact elements and a finite join of compact elements is compact, then by [3, Theorem 1.3], it follows that principal elements are always compact.

An element $a \in L$ is said to be *proper* if $a < 1$. An element $p < 1$ in L is said to be *prime* if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. In a C-lattice L, an element p is prime if and only if $ab \leq p$ implies either $a \leq p$ or $b \leq p$ for all $a, b \in L_*$. An element $m < 1$ in L is said to be *maximal* if $m < x \leq 1$ implies $x = 1$. It is easily seen that in C-lattices, maximal elements are prime elements.

The theory of C-lattices was initiated by R.P. Dilworth in his fundamental and ground breaking paper [9] based on the notion of a *principal element* e. Recall that an element $e \in L$ is said to be *principal* [9], if it satisfies the dual identities (i) $a \wedge be = ((a:e) \wedge b)e$ and (ii) $(ae \vee b:e) = (b:e) \vee a$. Elements satisfying the weaker identity (i$'$) $a \wedge e = (a:e)e$ obtained from (i) by setting $b = 1$ are called *weak meet principal.* An element $a \in L$ is said to be *locally principal* if a_m is principal in L_m for all maximal elements m of L. If $e \in L$ is weak meet principal and if $e \leq m$ for maximal element m of L, then for any $a_m \in L_m$, $a_m \leq e_m$ implies $a_m = (a \wedge e)_m = ((a:e)e)_m = (a:e)_m \circ e_m$, so e_m is weak meet principal in L_m, and hence by [3, Theorem 1.2], e_m is principal in L_m. Therefore, every weak meet principal element of L is locally principal.

An ideal I of R is said to be *quasi-principal* if it is a principal element of $L(R)$. An element $a \in L$ is said to be a *complemented element* if $a \vee b = 1$ and $ab = 0$ for some element $b \in L$ and a is called *nilpotent* if $a^n = 0$ for some positive integer n. For any $a, b \in L$, we say a and b are *comaximal* if $a \vee b = 1$. L is said to be *indecomposable* if 0 and 1 are the only complemented elements of L. If 0 is the only nilpotent element, then L is called *reduced.* An element $a \in L$ is said to be *weak invertible* if a is principal and $(0:a)$ is a complemented element of L, and a is said to be *invertible* if a is principal and $(0:a) = 0$. Weak invertible elements have been studied in [23] and [19]. An ideal I of R is said to be *weak invertible* [19] if it is a weak invertible element of $L(R)$. An ideal I of R is said to be *quasi-invertible* [19] if it is an invertible element of $L(R)$. For any $a \in L$, we denote $\sqrt{a} = \bigvee\{x \in L_* \mid x^n \leq a$ for some positive integer $n\}$. L is said to be a *domain* if the zero element is a prime element. Note that by [8, Theorem 1], in C-lattices, a compact locally principal element is principal.

L is said to be a *principal element lattice* if every element is principal. L is said to be an *almost principal element lattice* if L_m is a principal element lattice for each maximal element m of L. The classical example of a principal element lattice presented by Dilworth in [9] is the ideal structure of a Dedekind domain. For various characterizations of principal element lattices and almost principal element lattices, the reader is referred to [6], [20], [22], [15], [16] and [17]. L is said to be a *special principal element lattice* if it has a unique maximal element which is principal and every proper element is a power of the maximal element.

A lattice L is said to be an *M-lattice* if every element is weak meet principal. For more information on M-lattices, the reader is referred to [2], [13], and [28]. L is said to be a *regular lattice* if every compact element of L is a complemented element of L. For various characterizations of regular lattices, the reader is referred to [5]. L is an *M-normal lattice* [7] if every prime element contains a unique minimal prime element. Note that an M-normal lattice need not be an M-lattice. L is called a *Prüfer lattice* if every compact element is principal. L is called a *WI-lattice* if every compact element is weak invertible. Prüfer lattices have been studied in [3] and [17] and *WI*-lattices have been studied in [23] and [19].

Recall that an ideal I of R is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. R is a *multiplication ring* if every ideal is a multiplication ideal. Multiplication rings have been studied in [4] and [11]. It should be mentioned that R is a multiplication ring if and only if $L(R)$ is an M-lattice. R is a *general ZPI-ring*, if every ideal is a finite product of prime ideals of R. A ring R is said to be *arithmetical* if its ideal lattice is distributive. R is said to be a *WI-ring*, if every finitely generated ideal is weak invertible. Recall that R is called a *quasi-regular ring* if its classical ring of quotients is a Von-Neumann regular ring. Quasi-regular rings have been studied in [10] and [14]. It should be mentioned that quasi-regular rings are also known as *complemented rings.*

In this paper, we study weak complemented elements, weak invertible elements, and locally weak invertible elements in not necessarily modular C-lattices. For various examples of non modular C-lattices, the reader is referred to [1]. In Section 2, we prove that an indecomposable M-lattice is either a principal element domain or a special principal element lattice (see Theorem 2.2). We also introduce weak complemented elements and characterize reduced M-lattices in terms of weak complemented elements (see Theorem 2.7). In Section 3, we study weak invertible elements and locally weak invertible elements in C-lattices and characterize reduced Prüfer lattices, *WI*-lattices, reduced almost principal element lattices, and reduced principal element lattices in terms of locally weak invertible elements (see Theorem 3.3, Theorem 3.4, Theorem 3.5, and Theorem 3.10). Finally, we prove that if L is a reduced Prüfer lattice, then compact prime elements of L are either complemented elements or invertible maximal elements of L (see Theorem 3.13). As a consequence, we prove that if R is a reduced arithmetical ring, then finitely generated prime ideals of R are either complemented ideals or quasi-invertible maximal ideals of R. Further, if R is a *WI*-ring, then finitely generated prime ideals of R are either complemented ideals or invertible maximal ideals of R (see Corollary 3.14).

For general background and terminology, the reader may consult [3] and [21], and for general background and terminology in commutative ring theory, the reader is referred to [12] and [27].

2. Indecomposable M-lattices

In this section, we study weak complemented elements and characterize reduced M-lattices in terms of weak complemented elements.

We shall begin with the following lemma.

Lemma 2.1. *Let L be an M-lattice and let b be a nonzero idempotent element of L. Then b is the join of complemented elements of L.*

Proof. Let $c = \bigvee\{d \in L_* \mid d \le b$ and d is a nilpotent element of $L\}$. Choose any nonzero compact element y of L such that $y \le b$. By hypothesis, $y = bd$ for some $d \in L$. Note that $by = b^2d = bd = y$. As y is compact, we get $y = xy$ for some compact element $x \le b$. By hypothesis and by [8, Theorem 1 and Proposition 2], x is principal. Observe that x is a non-nilpotent element of L. If $x \le c$, then by [9, Property 2.16], x is a nilpotent element, so $x \not\le c$ and hence $c < b$. Also, it can be easily shown that $c = \sqrt{c} \wedge b$. Let $m = (c:x) \wedge b$ and $n = (c:m) \wedge b$. We claim that $m \wedge n = c$. Clearly,

$$m \wedge n = (c:x) \wedge b \wedge (c:m) = m \wedge (c:m)$$

and $c \le m \wedge (c:m) = m \wedge n$. Again, $m \wedge n \le n \le (c:m)$, so

$$(m \wedge n)^2 \le (c:m)(m \wedge n) \le (c:m)m \le c.$$

Consequently, $m \wedge n \le \sqrt{c} \wedge b = c$. This shows that $m \wedge n = c$. Also,

$$m \wedge x = (c:x) \wedge b \wedge x = (c:x) \wedge x \le c$$

since $\sqrt{c} \wedge b = c$. Next, we show that $1 = (c:x) \vee n$. Since $x \le c \vee x \vee m$ and $c \vee x \vee m$ is weak meet principal, it follows that $x = d(c \vee x \vee m)$ for some $d \in L$, and so $x = xb = db(c \vee x \vee m)$. Observe that $dbm \le x \wedge m \le c$, so $db \le (c:m) \wedge b = n$, and hence

$$x \le db(c \vee x \vee m) \le n(c \vee x \vee m) = nc \vee nx \vee mn \le c \vee nx$$

(as $nc \le c$ and $mn \le m \wedge n \le c$). Thus, $x \le c \vee nx$. Again, since $1 = (c \vee nx : x)$ and x is principal, it follows that $1 = n \vee (c:x)$.

Note that 1 is compact, so there exist compact elements $n_1, n_2 \in L$ such that $1 = n_1 \vee n_2$, $n_1 \le (c:x)$, and $n_2 \le n$. Since $n_1 n_2 \le (c:x) \wedge b = m$ and $n_1 n_2 \le n$, we have $n_1 n_2 \le m \wedge n = c$, and hence $n_1 n_2$ is a nilpotent element of L. Suppose $(n_1 n_2)^k = 0$ for some positive integer k. Then ${n_1}^k \vee {n_2}^k = 1$ and ${n_1}^k {n_2}^k = 0$, so ${n_2}^k$ is a complemented element of L and also ${n_2}^k \le b$. We show that ${n_2}^k \ne 0$. If ${n_2}^k = 0$, then $1 = {n_1}^k \le (c:x)$, and hence

$x \leq c$. This contradicts the fact that x is a non-nilpotent element of L. Let $b^* = \bigvee\{e \leq b \mid e$ is a complemented element of $L\}$. Since ${n_2}^k$ is a nonzero complemented element and ${n_2}^k \leq b$, it follows that $b^* \neq 0$. Since ${n_2}^k \leq b$ and $n_1 \leq (c{:}x)$, it follows that $x = ({n_1}^k \vee {n_2}^k)x = {n_1}^k x \vee {n_2}^k x \leq c \vee {n_2}^k$, so $x \leq a \vee e$ for some nilpotent $a \leq c$ and nonzero complemented element $e \leq b^*$. Suppose $a^l = 0$ for some positive integer l. Then by [9, Property 2.16], $x^l \leq (a \vee e)^l \leq a^l \vee e^l = e^l = e \leq b^*$ as e is idempotent. Note that $y = x^l y \leq e \leq b^*$, and hence $b = b^*$, completing the proof of the lemma. □

Theorem 2.2. *Suppose L is an indecomposable M-lattice. Then L is either a principal element domain or a special principal element lattice.*

Proof. Let m be a maximal element of L. Suppose the powers of m properly descend. By [22, Lemma 2], $m^\omega = \bigwedge_{k=1}^{\infty} m^k$ is a nonmaximal prime element of L. Next, we claim that m^ω is an idempotent element of L. Suppose $p > m^\omega$ is a maximal element of L. As p_p is principal in L_p, it follows that ${m^\omega}_p = p_p {m^\omega}_p$. Since ${m^\omega}_p$ is principal, we have $1_p = p_p \vee (0_p{:}{m^\omega}_p)$. As L_p is quasi-local, it follows that $1_p = (0_p{:}{m^\omega}_p)$, so ${m^\omega}_p = 0_p$. Suppose $m^\omega \not\leq q$ for some maximal element of L. Then ${m^\omega}_q = 1_q$. Therefore, m^ω is locally idempotent and hence idempotent. So by Lemma 2.1, $m^\omega = 0$. Consequently, L is a domain. It is well known that an M-lattice which is also a domain is a principal element domain (see [2, Theorem 4.1] or [13, Theorem 6 and corollary]). Suppose the powers of m do not properly descend. Then $m^k = m^{2k}$ for some positive integer k. So by hypothesis and Lemma 2.1, $m^k = 0$, and hence L is a quasi-local lattice. By [22, Lemma 4], every nonzero element is a power of m and principal. So, L is a special principal element lattice. □

An element $a \in L$ is a *σ-element* if for every compact element $x \leq a$, $a \vee (0{:}x) = 1$. It should be mentioned that a is a σ-element if and only if a is locally complemented (see [24, Proposition 1]). For more information on σ-elements, the reader is referred to [24].

Lemma 2.3. *Suppose L is an M-lattice and a is an element of L. Then $a = a^2$ if and only if a is a σ-element.*

Proof. Assume that $a = a^2$. Suppose $x \leq a$ is a compact element of L. By Lemma 2.1, we have $x \leq y$ for some complemented element $y \leq a$. Then $(0{:}y) \leq (0{:}x)$, so $a \vee (0{:}x) = 1$, and hence a is a σ-element.

Conversely, assume that a is a σ-element. It is enough if we show that a is locally idempotent. Suppose m is a maximal element of L. If $a \not\leq m$, then $a_m = 1_m$ in L_m. Suppose $a \leq m$. Choose any compact element $x \leq a$. Then $a \vee (0{:}x) = 1$, so $a_m \vee (0{:}x)_m = 1_m$. As L_m is quasi-local, it follows that $(0{:}x)_m = 1_m$, so $x_m = 0_m$. As L is compactly generated, it follows that $a_m = 0_m$, and hence a is a locally idempotent element of L. □

Lemma 2.4. *Suppose L is an M-lattice and m is a prime element of L that is both maximal and minimal. Assume that $(0{:}m) = (0{:}m)^2$. Then $m = m^2$.*

Proof. If $(0:m) \not\leq m$, then $m \vee (0:m) = 1$, so $m = m^2 \vee m(0:m) = m^2$. So assume that $(0:m) \leq m$. By Lemma 2.3, $(0:m)$ is a σ-element. We claim that $(0:m)_m = 0_m$ in L_m. Choose any compact element $x \leq (0:m)$. Since $(0:m)$ is a σ-element, we have $(0:x) \not\leq m$, so $xy = 0$ for some compact element $y \not\leq m$. Therefore, $x \leq 0_m$, and hence $(0:m)_m = 0_m$ in L_m. Again, for any maximal element $p \neq m$, we have $(0:m) \leq p$, so by the previous argument, $(0:m)_p = 0_p$ in L_p. Since in a C-lattice $a = b$ if and only if $a_p = b_p$ for all maximal elements p of L, it follows that $(0:m) = 0$, and hence $(0:m^k) = 0$ for all positive integers k. Now we show that $m = m^2$. Suppose $m^2 < m$. Then there exists a principal element $a \leq m$ such that $a \not\leq m^2$. As m is weak meet principal, it follows that $a = my$ for some $y \not\leq m$. As m is a minimal prime, by [2, Lemma 3.5], there exists a compact element $z \not\leq m$ such that $a^n z = 0$ for some positive integer n. Consequently, $y^n z \leq (0:m^n) = 0 \leq m$, a contradiction. Hence, $m = m^2$, completing the proof of the lemma. □

Lemma 2.5. *Let L be a reduced M-normal lattice and let $x \in L_*$. If $x \vee (0:x)$ is weak meet principal, then $(0:x)$ is a complemented element of L.*

Proof. By [7, Theorem 7], $(0:x)$ is a σ-element, so it is locally complemented, and hence it is an idempotent element. As $x \vee (0:x)$ is weak meet principal and not contained in any minimal prime, by [16, Lemma 9], $x \vee (0:x)$ is compact, so $x \vee (0:x) = x \vee a$ for some compact element $a \leq (0:x)$. Again, as $(0:x)$ is a σ-element, it follows that

$$(0:x) = (0:x) \wedge (x \vee a) = (0:x)(x \vee a) = (0:x)a \leq a$$

so $(0:x) = a$ which is compact. Since $(0:x)$ is compact and idempotent, by [21, Lemma 2], $(0:x)$ is principal and idempotent, and hence $(0:x)$ is a complemented element of L. □

Definition 2.6. An element $a \in L$ is said to be a *weak complemented element* if a is weak meet principal and $(0:a)$ is a σ-element.

Note that if L is a regular lattice, then by [24, Theorem 3], every element is a σ-element. Also, by [25, Remark 1] and [5, Theorem 4], every element is weak meet principal, and hence every element is a weak complemented element. Observe that by [8, Theorem 1 and Proposition 2] and [21, Lemma 2], an element $a \in L$ is weak invertible if and only if a is weak complemented and both a and $(0:a)$ are compact elements of L. Further, by Lemma 2.5, L is a reduced principal element lattice if and only if L is a reduced M-lattice in which every element is compact if and only if every element is weak invertible. The following Theorem 2.7 characterizes reduced M-lattices.

Theorem 2.7. *The following statements on L are equivalent:*

(i) *L is a reduced M-lattice.*
(ii) *Every element of L is a weak complemented element.*
(iii) *Every prime element of L is a weak complemented element.*

Proof. (i) $\Rightarrow$ (ii): Suppose L is a reduced M-lattice. Then for any maximal element m of L, L_m is an indecomposable principal element lattice, so by Theorem 2.2, every maximal element contains a unique minimal prime element, and hence L is an M-normal lattice. Let $a \in L$. Clearly, a is weak meet principal. We show that $(0\!:\!a)$ is a σ-element. Suppose x is a compact element such that $x \leq (0\!:\!a)$. By Lemma 2.5, $(0\!:\!x)$ is a complemented element of L. Let e be the complement of $(0\!:\!x)$. Since $ax = 0$, it follows that $a \leq (0\!:\!x)$, so $ae = 0$, and hence $e \leq (0\!:\!a)$. Therefore, $1 = e \vee (0\!:\!x) \leq (0\!:\!a) \vee (0\!:\!x)$. This shows that $(0\!:\!a)$ is a σ-element. Thus, a is a weak complemented element of L.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Since every prime element is weak meet principal, by [2, Theorem 4.3], L is an M-lattice. Now it is enough if we show that L_m is a domain for all maximal elements m of L. Let m be a maximal element of L. If m is nonminimal, then by Theorem 2.2, L_m is a principal element domain. Suppose m is both maximal and minimal. Then by Lemma 2.3, Lemma 2.4 and Theorem 2.2, L_m is a domain. This completes the proof of the theorem. □

For any $a \in R$, the principal ideal generated by a is denoted by (a).

Definition 2.8. An ideal $I \in L(R)$ is said to be a *weak complemented ideal* if I is a multiplication ideal and $((0)\!:\!I)$ is a σ-element of $L(R)$.

As a consequence of Theorem 2.7, we have the following result.

Corollary 2.9. *The following statements on R are equivalent:*

(i) *R is a reduced multiplication ring.*
(ii) *Every ideal of R is a weak complemented ideal.*
(iii) *Every prime ideal of R is a weak complemented ideal.*

Corollary 2.10. *L is a reduced principal element lattice if and only if every prime element is weak invertible.*

Proof. It is well known that L is a principal element lattice if and only if every prime element is principal. Now the result follows from Theorem 2.7 and the fact that compact σ-elements are complemented elements. □

Corollary 2.11. *R is a reduced general ZPI-ring if and only if every prime ideal is weak invertible.*

Proof. Note that by [15, Theorem 2.2], R is a reduced general *ZPI*-ring if and only if $L(R)$ is a reduced principal element lattice and every prime ideal is weak invertible if and only if every prime element of $L(R)$ is a weak invertible element of $L(R)$. Now the result follows from Corollary 2.10. □

3. Locally weak invertible elements in C-lattices

In this section, we study weak invertible elements and locally weak invertible elements in C-lattices and characterize reduced Prüfer lattices, *WI*-lattices, reduced almost principal element lattices, and reduced principal element lattices in terms of locally weak invertible elements. We shall begin with the following definition.

Definition 3.1. An element $a \in L$ is said to be *locally weak invertible* if for each maximal element m of L, a_m is a weak invertible element of L_m.

Observe that if $a \nleq m$ for some maximal element m of L, then $a_m = 1_m$ is weak invertible, so $a \in L$ is locally weak invertible if for each maximal element $m \geq a$, a_m is a weak invertible element of L_m. Note that an element $a \in L$ is locally weak invertible if and only if for any maximal element $m \geq a$, either $a_m = 0_m$ or a_m is invertible in L_m. Observe that weak invertible elements are locally weak invertible but the converse need not be true. For example, if L is a locally domain, then any weak meet principal element is locally weak invertible, but weak meet principal elements need not be compact. It can be easily verified that a compact element a is locally weak invertible if and only if a is weak complemented. Also, it not hard to show that if a and b are compact weak complemented elements, then ab is again a compact weak complemented element. In general, a locally weak invertible element need not be weak complemented. For example, let L be a domain which is an almost principal element domain, but not a principal element domain. Since L is not a principal element domain, it follows that L is not an M-lattice, so there exists at least one nonzero element a of L which is not weak meet principal. Note that a is not weak complemented but a is locally weak invertible. Also, note that a is not weak invertible.

It is not known whether a weak complemented element is locally weak invertible element or not.

Lemma 3.2. *Let L be a reduced lattice and let $a \in L_*$. Then the following statements are equivalent:*

(i) *a is weak invertible.*
(ii) *a is locally weak invertible and $(0\!:\!a)$ is compact.*
(iii) *There exists $b \in L_*$ such that $ab = 0$ and $a \vee b$ is invertible.*

Proof. (i) $\Rightarrow$ (ii): Suppose (i) holds. Assume that $m \geq a$ is a maximal element of L. By (i), $(0\!:\!a)$ is a complemented element, so $(0\!:\!a)_m = (0_m\!:\!a_m) = 1_m$ or $(0\!:\!a)_m = (0_m\!:\!a_m) = 0_m$ as a is a compact element of L. If $(0\!:\!a)_m = (0_m\!:\!a_m) = 1_m$, then $a_m = 0_m$. If $(0\!:\!a)_m = (0_m\!:\!a_m) = 0_m$, then a_m is invertible in L_m, so a is locally weak invertible. Again since $(0\!:\!a)$ is a complemented element, by [5, Lemma 6], $(0\!:\!a)$ is compact, so (ii) holds.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Since a and $(0\!:\!a)$ are compact elements, it follows that $a \vee (0\!:\!a)$ is compact. By [8, Theorem 1], it is enough if we show that $a \vee (0:a)$ is locally principal. Suppose $a \vee (0:a) \leq m$ for some maximal element m of L. As $(0:a) \leq m$, it follows that $(0_m:a_m) = (0:a)_m \neq 1_m$, so $(0_m:a_m) = (0:a)_m = 0_m$, and hence $(a \vee (0:a))_m = a_m$ which is principal in L_m. Therefore, $a \vee (0:a)$ is invertible.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. We show that a is locally principal. Let $m \geq a$ be a maximal element of L. If $b_m = 1_m$, then $a_m = 0_m$. Suppose $b_m \neq 1_m$. Then $a \vee b \leq m$. As $a \vee b$ is invertible, by [8, Proposition 2], $(a \vee b)_m$ is join irreducible. So either $(a \vee b)_m = a_m$ or $(a \vee b)_m = b_m$. Then either a_m or b_m is invertible. If b_m is invertible, then $(0:b)_m = 0_m$, so $a_m = 0_m$. Therefore, a is locally principal and hence principal as a is compact. Again note that $(0:a) \vee (0:b) \not\leq m$ for all maximal elements m of L, so $(0:a) \vee (0:b) = 1$. Also, $(0:a)(0:b) = (0:a) \wedge (0:b) = (0:a \vee b) = 0$. This shows that $(0:a)$ is a complemented element, and hence a is weak invertible. This completes the proof of the lemma. □

We now characterize reduced Prüfer lattices and *WI*-lattices as follows.

Theorem 3.3. *Every compact element of L is locally weak invertible if and only if L is a reduced Prüfer lattice.*

Proof. Suppose every compact element of L is locally weak invertible. We show that for any $x \in L_*$, $(0:x)$ is a σ-element. Let $x \in L_*$. Note that x is locally weak invertible. Let $m \geq x$ be a maximal element of L. Since x_m is weak invertible in L_m, it follows that either $x_m = 0_m$ or x_m is invertible, so either $(0:x)_m = (0_m:x_m) = 1_m$ or $(0:x)_m = (0_m:x_m) = 0_m$, and hence $(0:x)$ is locally complemented. Therefore, $(0:x)$ is a σ-element. We prove that L is a reduced lattice. Suppose $y \in L_*$ and assume that $y^2 = 0$. As $y \leq (0:y)$ and $(0:y)$ is a σ-element, by the definition of σ-element, we have $(0:y) = 1$, so $y = 0$, and hence L is reduced. Again if $a \in L_*$, then for any maximal element $m \geq a$, either $a_m = 0_m$ or a_m is invertible in L_m. So a is locally principal and hence principal. Consequently, L is a Prüfer lattice.

Conversely, assume that L is a reduced Prüfer lattice. Then by [23, Lemma 2], L is an M-normal lattice. Let $a \in L_*$. By hypothesis, a is principal. As L is an M-normal lattice, L_m is a domain for all maximal elements m of L. If $m \geq a$ is a maximal element of L, then either $a_m = 0_m$ or $(0_m:a_m) = (0:a)_m = 0_m$ in L_m. So a is locally weak invertible. This completes the proof of the theorem. □

Theorem 3.4. *The following statements on L are equivalent:*

(i) *L is a WI-lattice.*
(ii) *Every compact element is locally weak invertible and for every principal element $a \in L$, $(0:a)$ is compact.*
(iii) *For each $a \in L_*$, there exists $b \in L_*$ such that $ab = 0$ and $a \vee b$ is invertible.*

Proof. (i) $\Rightarrow$ (ii): This follows from the definition of a *WI*-lattice and from the fact that a complemented element is a compact element.

(ii) $\Rightarrow$ (iii): This follows from Lemma 3.2.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. We show that L is a reduced lattice. Suppose $a \in L_*$ and assume that $a^2 = 0$. By (iii), there exists $b \in L_*$ such that $ab = 0$ and $a \vee b$ is invertible. Then $a(a \vee b) = a^2 \vee ab = 0$, so $a = 0$ as $a \vee b$ is invertible. Therefore, L is a reduced lattice. Again by Lemma 3.2, every compact element is weak invertible. Hence, L is a *WI*-lattice and the proof is complete. □

Theorem 3.5. *Every element of L is locally weak invertible if and only if L is a reduced almost principal element lattice.*

Proof. Suppose every element of L is locally weak invertible. If a is locally weak invertible, then for any maximal element $m \geq a$, either $a_m = 0_m$ or a_m is invertible in L_m. So a is locally principal, and hence L is an almost principal element lattice. By Theorem 3.3, L is reduced.

Conversely, assume that L is a reduced almost principal element lattice. Then L_m is a domain for all maximal elements m of L. Further, if $a \in L$ and $a \leq m$ for some maximal element m of L, then either $a_m = 0_m$ or a_m is invertible in L_m. Consequently, every element of L is locally weak invertible and the proof is complete. □

Lemma 3.6. *Suppose a is a σ-element. Suppose there exist prime elements $p_1, p_2, \ldots, p_n$ such that for each maximal element m of L, $m \geq (0:a)$ implies $m \geq p_j$ for some $j \in \{1, 2, \ldots, n\}$. Then a is a complemented element of L.*

Proof. It is enough if we show that a is compact.

Case (i). Suppose $a \leq \bigwedge_{i=1}^{n} p_i$. If $a \leq m$ for some maximal element m of L, then for any compact element $x \leq a$, $(0:x) \nleq m$, so $(0:x)_m = 1_m$, and hence $x_m = 0_m$. Consequently, $a_m = 0_m$ in L_m. Suppose $a \nleq m$ for some maximal element m of L. Then $a_m = 1_m$, so $(0_m : a_m) = 0_m$. Clearly, $(0:a)_m = 0_m$ in L_m. As $(0:a) \leq m$, by hypothesis, $m \geq p_j$ for some $j \in \{1, 2, \ldots, n\}$, so $m \geq a$, a contradiction. Thus, $a_m = 0_m$ for all maximal elements m of L, and hence $a = 0$.

Case (ii). Suppose $a \nleq p_i$ for all i. We claim that $a \vee (0:a) = 1$. Suppose $a \vee (0:a) \leq m$ for some maximal element m of L. Since $(0:a) \leq m$, by hypothesis, $m \geq p_j$ for some $j \in \{1, 2, \ldots, n\}$. But $a_m = 0_m$ in L_m, so $a \leq p_j$, a contradiction. Therefore, a is a complemented element of L, and hence by [5, Lemma 6], a is compact.

Case (iii). Suppose $a \leq \bigwedge_{i=i}^{r} p_i$ and $a \nleq p_j$ for $j = r+1, \ldots, n$. We claim that a and p_j are comaximal for $j \in \{r+1, \ldots, n\}$. Suppose $a \vee p_j \leq m$ for some $j \in \{r+1, \ldots, n\}$ and for some maximal element m of L. Since $a_m = 0_m$ in L_m, it follows that $a \leq p_j$, a contradiction. So $a \vee p_j = 1$ for $j = r+1, \ldots, n$. Note that by [9, Property 2.14], $a \vee (\bigwedge_{j=r+1}^{n} p_j) = 1$. Therefore, $x \vee y = 1$ for some compact elements $x \leq a$ and $y \leq \bigwedge_{j=r+1}^{n} p_j$. Let m be a maximal element of L. If $a \leq m$, then $a_m = 0_m = x_m$ in L_m. If $a \nleq m$, then $(0:a) \leq m$, so $p_j \leq m$ for some $j \in \{r+1, \ldots, n\}$, and hence $y \leq m$. Therefore, $x \nleq m$, and hence $a_m = 1_m = x_m$ in L_m. This shows that $a_m = x_m$ for all maximal elements m of L, and hence $a = x$. Consequently, a is compact. This completes the proof of the lemma. $\square$

Lemma 3.7. *Suppose a is a compact locally weak invertible element of L. Assume that a has only finitely many minimal primes. Then a is weak invertible.*

Proof. Since a is a compact locally weak invertible element of L, it follows that a is principal. Also, $(0{:}a)$ is locally complemented, and hence $(0{:}a)$ is a σ-element. Suppose $p_1, p_2, \ldots, p_n$ are the minimal primes of a. If m is any maximal element and $m \geq (0{:}(0{:}a))$, then $m \geq a$, so $m \geq p_j$ for some $j \in \{1, 2, \ldots, n\}$. Therefore, by Lemma 3.6, $(0{:}a)$ is a complemented element, and hence a is weak invertible. $\square$

Remark 3.8. Since compact weak complemented elements are locally weak invertible, as a consequence of Lemma 3.7, it should be mentioned that compact weak complemented prime elements are weak invertible elements.

According to [18], a multiplicative lattice L_0 is said to satisfy the condition $(*)$ if there exists a multiplicatively closed set S of (not necessarily principal) elements which generate L_0 under joins such that every element of S is a finite meet of primary elements.

Lemma 3.9. *Suppose L satisfies the condition $(*)$. Then locally weak invertible elements of L are weak invertible elements.*

Proof. Suppose $a \in L$ is locally weak invertible. Since a is locally principal, by [18, Lemma 3], a is principal. As a is principal, by [18, Lemma 1], a has only finitely many minimal primes, and hence by Lemma 3.7, a is weak invertible. $\square$

We now give a new characterization for reduced principal element lattices.

Theorem 3.10. *L is a reduced principal element lattice if and only if L satisfies the condition $(*)$ and every prime element of L is locally weak invertible.*

Proof. The proof of the theorem follows from Corollary 2.10 and Lemma 3.9. $\square$

Corollary 3.11. *R is a reduced general ZPI-ring if and only if every principal ideal of R is a finite intersection of primary ideals and every prime ideal of R is locally weak invertible.*

Proof. The proof of the corollary follows from Theorem 3.10. $\square$

Lemma 3.12. *Let L be a Prüfer lattice and let p be a weak invertible prime element of L. Then p is either an invertible maximal element or p is a complemented element of L.*

Proof. Assume that p is not a complemented element of L. Since we have $(0{:}p)(0{:}(0{:}p)) = 0$, it follows that either $(0{:}p) \leq p$ or $(0{:}(0{:}p) \leq p$. As p is not a complemented element of L, it follows that $(0{:}p) = 0$. Suppose $p < m$

for some maximal element m of L. Choose a principal element $a \leq m$ such that $a \not\leq p$. Then by hypothesis, $(p \vee a)$ is principal. By [8, Proposition 2], $(p \vee a)_m$ is join irreducible in L_m. As $a \not\leq p$, it follows that $(p \vee a)_m = a_m$ in L_m. As a_m is principal and $a_m \not\leq p_m$, it follows that $p_m = a_m p_m$, and hence by [3, Theorem 1.4], $p_m = 0_m$, a contradiction since $(0:p) = 0$. Therefore, p is an invertible maximal element of L. □

Finally, we prove that in a reduced Prüfer lattice, compact primes are either complemented elements or invertible maximal elements.

Theorem 3.13. *Suppose L is a reduced Prüfer lattice. Then compact prime elements of L are either complemented elements or invertible maximal elements of L.*

Proof. Let p be a compact prime element of L. By Theorem 3.3 and by Lemma 3.7, p is weak invertible. Now the result follows from Lemma 3.12. □

Corollary 3.14. *If R is a reduced arithmetical ring, then finitely generated prime ideals of R are either complemented ideals or quasi-invertible maximal ideals of R. Further, if R is a WI-ring, then finitely generated prime ideals of R are either complemented ideals or invertible maximal ideals of R.*

Proof. If R is a reduced arithmetical ring, then by Theorem 3.13, finitely generated prime ideals of R are either complemented ideals or quasi-invertible maximal ideals of R. Now assume that R is a *WI*-ring. Then by [19, Theorem 3.3], R is a reduced arithmetical ring and a quasi-regular ring. Again by [12, Theorem 4.5] and [12, Lemma 18.1, page 110], an ideal I of R is quasi-invertible if and only if it is invertible, and hence finitely generated prime ideals of R are either complemented ideals or invertible maximal ideals of R. □

Acknowledgement. In closing, I would like to express my gratitude for the assistance provided by the anonymous referee of this paper.

References

[1] Alarcon, F., Anderson, D.D.: Commutative semirings and their lattices of ideals. Houston J. Math. **20**, 571–590 (1994)

[2] Alarcon, F., Anderson, D.D., Jayaram, C.: Some results on abstract commutative ideal theory. Period. Math. Hungar. **30**, 1–26 (1995)

[3] Anderson, D.D.: Abstract commutative ideal theory without chain condition. Algebra Universalis **6**, 131–145 (1976)

[4] Anderson, D.D.: Multiplication ideals, multiplication rings and the ring $R(X)$. Canad. J. Math. **28**, 760–768 (1976)

[5] Anderson, D.D., Jayaram, C.: Regular lattices. Studia Sci. Math. Hungar. **30**, 379–388 (1995)

[6] Anderson, D.D., Jayaram, C.: Principal element lattices. Czechoslovak Math. J. **46**, 99–109 (1996)

[7] Anderson, D.D., Jayaram, C., Phiri, P.A.: Baer lattices. Acta. Sci. Math. (Szeged) **59**, 61–74 (1994)

[8] Anderson, D.D., Johnson, E.W.: Dilworth's principal elements. Algebra Universalis **36**, 392–404 (1996)

[9] Dilworth, R.P.: Abstract commutative ideal theory. Pacific J. Math. **12**, 481–498 (1962)
[10] Evans, M.W.: On Commutative P.P. Rings. Pacific J. Math. **41**, 687–697 (1972)
[11] Gilmer, R.W., Mott, J.L.: Multiplication rings as rings in which ideals with prime radical are primary. Trans. Amer. Math. Soc. **114**, 40–52 (1965)
[12] Huckaba, J.A.: Commutative rings with zero divisors. Marcel Dekker, New York (1988)
[13] Janowitz, M.F.: Principal multiplicative lattices. Pacific J. Math. **33**, 653–656 (1970)
[14] Jayaram, C.: Baer ideals in commutative semiprime rings. Indian J. Pure Appl. Math. **15**(8), 855–864 (1984)
[15] Jayaram, C.: 2-Join decomposition lattices. Algebra Universalis **45**, 07–13 (2001)
[16] Jayaram, C.: ℓ-prime elements in multiplicative lattices. Algebra Universalis **48**, 117–127 (2002)
[17] Jayaram, C.: Primary elements in Prüfer lattices, Czechoslovak Math. J. **52** (127), 585–593 (2002)
[18] Jayaram, C.: Almost π-lattices, Czechoslovak Math. J. **54** (129), 119–130 (2004)
[19] Jayaram, C.: Regular elements in multiplicative lattices. Algebra Universalis **59**, 73–84 (2008)
[20] Jayaram, C., Johnson, E.W.: Some results on almost principal element lattices. Period. Math. Hungar. **31**, 33–42 (1995)
[21] Jayaram, C., Johnson, E.W.: s-prime elements in multiplicative lattices, Period. Math. Hungar. **31**, 201–208 (1995)
[22] Jayaram, C., Johnson, E.W.: Strong compact elements in multiplicative lattices. Czechoslovak Math. J. **47**(122), 105–112 (1997)
[23] Jayaram, C., Johnson, E.W.: Dedekind lattices. Acta Sci. Math. (Szeged) **63**, 367–378 (1997)
[24] Jayaram, C., Johnson, E.W.: σ-elements in multiplicative lattices. Czechoslovak Math. J. **48**(123), 641–651 (1998)
[25] Jayaram, C., Tekir, Ü., Yetkin, E.: 2-Absorbing and weakly 2-Absorbing elements in multiplicative lattices. Comm. Algebra **42**, 2338–2353 (2014)
[26] Johnson, J.A., Sherette, G.R.: Structural properties of a new class of CM-lattices. Canad. J. Math. **38**, 552–562 (1986)
[27] Larsen, M.D., McCarthy, P.J.: Multiplicative Theory of Ideals. Academic Press, New York-London (1971)
[28] McCarthy, P. J.: Note on abstract commutative ideal theory. Amer. Math. Monthly. **74**, 706–707 (1967)

C. Jayaram
Department of CMP, University of the West Indies, Cave Hill Campus, P.O. Box 64, Bridgetown, Barbados
e-mail: jayaram.chillumu@cavehill.uwi.edu