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Universal varieties of quasi-Stone algebras

M. E. Adams, W. Dziobiak, and H. P. Sankappanavar

Dedicated to Ervin Fried and Jiří Sichler

Abstract. The lattice of varieties of quasi-Stone algebras ordered by inclusion is an $\omega+1$ chain. It is shown that the variety $\mathbf{Q_{2,2}}$ (of height 13) is finite-to-finite universal (in the sense of Hedrlín and Pultr). Further, it is shown that this is sharp; namely, the variety $\mathbf{Q_{3,1}}$ (of height 12) is not finite-to-finite universal and, hence, no proper subvariety of $\mathbf{Q_{2,2}}$ is finite-to-finite universal. In fact, every proper subvariety of $\mathbf{Q_{2,2}}$ fails to be universal. However, $\mathbf{Q_{1,2}}$ (the variety of height 9) is shown to be finite-to-finite universal relative to $\mathbf{Q_{2,1}}$ (the variety of height 8). This too is sharp; namely, no proper subvariety of $\mathbf{Q_{1,2}}$ is finite-to-finite relatively universal. Consequences of these facts are discussed.

1. Quasi-Stone algebras

As introduced in [18], an algebra $L = (L; \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a *quasi-Stone algebra* if

(i) $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice;
(ii) $0' = 1$ and $1' = 0$;
(iii) $(x \vee y)' = x' \wedge y'$;
(iv) $(x \wedge y')' = x' \vee y''$;
(v) $x \leq x''$;
(vi) $x' \vee x'' = 1$.

For $m \in \omega$, B_m denotes the Boolean lattice with m atoms and $\widehat{B}_m$ denotes the lattice $B_m \oplus \{1_m\}$ where 1_m is a new element. For $m, n \in \omega$, $Q_{m,n}$ denotes the quasi-Stone algebra $(\widehat{B}_m \times B_n; \vee, \wedge, ', (0,0), (1_m, 1))$ where $(x, y)' = (0, 0)$ if $(x, y) \neq (0, 0)$, and $(1_m, 1)$ otherwise.

As shown in [18], the lattice of varieties of quasi-Stone algebras forms an $\omega+1$ chain (see Figure 1) where $\mathbf{T}$, $\mathbf{Q_{m,n}}$, and $\mathbf{QS}$ denote the trivial variety, the variety $\mathbf{V}(Q_{m,n})$ of quasi-Stone algebras generated by $Q_{m,n}$, and the variety of all quasi-Stone algebras, respectively.

A *graph* $G = (V, E)$ is a set of *vertices* V and a set of *edges* E consisting of 2-element subsets of V. For graphs $G = (V, E)$ and $H = (W, F)$, a mapping $\phi\colon G \to H$ is *compatible* providing, for every $\{x, y\} \in E$, $\{\phi(x), \phi(y)\} \in F$.

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$$\begin{array}{l}
\mathbf{T} \subset \\
\mathbf{Q_{0,0}} \subset \\
\mathbf{Q_{1,0}} \subset \mathbf{Q_{0,1}} \subset \\
\mathbf{Q_{2,0}} \subset \mathbf{Q_{1,1}} \subset \mathbf{Q_{0,2}} \subset \\
\mathbf{Q_{3,0}} \subset \mathbf{Q_{2,1}} \subset \mathbf{Q_{1,2}} \subset \mathbf{Q_{0,3}} \subset \\
\mathbf{Q_{4,0}} \subset \mathbf{Q_{3,1}} \subset \mathbf{Q_{2,2}} \subset \mathbf{Q_{1,3}} \subset \mathbf{Q_{0,4}} \subset \cdots \subset \mathbf{QS}
\end{array}$$

FIGURE 1. The $\omega + 1$ chain of varieties of quasi-Stone algebras

The class of all graphs with all compatible mappings forms a category. This category will be denoted by $\mathbf{G}$.

A variety $\mathbf{V}$ is *universal* provided every category of algebras of finite type is isomorphic to a full subcategory of $\mathbf{V}$, or equivalently, the category $\mathbf{G}$ is isomorphic to a full subcategory of $\mathbf{V}$, see Hedrlín and Pultr [13] (as well as Pultr and Trnková [17]). If there exists a functor $\mathbf{\Phi}\colon \mathbf{G} \to \mathbf{V}$ which establishes that $\mathbf{V}$ is universal and, in addition, sends finite graphs to finite algebras, then $\mathbf{V}$ is said to be *finite-to-finite universal.*

The following will be shown in Section 3.

Theorem 1.1. $\mathbf{Q_{2,2}}$ *is finite-to-finite universal.*

For a quasi-Stone algebra L, let $\mathrm{End}(L)$ denote the monoid of endomorphisms of L under composition. The following is an immediate consequence of Theorem 1.1 together with known properties of graphs.

Corollary 1.2. *For every monoid M and cardinal $\kappa \geq |M| + \omega$, there exists a family of quasi-Stone algebras $(L_i : i \in I)$ such that*

(i) $L_i \in \mathbf{Q_{2,2}}$ *for* $i \in I$,
(ii) $L_i \ncong L_j$ *for distinct* $i, j \in I$,
(iii) $\mathrm{End}(L_i) \cong M$ *for* $i \in I$,
(iv) $|I| = 2^\kappa$ *and* $|L_i| = \kappa$ *for* $i \in I$.

Moreover, if $|M|$ is finite, then there also exists a countably infinite family of finite quasi-Stone algebras $(L_i : i \in I)$ satisfying (i), (ii), *and* (iii).

A class of algebras with a common finite signature is a *variety* (or an *equational class*) if it is defined by a set of identities (or equations) of the form $p = q$ and a *quasivariety* (or an *implicational class*) if it is defined by a set of quasi-identities (or implications) of the form $(p_0 = q_0 \wedge \cdots \wedge p_{n-1} = q_{n-1}) \Rightarrow p = q$. Recall that for a class $\mathbf{K}$ of algebras with common finite signature, the variety generated by $\mathbf{K}$ is $\mathbf{HSP}(\mathbf{K})$ (where $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ denote the operators of all homomorphic images, subalgebras, and products, respectively) and the quasivariety generated by $\mathbf{K}$ is $\mathbf{ISPP_u}(\mathbf{K})$ (where $\mathbf{I}$ and $\mathbf{P_u}$ denote the operators of all isomorphic copies and ultraproducts, respectively).

For a variety $\mathbf{V}$, let $L_V(\mathbf{V})$ denote the lattice of subvarieties of $\mathbf{V}$ ordered by inclusion. Then as noted above (and shown in [18]), for any variety $\mathbf{V}$ of quasi-Stone algebras, $L_V(\mathbf{V})$ is a chain. Analogously, let $L_Q(\mathbf{V})$ denote the lattice of subquasivarieties of $\mathbf{V}$ ordered by inclusion. Then another immediate consequence of Theorem 1.1 is that $|L_Q(\mathbf{Q_{2,2}})| = 2^\omega$. In fact, it follows from Theorem 1.1 and the main result of [1] that the ideal lattice of a free lattice on ω free generators is embeddable into $L_Q(\mathbf{Q_{2,2}})$ and that $\mathbf{Q_{2,2}}$ is Q-universal in the sense of Sapir [19].

That Theorem 1.1 is sharp will be shown in Section 4, by establishing that the following holds for $\mathbf{Q_{3,1}}$, the largest proper subvariety of $\mathbf{Q_{2,2}}$, and hence, also holds for any proper subvariety of $\mathbf{Q_{2,2}}$.

Theorem 1.3. $\mathbf{Q_{3,1}}$ *is not universal.*

The justification of Theorem 1.3 will consist of showing that, up to isomorphism, there are only two algebras in $\mathbf{Q_{3,1}}$ with a trivial endomorphism monoid, from which it follows immediately that neither $\mathbf{Q_{3,1}}$ nor any subvariety is universal.

Investigation of varieties of semigroups led Demlová and Koubek (see [6], [7], [8], and [9]) to introduce a notion of relatively universal. Precursors of this notion may be found as far back as Sichler [20].

A variety $\mathbf{V}$ is *relatively universal* to a proper subvariety $\mathbf{W}$ (or, briefly, $\mathbf{W}$-*universal*) if there is a faithful functor $\mathbf{\Psi}\colon \mathbf{G} \to \mathbf{V}$ such that $\mathrm{Im}(\mathbf{\Psi}(f))$ belongs to $\mathbf{W}$ for no compatible mapping f (that is, if $f\colon G \to G'$ is a compatible map, then the image of $\mathbf{\Psi}(G)$ under $\mathbf{\Psi}(f)$ does not belong to $\mathbf{W}$) and if $\phi\colon \mathbf{\Psi}(G) \to \mathbf{\Psi}(G')$ is a homomorphism, where G and G' are graphs, then either $\mathrm{Im}(\phi)$ belongs to $\mathbf{W}$ (that is, the image of $\mathbf{\Psi}(G)$ under $\mathbf{\Psi}(f)$ belongs to $\mathbf{W}$) or $\phi = \mathbf{\Psi}(f)$ for a compatible mapping $f\colon G \to G'$. If, in addition, $\mathbf{\Psi}$ assigns finite algebras to finite graphs, $\mathbf{V}$ is said to be *finite-to-finite* $\mathbf{W}$-*universal.*

The following will be shown in Section 5.

Theorem 1.4. $\mathbf{Q_{1,2}}$ *is finite-to-finite* $\mathbf{Q_{2,1}}$*-universal.*

In [3] (see the remarks in Section 6 below), quasivarieties of quasi-Stone algebras are investigated. In particular, it is shown in [3] that $L_Q(\mathbf{Q_{2,1}})$ is countably infinite. This together with the following observation proves that Theorem 1.4 is sharp, in the sense that no proper subvariety of $\mathbf{Q_{1,2}}$ is finite-to-finite relatively universal.

Proposition 1.5. *If* $\mathbf{V}$ *is a variety of algebras of finite signature that is finite-to-finite* $\mathbf{W}$*-universal, then* $|L_Q(\mathbf{V})| = 2^\omega$.

Proof. Let $\mathbf{\Psi}\colon \mathbf{G} \to \mathbf{V}$ be a faithful functor establishing that $\mathbf{V}$ is finite-to-finite $\mathbf{W}$-universal. Let $(G_i : i < \omega)$ be a family of finite graphs having the property that $i = j$ if and only if there is a compatible mapping from G_i to G_j; such a family does exist. The identity map $\mathrm{id}_{G_i}\colon G_i \to G_i$ is a compatible mapping. Thus, $\mathrm{Im}(\mathbf{\Psi}(\mathrm{id}_{G_i})) \notin \mathbf{W}$. Hence, for all $i < \omega$, $\mathbf{\Psi}(G_i) \notin \mathbf{W}$ because by the definition of a functor, $\mathbf{\Psi}(\mathrm{id}_{G_i}) = \mathrm{id}_{\mathbf{\Psi}(G_i)}$.

We will show that if $\mathbf{\Psi}(G_i) \in \mathbf{ISPP_u}(\{\mathbf{\Psi}(G_j) : j \in J\})$, where $J \subseteq \omega$, then $i \in J$. Let $\mathbf{\Psi}(G_i) \in \mathbf{ISPP_u}(\{\mathbf{\Psi}(G_j) : j \in J\})$. Since $\mathbf{V}$ is of finite signature and $\mathbf{\Psi}(G_i)$ is finite, there is a finite subset K of J and a family of homomorphisms, $(\phi_k \colon \mathbf{\Psi}(G_i) \to \mathbf{\Psi}(G_k) : k \in K)$, which separates the elements of $\mathbf{\Psi}(G_i)$. If $\mathrm{Im}(\phi_k) \in \mathbf{W}$ for all $k \in K$, then $\mathbf{\Psi}(G_i) \in \mathbf{W}$. So, $\mathrm{Im}(\phi_k) \notin \mathbf{W}$ for some $k \in K$, and therefore, $\phi = \mathbf{\Psi}(f_k)$ for some compatible mapping $f_k \colon G_i \to G_k$. Hence, $i = k$, that is, $i \in J$. Thus, $|L_Q(\mathbf{V})| = 2^\omega$. □

2. Duality

Priestley [16] developed a duality for the category of non-trivial $(0,1)$-distributive lattices. Subsequently, Cignoli [5] (cf. Halmos [12]) derived an analogous duality for the category of non-trivial Q-distributive lattices. Later Gaitán [10] adapted Cignoli's duality to quasi-Stone algebras. In [2], a variant of Cignoli's duality was given, which we believe is somewhat easier to use. In this section, we present a duality for quasi-Stone algebras based on [2].

A *Priestley space* is a triple $(P; \leq, \tau)$ such that $(P; \leq)$ is a partially ordered set, $(P; \tau)$ is a compact topological space, and the triple is *totally order-disconnected* (that is, for $x, y \in P$, if $x \not\leq y$, then there exists a clopen order-filter $X \subseteq P$ such that $x \in X$ and $y \notin X$.)

Priestley showed that the category of non-trivial $(0,1)$-distributive lattices with $(0,1)$-lattice homomorphisms is dually equivalent to the category of Priestley spaces with continuous order-preserving maps as morphisms.

Pertinent to what follows, recall that for a Priestley space $(P; \leq, \tau)$, if X is a closed subset of P, then for every $x \in X$, there exists $y \in X$ such that y is maximal in $(X; \leq\restriction X)$ and $x \leq y$. The set of maximal elements in $(X; \leq\restriction X)$ will be denoted by $\mathrm{Max}(X)$.

If E is an equivalence relation on a set P and $X \subseteq P$, then $E(X)$ denotes $\{y : yEx \text{ for some } x \in X\}$; for $x \in P$, let $E(x)$ denote $E(\{x\})$.

A *QS-space* $(P; \leq, \tau, E)$ is a Priestley space $(P; \leq, \tau)$ together with an equivalence relation E defined on P such that

(i) for x and $y \in P$, if $x \leq y$ or $y \leq x$, then $E(x) = E(y)$ (that is, $E(x)$, is both an order-ideal and an order-filter for every $x \in P$),
(ii) if $X \subseteq P$ is a clopen order-filter, then $E(X)$ is clopen, and
(iii) for $x \in P$, $E(x)$ is a closed subset of P.

For QS-spaces $(P; \leq, \tau, E)$ and $(P'; \leq', \tau', E')$, a mapping $\varphi \colon P \to P'$ is a *QS-map* if it is a continuous order-preserving map such that

(i) for x and $y \in P$, $\varphi(x)E'\varphi(y)$ whenever xEy, and
(ii) if z is a maximal element of $E'(\varphi(x))$, then $z = \varphi(y)$ for some $y \in E(x)$, that is, $\mathrm{Max}(E'(\varphi(x))) \subseteq \varphi(\mathrm{Max}(E(x)))$.

If X is a closed subset of a Priestley space $(P; \leq, \tau)$, then $(X; \leq\restriction X, \tau\restriction X)$ is also a Priestley space. However, if X is a closed subset of a QS-space $(P; \leq, \tau, E)$, then $(X; \leq\restriction X, \tau\restriction X, E\restriction X)$ need not be a QS-space. As suggested by the the definition of a QS-map, define $(X; \leq\restriction X, \tau\restriction X, E\restriction X)$ to be a QS-*subspace* of a QS-space $(P; \leq, \tau, E)$ provided that X is a closed subset of P and for $x \in X$, every maximal element of $E(x)$ is also an element of X.

The category $\mathbf{QS}$ of all non-trivial quasi-Stone algebras whose morphisms are all homomorphisms is dually equivalent to the category $\mathbf{S}$ of all QS-spaces whose morphisms are all QS-maps. The contravariant functors $\mathcal{S}\colon \mathbf{QS} \to \mathbf{S}$ and $\mathcal{Q}\colon \mathbf{S} \to \mathbf{QS}$ and the pair of natural isomorphisms $\sigma\colon 1_{\mathbf{QS}} \cong \mathcal{QS}$ and $\varepsilon\colon 1_{\mathbf{S}} \cong \mathcal{SQ}$ that establish a dual equivalence between $\mathbf{QS}$ and $\mathbf{S}$ are defined as follows:

With each object L of $\mathbf{QS}$ is associated a QS-space $\mathcal{S}(L) = (S(L); \leq, \tau, E)$ where $S(L)$ is the set of all prime filters of L, $\leq$ is set inclusion, τ has as a sub-basis all subsets of $S(L)$ of the form $\{x \in S(L) : a \in x\}$, where $a \in L$, and their complements, and for x and $y \in S(L)$, $(x, y) \in E$ if and only if $x \cap \{z'\colon z \in L\} = y \cap \{z'\colon z \in L\}$. With each morphism $f\colon L \to L'$ in $\mathbf{QS}$ is associated a QS-map $\mathcal{S}(f)\colon S(L') \to S(L)$ such that $\mathcal{S}(f)(x) = f^{-1}(x)$ for $x \in S(L')$.

With each object $(P; \leq, \tau, E)$ of $\mathbf{S}$ is associated a quasi-Stone algebra

$$\mathcal{Q}(P) = (Q(P); \cup, \cap, ', \emptyset, P)$$

where $Q(P)$ is the set of all clopen order-filters of P, $\cup$ and $\cap$ are set-theoretical union and intersection, respectively, and for $X \in Q(P)$, $X' = Q(P) \setminus E(X)$. With each morphism $\varphi\colon P \to P'$ in $\mathbf{S}$ is associated a QS-homomorphism $\mathcal{Q}(\varphi)\colon Q(P') \to Q(P)$ such that for $X \in Q(P')$, we have $\mathcal{Q}(\varphi)(X) = \varphi^{-1}(X)$. The natural isomorphisms $\sigma\colon 1_{\mathbf{QS}} \cong \mathcal{QS}$ and $\varepsilon\colon 1_{\mathbf{S}} \cong \mathcal{SQ}$ are given by $\sigma(L)(a) = \{x \in S(L) : a \in x\}$ and $\varepsilon(P)(x) = \{X \in Q(P) : x \in X\}$.

Proposition 2.1 (Gaitán [10])**.** *The category* $\mathbf{QS}$ *is dually equivalent to the category* $\mathbf{S}$*. The dual equivalence is given by the pair of contravariant functors* $\mathcal{S}\colon \mathbf{QS} \to \mathbf{S}$ *and* $\mathcal{Q}\colon \mathbf{S} \to \mathbf{QS}$ *and by the pair of natural isomorphisms* $\sigma\colon 1_{\mathbf{QS}} \cong \mathcal{QS}$ *and* $\varepsilon\colon 1_{\mathbf{S}} \cong \mathcal{SQ}$*. Furthermore, one-to-one and onto morphisms in the category* $\mathbf{QS}$ *correspond, respectively, to onto and one-to-one order-preserving morphisms in* $\mathbf{S}$*.*

Since those varieties $\mathbf{V}$ of quasi-Stone algebras for which $\mathbf{V} \subseteq \mathbf{Q_{2,2}}$ are of particular interest here, the QS-spaces $\mathcal{S}(Q_{m,n})$ of the quasi-Stone algebras $Q_{m,n}$ for which $\mathbf{V}(Q_{m,n}) \subseteq \mathbf{Q_{2,2}}$ are diagrammed in Figure 2. It is to be understood that each of the QS-spaces in Figure 2 have precisely one equivalence class which is the entire space.

Finally, for a QS-space $(P; \leq, \tau, E)$ and $X \subseteq P$, let $\mathrm{Max}(X)$ denote the set of maximal elements of $[X)$, where $[X) = \{y : x \leq y$ for some $x \in X\}$; in the event that $X = \{x\}$, denote $[X)$ by $[x)$ and $\mathrm{Max}(X)$ by $\mathrm{Max}(x)$. Similarly, set $(X] = \{y : y \leq x$ for some $x \in X\}$, and in the event that $X = \{x\}$, denote $(X]$ by $(x]$. If $x \notin \mathrm{Max}(x)$, then x is said to be a *type n element* if $n = |\,\mathrm{Max}(x)|$.

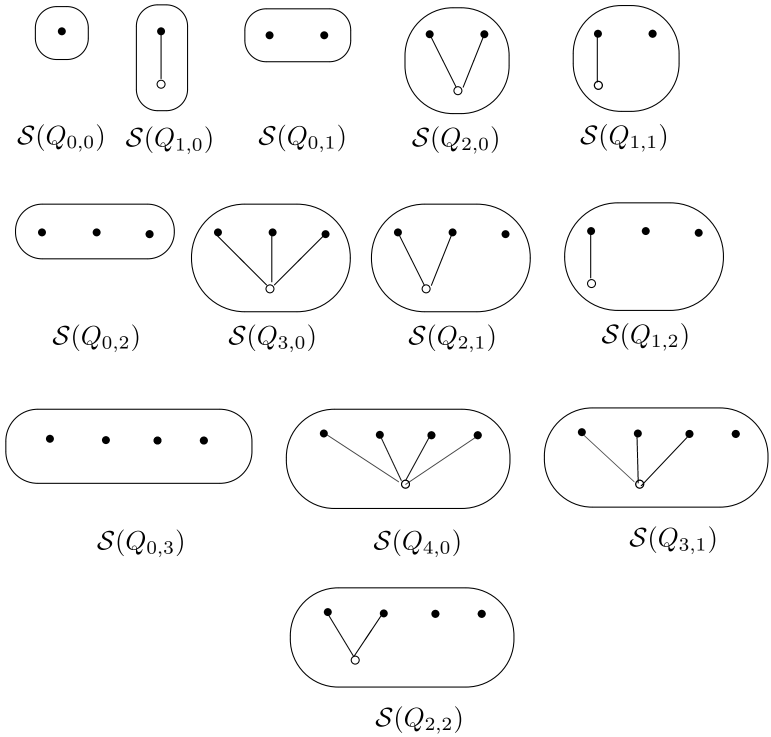

Figure 2

3. The variety $\mathbf{Q_{2,2}}$ (Theorem 1.1)

Throughout this section, $\mathbf{P_5}$ is the category whose objects are structures $(X; \leq, \tau, b_0, b_1, b_2, b_3, b_4)$, where $(X; \leq, \tau)$ is a Priestley space and for $0 \leq i < 5$, b_i is a minimal element in $(X; \leq)$, each set $\{b_i\}$ is clopen in $(X; \tau)$, and for $B = \{b_i : 0 \leq i < 5\}$, $[B) = X$, while $|(x] \cap B| > 1$ for any $x \in X \setminus B$. Morphisms between objects

$$(X; \leq, \tau, b_0, b_1, b_2, b_3, b_4) \text{ and } (X'; \leq', \tau', b'_0, b'_1, b'_2, b'_3, b'_4)$$

are mappings $f \colon X \to X'$ which are morphisms between Priestley spaces $(X; \leq, \tau)$ and $(X'; \leq', \tau')$ satisfying $f(b_i) = b'_i$ for every $0 \leq i < 5$.

In [4] (see also Koubek [14] and Goralčík, Koubek, and Sichler [11]), it is shown that $\mathbf{P_5}$ contains a full subcategory dually isomorphic to the category $\mathbf{G}$, for which finite graphs correspond to finite members of $\mathbf{P_5}$. Thus, in view of Proposition 2.1, in order to establish Theorem 1.1, it is sufficient to construct a functor $\mathbf{\Phi} \colon \mathbf{P_5} \to \mathbf{S}$ which is faithful and full, assigns finite QS-spaces to finite members of $\mathbf{P_5}$, and which satisfies $\mathcal{Q}(\mathbf{\Phi}(X)) \in \mathbf{Q_{2,2}}$ for every object X of $\mathbf{P_5}$. Such a functor is constructed below.

First, the formal definition of $\mathbf{\Phi}$ will be given, then an indication of the underlying idea behind $\mathbf{\Phi}$, followed by a verification that it fulfils the necessary properties.

For each $(X; \leq, \tau, b_0, b_1, b_2, b_3, b_4) \in \mathbf{P_5}$, define $\mathbf{\Phi}(X) = (P; \leq, \tau, E)$ by

$$P = (X \setminus B) \cup A \cup C \cup D_0 \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5),$$

where

$$A = \{a_i : 0 \leq i < 4\}, \quad C = \{c_i : 0 \leq i < 4\},$$
$$D_0 = \{d_{0,i} : 0 \leq i < 3\}, \quad D_1 = \{d_{1,i} : 0 \leq i < 27\},$$
$$D_2 = \{d_{2,i} : 0 \leq i < 29\}, \quad \text{and} \quad E_i = \{e_{i,j} : 0 \leq j < 15\}, \text{ for } 0 \leq i < 5.$$

The partial order $(P; \leq)$ is the least order such that $\mathrm{Max}(P) = A$ and $P \setminus A$ has precisely four elements of type 3, namely, elements of the set C, while all other elements are of type 2.

- The type 3 elements are such that $c_0 \leq a_0, a_1, a_2$, while $c_1, c_2 \leq a_1, a_2, a_3$, and $c_3 \leq a_0, a_2, a_3$.
- The type 2 elements are such that $x \leq a_2, a_3$ for $x \in D_0$, while $x \leq a_1, a_2$ for $x \in (X \setminus B) \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$.

Each of D_0, D_1, D_2, E_0, E_1, E_2, E_3, and E_4 are fences where

- $d_{0,1} \leq d_{0,0}, d_{0,2}$ for D_0,
- $d_{1,2i+1} \leq d_{1,2i}, d_{1,2i+2}$ where $0 \leq i < 13$ for D_1,
- $d_{2,2i+1} \leq d_{2,2i}, d_{2,2i+2}$ where $0 \leq i < 14$ for D_2,
- and if E_0, E_1, E_2, E_3, and E_4, then $e_{i,2j+1} \leq e_{i,2j}, e_{i,2j+2}$ for $0 \leq i < 5$ and $0 \leq j < 7$.

The fences E_0, E_1, E_2, E_3, and E_4 are connected to the fences D_1 and D_2 and the fences D_0, D_1, and D_2 are connected to C, the elements of type 3, as follows

- $d_{1,9+2i} \leq e_{i,0}$, and $d_{2,9+2i} \leq e_{i,14}$ for $0 \leq i < 5$,
- $c_0 \leq d_{1,0}, d_{2,0}$, while $c_1 \leq d_{1,26}$, $c_2 \leq d_{2,28}$, $c_2 \leq d_{0,0}$, and $c_3 \leq d_{0,2}$.

Elements of $X \setminus B$ are connected to the fences E_0, E_1, E_2, E_3, and E_4 by

- $e_{i,7} \leq x$ iff $b_i \leq x$ for $0 \leq i < 5$ and $x \in X \setminus B$.

While, finally, for elements of $X \setminus B$,

- $x \leq y$ in P iff $x \leq y$ in X for $x, y \in X \setminus B$.

Figure 3 visualizes the partial order on the part $A \cup C \cup D_0$ of the domain of $\mathbf{\Phi}(X)$. The solid points on Figure 3 indicate that a_1 and a_2 are connected to every element x of the remaining part of the domain of $\mathbf{\Phi}(X)$, that is, the part $(X \setminus B) \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$ as follows: $x \leq a_1, a_2$. The arrow on Figure 3 from c_1 to a_3 abbreviates the order relation $c_1 \leq a_3$.

Figure 4 helps to visualize the partial order on the part

$$C \cup (X \setminus B) \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$$

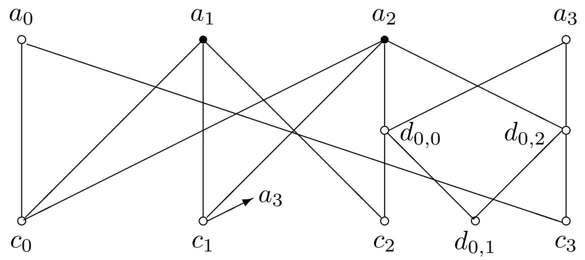

FIGURE 3

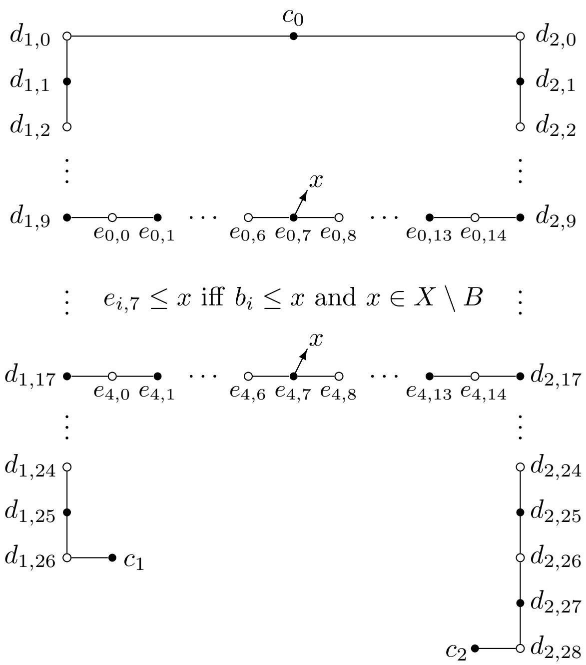

FIGURE 4

of the domain of $\mathbf{\Phi}(X)$ and together with Figure 3 shows how it is linked with the partial order on the part $A \cup C \cup D_0$. It is linked by the elements c_0, c_1, c_2 and by a_1 and a_2 (see Figure 3). The partial order on Figure 4 one should read as follows: $a \leq b$ iff the point assigned to a is solid while the point assigned to b is empty. The arrows on Figure 4 from $e_{0,7}$ to x and from $e_{4,7}$ to x abbreviate the order relation $e_{i,7} \leq x$, where $0 \leq i < 5$ and x is an element of $X \setminus B$ with $b_i \leq x$.

The topology τ on the domain P of $\mathbf{\Phi}(X)$ is the union topology of the clopen subspace $X \setminus B$ of X together with the discrete topology on the finite set $A \cup C \cup D_0 \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$.

The equivalence relation E on P is defined by xEy for all $x, y \in P$. Clearly, $\mathbf{\Phi}(X) = (P; \leq, \tau, E)$ is finite whenever X is. Clearly too, since P has only one equivalence class, $\mathbf{\Phi}(X)$ satisfies (i), (ii), and (iii) of the requirements for a Priestley space to be a QS-space. To see that $\mathbf{\Phi}(X)$ is a Priestley space, suppose that $x \not\leq y$ for some $x, y \in P$. If $x \notin X \setminus B$, then $[x) \cap (X \setminus B) = \emptyset$ unless $x = e_{i,7}$ for some $0 \leq i < 5$. That is, $[x)$ is a finite union of discrete points and, hence, clopen. On the other hand, if $x = e_{i,7}$ for some $0 \leq i < 5$, then we have $[x) = \{e_{i,6}, e_{i,7}, e_{i,8}\} \cup ([b_i) \cap (X \setminus B)) \cup \{a_1, a_2\}$. If $y \notin X \setminus B$, then $[x) = \{e_{i,6}, e_{i,7}, e_{i,8}\} \cup (X \setminus B) \cup \{a_1, a_2\}$ is a clopen order-filter which contains x but not y. If $y \in X \setminus B$, then $b_i \not\leq y$, and there exists a clopen order-filter $F \subseteq X$ such that $[b_i) \subseteq F$ and $y \notin F$, in which case we have that $\{e_{i,6}, e_{i,7}, e_{i,8}\} \cup (F \setminus B) \cup \{a_1, a_2\}$ is a clopen order-filter of P which contains x and not y. If $x \in X \setminus B$, then either $y \notin X \setminus B$ and $(X \setminus B) \cup \{a_1, a_2\}$ will suffice, or $y \in X \setminus B$ and there exists a clopen order-filter $F \subseteq X$ such that $x \in F$ and $y \notin F$, as such, $(F \setminus B) \cup \{a_1, a_2\}$ will serve.

For a morphism $f \colon X \to X'$ in $\mathbf{P_5}$, let $\mathbf{\Phi}(f) \colon P \to P'$ be given by $\mathbf{\Phi}(f)(x) = f(x)$ for $x \in X \setminus B$ and the identity otherwise. Clearly, $\mathbf{\Phi}(f)$ is a QS-map.

Informally, the functor $\mathbf{\Phi}$ works in the following way. For a QS-map $\phi \colon \mathbf{\Phi}(X) \to \mathbf{\Phi}(X')$, since $\mathrm{Max}(\mathbf{\Phi}(X)) = \mathrm{Max}(\mathbf{\Phi}(X'))$ is a 4-element set A, ϕ is 1-1 on it. Further, as $\mathbf{\Phi}(X)$ has no type 1 element, $\phi(\mathbf{\Phi}(X) \setminus A) \subseteq (\mathbf{\Phi}(X') \setminus A)$, and given that $\mathbf{\Phi}(X)$ has no element of type 4, $\phi(C) \subseteq C$ for the set C of type 3 elements. All other elements are type 2. The fences D_0, D_1, and D_2 are of varying length and connect different members of C. They ensure that ϕ is the identity on A, C, D_0, D_1, and D_2. The fences D_1 and D_2 have greater length, guaranteeing that ϕ is also the identity on the 5 fences $(E_i \colon 0 \leq i < 5)$ which are strung between them. By identifying $e_{i,7}$ with b_i for each $0 \leq i < 5$, it follows that $\phi(X \setminus B) \subseteq (X' \setminus B)$, thereby forcing ϕ to mimic $\mathbf{\Phi}(f)$ for some $f \colon X \to X'$. A careful proof of Theorem 1.1 follows.

The following lemma shows that the quasi-Stone algebra represented by each $\mathbf{\Phi}(X)$ belongs to $\mathbf{Q_{2,2}}$.

Lemma 3.1. *For $X \in \mathbf{P_5}$, $\mathcal{Q}(\mathbf{\Phi}(X)) \in \mathbf{Q_{2,2}}$.*

Proof. For $P = \mathbf{\Phi}(X)$, observe that $\mathrm{Max}(P) = A$ and that for $x \in P \setminus A$, $|\mathrm{Max}(x)| \geq 2$. Thus, for each $x \in P \setminus A$, there is a QS-map $\psi_x \colon \mathcal{S}(Q_{2,2}) \to P$ with $\psi_x(\mathcal{S}(Q_{2,2})) = \{x\} \cup A$, where $\mathcal{S}(Q_{2,2})$ is the QS-space of the algebra $Q_{2,2}$ diagrammed in Figure 2; i.e., the set of QS-maps $(\mathcal{Q}(\psi_x) : x \in P \setminus A)$ represent homomorphisms from the quasi-Stone algebra represented by $\mathcal{Q}(\mathbf{\Phi}(X))$ to $Q_{2,2}$ which separate the elements of the algebra. Since $Q_{2,2}$ generates the variety $\mathbf{Q_{2,2}}$, the quasi-Stone algebra $\mathcal{Q}(\mathbf{\Phi}(X))$ represented by the QS-space $\mathbf{\Phi}(X)$ belongs to $\mathbf{Q_{2,2}}$. □

Clearly, $\mathbf{\Phi}\colon \mathbf{P_5} \to \mathbf{S}$ is faithful. The following lemma shows that $\mathbf{\Phi}$ is full.

Lemma 3.2. *For $X, X' \in \mathbf{P_5}$, if $\phi\colon \mathbf{\Phi}(X) \to \mathbf{\Phi}(X')$ is a QS-map, then there exists a morphism $f\colon X \to X'$ such that $\phi = \mathbf{\Phi}(f)$.*

Proof. Let $P = \mathbf{\Phi}(X)$ and $P' = \mathbf{\Phi}(X')$. Since ϕ is a QS-map and $\mathrm{Max}(P) = \mathrm{Max}(P') = A$, $\phi(A) = A$ follows from the definition of QS-map. Consequently, $\phi(P \setminus A) \subseteq P' \setminus A$ because for $x \in P \setminus A$, $|\mathrm{Max}(x)| \geq 2$. Moreover, since for $x \in P \setminus C$, $|\mathrm{Max}(x)| \leq 2$, and for $x \in C$, $|\mathrm{Max}(x)| = 3$, it follows that $\phi(C) \subseteq C$. As $\mathrm{Max}(c_2) \neq \mathrm{Max}(c_3)$, $\phi(c_2) \neq \phi(c_3)$. Since D_0 is the shortest path between c_2 and c_3 in the comparability graph of $(P \setminus A; \leq)$ and since this path is shorter than any other path between distinct c_i and c_j, $\phi(\{c_2, c_3\}) = \{c_2, c_3\}$.

As $\mathrm{Max}(c_0) \neq \mathrm{Max}(c_2)$ and $\mathrm{Max}(c_0) \neq \mathrm{Max}(c_3)$, $\mathrm{Max}(\phi(c_0)) \neq \mathrm{Max}(\phi((c_2))$ and $\mathrm{Max}(\phi(c_0)) \neq \mathrm{Max}(\phi(c_3))$. Since $\phi(\{c_2, c_3\}) = \{c_2, c_3\}$, this means that $\mathrm{Max}(\phi(c_0)) \neq \{a_1, a_2, a_3\}$ and $\mathrm{Max}(\phi(c_0)) \neq \{a_0, a_2, a_3\}$. Notice that we have $\mathrm{Max}(c_0) = \{a_0, a_1, a_2\}$, $\mathrm{Max}(c_1) = \mathrm{Max}(c_2) = \{a_1, a_2, a_3\}$, and $\mathrm{Max}(c_3) = \{a_0, a_2, a_3\}$. So, as $\phi(C) \subseteq C$, it follows that $\phi(c_0) = c_0$.

We show that $\phi(c_2) = c_2$ and $\phi(c_3) = c_3$. Supposing otherwise, we have by $\phi(\{c_2, c_3\}) = \{c_2, c_3\}$ that $\phi(c_2) = c_3$. Since $\mathrm{Max}(c_1) = \mathrm{Max}(c_2)$, we have $\mathrm{Max}(\phi(c_1)) = \mathrm{Max}(c_3)$. So, as $\phi(C) \subseteq C$ and for $i < 4$, $\mathrm{Max}(c_i) = \mathrm{Max}(c_3)$ iff $i = 3$, we conclude that $\phi(c_1) = c_3$. However, D_1 is the shortest path between c_0 and c_1 in the comparability graph of $(P \setminus A; \leq)$ and it is shorter than any path between c_0 and c_3. Thus, $\phi(c_2) = c_2$ and $\phi(c_3) = c_3$.

We now show that $\phi(c_1) = c_1$. As we have $\mathrm{Max}(\phi(c_0)) = \{a_0, a_1, a_2\}$ and $\mathrm{Max}(\phi(c_3)) = \{a_0, a_2, a_3\}$, and as $\mathrm{Max}(\phi(c_1)) = \{a_1, a_2, a_3\}$ because $\mathrm{Max}(c_1) = \mathrm{Max}(c_2)$, we conclude that $\phi(c_1) \neq \phi(c_0)$ and $\phi(c_1) \neq \phi(c_3)$. Thus, $\phi(c_1) \in \{c_1, c_2\}$. Now, since the shortest path between c_0 and c_2 in the comparability graph of $(P \setminus A; \leq)$ is longer than that between c_0 and c_1, it follows from $\phi(c_0) = c_0$ that $\phi(c_1) = c_1$.

We have shown that $\phi(c_i) = c_i$ for $0 \leq i < 4$. Thus: for $0 \leq i < 3$, $\phi(d_{0,i}) = d_{0,i}$ since D_0 is the unique shortest path between c_2 and c_3 in the comparability graph of $(P \setminus A; \leq)$; for $0 \leq i < 27$, $\phi(d_{1,i}) = d_{1,i}$ since D_1 is the unique shortest path between c_0 and c_1; for $0 \leq i < 29$, $\phi(d_{2,i}) = d_{2,i}$ since D_2 is the unique shortest path between c_0 and c_2. Furthermore, for $0 \leq i < 5$ and $0 \leq j < 15$, $\phi(e_{i,j}) = e_{i,j}$ since E_i is the shortest path between $d_{1,9+2i}$ and $d_{2,9+2i}$ in the comparability graph of $(P \setminus A; \leq)$.

We now show that $\phi(x) \in X' \setminus B$ for $x \in X \setminus B$. Let $x \in X \setminus B$. Then by $|(x] \cap B| > 1$, there are $0 \leq i \neq j < 5$ such that $b_i \leq x$ and $b_j \leq x$ in $(X; \leq)$. This implies that $e_{i,7} \leq x$ and $e_{j,7} \leq x$ in $(P; \leq)$. As $\phi(e_{i,7}) = e_{i,7}$ and $\phi(e_{j,7}) = e_{j,7}$, we have $e_{i,7} \leq \phi(x)$ and $e_{j,7} \leq \phi(x)$ in $(P'; \leq)$. But $X \setminus B \subseteq P \setminus A$ and $\phi(P \setminus A) \subseteq P' \setminus A$. So $\phi(X \setminus B) \subseteq P' \setminus A$, and therefore $\phi(x) \notin A$. Since for $y \in P'$, if $e_{i,7} \leq y$ and $e_{j,7} \leq y$, then $y \in \{a_1, a_2\}$ or $y \in X' \setminus B$, it follows that $\phi(x) \in X' \setminus B$.

Finally, as $\mathrm{Max}(c_0)\cap\mathrm{Max}(c_1) = \{a_1, a_2\}$ and $\mathrm{Max}(c_0)\cap\mathrm{Max}(c_3) = \{a_0, a_2\}$, $\phi(\{a_1, a_2\}) = \{a_1, a_2\}$ and $\phi(\{a_0, a_2\}) = \{a_0, a_2\}$. In particular, $\phi(a_2) = a_2$. It then follows that $\phi(a_0) = a_0$ and $\phi(a_1) = a_1$, from which $\phi(a_3) = a_3$.

To summarize, $\phi = \Phi(f)$ where $f\colon X \to X'$ is given by $f = \phi\restriction_{(X\setminus B)}$ on $X \setminus B$ and the identity on B. $\square$

4. The variety $\mathbf{Q_{3,1}}$ (Theorem 1.3)

An algebra is called *endomorphism rigid* or just *rigid* if the identity map is its only endomorphism. To establish Theorem 1.3, it is sufficient to show that, up to isomorphism, $\mathbf{Q_{3,1}}$ contains only two rigid algebras, the one-element and two-element algebras (since any finite-to-finite universal variety contains a proper class of non-isomorphic rigid algebras).

Throughout this section, $(P; \leq, \tau, E)$ denotes the QS-space of a quasi-Stone algebra $L \in \mathbf{Q_{3,1}}$ that is rigid.

Since $\mathbf{Q_{3,1}}$ is generated by the algebra $Q_{3,1}$ (the QS-space $\mathcal{S}(Q_{3,1})$ is diagrammed in Figure 2), each $x \in P$ is in the QS-map image of some QS-map $\psi_x\colon \mathcal{S}(Q_{3,1}) \to P$. In particular, for $x \in P$, $|\,\mathrm{Max}(E(x))| \leq 4$ and, in the event that $|\,\mathrm{Max}(E(x))| = 4$, either $|\,\mathrm{Max}(x)| = 3$ or 4, or $x \in \mathrm{Max}(E(x))$. Moreover, since $\mathbf{Q_{1,2}}$ is generated by the algebra $Q_{1,2}$ (the QS-space $\mathcal{S}(Q_{1,2})$ is diagrammed in Figure 2), it follows that $L \notin \mathbf{Q_{1,2}}$ if and only if $|\,\mathrm{Max}(E(x))| = 4$ for some $x \in P$.

The proof of Theorem 1.3 falls naturally into three cases. In Lemma 4.1, it is shown that $L \in \mathbf{Q_{1,2}}$, then in Lemma 4.2, that $L \in \mathbf{Q_{1,1}}$, and finally, in Lemma 4.3, that L is a one-element or a two-element algebra. In each case, the proof is similar but different. We have tried to emphasise the similarities in order to assist the reader with the technical details.

The basic idea behind the proof of Lemma 4.1 is as follows: By the above remarks, for $L \notin \mathbf{Q_{1,2}}$, there are two possibilities: there exists $x \in P$ such that $|\,\mathrm{Max}(x)| = 4$ or, if such an x does not exist, there exists $x \in P$ such that $|\,\mathrm{Max}(E(x))| = 4$ where either $|\,\mathrm{Max}(x)| = 3$ or $\mathrm{Max}(x) = \{x\}$. Consider the the first case. Set $\mathrm{Max}(x) = \{a_0, a_1, a_2, a_3\}$ and denote x by $a_{0,1,2,3}$. If P were finite, then for $A = E(x)$, the mapping $\phi\colon P \to P$, given by

$$\phi(x) = \begin{cases} x & \text{if } x = a_i \text{ for some } 0 \leq i < 4, \\ a_{0,1,2,3} & \text{if } x \in A \setminus \{a_0, a_1, a_2, a_3\}, \\ x & \text{otherwise,} \end{cases}$$

is a QS-map. (Recall that A is both an order-ideal as well as an order-filter.) As L is rigid, ϕ must be trivial, but then there is a non-trivial $\psi\colon \phi(P) \to \phi(P)$ given, for example, by $\psi(a_0) = a_1$, $\psi(a_1) = a_0$, and $\psi(x) = x$ otherwise, a contradiction. If P is not finite, then greater care must be taken. Again, let $\mathrm{Max}(x) = \{a_0, a_1, a_2, a_3\}$ and denote x by $a_{0,1,2,3}$. The objective now is to choose a clopen set $A \subseteq P$ such that $E(A) = A$ (whereupon A is an order-ideal as well as an order-filter), which may be partitioned into five clopen sets $(A_i \colon 0 \leq i < 4)$ and $A \setminus \bigcup(A_i \colon 0 \leq i < 4)$ such that for $0 \leq i, j < 4$, we have $a_i \in A_i$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\mathrm{Max}(A) = \bigcup(A_i \colon 0 \leq i < 4)$, and $E(A_i) = E(A_j) = A$. It will follow that $\phi \colon P \to P$, given by

$$\phi(x) = \begin{cases} a_i & \text{if } x \in A_i \text{ for some } 0 \leq i < 4, \\ a_{0,1,2,3} & \text{if } x \in A \setminus \bigcup(A_i \colon 0 \leq i < 4), \\ x & \text{otherwise,} \end{cases}$$

is a QS-map, and as in the finite case, $\psi \colon \phi(P) \to \phi(P)$ will show that L is not rigid. Consider the second case, where there does not exist an $x \in P$ such that $|\mathrm{Max}(x)| = 4$ and, in particular, for every $x \in P$ such that $|\mathrm{Max}(E(x))| = 4$, either $|\mathrm{Max}(x)| = 3$ or $\mathrm{Max}(x) = \{x\}$. Choose such an x and set $\mathrm{Max}(x) = \{a_0, a_1, a_2, a_3\}$. For distinct $0 \leq i, j, k < 4$, if $(a_i] \cap (a_j] \cap (a_k] \neq \emptyset$, choose some distinguished element $a_{i,j,k} \in (a_i] \cap (a_j] \cap (a_k]$. If P were finite, then for $A = E(x)$, the mapping $\phi \colon P \to P$, given by

$$\phi(x) = \begin{cases} x & \text{if } x = a_i \text{ for some } 0 \leq i < 4, \\ a_{i,j,k} & \text{if } x \in (a_i] \cap (a_j] \cap (a_k] \text{ for distinct } 0 \leq i, j, k < 4, \\ x & \text{otherwise,} \end{cases}$$

would be a QS-map. Since L is rigid, ϕ must be trivial, but then, as before, there exists a suitable non-trivial $\psi \colon \phi(P) \to \phi(P)$, which must be chosen more carefully this time, depending on which, if any, distinct $0 \leq i, j, k < 4$ are such that $(a_i] \cap (a_j] \cap (a_k] \neq \emptyset$, thereby leading to a contradiction. If P is not finite, then even greater care must be taken than before. The objective now is to choose a clopen set $A \subseteq P$ such that $E(A) = A$ and which may be partitioned into between four and eight clopen sets $(A_i \colon 0 \leq i < 4)$ and $(A_{i,j,k} \colon 0 \leq i, j, k < 4$ are distinct) (the size of the partition depending on whether or not $(a_i] \cap (a_j] \cap (a_k] \neq \emptyset$ for particular $0 \leq i, j, k < 4$) such that for distinct $0 \leq i, j, k < 4$, $a_i \in A_i$, $a_{i,j,k} \in A_{i,j,k}$, $\mathrm{Max}(A) = \bigcup(A_i \colon 0 \leq i < 4)$, and $E(A_i) = E(A_j) = A$. It will follow that $\phi \colon P \to P$ given by

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \leq i < 4,\ x \in A_i, \\ a_{i,j,k} & \text{if, for distinct } 0 \leq i, j, k < 4,\ x \in A_{i,j,k}, \\ x & \text{otherwise,} \end{cases}$$

is a QS-map and, as in the finite case, a suitable $\psi \colon \phi(P) \to \phi(P)$ will show that L is not rigid. To ensure that everything works as anticipated, the family $(A_i \colon 0 \leq i < 4)$ must be chosen carefully. First, a family $(B_i \colon 0 \leq i < 4)$ is chosen in the same way as the family $(A_i \colon 0 \leq i < 4)$ was chosen in the first case; in turn, the family $(B_i \colon 0 \leq i < 4)$ is refined to choose a suitable family $(A_i \colon 0 \leq i < 4)$. In both cases, we will observe that for $0 \leq i < 4$, A_i is an antichain. This fact is not explicitly used in the proof of Lemma 4.1, rather it is to clarify a difference between the proof of Lemma 4.1 and those of Lemma 4.2 and Lemma 4.3, where such a choice may not be possible.

Lemma 4.1. $L \in \mathbf{Q_{1,2}}$.

Proof. Suppose on the contrary that $L \notin \mathbf{Q_{1,2}}$. Then, as noted above, there exists an x in P such that $|\operatorname{Max}(E(x))| = 4$ and either $|\operatorname{Max}(x)| = 3$ or 4, or $x \in \operatorname{Max}(E(x))$.

Case 1. There exists an $x \in P$ such that $|\operatorname{Max}(E(x))| = 4$ and $|\operatorname{Max}(x)| = 4$.

Let the x be denoted by $a_{0,1,2,3}$ and let $\operatorname{Max}(E(x)) = \{a_i : 0 \le i < 4\}$. Since $(P; \le, \tau)$, being a Priestley space, is totally order-disconnected, there are mutually disjoint clopen order-filters A'_i, where $0 \le i < 4$, such that $a_i \in A'_i$. For $0 \le i < 4$, let $A_i = A'_i \cap \bigcap(E(A'_j) : 0 \le j < 4 \text{ and } j \neq i)$.

By the definition of a QS-space, every set $E(A'_j)$ is clopen. Further, if $z \ge y \in E(A'_j)$, then $z \in E(y)$ and $z \in E(A'_j)$. That is, each $E(A'_j)$ is an order-filter. Obviously, $a_i E a_j$ for $0 \le i, j < 4$, so $a_i \in E(A'_j)$ for any $0 \le i, j < 4$. Thus, $(A_i : 0 \le i < 4)$ are mutually disjoint clopen order-filters and $a_i \in A_i$. We prove the following:

For $y \in P$, if $E(y) \cap A_i \neq \emptyset$ for some $i < 4$, then

(1) $E(y) \cap A_j \neq \emptyset$ for all $j < 4$,
(2) $|\operatorname{Max}(E(y))| = 4$, and
(3) $|\operatorname{Max}(E(y)) \cap A_j| = 1$ for all $j < 4$.

Assume $E(y) \cap A_i \neq \emptyset$ for some $i < 4$. Since A_i is an order-filter, $z_i \in A_i$ for some $z_i \in \operatorname{Max}(E(y))$. Thus, $z_i \in A'_i$ and $z_i \in E(A'_j)$ for all $j \neq i$. Consequently, $y \in E(A'_j)$ for all $j \neq i$. Thus, for every $j \neq i$, there exists $w_j \in A'_j$ such that yEw_j, that is $w_j \in E(y)$. For each $j \neq i$, let $z_j \in \operatorname{Max}(E(y))$ be such that $w_j \le z_j$. Since A'_j is an order-filter, $z_j \in A'_j$. As $z_k \in E(y)$ and $z_k \in A'_k$ for all $k < 4$, we have $\{z_0, z_1, z_2, z_3\} \subseteq E(A'_k)$ for all $k < 4$. Thus, $z_j \in E(y) \cap A_j$ for all $j < 4$. This proves (1). (2) and (3) follow from (1) and the facts that $|\operatorname{Max}(E(y))| \le 4$ and that $(A_i : 0 \le i < 4)$ are mutually disjoint order-filters.

The following are true.

(4) $E(A_i) = E(A_j)$ for all $0 \le i, j < 4$, and
(5) $(A_i : 0 \le i < 4)$ are antichains in $(P; \le)$.

Then (4) follows from (1). To see (5), recall that $L \in \mathbf{Q_{3,1}}$. So, for every $y \in P$ with $y \notin \operatorname{Max}(E(y))$, the following holds: $|\operatorname{Max}(y) \cap \operatorname{Max}(E(y))| \ge 2$, or $|\operatorname{Max}(y)| = 1$ and $|\operatorname{Max}(E(y))| \le 3$. Thus, (5) follows from (3) and the fact that $(A_i : 0 \le i < 4)$ are mutually disjoint order-filters.

Now, set $A = E(A_i) = E(A_j)$ for $0 \le i, j < 4$, and define the mapping $\phi\colon P \to P$ as follows:

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \le i < 4,\ x \in A_i, \\ a_{0,1,2,3} & \text{if } x \in A \setminus \bigcup(A_i : 0 \le i < 4), \\ x & \text{otherwise.} \end{cases}$$

Since A is clopen as too are A_i for $0 \leq i < 4$, ϕ is continuous. Let $x \leq y$ in $(P; \leq, \tau, E)$. Since $(P; \leq, \tau, E)$ is a QS-space, $E(x) = E(y)$. So, x belongs to A if and only if y does. Hence, $\phi(x) = x \leq y = \phi(y)$ in case $x \notin A$. So, let both x and y belong to A. If $x \in \bigcup(A_i : 0 \leq i < 4)$, then, by (5) and $x \leq y$, we have $x = y$, and so $\phi(x) \leq \phi(y)$. If $x, y \in A \setminus \bigcup(A_i : 0 \leq i < 4)$, then $\phi(x) = a_{0,1,2,3} = \phi(y)$, and so $\phi(x) \leq \phi(y)$. If $x \in A \setminus \bigcup(A_i : 0 \leq i < 4)$ and $y \in \bigcup(A_i : 0 \leq i < 4)$, then $\phi(x) = a_{0,1,2,3}$ and $\phi(y) = a_i$ for some $0 \leq i < 4$. As $a_{0,1,2,3} \leq a_i$ in $(P; \leq)$ for all $i < 4$, we have that $\phi(x) \leq \phi(y)$, proving that ϕ is order-preserving. By (1)–(5), $\phi(\text{Max}(E(x))) \supseteq \text{Max}(E(\phi(x)))$ for all $x \in P$. Thus, ϕ is a QS-map.

Recall that L has been assumed to be rigid and that $(P; \leq, \tau, E)$ is a QS-space of L. Thus, if ϕ is not the identity map, then L is not rigid, and we get a contradiction. So, let us assume that $\phi = id$. This means that $A_i = \{a_i\}$ for $0 \leq i < 4$ and $A \setminus \bigcup(A_i : 0 \leq i < 4) = \{a_{0,1,2,3}\}$. In particular, the singleton sets $\{a_i\}$, where $i < 4$, are clopen. This is why the map $\psi \colon P \to P$, given by

$$\psi(x) = \begin{cases} a_1 & \text{if } x = a_0, \\ a_0 & \text{if } x = a_1, \\ x & \text{otherwise}, \end{cases}$$

is continuous. Obviously, it is order-preserving and satisfies the remaining requirements of a QS-map. Evidently, $\phi \neq id$, contradicting the rigidity of L.

In view of Case 1 just considered, we may assume that for every $x \in P$, if $|\text{Max}(E(x))| = 4$, then either $|\text{Max}(x)| = 3$ or $x \in \text{Max}(E(x))$.

Case 2. There exists an $x \in P$ such that $|\text{Max}(E(x))| = 4$ and either $|\text{Max}(x)| = 3$ or $x \in \text{Max}(E(x))$.

Choose one such $x \in P$ and let $\text{Max}(E(x)) = \{a_i : 0 \leq i < 4\}$. By hypothesis, if $x \in \bigcup((a_i] \setminus \{a_i\} : 0 \leq i < 4)$, then $x \in (a_i] \cap (a_j] \cap (a_k]$ for distinct $0 \leq i, j, k < 4$. Whenever $(a_i] \cap (a_j] \cap (a_k]$ is non-empty for distinct $0 \leq i, j, k < 4$, choose an element of it and denote it by $a_{i,j,k}$; note that $a_{i,j,k} \not\leq a_l$ for $l \neq i, j, k$.

As in Case 1, for $0 \leq i < 4$, choose mutually disjoint clopen order-filters B'_i such that $a_i \in B'_i$ and set $B_i = B'_i \cap \bigcap(E(B'_j) : 0 \leq j < 4 \text{ and } j \neq i)$.

It follows, just as in Case 1, that $(B_i : 0 \leq i < 4)$ are mutually disjoint clopen order-filters, $a_i \in B_i$ for $0 \leq i < 4$, and for $y \in P$, if $E(y) \cap B_i \neq \emptyset$ for some $i < 4$, then

(1) $E(y) \cap B_j \neq \emptyset$ for all $j < 4$,
(2) $|\text{Max}(E(y))| = 4$,
(3) $|\text{Max}(E(y)) \cap B_j| = 1$ for all $j < 4$,
(4) $E(B_i) = E(B_j)$ for all $0 \leq i, j < 4$, and
(5) $(B_i : 0 \leq i < 4)$ are antichains in $(P; \leq)$.

Set $B = E(B_i) = E(B_j)$ for $0 \leq i, j < 4$. By (2) and (3), $y \notin \text{Max}(P)$ for $y \in B \setminus \bigcup(B_i : 0 \leq i < 4)$, and, by hypothesis, $|\text{Max}(y)| = 3$. In particular, $y \in (B_i] \cap (B_j] \cap (B_k]$ for some distinct $0 \leq i, j, k < 4$, while $y \notin (B_l]$ for

$l \neq i,j,k$. Since $(B_i : 0 \leq i < 4)$ are clopen and $(B_i] \cap (B_j] \cap (B_k]$ is closed for distinct $0 \leq i,j,k < 4$, we have

(6) $(B_i : 0 \leq i < 4) \cup ((B_i] \cap (B_j] \cap (B_k] : \text{for distinct } 0 \leq i,j,k < 4)$ is a clopen partition of B.

Suppose we have $y \in (B_i] \cap (B_j] \cap (B_k]$ for some distinct $0 \leq i,j,k < 4$, but $(a_i] \cap (a_j] \cap (a_k] = \emptyset$. Since $y \leq b_l \in B_l$ but $y \nleq a_l \in B_l$ for some $l = i, j$ or k, so $\mathrm{Max}(E(y)) \cap \{a_i : 0 \leq i < 4\} = \emptyset$ by (3). In particular, there exists a clopen order-filter Y such that $y \in Y$ and $Y \cap \{a_i : 0 \leq i < 4\} = \emptyset$. By compactness, there exists a clopen order-filter $B'_{i,j,k}$ such that

$$B'_{i,j,k} \supseteq (B_i] \cap (B_j] \cap (B_k] \text{ and } B'_{i,j,k} \cap \{a_i : 0 \leq i < 4\} = \emptyset.$$

Since $B'_{i,j,k}$ is an order-filter, that $E(x) \cap B'_{i,j,k} = \emptyset$. Set $B_{i,j,k} = E(B'_{i,j,k})$, which, since $B'_{i,j,k}$ is a clopen order-filter, is also a clopen order-filter.

For $0 \leq i < 4$, let

$$A'_i = B_i \setminus \bigcup(B_{j,k,l} : (a_j] \cap (a_k] \cap (a_l] = \emptyset \text{ for distinct } 0 \leq j,k,l < 4).$$

Then $(A'_i : 0 \leq i < 4)$ are mutually disjoint clopen order-filters with $a_i \in A'_i$. Set $A_i = A'_i \cap \bigcap(E(A'_j) : 0 \leq j < 4 \text{ and } j \neq i)$. As above, $(A_i : 0 \leq i < 4)$ are mutually disjoint clopen order-filters, $a_i \in A_i$ for $0 \leq i < 4$, and, for $y \in P$, if $E(y) \cap A_i \neq \emptyset$ for some $i < 4$, then

(1) $E(y) \cap A_j \neq \emptyset$ for all $j < 4$,
(2) $|\mathrm{Max}(E(y))| = 4$,
(3) $|\mathrm{Max}(E(y)) \cap A_j| = 1$ for all $j < 4$,
(4) $E(A_i) = E(A_j)$ for all $0 \leq i,j < 4$, and
(5) $(A_i : 0 \leq i < 4)$ are antichains in $(P; \leq)$.

Set $A = E(A_i) = E(A_j)$ for $0 \leq i,j < 4$ and for distinct $0 \leq i,j,k < 4$, let $A_{i,j,k}$ denote the closed set $(A_i] \cap (A_j] \cap (A_k]$. Then

(6) $(A_i : 0 \leq i < 4) \cup (A_{i,j,k} : \text{for distinct } 0 \leq i,j,k < 4)$ is a clopen partition of A,
(7) for distinct $0 \leq i,j,k < 4$, if $A_{i,j,k} \neq \emptyset$, then $(a_i] \cap (a_j] \cap (a_k] \neq \emptyset$, and
(8) for distinct $0 \leq i,j,k < 4$ and $0 \leq p,q,r < 4$, if $y \leq z$, $y \in A_{i,j,k}$, and $z \in A_{p,q,r}$, then $\{i,j,k\} = \{p,q,r\}$.

If $y \in A_{i,j,k}$ and $l \in \{i,j,k\}$, $y \leq b_l \in A_l \subseteq A'_l \subseteq B_l$. If $(a_i] \cap (a_j] \cap (a_k] = \emptyset$, then $y \in B'_{i,j,k}$ and $\mathrm{Max}(E(y)) \subseteq B_{i,j,k}$. Hence, for any $0 \leq l < 4$, we have that $\mathrm{Max}(E(y)) \cap A'_l = \emptyset$, that is, $\mathrm{Max}(E(y)) \cap A_l = \emptyset$, contradicting $y \in A$ and verifying (7); (8) follows from (1) and (2).

Consider the mapping $\phi\colon P \to P$ given by

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \leq i < 4,\ x \in A_i, \\ a_{i,j,k} & \text{if, for distinct } 0 \leq i,j,k < 4,\ x \in A_{i,j,k}, \\ x & \text{otherwise.} \end{cases}$$

By (6), ϕ is continuous. Let $x \leq y$ in $(P; \leq, \tau)$. Since $E(x) = E(y)$, $x \in A$ iff $y \in A$. As in Case (1), if $x \notin A$, then $\phi(x) = x \leq y = \phi(y)$. Assume

$x, y \in A$. Again by (5), if $x \in \bigcup(A_i : 0 \le i < 4)$, then $x = y$ and it follows that $\phi(x) \le \phi(y)$. If $x, y \in A \setminus \bigcup(A_i : 0 \le i < 4)$, then by (8), both $x, y \in A_{i,j,k}$ for some distinct $0 \le i, j, k < 4$, from which $\phi(x) = \phi(y) = a_{i,j,k}$ by (7). If $x \in A \setminus \bigcup(A_i : 0 \le i < 4)$ and $y \in \bigcup(A_i : 0 \le i < 4)$, then $x \in A_{i,j,k}$ for some distinct $0 \le i, j, k < 4$ and $y \in A_l$ for some $l \in \{i, j, k\}$. In particular, $\phi(x) = a_{i,j,k}$ and $\phi(y) = a_l$, where $a_{i,j,k} < a_l$. By (1), ϕ is a QS-map.

Were ϕ non-identity, it would violate the rigidity of L. So suppose otherwise, that ϕ is the identity map. Observe then that by (6), each non-empty set of $(A_i : 0 \le i < 4) \cup (A_{i,j,k} :$ for distinct $0 \le i, j, k < 4)$ is an isolated point of $(P; \tau)$. First consider the case that there exist distinct p and q such that whenever $A_{i,j,k} \ne \emptyset$, we have $p, q \in \{i, j, k\}$. That is, either $A_{i,j,k} = \emptyset$ for all distinct $0 \le i, j, k < 4$ or $A_{p,q,k} \ne \emptyset$ for some $k \ne p, q$, where there are two possible values for k and $A_{p,q,k} \ne \emptyset$ may hold for either one or both. Let ζ denote the permutation on $\{i : 0 \le i < 4\}$ given by $\zeta(p) = q$, $\zeta(q) = p$, and the identity otherwise. Then $\psi \colon P \to P$, given by

$$\psi(x) = \begin{cases} a_{\zeta(i)} & \text{if } x = a_i, \\ a_{\zeta(i),\zeta(j),\zeta(k)} & \text{if } x = a_{i,j,k}, \\ x & \text{otherwise,} \end{cases}$$

yields a non-identity QS-map on P. Failing this, $A_{i,j,k} \ne \emptyset$ for some distinct $0 \le i, j, k < 4$ and either there exists precisely one p such that $p \in \{i, j, k\}$ whenever $A_{i,j,k} \ne \emptyset$ or no such p exists. If a unique p exists, then $A_{p,j,k} \ne \emptyset$ for all valid choices of j and k, of which there are three. In this case, choose distinct j and k both of which are distinct from p and let ζ denote the permutation on $\{i : 0 \le i < 4\}$ given by $\zeta(j) = k$, $\zeta(k) = j$, with the identity elsewhere. Should no such p exist, then $A_{i,j,k} \ne \emptyset$ for all distinct $0 \le i, j, k < 4$. In this case, simply choose distinct j and k, and again let ζ denote the permutation on $\{i : 0 \le i < 4\}$ given by $\zeta(j) = k$, $\zeta(k) = j$, with the identity elsewhere. In either case, ψ as given above but with the modified ζ provides a non-trivial QS-map, contradicting the rigidity of L. □

Informally, the proof of Lemma 4.1 shows that if for some $x \in P$, we have $|\operatorname{Max}(E(x))| = 4$, then L is not rigid. Now Lemma 4.2 will show that if for some $x \in P$, $|\operatorname{Max}(E(x))| = 3$, then it is also the case that L is not rigid. As with Lemma 4.1, there are two cases. First there exists $x \in P$ such that $|\operatorname{Max}(x)| = 3$ while next, for no $x \in P$ is it the case that $|\operatorname{Max}(x)| = 3$, but there is an $x \in P$ such that $|\operatorname{Max}(E(x))| = 3$ (from which it follows that $|\operatorname{Max}(x)| = 1$ or 2, or $x \in \operatorname{Max}(E(x))$). As in Lemma 4.1, the objective is to choose a suitable clopen set $A \subseteq P$ for which $E(A) = A$ and to partition it with a view to defining QS-maps ϕ and ψ. As in Lemma 4.1, the choice of A is more subtle in the second case. With clarity in mind, we have presented the proof in a similar format to that of Lemma 4.1. So too for the proof of Lemma 4.3, which shows that if for all $x \in P$, $|\operatorname{Max}(E(x))| \le 2$, then L is a one-element or two-element algebra.

Lemma 4.2. $L \in \mathbf{Q}_{1,1}$.

Proof. Suppose $L \notin \mathbf{Q}_{1,1}$; then by Lemma 4.1, $|\operatorname{Max}(E(x))| < 4$ for $x \in P$, but by hypothesis, $|\operatorname{Max}(E(x))| = 3$ for some $x \in P$.

Case 1. There exists $x \in P$ such that $|\operatorname{Max}(E(x))| = 3$ and $|\operatorname{Max}(x)| = 3$.

Let the x be denoted $a_{0,1,2}$ and set $\operatorname{Max}(E(x)) = \{a_i : 0 \le i < 3\}$. As in Case 1 of Lemma 4.1, for $0 \le i < 3$, choose mutually disjoint clopen order-filters A'_i such that $a_i \in A'_i$ and again set

$$A_i = A'_i \cap \bigcap (E(A'_j) : 0 \le j < 3 \text{ and } j \ne i).$$

Arguing as before, $(A_i : 0 \le i < 3)$ are mutually disjoint clopen order-filters, $a_i \in A_i$ for every $0 \le i < 3$, and for $y \in P$, if $E(y) \cap A_i \ne \emptyset$ for some $i < 3$, then

(1) $E(y) \cap A_j \ne \emptyset$ for all $j < 3$,
(2) $|\operatorname{Max}(E(y))| = 3$,
(3) $|\operatorname{Max}(E(y)) \cap A_j| = 1$ for all $j < 3$, and
(4) $E(A_i) = E(A_j)$ for all $0 \le i, j < 3$.

However, the presence of type 1 elements means that it no longer need be the case that each A_i is an antichain.

Set $A = E(A_i) = E(A_j)$ for $0 \le i, j < 3$, and let the mapping $\phi \colon P \to P$ be given by

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \le i < 3,\ x \in A_i, \\ a_{0,1,2} & \text{if } x \in A \setminus \bigcup (A_i : 0 \le i < 3), \\ x & \text{otherwise.} \end{cases}$$

As A is clopen, as is A_i for each $0 \le i < 4$, so ϕ is continuous. Let $x \le y$ in $(P; \le, \tau, E)$. Since $E(x) = E(y)$, x belongs to A if and only if y does. Hence, $\phi(x) = x \le y = \phi(y)$ whenever $x \notin A$. Assume both x and y belong to A. If $x \in A_i$ for some $0 \le i < 3$, then $y \in A_i$ and $\phi(x) = a_i = \phi(y)$, whence $\phi(x) \le \phi(y)$. If $x \in A \setminus \bigcup (A_i : 0 \le i < 3)$, then $\phi(x) = a_{0,1,2}$. Either $y \in A \setminus \bigcup (A_i : 0 \le i < 3)$ and $\phi(y) = a_{0,1,2} = \phi(x)$, or $y \in A_i$ for some $0 \le i < 3$ and $\phi(y) = a_i$. Either way, $\phi(x) \le \phi(y)$ since $a_{0,1,2} \le a_i$ for every $0 \le i < 3$. By (1)–(4), $\phi(\operatorname{Max}(E(x))) \supseteq \operatorname{Max}(E(\phi(x)))$ for all $x \in P$, showing that ϕ is a QS-map.

Since L is rigid, ϕ must be the identity map. That is, the singleton sets $A_i = \{a_i\}$ for $0 \le i < 3$ and $A \setminus \bigcup (A_i : 0 \le i < 3) = \{a_{0,1,2}\}$ are clopen. Hence, the map $\psi \colon P \to P$, given by

$$\psi(x) = \begin{cases} a_1 & \text{if } x = a_0, \\ a_0 & \text{if } x = a_1, \\ x & \text{otherwise,} \end{cases}$$

is continuous. Obviously, it is order-preserving and satisfies the requirements of a QS-map. Evidently, $\phi \ne id$, contradicting the rigidity of L.

In view of Case 1 just considered, we may assume that for every $x \in P$, if $|\operatorname{Max}(E(x))| = 3$, then either $|\operatorname{Max}(x)| = 1$ or 2 or $x \in \operatorname{Max}(E(x))$.

Case 2. There exists an $x \in P$ such that $|\operatorname{Max}(E(x))| = 3$ and either $|\operatorname{Max}(x)| = 1$ or 2, or $x \in \operatorname{Max}(E(x))$.

Choose such an $x \in P$ and let $\operatorname{Max}(E(x)) = \{a_i : 0 \leq i < 3\}$. By hypothesis, if $x \in \bigcup((a_i] \setminus \{a_i\} : 0 \leq i < 3)$, then $x \in (a_i]$ for some $0 \leq i < 3$ and $x \notin (a_j]$ for any $0 \leq j < 3$ with $i \neq j$, or $x \in (a_i] \cap (a_j]$ for some distinct $0 \leq i, j < 3$ and $x \notin (a_k] \cap (a_l]$ for any distinct $0 \leq k, l < 3$ with $\{i, j\} \neq \{k, l\}$. Whenever $(a_i] \cap (a_j]$ is non-empty for distinct $0 \leq i, j < 3$, choose an element of it and denote it by $a_{i,j}$, observing that $a_{i,j} \not\leq a_k$ for $k \neq i, j$.

Just as in Case 1 above and mimicking Case 2 of Lemma 4.1, for $0 \leq i < 3$, choose mutually disjoint clopen order-filters B'_i such that $a_i \in B'_i$ and set

$$B_i = B'_i \cap \bigcap(E(B'_j) : 0 \leq j < 3 \text{ and } j \neq i).$$

It follows, just as before, that $(B_i : 0 \leq i < 3)$ are mutually disjoint clopen order-filters, $a_i \in B_i$ for $0 \leq i < 3$, and for $y \in P$, if $E(y) \cap B_i \neq \emptyset$ for some $i < 3$, then

(1) $E(y) \cap B_j \neq \emptyset$ for all $j < 3$,
(2) $|\operatorname{Max}(E(y))| = 3$,
(3) $|\operatorname{Max}(E(y)) \cap B_j| = 1$ for all $j < 3$, and
(4) $E(B_i) = E(B_j)$ for all $0 \leq i, j < 3$.

As in Case 1, the presence of type 1 elements means that no B_i need be an antichain. However, since B_i is an order-filter, it follows from (1) - (3) that for $y \in B_i$, either $y \in \operatorname{Max}(P)$ or it is of type 1.

Set $B = E(B_i) = E(B_j)$ for $0 \leq i, j < 3$.

By (2) and (3), for $y \in B \setminus \bigcup(B_i : 0 \leq i < 3)$, $y \notin \operatorname{Max}(P)$, and so $|\operatorname{Max}(y)| = 1$ or 2. If $|\operatorname{Max}(y)| = 2$, then $y \in (B_i] \cap (B_j]$ for some distinct $0 \leq i, j < 3$, while $y \notin (B_k]$ for $k \neq i, j$. Although $(B_i] \cap (B_j]$ is a closed order-ideal for distinct $0 \leq i, j < 3$, it need not be clopen, and unlike in Lemma 4.1, we may not conclude that

$$(B_i : 0 \leq i < 3) \cup ((B_i] \cap (B_j] : \text{ for distinct } 0 \leq i, j, k < 3)$$

is a clopen partition of B. However, initially, we may still proceed as in Case 2 of Lemma 4.1.

Suppose $y \in (B_i] \cap (B_j]$ for some distinct $0 \leq i, j < 3$, but $(a_i] \cap (a_j] = \emptyset$. Since $y \leq b_k \in B_k$, but $y \not\leq a_k \in B_k$ for some $k = i$ or j, so by (3), we have $\operatorname{Max}(E(y)) \cap \{a_i : 0 \leq i < 3\} = \emptyset$. In particular, there exists a clopen order-filter Y such that $y \in Y$ and $Y \cap \{a_i : 0 \leq i < 3\} = \emptyset$. By compactness, there exists a clopen order-filter $B'_{i,j}$ such that

$$B'_{i,j} \supseteq (B_i] \cap (B_j] \text{ and } B'_{i,j} \cap \{a_i : 0 \leq i < 3\} = \emptyset.$$

Observe, once more, that $E(x) \cap B'_{i,j} = \emptyset$ since $B'_{i,j}$ is an order-filter. Now set $B_{i,j} = E(B'_{i,j})$, which, since $B'_{i,j}$ is a clopen order-filter, is also a clopen order-filter.

For $0 \leq i < 3$, let

$$A'_i = B_i \setminus \bigcup(B_{j,k} : (a_j] \cap (a_k] = \emptyset \text{ for distinct } 0 \leq j,k < 3).$$

Then $(A'_i : 0 \leq i < 3)$ are mutually disjoint clopen order-filters such that $a_i \in A'_i$. Set

$$A_i = A'_i \cap \bigcap(E(A'_j) : 0 \leq j < 3 \text{ and } j \neq i).$$

Yet again, $(A_i : 0 \leq i < 3)$ are mutually disjoint clopen order-filters, $a_i \in A_i$ for $0 \leq i < 3$, and for $y \in P$, if $E(y) \cap A_i \neq \emptyset$ for some $i < 3$, then

(1) $E(y) \cap A_j \neq \emptyset$ for all $j < 3$,
(2) $|\operatorname{Max}(E(y))| = 3$,
(3) $|\operatorname{Max}(E(y)) \cap A_j| = 1$ for all $j < 3$, and
(4) $E(A_i) = E(A_j)$ for all $0 \leq i,j < 3$.

Set $A = E(A_i) = E(A_j)$ for $0 \leq i,j < 3$.

Suppose, for distinct $0 \leq i,j < 3$, $y \in (A_i] \cap (A_j]$; then by hypothesis, y is a type 2 element. In particular, $y \notin A_k$ for any $0 \leq k < 3$ and there exists a clopen order-ideal containing y whose intersection with the clopen order-filter $\bigcup(A_k : 0 \leq k < 3)$ is empty. Further, since y is a type 2 element, $y \notin (A_k]$ for $k \neq i,j$. That is, there exists a clopen order ideal containing y whose intersection with the closed order-ideal $(A_k] \cap (A_l]$ is empty whenever $\{i,j\} \neq \{k,l\}$ for distinct $0 \leq k,l < 3$. Since A is also a clopen order-ideal, by compactness there is a clopen order-ideal $A_{i,j}$ such that $(A_i] \cap (A_j] \subseteq A_{i,j} \subseteq A$ for which $A_{i,j} \cap A_{k,l} = \emptyset$ whenever $\{i,j\} \neq \{k,l\}$ for distinct $0 \leq k,l < 3$. Proceed inductively to determine a family $(A_{i,j} : \text{for distinct } 0 \leq i,j < 3)$ such that

(5) for distinct $0 \leq i,j < 3$, we have that $A_{i,j}$ is a clopen order-ideal, $(A_i] \cap (A_j] \subseteq A_{i,j} \subseteq A$, $A_{i,j} \cap \bigcup(A_k : 0 \leq k < 3) = \emptyset$, and $A_{i,j} \cap A_{k,l} = \emptyset$ whenever $\{i,j\} \neq \{k,l\}$ for distinct $0 \leq k,l < 3$,
(6) for distinct $0 \leq i,j < 3$, if $A_{i,j} \neq \emptyset$, then $(a_i] \cap (a_j] \neq \emptyset$, and
(7) for distinct $0 \leq i,j < 3$ and $0 \leq k,l < 3$, if $y \leq z$, $y \in A_{i,j}$ and $z \in A_{k,l}$, then $\{i,j\} = \{k,l\}$.

The same argument as is given in Case 2 of Lemma 4.1 establishes (6). Namely, if $y \in A_{i,j}$, then for $l = i$ or j, $y \leq b_l \in A_l \subseteq A'_l \subseteq B_l$. If we have $(a_i] \cap (a_j] = \emptyset$, then $y \in B'_{i,j}$ and $\operatorname{Max}(E(y)) \subseteq B_{i,j}$. Hence, for any $0 \leq l < 3$, $\operatorname{Max}(E(y)) \cap A'_l = \emptyset$, that is, $\operatorname{Max}(E(y)) \cap A_l = \emptyset$, contradicting $y \in A$; (7) follows from (1) and (2).

Set $A^+ = A \setminus \bigcup(A_i : 0 \leq i < 3) \cup \bigcup(A_{i,j} : \text{for distinct } 0 \leq i,j < 3)$.

For $y \in A$ of type 2, $y \in A_{i,j}$ for some distinct $0 \leq i,j < 3$. In particular, by (3), if $y \in A^+$, then y is of type 1. That is, $y \in (A_i]$ for some $0 \leq i < 3$, which, by (3) again, implies that $y \notin (A_j]$ for $j \neq i$. Set $A_i^+ = A^+ \cap (A_i]$ for $0 \leq i < 3$. Since A^+ is clopen and $(A_i]$ is closed, A_i^+ is closed. Thus, $(A_i^+ : 0 \leq i < 3)$ partitions A^+ into closed subsets. In other words,

(8) $(A_i : 0 \leq i < 3) \cup (A_i^+ : 0 \leq i < 3) \cup (A_{i,j} : \text{for distinct } 0 \leq i,j < 3)$ is a clopen partition of A.

Consider the mapping $\phi\colon P \to P$ given by

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \le i < 3,\ x \in A_i \cup A_i^+, \\ a_{i,j} & \text{if, for distinct } 0 \le i,j < 3,\ x \in A_{i,j}, \\ x & \text{otherwise.} \end{cases}$$

By (8), ϕ is continuous. Let $x \le y$ in $(P; \le, \tau)$. Since $x \in A$ iff $y \in A$, if $x \notin A$, then $\phi(x) = x \le y = \phi(y)$. Assume $x, y \in A$. If $x \in A_i$ for some $0 \le i < 3$, then $y \in A_i$, $\phi(x) = a_i = \phi(y)$, and $\phi(x) \le \phi(y)$. If $x \in A_i^+$ for some $0 \le i < 3$, then $y \in A_i \cup A_i^+$ since y is maximal or of type 1, while $y \notin (A_j]$ for any $j \ne i$. So, again, $\phi(x) = a_i = \phi(y)$ and $\phi(x) \le \phi(y)$. If $x \in A_{i,j}$ for distinct $0 \le i, j < 3$, then $y \in A_{i,j}$ and $\phi(x) = a_{i,j} = \phi(y)$, or $y \notin A_{i,j}$. In the latter case, by (7), $y \in A_k$ or A_k^+ for some $0 \le k < 3$. Since $x \le y$, $y \in (A_i]$ or $(A_j]$, that is, $y \in A_i, A_j, A_i^+$, or A_j^+. Either way, $\phi(x) = a_{i,j} \le a_i$ or a_j while $\phi(y) = a_i$ or a_j. That is, $\phi(x) \le \phi(y)$. By (1)–(3), ϕ is a QS-map.

Were ϕ non-trivial, it would violate the rigidity of L. So suppose otherwise, that ϕ is trivial. Then by (8), each non-empty set of

$$(A_i : 0 \le i < 3) \cup (A_{i,j} : \text{for distinct } 0 \le i, j < 3)$$

would be an isolated point of $(P; \tau)$.

If there exist distinct p and q such that $p, q \in \{i, j\}$ whenever $A_{i,j} \ne \emptyset$, let ζ denote the permutation $\zeta(p) = q$, $\zeta(q) = p$, and the identity otherwise. Alternatively, there is at most one such p, in which case let ζ denote the permutation $\zeta(k) = l$, $\zeta(l) = k$, and the identity otherwise, where k and l are distinct and distinct from p should it exist. Then the mapping $\psi\colon P \to P$, given by

$$\psi(x) = \begin{cases} a_{\zeta(i)} & \text{if } x = a_i, \\ a_{\zeta(i),\zeta(j)} & \text{if } x = a_{i,j}, \\ x & \text{otherwise,} \end{cases}$$

provides a non-trivial QS-map, contradicting the rigidity of L. □

Thus, by Lemma 4.2, the proof of Theorem 1.3 will be complete once we have shown the following.

Lemma 4.3. *If $L \in \mathbf{Q}_{1,1}$, then L is either a one-element or a two-element algebra.*

Proof. Since, by assumption, $L \in \mathbf{Q}_{1,1}$, so $|\operatorname{Max}(E(x))| \le 2$ for all $x \in P$. If $L \notin \mathbf{Q}_{1,0}$, then $|\operatorname{Max}(E(x))| = 2$ for some $x \in P$ and either $|\operatorname{Max}(x)| = 1$ or 2, or $x \in \operatorname{Max}(E(x))$.

Case 1. There exists $x \in P$ such that $|\operatorname{Max}(E(x))| = 2$.

If for some $x \in P$, we have $|\operatorname{Max}(x)| = 2$, let that be the choice of x. Set $\operatorname{Max}(E(x)) = \{a_0, a_1\}$. Proceeding as before, for $0 \le i < 2$, choose mutually disjoint clopen order-filters A_i' such that $a_i \in A_i'$, and set

$$A_i = A_i' \cap \bigcap (E(A_j') : 0 \le j < 2 \text{ and } j \ne i).$$

Once more, $(A_i : 0 \leq i < 2)$ are mutually disjoint clopen order-filters, $a_i \in A_i$ for each $0 \leq i < 2$, and for $y \in P$, if $E(y) \cap A_i \neq \emptyset$ for some $i < 2$, then

(1) $E(y) \cap A_j \neq \emptyset$ for all $j < 2$,
(2) $|\operatorname{Max}(E(y))| = 2$,
(3) $|\operatorname{Max}(E(y)) \cap A_j| = 1$ for all $j < 2$, and
(4) $E(A_i) = E(A_j)$ for all $0 \leq i, j < 2$.

Set $A = E(A_i) = E(A_j)$ for $0 \leq i, j < 2$.

Case 1a. There exists $x \in P$ such that $|\operatorname{Max}(x)| = 2$.

Denote x by $a_{0,1}$ and let the mapping $\phi\colon P \to P$ be given by

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \leq i < 2,\ x \in A_i, \\ a_{0,1} & \text{if } x \in A \setminus \bigcup(A_i : 0 \leq i < 2), \\ x & \text{otherwise.} \end{cases}$$

As argued before, ϕ is a QS-map. In the event that ϕ is trivial, the map $\psi\colon P \to P$, given by

$$\psi(x) = \begin{cases} a_1 & \text{if } x = a_0, \\ a_0 & \text{if } x = a_1, \\ x & \text{otherwise,} \end{cases}$$

provides a non-trivial QS-map, contradicting the rigidity of L.

Case 1b. For all $x \in P$, we have $|\operatorname{Max}(x)| = 1$, that is, x is either a type 1 element or $x \in \operatorname{Max}(E(x))$.

Then $(A_0]$ and $(A_1]$ partition A into two closed sets, which are *de facto* clopen. Or, consistent with earlier notation, for $0 \leq i < 2$, set

$$A^+ = A \setminus \bigcup(A_i : 0 \leq i < 2) \text{ and } A_i^+ = A^+ \cap (A_i],$$

whereupon $(A_i, A_i^+ : 0 \leq i < 2)$ is a clopen partition of A. It follows that the mapping $\phi\colon P \to P$, given by

$$\phi(x) = \begin{cases} a_i & \text{if, for some } 0 \leq i < 2,\ x \in A_i \cup A_i^+, \\ x & \text{otherwise,} \end{cases}$$

is a QS-map. Should ϕ be trivial, then the QS-map $\psi\colon P \to P$, given by

$$\psi(x) = \begin{cases} a_1 & \text{if } x = a_0, \\ a_0 & \text{if } x = a_1, \\ x & \text{otherwise,} \end{cases}$$

is not, showing that L is not in fact rigid.

In view of Case 1, $L \in \mathbf{Q}_{1,0}$, whereupon $|\operatorname{Max}(E(x))| = 1$ for all $x \in P$.

Case 2. For all $x \in P$, $|\operatorname{Max}(E(x))| = 1$.

For any $x \in P$, let a denote the unique element of $\operatorname{Max}(E(x))$. Then the QS-map $\phi\colon P \to P$, given by $\phi(x) = a$ for all x, is non-trivial unless P is a singleton, that is, L is a two-element algebra. The only other possibility is that L is a one-element algebra (for which a dual space does not exit), both of which are clearly rigid. □

5. The variety $\mathbf{Q_{1,2}}$ (Theorem 1.4)

The category $\mathbf{P_5}$ which is used in this section is defined in Section 3.

In order to establish Theorem 1.4, it is sufficient to construct a faithful functor $\mathbf{\Psi}\colon \mathbf{P_5} \to \mathbf{S}$ which has the following properties:

(i) for every object X of $\mathbf{P_5}$, $\mathbf{\Psi}(X)$ is finite if X is,

(ii) $\mathcal{Q}(\mathbf{\Psi}(X)) \in \mathbf{Q_{1,2}}$ for every object X of $\mathbf{P_5}$,

(iii) $\mathcal{Q}(\mathrm{Im}(\mathbf{\Psi}(f))) \in \mathbf{Q_{2,1}}$ for no morphism f of $\mathbf{P_5}$,

(iv) if $\psi\colon \mathbf{\Psi}(X) \to \mathbf{\Psi}(X')$ is a QS-map, where X and X' are objects of $\mathbf{P_5}$, then either $\mathcal{Q}(\mathrm{Im}(\psi)) \in \mathbf{Q_{2,1}}$ or $\psi = \mathbf{\Psi}(f)$ for some morphism $f\colon X \to X'$ of $\mathbf{P_5}$.

Such a functor is constructed below. As in Section 3 (which is similar in spirit), first a formal definition of the functor $\mathbf{\Psi}$ will be given, then an informal description of the idea lying behind it, followed by a justification that it indeed has the properties required to establish Theorem 1.4.

For $(X; \leq, \tau, b_0, b_1, b_2, b_3, b_4) \in \mathbf{P_5}$ where, as before, $B = \{b_i : 0 \leq i < 5\}$, define $\mathbf{\Psi}(X) = (P; \leq, \tau, E)$ as follows:

$$P = (X \setminus B) \cup A \cup \{a\} \cup A_{0,1} \cup A_{1,2} \cup C \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5),$$

where
$$A = \{a_i : 0 \leq i < 3\}, \quad A_{0,1} = \{a_{0,1}, a_{0,1}^{-1}, a_{0,1}^{0}\},$$
$$A_{1,2} = \{a_{1,2}, a_{1,2}^{-1}, a_{1,2}^{0}, a_{1,2}^{1}, a_{1,2}^{2}\}, \quad C = \{c_i : 0 \leq i < 3\},$$
$$D_1 = \{d_{1,i} : 0 \leq i < 27\}, \quad D_2 = \{d_{2,i} : 0 \leq i < 29\}, \quad \text{and}$$
$$E_i = \{e_{i,j} : 0 \leq j < 15\} \text{ for } 0 \leq i < 5.$$

The partial order $(P; \leq)$ is the least order such that $\mathrm{Max}(P) = A$ and $P \setminus A$ has precisely three elements of type 3, namely, elements of the set C, while all other elements, with the exception of a which is of type 1, are of type 2.

For the type 3 elements and the type 1 element,

- $c_0, c_1, c_2 \leq a_0, a_1, a_2$, while $a \leq a_1$.

The type 2 elements fall into three groups. For $x \in A_{0,1}$,

- $x \leq a_0, a_1$, while $c_0 \leq a_{0,1}, a_{0,1}^{-1}$, and $a_{0,1}^{0} \leq a_{0,1}^{-1}, a$.

Note that, in particular, $a_{0,1}^{-1}, a_{0,1}^{0}, a$ is a fence. Analogously, for $x \in A_{1,2}$,

- $x \leq a_1, a_2$, while $c_2 \leq a_{1,2}, a_{1,2}^{-1}$, $a_{1,2}^{0} \leq a_{1,2}^{-1}, a_{1,2}^{1}$, and $a_{1,2}^{2} \leq a_{1,2}^{1}, a$.

Note that, in particular, $a_{1,2}^{-1}, a_{1,2}^{0}, a_{1,2}^{1}, a_{1,2}^{2}, a$ is a fence. Moreover,

- $c_1 \leq a_{0,1}, a_{1,2}$.

Finally,

- $x \leq a_0, a_2$ for $x \in (X \setminus B) \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$.

As in Section 3, D_1, D_2, E_0, E_1, E_2, E_3, and E_4 are fences where

- $d_{1,2i+1} \leq d_{1,2i}, d_{1,2i+2}$ where $0 \leq i < 13$ for D_1,
- $d_{2,2i+1} \leq d_{2,2i}, d_{2,2i+2}$ where $0 \leq i < 14$ for D_2,

and, for E_0, E_1, E_2, E_3, and E_4,

- $e_{i,2j+1} \leq e_{i,2j}, e_{i,2j+2}$ for $0 \leq i < 5$ and $0 \leq j < 7$.

Again as in Section 3, the fences E_0, E_1, E_2, E_3, and E_4 are connected to the fences D_1 and D_2 by

- $d_{1,9+2i} \leq e_{i,0}$, and $d_{2,9+2i} \leq e_{i,14}$ for $0 \leq i < 5$.

The fences D_1 and D_2 are connected to C, the elements of type 3, by

- $c_0 \leq d_{1,0}, d_{2,0}$, while $c_1 \leq d_{1,26}$, and $c_2 \leq d_{2,28}$.

Again, as in Section 3, elements of $X \setminus B$ are connected to the fences E_0, E_1, E_2, E_3, and E_4 by

- $e_{i,7} \leq x$ iff $b_i \leq x$ for $0 \leq i < 5$ and $x \in X \setminus B$.

While, finally, for elements of $X \setminus B$,

- $x \leq y$ in P iff $x \leq y$ in X for $x, y \in X \setminus B$.

For a given $X \in \mathbf{P_5}$, the domains of the objects $\mathbf{\Phi}(X)$ and $\mathbf{\Psi}(X)$ of the previous functor $\mathbf{\Phi}$ and the current $\mathbf{\Psi}$ share a common part. The common part is $\{c_0, c_1, c_2\} \cup (X \setminus B) \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$ and the partial order on it is visualized in Figure 4.

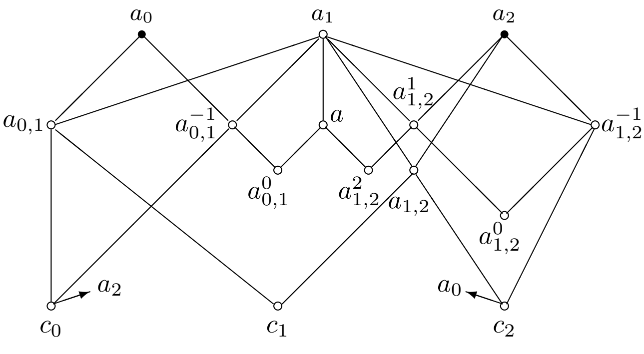

FIGURE 5

Figure 5 visualizes the partial order of the part $A \cup \{a\} \cup A_{0,1} \cup A_{1,2} \cup C$ of the domain of $\mathbf{\Psi}(X)$. The solid points on Figure 5 indicate that a_0 and a_2 are connected to every element x of the remaining part of the domain of $\mathbf{\Psi}(X)$, that is, the part $(X \setminus B) \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$ (see Figure 4) as follows: $x \leq a_0, a_2$. The arrows on Figure 5 from c_0 to a_2 and from c_2 to a_0 abbreviate the following order relations $c_0 \leq a_2$ and $c_2 \leq a_0$, respectively.

The topology τ on the domain P of $\mathbf{\Psi}(X)$ is the union topology of the clopen subspace $X \setminus B$ of X together with the discrete topology on the finite set

$$A \cup \{a\} \cup A_{0,1} \cup A_{1,2} \cup C \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5).$$

The equivalence relation on P is defined by xEy for all $x, y \in P$. As in Section 3, $\mathbf{\Psi}(X) = (P; \leq, \tau, E)$ is finite whenever X is, and since P again has only one equivalence class, $\mathbf{\Psi}(X)$ satisfies (i), (ii), and (iii) of the requirements for a Priestley space to be a QS-space. To see that $\mathbf{\Psi}(X)$ is a Priestley space, suppose that $x \not\leq y$ for some $x, y \in P$. The argument is similar to that of Section 3. If $x \notin X \setminus B$, then $[x) \cap (X \setminus B) = \emptyset$ unless $x = e_{i,7}$ for some $0 \leq i < 5$. That is, $[x)$ is a finite union of discrete points, and hence clopen. On the other hand, if $x = e_{i,7}$ for some $0 \leq i < 5$, then

$$[x) = \{e_{i,6}, e_{i,7}, e_{i,8}\} \cup ([b_i) \cap (X \setminus B)) \cup \{a_0, a_2\}.$$

If $y \notin X \setminus B$, then $[x) = \{e_{i,6}, e_{i,7}, e_{i,8}\} \cup (X \setminus B) \cup \{a_0, a_2\}$ is a clopen order-filter which contains x but not y. If $y \in X \setminus B$, then $b_i \not\leq y$, and there exists a clopen order-filter $F \subseteq X$ such that $[b_i) \subseteq F$ and $y \notin F$, in which case

$$\{e_{i,6}, e_{i,7}, e_{i,8}\} \cup (F \setminus B) \cup \{a_0, a_2\}$$

is a clopen order-filter of P which contains x and not y. If $x \in X \setminus B$, then either $y \notin X \setminus B$ and $(X \setminus B) \cup \{a_0, a_2\}$ will suffice, or $y \in X \setminus B$ and there exists a clopen order-filter $F \subseteq X$ such that $x \in F$ and $y \notin F$, in which case $(F \setminus B) \cup \{a_0, a_2\}$ will serve.

For a morphism $f\colon X \to X'$ in $\mathbf{P_5}$, let $\mathbf{\Psi}(f)\colon P \to P'$ be given by $\mathbf{\Psi}(f)(x) = f(x)$ for $x \in X \setminus B$ and the identity otherwise. Clearly, $\mathbf{\Psi}(f)$ is a QS-map. Thus, $\mathbf{\Psi}\colon \mathbf{P_5} \to \mathbf{S}$ is functor which is easily seen to be faithful.

Informally, the functor $\mathbf{\Psi}$ works as follows. Let $\psi\colon \mathbf{\Psi}(X) \to \mathbf{\Psi}(X')$ be a QS-map such that $\psi(\mathbf{\Psi}(X)) \notin \mathbf{Q_{2,1}}$. Since $\mathrm{Max}(\mathbf{\Psi}(X)) = \mathrm{Max}(\mathbf{\Psi}(X'))$ is a 3-element set A, ψ is 1-1 on it. Although $\mathbf{\Psi}(X)$ now has only 3 maximal elements, it does have a type 1 element a. Were $\psi(a) \in \mathrm{Max}(\mathbf{\Psi}(X'))$, then it would follow that $\psi(\mathbf{\Psi}(X)) \in \mathbf{Q_{2,1}}$, violating the hypothesis. Thus, we have $\psi(a) = a$, and hence, $\psi(a_1) = a_1$. Further, as $\mathbf{\Psi}(X)$ has no type 1 element other than a, $\psi(\mathbf{\Psi}(X) \setminus A) \subseteq (\mathbf{\Psi}(X') \setminus A)$ and $\psi(C) \subseteq C$ for the set C of type 3 elements. All other elements are type 2. Again, fences of varying length connect different members of C as well as a. Namely, c_1 is connected to c_0 by the fence $c_1, a_{0,1}, c_0$, and to c_2 by the fence $c_1, a_{1,2}, c_2$, while $c_0, a_{0,1}^{-1}, a_{0,1}^{0}, a$ is a fence connecting c_0 to a, and $c_2, a_{1,2}^{-1}, a_{1,2}^{0}, a_{1,2}^{1}, a_{1,2}^{2}, a$ is a fence connecting c_2 to a. The fences D_1 and D_2 connect c_0 to c_1 and c_2, respectively. Together, they ensure that ψ is the identity on A, $\{a\}$, $A_{0,1}$, $A_{1,2}$, C, D_1, and D_2. Again, the fences D_1 and D_2 have greater length, guaranteeing that ψ is also the identity on the 5 fences $(E_i\colon 0 \leq i < 5)$ which are strung between them. By identifying $e_{i,7}$ with b_i for each $0 \leq i < 5$, it follows that $\psi(X \setminus B) \subseteq (X' \setminus B)$ and forces ψ to mimic $\mathbf{\Psi}(f)$ for some $f\colon X \to X'$. A justification follows.

The required property (i) of $\mathbf{\Psi}$ is obvious. The property (ii) of $\mathbf{\Psi}$ is established by the following:

Lemma 5.1. *For $X \in \mathbf{P_5}$, $\mathcal{Q}(\mathbf{\Psi}(X)) \in \mathbf{Q_{1,2}}$.*

Proof. Observe that $\mathrm{Max}(P) = A$, where $|A| = 3$. Thus, for each $x \in P \setminus A$, there is a QS-map $\psi_x : \mathcal{S}(Q_{1,2}) \to P$ such that $\psi_x(\mathcal{S}(Q_{1,2})) = \{x\} \cup A$, where $\mathcal{S}(Q_{1,2})$ denotes the QS-space diagrammed in Figure 2. That is, the family of QS-maps $(\psi_x : x \in P \setminus A)$ represent homomorphisms from $\mathcal{Q}(\mathbf{\Psi}(X))$ to $Q_{1,2}$ which separate the elements of the algebra. Since $Q_{1,2}$ generates $\mathbf{Q_{1,2}}$, the quasi-Stone algebra $\mathcal{Q}(\mathbf{\Psi}(X))$ belongs to $\mathbf{Q_{1,2}}$. □

Since for any morphism f of $\mathbf{P_5}$, $\mathbf{\Psi}(f)(a) = a$ and a is an element of type 1 for which $\mathrm{Max}(a) = A$, so $\mathcal{Q}(\mathrm{Im}(\mathbf{\Psi}(f))) \notin \mathbf{Q_{2,1}}$. Thus, $\mathbf{\Psi}$ has the property (iii). The property (iv) of $\mathbf{\Psi}$ is established by the following:

Lemma 5.2. *For $X, X' \in \mathbf{P_5}$, if $\psi : \mathbf{\Psi}(X) \to \mathbf{\Psi}(X')$ is a QS-map such that $\mathcal{Q}(\mathrm{Im}(\psi)) \notin \mathbf{Q_{2,1}}$, then there exists a morphism $f : X \to X'$ such that $\psi = \mathbf{\Psi}(f)$.*

Proof. Let $P = \Psi(X)$ and $P' = \Psi(X')$. Since $\psi : P \to P'$ is a QS-map, $\psi(A) = A$, and since $\mathcal{Q}(\mathrm{Im}(\psi))$ belongs to $\mathbf{Q_{1,2}} \setminus \mathbf{Q_{2,1}}$ (see 5.1 above), it must be the case that $a \in \psi(P)$. This is so because the set $\{a, a_0, a_1, a_2\}$ forms the only QS-subspace in X' that is isomorphic to $\mathcal{S}(Q_{1,2})$ (see Figure 2). Further, since a is the only element of type 1, so $a \in \psi(P)$, and since ψ is order-preserving, then it must be the case that $\psi(a) = a$, and hence $\psi(a_1) = a_1$.

As all elements of $P \setminus (A \cup \{a\})$ are of type 2 or 3, we have

$$\psi(P \setminus (A \cup \{a\}) \subseteq P' \setminus (A \cup \{a\}),$$

and as C contains precisely the elements of type 3, it follows that $\psi(C) \subseteq C$. Since $(a] = \{a, a^0_{0,1}, a^2_{1,2}\}$ and $\psi(a) = a$, so $\psi(\{a, a^0_{0,1}, a^2_{1,2}\}) \subseteq \{a, a^0_{0,1}, a^2_{1,2}\}$. Moreover, since a is the only element of type 1, $\psi(\{a^0_{0,1}, a^2_{1,2}\}) \subseteq \{a^0_{0,1}, a^2_{1,2}\}$. As $\psi(P \setminus (A \cup \{a\}) \subseteq P' \setminus (A \cup \{a\})$, so $[\{a^0_{0,1}, a^2_{1,2}\}) \setminus A = \{a^{-1}_{0,1}, a^0_{0,1}, a, a^2_{1,2}, a^1_{1,2}\}$ and $a^0_{0,1} \le a^{-1}_{0,1}$, implying $\psi(a^{-1}_{0,1}) \in \{a^{-1}_{0,1}, a^0_{0,1}, a^2_{1,2}, a^1_{1,2}\}$. However, $c_0 \le a^{-1}_{0,1}$ and c_0 is of type 3, yet $(\{a^0_{0,1}, a^2_{1,2}, a^1_{1,2}\}]$ contains no element of type 3. Thus, $\psi(a^{-1}_{0,1}) = a^{-1}_{0,1}$, and therefore $\psi(c_0) = c_0$ because c_0 is the only element of type 3 in $(a^{-1}_{0,1}]$.

Since $\mathrm{Max}(a^{-1}_{0,1}) = \{a_0, a_1\}$ and $\psi(a_1) = a_1$, $\psi(a_0) = a_0$, and hence as $\psi(A) = A$, so $\psi(a_2) = a_2$. Since $a^0_{0,1}$ is the unique element both in P and P' with $a^0_{0,1} \le a^{-1}_{0,1}$ and $a^0_{0,1} \le a$, it follows that $\psi(a^0_{0,1}) = a^0_{0,1}$, and as $\psi(\{a^0_{0,1}, a^2_{1,2}\}) \subseteq \{a^0_{0,1}, a^2_{1,2}\}$, it also follows that $\psi(a^2_{1,2}) = a^2_{1,2}$. Notice that $\mathrm{Max}(a^0_{0,1}) = \{a_0, a_1\}$ and $\mathrm{Max}(a^2_{1,2}) = \{a_1, a_2\}$. Note too that $[a^2_{1,2}) = \{a^2_{1,2}, a, a^1_{1,2}, a_1, a_2\}$ and $\mathrm{Max}(a^2_{1,2}) = \{a_1, a_2\}$. So $\psi(x) = x$ for $x \in \{a^2_{1,2}, a, a_1, a_2\}$ (see above). Thus, $\psi(a^1_{1,2}) \in \{a^1_{1,2}, a^2_{1,2}\}$. If $\psi(a^1_{1,2}) = a^2_{1,2}$, then $\psi(a^0_{1,2}) = a^2_{1,2}$ since $(a^2_{1,2}] = \{a^2_{1,2}\}$. But this implies $\psi(a^{-1}_{1,2}) \ge a^2_{1,2}$, that is, $\psi(a^{-1}_{1,2}) \in \{a^1_{1,2}, a^2_{1,2}\}$, since $\mathrm{Max}(a^{-1}_{1,2}) = \{a_1, a_2\}$. However, $c_2 \leq a^{-1}_{1,2}$, while $(\{a^1_{1,2}, a^2_{1,2}\}] \cap C = \emptyset$. Since $\psi(c_2) \in C$, this is impossible, and it follows that $\psi(a^1_{1,2}) = a^1_{1,2}$.

As $a^0_{1,2} \leq a^1_{1,2}$, so $\psi(a^0_{1,2}) \leq \psi(a^1_{1,2}) = a^1_{1,2}$ and $\psi(a^0_{1,2}) \in \{a^0_{1,2}, a^1_{1,2}, a^2_{1,2}\}$. The case $\psi(a^0_{1,2}) = a^2_{1,2}$ is excluded because, as above, if $\psi(a^0_{1,2}) = a^2_{1,2}$, then $\psi(a^{-1}_{1,2}) \in \{a^1_{1,2}, a^2_{1,2}\}$, which, since $c_2 \leq a^{-1}_{1,2}$, $(\{a^1_{1,2}, a^2_{1,2}\}] \cap C = \emptyset$, and $\psi(c_2) \in C$, is not possible. Thus, $\psi(a^0_{1,2}) = a^0_{1,2}$.

However, $\psi(a^0_{1,2}) = a^0_{1,2}$ together with $[a^0_{1,2}) \setminus A = \{a^{-1}_{1,2}, a^0_{1,2}, a^1_{1,2}\}$ imply $\psi(a^{-1}_{1,2}) \in \{a^{-1}_{1,2}, a^0_{1,2}, a^1_{1,2}\}$. The case $\psi(a^{-1}_{1,2}) = a^0_{1,2}$ is impossible because $a^0_{1,2}$ is minimal in X' and $c_2 \leq a^{-1}_{1,2}$. That is, $\psi(a^{-1}_{1,2}) = a^0_{1,2}$ would imply $\psi(c_2) = \psi(a^{-1}_{1,2}) = a^0_{1,2}$; recall that c_2 is of type 3. If $\psi(a^{-1}_{1,2}) = a^1_{1,2}$, then as $c_2 \leq a^{-1}_{1,2}$, we would have $\psi(c_2) \in \{a^0_{1,2}, a^1_{1,2}, a^2_{1,2}\}$ which is impossible because each element of $\{a^0_{1,2}, a^1_{1,2}, a^2_{1,2}\}$ is of type 2. Thus, $\psi(a^{-1}_{1,2}) = a^{-1}_{1,2}$ and $\psi(c_2) = c_2$.

Notice that $[c_2) \setminus A = \{c_2, a^{-1}_{1,2}, a_{1,2}\}$. Thus, $\psi(a_{1,2}) \in \{c_2, a^{-1}_{1,2}, a_{1,2}\}$ as $\psi(c_2) = c_2$. The case $\psi(a_{1,2}) = c_2$ implies $\psi(c_1) = c_2$ because c_2 is minimal in X' and $c_1 \leq a_{1,2}$. The argument that $\psi(c_1) = c_2$ leads to a contradiction will be referred to several times below. It proceeds as follows. Since $c_1 \leq a_{0,1}$, so $c_2 = \psi(c_1) \leq \psi(a_{0,1})$; i.e., $\psi(a_{0,1}) \in [c_2)$. Either $\psi(a_{0,1}) \in \{a_{1,2}, a^{-1}_{1,2}, a_1, a_2\}$ or $\psi(a_{0,1}) = c_2$. But $c_0 \leq a_{0,1}$, so $\psi(a_{0,1}) = c_2$ implies that $c_0 = \psi(c_0) = c_2$, a contradiction. Thus, $\psi(a_{0,1}) \in \{a_{1,2}, a^{-1}_{1,2}, a_1, a_2\}$. But $a_{0,1} \leq a_0$ means that $\psi(a_{0,1}) \leq a_0 = \psi(a_0)$, while $a_0 \notin [\{a_{1,2}, a^{-1}_{1,2}, a_1, a_2\}) = \{a_{1,2}, a^{-1}_{1,2}, a_1, a_2\}$, a contradiction. In other words, $\psi(c_1) \neq c_2$. In this instance, it now follows that $\psi(a_{1,2}) \neq c_2$. Thus, $\psi(a_{1,2}) = a^{-1}_{1,2}$ or $\psi(a_{1,2}) = a_{1,2}$. We will show that the case $\psi(a_{1,2}) = a^{-1}_{1,2}$ is impossible. Suppose that $\psi(a_{1,2}) = a^{-1}_{1,2}$. Then by $c_1 \leq a_{1,2}$ and $(a^{-1}_{1,2}] = \{a^{-1}_{1,2}, a^0_{1,2}, c_2\}$, $\psi(c_1) \in \{\psi(a^{-1}_{1,2}), \psi(a^0_{1,2}), \psi(c_2)\}$. But $\psi(x) = x$ for $x \in \{a^{-1}_{1,2}, a^0_{1,2}, c_2\}$. So, $\psi(c_1) \in \{a^{-1}_{1,2}, a^0_{1,2}, c_2\}$. The case $\psi(c_1) \in \{a^{-1}_{1,2}, a^0_{1,2}\}$ is impossible because c_1 is of type 3 and $\psi(A) = A$. But, as just argued, $\psi(c_1) = c_2$ leads to a contradiction. Thus, $\psi(a_{1,2}) = a_{1,2}$.

From $\psi(a_{1,2}) = a_{1,2}$, it follows that $\psi(c_1) \in \{c_1, c_2\}$. This is because c_1 and c_2 are the only elements of type 3 in $(a_{1,2}]$. As argued above, $\psi(c_1) = c_2$ is not possible, and so $\psi(c_1) = c_1$.

Since $a_{0,1}$ is the only non-maximal element with $c_0, c_1 \leq a_{0,1}$, $\psi(a_{0,1}) = a_{0,1}$.

Summarizing, we have shown that $\psi(x) = x$ for $x \in A \cup \{a\} \cup A_{0,1} \cup A_{1,2} \cup C$.

Let $A_{0,2} = (X \setminus B) \cup C \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$ and $A'_{0,2} = (X' \setminus B) \cup C \cup D_1 \cup D_2 \cup \bigcup(E_i : 0 \leq i < 5)$. Observe that $\psi(A_{0,2}) \subseteq A'_{0,2}$. Indeed, if $x \in A_{0,2}$, then $x \leq a_0, a_2$. So, as $\psi(a_0) = a_0$ and $\psi(a_2) = a_2$, $\psi(x) \leq a_0, a_2$, and hence $\psi(x) \in A'_{0,2}$. Now notice that D_1 is the unique shortest path between c_0 and c_1 in the comparability graph of $(A_{0,2}; \leq)$, as well as in the comparability graph of $(A'_{0,2}; \leq)$. So, as $\psi(c_0) = c_0$, $\psi(c_1) = c_1$, and $\psi(A_{0,2}) \subseteq A'_{0,2}$, it follows that $\psi(x) = x$ for $x \in D_1$. Similarly, since $\psi(c_2) = c_2$ and D_2 is the unique shortest path between c_0 and c_2 in the comparability graph of $(A_{0,2};\leq)$ and in the comparabilty graph of $(A'_{0,2};\leq)$, it follows $\psi(x)=x$ for $x\in D_2$.

In order to finish the proof that $\psi=\Psi(f)$ for some morphism $f\colon X\to X'$, we still need to show that $\psi(x)\in X'\setminus B$ for $x\in X\setminus B$. But this can be shown in exactly the same way as in the proof of Lemma 3.2. Thus, $\psi=\Psi(f)$ where $f\colon X\to X'$ is given $f=\psi\restriction_{(X\setminus B)}$ on $X\setminus B$ and the identity on B. □

6. Concluding remarks

In [3], quasivarieties of quasi-Stone algebras are considered. The connection between [3] and this paper lies in another notion of universality, one due to Sapir [19].

A variety $\mathbf{V}$ of algebras of finite type is *Q-universal* providing that for any quasivariety $\mathbf{M}$ of finite type, $L_Q(\mathbf{M})$ is a homomorphic image of a sublattice of $L_Q(\mathbf{V})$, see Sapir [19].

As shown in [1], every finite-to-finite universal variety contains the ideal lattice of a free lattice on countably many generators as a sublattice, and hence is Q-universal.

By Theorem 1.1, $\mathbf{Q_{2,2}}$ is Q-universal. It follows then that $|L_Q(\mathbf{Q_{2,2}})|=2^\omega$ (which we already know from Proposition 1.5) and that $L_Q(\mathbf{Q_{2,2}})$ fails to satisfy any non-trivial lattice identity.

Although we already know that $|L_Q(\mathbf{Q_{1,2}})|=2^\omega$, it is shown in [3] that for its largest proper subvariety $\mathbf{Q_{2,1}}$, $L_Q(\mathbf{Q_{2,1}})$ is countably infinite and that for its largest proper subvariety $\mathbf{Q_{3,0}}$, $L_Q(\mathbf{Q_{3,0}})$ is finite. We will achieve this, in [3], by giving a complete description of the *critical* algebras in each subvariety of $L_Q(\mathbf{Q_{2,1}})$. (Recall that any locally finite quasivariety is determined by its critical algebras, and quasi-Stone algebras are locally finite.) Of particular interest to us in [3] is the lattice $L_Q(\mathbf{Q_{1,2}})$.

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M. E. Adams
Department of Mathematics, State University of New York, New Paltz, NY 12561, USA
e-mail: adamsm@newpaltz.edu

W. Dziobiak
Department of Mathematical Sciences, University of Puerto Rico, Mayagüez, PR 0061, USA
e-mail: w.dziobiak@gmail.com

H. P. Sankappanavar
Department of Mathematics, State University of New York, New Paltz, NY 12561, USA
e-mail: sankapph@newpaltz.edu