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Commutators for near-rings: Huq $\neq$ Smith

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This article is dedicated to the memory of Ervin Fried

Abstract. It is shown that the Huq and the Smith commutators do not coincide in the variety of near-rings.

1. Introduction

As the title shows, the paper is devoted to commutators of ideals (normal subobjects) in the variety (category) of near-rings, and its main purpose is to present a counter-example, due to the third named author, showing that, in the case of near-rings, the Huq and the Smith commutators need not coincide. For readers less familiar with these commutators, let us recall the following.

What we call the *Huq commutator* is a category-theoretic concept introduced by Huq [14]. In the case of a semi-abelian [16] variety $\mathbf{C}$ of universal algebras, such as the varieties of groups, rings, or near-rings, it can be defined as follows. Given X in $\mathbf{C}$ and normal subalgebras A and B of X, the Huq commutator $[A, B]_H$ is the smallest normal subalgebra C of X such that the canonical homomorphism $A * B \to X/C$ factors through the canonical homomorphism $A * B \to A \times B$. Briefly, the existence of such a factorization means that the canonical homomorphism $A \times B \to X/C$ is well defined. Here $A * B$ stands for the free product (in categorical terms, the coproduct or sum) of A and B.

The *Smith commutator* is a concept originally introduced by Smith [20] for congruences in a Mal'tsev (that is, congruence permutable) variety. Together with its various generalizations, this notion is well known not only in universal algebra but also in category theory (see e.g., [17] and references therein). In the formulation given in [15], for an algebra X in a Mal'tsev variety with Mal'tsev term $p(x, y, z)$ and two congruences α and β on X, the commutator $[\alpha, \beta]_S$ is the smallest congruence θ on X for which the function

$$p\colon \{(x, y, z) \mid (x, y) \in \alpha \text{ and } (y, z) \in \beta\} \to X/\theta$$

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sending (x, y, z) to the θ-class of $p(x, y, z)$ is a homomorphism.

When X belongs to a semi-abelian variety $\mathbf{C}$ (and in some more general situations), there is a one-to-one correspondence between the normal subalgebras and the congruences on X. From a superficial glance, this may suggest that the congruence approach should give the same results everywhere as the ideal (normal subalgebra) approach, that is, for normal subalgebras A and B of X and their corresponding congruences α and β on X, the congruence corresponding to $[A, B]_H$ coincides with the Smith commutator $[\alpha, \beta]_S$. Indeed, this is the case for pointed strongly protomodular exact categories [4] and for action accessible categories [6, 19]. Well-known examples are the varieties of groups, Lie algebras, associative algebras, non-unital rings, and (pre)crossed modules of groups. However, this is not the case in general, as is suggested, in a sense, already by the commutator constructions of Higgins [13] (much earlier than the Huq and the Smith commutators were introduced). The first explicit counter-example ('digroups' in the sense of Bourn: two independent group structures on the same set with the same identity element—notice that this term is used with a different meaning in Loday's theory of dimonoids) was constructed much later in a joint work of the first named author and Bourn (unpublished, but later mentioned, first in [3], in the form of an observation on change-of-base functors for split extensions). Another counter-example (loops) was given recently by Hartl and van der Linden [12]. The question of when these two commutators coincide, is of sufficient importance to justify a condition "Smith = Huq" in universal algebra around which several theories have been developed, see for example [18]. The validity of this condition facilitates work in homological algebra; for, we mention that it allows one to know that all abelian extensions are torsors in an appropriate sense [5]. This immediately gives what Gerstenhaber [9] calls Baer Extension Theory, not using congruence counterparts of the constructions involved. Note, however, that congruences cannot be avoided, not only in the general congruence permutable case considered in the last chapter of J. D. H. Smith [20], but even in the general semi-abelian case.

Let us recall that a near-ring N is a system $N = (N, 0, +, -, \cdot)$ in which $(N, 0, +, -)$ is a group (not necessarily commutative), $(N, \cdot)$ is a semigroup (with $x \cdot y$ written as xy), and the right distributive law $(x + y)z = xy + xz$ holds. Notice that $0x = 0$ is an identity in near-rings but $x0 = 0$ need not be valid. In the semi-abelian variety of near-rings, the normal subalgebras are called ideals, and $A \lhd N$ if and only if A is a normal subgroup of $(N, 0, +, -)$ with an and $n(a + m) - nm$ in A for all $a \in A$ and $n, m \in N$. The next two sections give more information on these two commutators for near-rings, while the last section presents our counter-example.

Throughout this paper, N denotes a near-ring, A and B ideals of N, and α and β the corresponding congruences. Furthermore, we shall write $[A, B]_H$ for the Huq commutator of A and B and $[A, B]_S$ for the ideal corresponding to the Smith commutator $[\alpha, \beta]_S$.

2. The Huq commutator for near-rings

Apart from the two commutator operations we are interested in, we introduce two more operations on ideals, namely:

(1) $[A, B]_G$, the ideal of N generated by the usual group-theoretic commutator of A and B considered as subgroups of the additive group of N; that is, $[A, B]_G$ is the ideal of N generated by the set

$$\{a + b - a - b \mid a \in A, b \in B\};$$

(2) $A \bullet B$, the ideal of N generated by the set

$$\{a(b + a') - aa' \mid a, a' \in A \text{ and } b \in B\}.$$

For our ideals A and B, $[A, B]_H$ is the smallest ideal of N for which the canonical map $\theta_0 \colon A \times B \to N/[A, B]_H$ is a near-ring homomorphism; the subscript 0 indicates here that we are dealing with ideals, that is, with congruence classes of 0; later we shall deal with congruences themselves. The homomorphism θ_0 must send elements of the form $(a, 0)$ and $(0, b)$ to the classes of a and b, respectively, and so

$$\theta_0(a, b) = a + b + [A, B]_H,$$

as follows from $(a, b) = (a, 0) + (0, b)$. This formula easily gives the next result.

Theorem 2.1. $[A, B]_H = [A, B]_G \vee (A \bullet B) \vee (B \bullet A)$ *in the lattice of sub-near-rings of* N (*or, equivalently, in the lattice of ideals of* N). *That is,* $[A, B]_H$ *is the ideal of* N *generated by all elements of the form* $a + b - a - b$, $a(b + a') - aa'$, *and* $b(a + b') - bb'$, *where* a *and* a' *are in* A, *and* b *and* b' *are in* B.

Proof. Just observe the following.

– The map θ_0 preserves addition if and only if $[A, B]_G \subseteq [A, B]_H$.

– The map θ_0 preserves multiplication if and only if $aa' + bb' - (a + b)(a' + b')$ is in $[A, B]_H$ for all $a, a' \in A$ and $b, b' \in B$.

– These preservation properties hold since θ_0 is a homomorphism.

– As follows from the right distributive law and the fact that $[A, B]_H$ is an ideal in N containing $[A, B]_G$, all elements of the form $aa' + bb' - (a + b)(a' + b')$, with $a, a' \in A$ and $b, b' \in B$, are in $[A, B]_H$ if and only if all elements of the forms $a(b + a') - aa'$ and $b(a + b') - bb'$, again with $a, a' \in A$ and $b, b' \in B$, are in $[A, B]_H$. □

3. The Smith commutator for near-rings

As experience with the Smith commutator theory shows, and as even suggested, in a sense, by classical affine geometry (see e.g., [11]), the suitable congruence counterpart of the map θ_0 is the map

$$\theta \colon \{(x, y, z) \in N^3 \mid x - y \in A \text{ and } y - z \in B\} \to N/[A, B]_S \qquad (3.1)$$

defined by $\theta(x, y, z) = x - y + z$ where $[A, B]_S$ is the smallest ideal of N for which θ is a near-ring homomorphism. This gives a simple characterization of the Smith commutator, perfectly analogous to the definition of the Huq commutator, and explicitly mentioned in [15] (referring to [17]) in a more general context.

The next theorem will be a counterpart of Theorem 2.1. In order to formulate it, we introduce two more operations on ideals A, B, C, and D of N; this time a ternary and a quaternary operation, respectively.

– $\mathcal{C}(A, B, C)$ is the ideal of N generated by the set

$$\{a(b + c) - ac \mid a \in A, b \in B, c \in C\};$$

note that $\mathcal{C}(A, B, A) = A \bullet B$.

– $\mathcal{C}'(A, B, C, D)$ is the ideal of N generated by the set

$$\{a(b + c + d) - a(c + d) + ad - a(b + d) \mid a \in A, b \in B, c \in C, d \in D\}.$$

Theorem 3.1. $[A, B]_S = [A, B]_G \vee \mathcal{C}(A, B, N) \vee \mathcal{C}(B, A, N) \vee \mathcal{C}'(N, A, B, N)$ *in the lattice of sub-near-rings of* N (*or, equivalently, in the lattice of ideals of* N). *That is,* $[A, B]_S$ *is the ideal of* N *generated by all elements of the forms*

$$a+b-a-b,\ a(b+x)-ax,\ b(a+x)-bx,\ x(a+b+y)-x(b+y)+xy-x(a+y) \quad (3.2)$$

where $a \in A$, $b \in B$ *and* $x, y \in N$.

Proof. We begin as in the proof of Theorem 2.1. Being a homomorphism, θ (defined by (3.1)) preserves addition and multiplication. Preservation of addition is equivalent to $[A, B]_G \subseteq [A, B]_S$ or, in other words, that all elements of the form $a+b-a-b$ with $a \in A$ and $b \in B$ are in $[A, B]_S$. Next, θ preserves multiplication if and only if $[A, B]_S$ contains all elements of the form

$$xx' - yy' + zz' - (x - y + z)(x' - y' + z') \quad (3.3)$$

with $x - y$ and $x' - y'$ in A, and $y - z$ and $y' - z'$ in B. Denoting $x - y$, $x' - y'$, $y - z$, and $y' - z'$ by a, a', b, and b', respectively, we can rewrite (3.3) as

$$(a + b + z)(a' + b' + z') - (b + z)(b' + z') + zz' - (a + z)(a' + z'), \quad (3.4)$$

and then, using the right distributive law, as

$$\begin{aligned} &a(a' + b' + z') + b(a' + b' + z') + z(a' + b' + z') \\ &\quad - z(b' + z') - b(b' + z') + zz' - z(a' + z') - a(a' + z'). \end{aligned} \quad (3.5)$$

We need to show that given a congruence $\sim$ on N with $a + b \sim b + a$ for all a in A and b in B, all elements of the forms (3.2) are congruent to 0 if and only if all elements of the form (3.5) are congruent to 0.

"If": Just note that in the cases $a' = b = z = 0$, $a = b' = z = 0$, and $a = b = 0$, the expression (3.5) reduces to $a(b' + z') - az'$, $b(a' + z') - bz'$, and $z(a' + b' + z') - z(b' + z') + zz' - z(a' + z')$, respectively, which gives the expressions in (3.2).

"Only if": Assuming that all elements of the forms (3.2) are congruent to 0, for the expression (3.5), we obtain in turn:

$$\begin{aligned} & a(a'+b'+z')+b(a'+b'+z')+z(a'+b'+z')-z(b'+z') \\ & \qquad\qquad -b(b'+z')+zz'-z(a'+z')-a(a'+z') \\ \sim\ & a(a'+b'+z')+b(a'+b'+z')-b(b'+z')+z(a'+b'+z') \\ & \qquad\qquad -z(b'+z')+zz'-z(a'+z')-a(a'+z') \end{aligned}$$

(since $z(a'+b'+z')-z(b'+z')$ is in A and $-b(b'+z')$ is in B, whence these elements commute up to $[A,B]_G$)

$$\sim a(a'+b'+z')+b(a'+b'+z')-b(b'+z')-a(a'+z')$$

(since $z(a'+b'+z')-z(b'+z')+zz'-z(a'+z')\sim 0$)

$$\sim a(a'+b'+z')-a(a'+z')\sim 0$$

(since $b(a'+b'+z')-b(b'+z')\sim 0$). □

4. Huq ≠ Smith

As mentioned in the Introduction, the purpose of this section is to give an example of a near-ring N with ideals A and B for which $[A,B]_S \neq [A,B]_H$. Since the inclusion $[A,B]_H \subseteq [A,B]_S$ (trivially) holds in general, inequality here means strict inclusion.

Example. We take $N=\Psi$, the near-ring constructed in [21] using an idea of Betsch and Kaarli [1]. Its underlying group is $M^3 = M\times M\times M$ where M is any abelian group with a nonzero proper subgroup K, and its multiplication is defined by

$$(m_1,m_2,m_3)(n_1,n_2,n_3)=\begin{cases}(m_2,0,0), & \text{if } n_2\neq 0\neq n_3,\\ (0,0,0), & \text{otherwise.}\end{cases}$$

We then take $A=M\times K\times\{0\}=\{(m_1,m_2,m_3)\in M^3 \mid m_2\in K \text{ and } m_3=0\}$ and $B=M\times\{0\}\times M=\{(m_1,m_2,m_3)\in M^3\mid m_2=0\}$. Then:

(a) $[A,B]_G=\{0\}$ since M^3 is an abelian group.

(b) $\mathcal{C}(A,B,N)=K\times\{0\}\times\{0\}=\{(m_1,m_2,m_3)\in M^3\mid m_1\in K$ and $m_2=0=m_3\}$. Indeed, on the one hand, $\mathcal{C}(A,B,N)\subseteq K\times\{0\}\times\{0\}$ by the definition of multiplication in N, and, on the other hand, for every non-zero $k\in K$, we have that $\mathcal{C}(A,B,N)$ contains

$$(k,0,0)=(0,-k,0)[(0,0,k)+(0,k,-k)]-(0,-k,0)(0,k,-k),$$

and also $\mathcal{C}(A,B,A)=K\times\{0\}\times\{0\}$.

(c) $\mathcal{C}(B,A,N)=\{0\}\times\{0\}\times\{0\}$ since $bx=0$ for every $b\in B$ and every $x\in N$, and also $\mathcal{C}(B,A,B)=\{0\}\times\{0\}\times\{0\}$.

(d) $\mathcal{C}'(N,A,B,N) = M\times\{0\}\times\{0\} = \{(m_1,m_2,m_3)\in M^3 \mid m_2=m_3=0\}$. Indeed, on the one hand, $xy \in M\times\{0\}\times\{0\}$ for every x and y in N, making the inclusion $\mathcal{C}'(N,A,B,N)\subseteq M\times\{0\}\times\{0\}$ obvious; on the other hand, for every non-zero $m\in M$, we choose any non-zero $k\in K$, and we have

$$\begin{aligned}(m,0,0) &= (0,m,0)[(0,k,0)+(0,0,m)+(0,0,0)]-(0,m,0)[(0,0,m)+(0,0,0)]\\ &\quad + (0,m,0)(0,0,0)-(0,m,0)[(0,k,0)+(0,0,0)]\in\mathcal{C}'(N,A,B,N).\end{aligned}$$

Therefore, $[A,B]_S = M\times\{0\}\times\{0\}$ by Theorem 3.1. At the same time, using Theorem 2.1 and the calculation above, we obtain

$$\begin{aligned}[A,B]_H &= [A,B]_G \vee (A\bullet B)\vee(B\bullet A)\\ &= [A,B]_G\vee\mathcal{C}(A,B,A)\vee\mathcal{C}(B,A,B) = K\times\{0\}\times\{0\}.\end{aligned}$$

That is, $[A,B]_H\neq[A,B]_S$, as desired.

Remarks. (a) Obviously, the same (counter-)example can be used in the category of *zero-symmetric near-rings*, that is, those near-rings X in which $x0=0$ for every $x\in X$; the variety of near-rings in which the constants form an ideal, cf. [7] or [8]; or we could even require all near-rings to have commutative addition, and/or to satisfy the identity $xyz=0$.

(b) As mentioned in the example above, we have $xy\in M\times\{0\}\times\{0\}$ for every x and y in N, which implies $[N,N]_S\subseteq M\times\{0\}\times\{0\}$ (which is in fact equality, since we know that $[A,B]_S = M\times\{0\}\times\{0\}$). On the other hand, $xy=(0,0,0)=0$ for every $x\in N$ and $y\in M\times\{0\}\times\{0\}$, which implies $[N, M\times\{0\}\times\{0\}]_S = 0$. This shows that N is a nilpotent object of class 2.

(c) We do not fully understand the role and behaviour of the operations $\bullet$, $\mathcal{C}$ and $\mathcal{C}'$; further investigations, including comparisons with weighted commutators [6], may yield more information here.

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