Algebra Univers. 74 (2015) 333–350
DOI 10.1007/s00012-015-0356-x
Published online August 12, 2015

Algebra Universalis

The finite basis problem for Kauffman monoids

K. Auinger, Yuzhu Chen, Xun Hu, Yanfeng Luo, and M. V. Volkov

Dedicated to Brian Davey on the occasion of his 65th birthday

Abstract. We prove a sufficient condition under which a semigroup admits no finite identity basis. As an application, it is shown that the identities of the Kauffman monoid $\mathcal{K}_n$ are nonfinitely based for each $n \geq 3$. This result holds also for the case when $\mathcal{K}_n$ is considered as an involution semigroup under either of its natural involutions.

1. Introduction

Temperley and Lieb [22], motivated by some graph-theoretic problems in statistical mechanics, introduced what are now called the *Temperley–Lieb algebras*. These are associative linear algebras with 1 over a commutative ring R. Given an integer $n \geq 2$ and a scalar $\delta \in R$, the Temperley–Lieb algebra $\mathcal{TL}_n(\delta)$ is generated by elements $h_1, \dots, h_{n-1}$ subject to the relations

$$h_i h_j = h_j h_i \qquad \text{if } |i-j| \geq 2, \text{ for } i, j = 1, \dots, n-1; \tag{1.1}$$

$$h_i h_j h_i = h_i \qquad \text{if } |i-j| = 1, \text{ for } i, j = 1, \dots, n-1; \tag{1.2}$$

$$h_i h_i = \delta h_i \qquad \text{for each } i = 1, \dots, n-1. \tag{1.3}$$

The relations (1.1)–(1.3) are 'multiplicative', i.e., they do not involve addition. This observation suggests introducing a monoid whose monoid algebra over R could be identified with $\mathcal{TL}_n(\delta)$. A tiny obstacle is the presence of the scalar δ in (1.3), but it can be bypassed by adding a new generator c that imitates δ. This way, one comes to the monoid $\mathcal{K}_n$ with n generators $c, h_1, \dots, h_{n-1}$ subject to the relations (1.1), (1.2), and the relations

$$h_i h_i = c h_i = h_i c \qquad \text{for each } i = 1, \dots, n-1, \tag{1.4}$$

which both mimic (1.3) and mean that c behaves like a scalar. The monoids $\mathcal{K}_n$ are called the *Kauffman monoids* after Kauffman [15] who independently

Presented by M. Jackson.
Received April 15, 2014; accepted in final form July 30, 2014.
2010 *Mathematics Subject Classification*: Primary: 20M07.
Key words and phrases: semigroup, involution semigroup, semigroup identity, variety, finite basis problem, Kauffman monoid, wire monoid, Rees matrix semigroup.
Yuzhu Chen, Xun Hu, Yanfeng Luo have been partially supported by the Natural Science Foundation of China (projects no. 10971086, 11371177). M. V. Volkov acknowledges support from the Presidential Programme "Leading Scientific Schools of the Russian Federation", project no. 5161.2014.1, the Russian Foundation for Basic Research, project no. 14-01-00524, and the Ministry of Education and Science of the Russian Federation, project no. 1.1999.2014/K..

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invented these monoids as geometric objects. (The name was suggested by Borisavljević, Došen and Petrić [6]; in the literature one also meets the name *Temperley–Lieb–Kauffman monoids*, see, e.g., [5]. Kauffman himself used the term *connection monoids.*) It turns out that Kauffman monoids play a major role in several 'fashionable' parts of mathematics such as knot theory, low-dimensional topology, topological quantum field theory, quantum groups, etc. As algebraic objects, these monoids belong to the family of so-called diagram or Brauer-type monoids that originally arose in representation theory and have gained much attention recently among semigroup theorists. In particular, the first-named author (solo and with collaborators) has considered universal-algebraic aspects of some monoids from this family such as the finite basis problem for their identities or the identification of the pseudovarieties generated by certain series of such monoids, see, e.g., [1, 4]. In the present paper, we follow this line of research and investigate the finite basis problem for the identities holding in Kauffman monoids.

Whilst it is not clear whether a study of the identities of Kauffman monoids may be of any use for any of their non-algebraic applications, such a study constitutes an interesting challenge from the algebraic viewpoint since—in contrast to other types of diagram monoids—Kauffman monoids are infinite. We recall that there exist several powerful methods to attack the finite basis problem for *finite* semigroups (see the survey [23] for an overview), but, to the best of our knowledge, so far the problem has been solved for only one natural family of concrete infinite semigroups that contains semigroups satisfying a nontrivial identity, namely, for non-cyclic one-relator semigroups and monoids [21]. Here, we prove that for each $n \geq 3$, the identities of the monoid $\mathcal{K}_n$ are not finitely based. The monoid $\mathcal{K}_2$ is commutative, and thus, its identities are finitely based.

The paper is structured as follows. In Section 2, we present geometric definitions for some classes of diagram monoids including Kauffman monoids and so-called Jones monoids. We also summarize properties of Kauffman and Jones monoids which are essential for the proofs of our main results. Section 3 contains a new sufficient condition under which a semigroup admits no finite identity basis. In Section 4, this condition is applied to the monoid $\mathcal{K}_n$ with $n \geq 3$, thus showing that the identities of $\mathcal{K}_n$ are nonfinitely based; we also observe that the same result holds for the case when $\mathcal{K}_n$ is considered as an involution semigroup under either of its natural involutions. Besides that, we demonstrate a further application of our sufficient condition.

The fact that the identities of $\mathcal{K}_n$ with $n \geq 4$ are nonfinitely based was announced by the last-named author in his invited lecture at the 3rd Novi Sad Algebraic Conference held in August 2009. Slides of this lecture (see `http://csseminar.imkn.urfu.ru/SLIDES/nsac2009/volkov_nsac.pdf`) included an outline of the proof for $n \geq 4$ as well as an explicit mentioning that the case $n = 3$ was left open. This case has been recently analyzed by the first-named author and, independently and by completely different methods, by the three 'middle-named' authors of the present paper: it turns out that also the identities of $\mathcal{K}_3$ are nonfinitely based. Naturally, the authors have decided to join their results into a single article, and so the present paper has been written. The unified proof presented here is based on the approach by the first-named and the last-named authors. The alternative approach by the three 'middle-named' is of a syntactic flavor; it also has some further applications that will be published in a separate paper.

2. Diagrams and their multiplication

The primary aim of this section is to present a geometric definition for a series of diagram monoids which we call the *wire monoids* $\mathcal{W}_n$, $n \geq 2$. Each Kauffman monoid $\mathcal{K}_n$ can be identified with a natural submonoid of the corresponding wire monoid $\mathcal{W}_n$ so that a geometric definition for the Kauffman monoids appears as a special case. The reader should be advised that even though this geometric definition certainly clarifies the nature of Kauffman monoids and is crucial to their connections to various parts of mathematics, knowing it is not really necessary for understanding the proofs in the present paper. Therefore, those readers who are mainly interested in the finite basis problem for $\mathcal{K}_n$ may skip the 'geometric part' of this section and rely on the definition of Kauffman monoids in terms of generators and relations as stated in the introduction and on a similar definition of Jones monoids given at the end of the section.

We fix an integer $n \geq 2$ and define the wire monoid $\mathcal{W}_n$. Let

$$[n] := \{1, \dots, n\}, \quad [n]' := \{1', \dots, n'\}$$

be two disjoint copies of the set of the first n positive integers. The base set of $\mathcal{W}_n$ is the set of all pairs $(\pi; d)$ where π is a partition of the $2n$-element set $[n] \cup [n]'$ into 2-element blocks and d is a non-negative integer referred to as the *number of circles*. Such a pair is represented by a *wire diagram* as shown in Figure 1. We draw a rectangular 'chip' with $2n$ 'pins' and represent the elements of $[n]$ by pins on the left hand side of the chip (*left pins*) while the elements of $[n]'$ are represented by pins on the right hand side of the chip (*right pins*). Usually we omit the numbers $1, 2, \dots$ in our illustrations. Now, for $(\pi; d) \in \mathcal{W}_n$, we represent the number d by d closed curves ('circles') drawn somewhere within the chip and each block of the partition π is represented by a line referred to as a *wire*. Thus, each wire connects two pins; it is called an *ℓ-wire* if it connects two left pins, an *r-wire* if it connects two right pins, and a *t-wire* if it connects a left pin with a right pin. The wire diagram in Figure 1 corresponds to the pair

$$\Big(\{\{1,5'\},\{2,4\},\{3,5\},\{6,9'\},\{7,9\},\{8,8'\},\{1',2'\},\{3',4'\},\{6',7'\}\};\, 3\Big).$$

Next, we explain the multiplication in $\mathcal{W}_n$. Pictorially, in order to multiply two chips, we 'shortcut' the right pins of the first chip with the corresponding

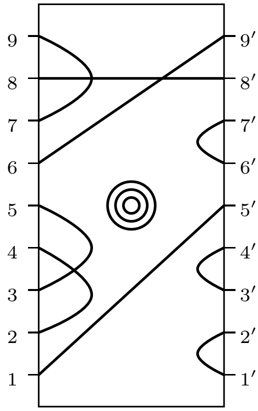


FIGURE 1. Wire diagram representing an element of $\mathcal{W}_9$

left pins of the second chip. Thus, we obtain a new chip whose left (respectively, right) pins are the left (respectively, right) pins of the first (respectively, second) chip and whose wires are sequences of consecutive wires of the factors, see Figure 2. All circles of the factors are inherited by the product; in addition, some extra circles may arise from r-wires of the first chip combined with ℓ-wires of the second chip.

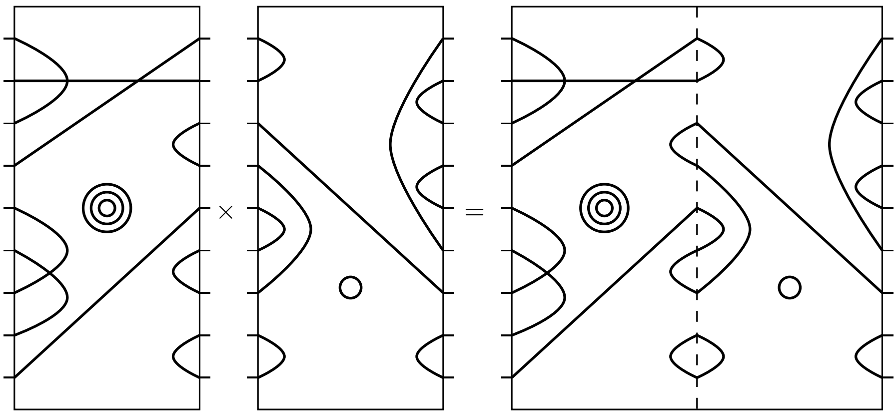


FIGURE 2. Multiplication of wire diagrams

In more precise terms, if $\xi = (\pi_1; d_1)$ and $\eta = (\pi_2; d_2)$, then a left pin p and a right pin q' of the product $\xi\eta$ are connected by a t-wire if and only if one of the following conditions holds:

- p —— u' is a t-wire in ξ and u —— q' is a t-wire in η for some $u \in [n]$;

- for some $s > 1$ and some $u_1, v_1, u_2, \ldots, v_{s-1}, u_s \in [n]$ (all pairwise distinct), p — u'_1 is a t-wire in ξ and u_s — q' is a t-wire in η, while all u_i — v_i are ℓ-wires in η and all v'_i — u'_{i+1} are r-wires in ξ.

An analogous characterization holds for the ℓ-wires and r-wires of the product. Each extra circle of $\xi\eta$ corresponds to a sequence $u_1, v_1, \ldots, u_s, v_s \in [n]$ with $s \geq 1$ and pairwise distinct $u_1, v_1, \ldots, u_s, v_s$ such that all u_i — v_i are ℓ-wires in η, while all v'_i — u'_{i+1} and v'_s — u'_1 are r-wires in ξ.

It easy to see that the above defined multiplication in $\mathcal{W}_n$ is associative and that the chip with 0 circles and the horizontal t-wires 1 — $1', \ldots, n$ — n' is the identity element with respect to the multiplication. Thus, $\mathcal{W}_n$ is a monoid; $\mathcal{W}_n$ also admits two natural unary operations. The first of them geometrically amounts to the reflection of each chip along its vertical symmetry axis. To formally introduce this reflection, consider the permutation * on $[n] \cup [n]'$ that swaps primed with unprimed elements, that is, set

$$k^* := k',\ (k')^* := k \text{ for all } k \in [n].$$

Then define $(\pi; d)^* := (\pi^*; d)$, where

$$\pi^* := \big\{\{x^*, y^*\} \mid \{x, y\} \text{ is a block of } \pi\big\}.$$

It is easy to verify that

$$\xi^{**} = \xi,\ (\xi\eta)^* = \eta^*\xi^* \quad \text{for all} \quad \xi, \eta \in \mathcal{W}_n,$$

hence the operation $\xi \mapsto \xi^*$ is an *involution* of $\mathcal{W}_n$. The second unary operation on $\mathcal{W}_n$ rotates each chip by the angle of 180 degrees. To define it formally, let

$$\alpha := \Big(\big\{\{1, n'\}, \{2, (n-1)'\}, \ldots, \{n, 1'\}\big\};\, 0\Big)$$

and define the unary operation $^\rho\colon \mathcal{W}_n \to \mathcal{W}_n$ by $\xi^\rho := \alpha\xi^*\alpha$. Since $\alpha^* = \alpha$ and $\alpha^2 = 1$, we get that $\xi \mapsto \xi^\rho$ is also an involution on $\mathcal{W}_n$. We refer to the involutions * and $^\rho$ as the *reflection* and the *rotation*, respectively.

Kauffman [15] defined the *connection monoid* $\mathcal{C}_n$ as the submonoid of the wire monoid $\mathcal{W}_n$ consisting of all elements of $\mathcal{W}_n$ that have a representation as a chip whose wires do not cross. He has shown that $\mathcal{C}_n$ is generated by the *hooks* $h_1, \ldots, h_{n-1}$, where

$$h_i := \Big(\big\{\{i, i+1\}, \{i', (i+1)'\}, \{j, j'\} \mid \text{for all } j \neq i, i+1\big\};\, 0\Big),$$

and the *circle* $c := \Big(\big\{\{j, j'\} \mid \text{for all } j = 1, \ldots, n\big\};\, 1\Big)$, see Figure 3 for an illustration. It is immediate to check that the generators $h_1, \ldots, h_{n-1}, c$ satisfy the relations (1.1), (1.2), and (1.4), whence there exists a homomorphism from the Kauffman monoid $\mathcal{K}_n$ onto the connection monoid $\mathcal{C}_n$. In fact, this homomorphism turns out to be an isomorphism between $\mathcal{K}_n$ and $\mathcal{C}_n$; a proof was outlined in [15] and presented in full detail in [6].

Observe that the set $\{h_1, \ldots, h_{n-1}, c\}$ is closed under both the reflection and the rotation in $\mathcal{W}_n$: the reflection fixes each generator, while the rotation fixes c and maps h_i to h_{n-i} for each $i = 1, \ldots, n-1$. Therefore, the submonoid

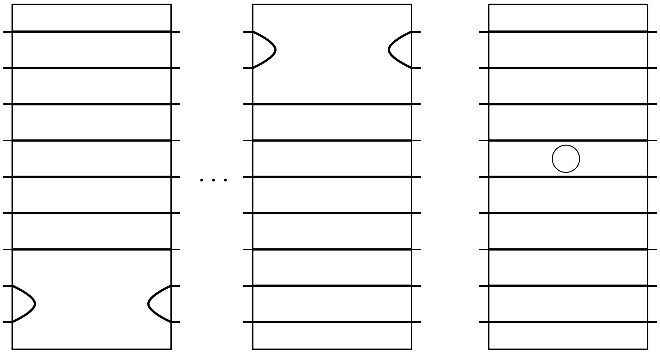


FIGURE 3. The hooks $h_1, \dots, h_8$ and the circle c in $\mathcal{C}_9$

$\mathcal{C}_n$ generated by $\{h_1, \dots, h_{n-1}, c\}$ is also closed under these involutions that, of course, transfer to the isomorphic monoid $\mathcal{K}_n$, as well. The reader who prefers to have a 'picture-free' definition of the two involutions in Kauffman monoids may observe that the relations (1.1), (1.2), and (1.4) are left-right symmetric: each of these relations coincides with its mirror image. Therefore, the map that fixes each generator of the monoid $\mathcal{K}_n$ uniquely extends to an involution of $\mathcal{K}_n$; clearly, this extension is nothing but the reflection *, and this gives a purely syntactic definition of the latter. In a similar way, one can give a syntactic definition of the rotation $^\rho$: it is the unique involutary extension of the map that fixes c and swaps h_i and h_{n-i} for each $i = 1, \dots, n-1$.

Since the involutions $\xi \mapsto \xi^*$ and $\xi \mapsto \xi^\rho$ (especially the first one) are essential for many applications of Kauffman monoids, we find it appropriate to extend our study of the finite basis problem for the identities holding in $\mathcal{K}_n$ also to their identities as algebras of type (2,1), with the reflection or the rotation in the role of the unary operation. The corresponding question was stated in the last-named author's lecture mentioned in the introduction; here we will give a complete answer to it.

Let us return for a moment to the wire monoid $\mathcal{W}_n$. Denote by $\mathcal{B}_n$ the set of all $2n$-pin chips without circles, in other words, the set of all partitions of $[n] \cup [n]'$ into 2-element blocks. Observe that this set is finite. We define the multiplication of two chips in $\mathcal{B}_n$ as follows: we multiply the chips as elements of $\mathcal{W}_n$ and then reduce the product to a chip in $\mathcal{B}_n$ by removing all circles. This multiplication makes $\mathcal{B}_n$ a monoid known as the *Brauer monoid*: the monoids $\mathcal{B}_n$ were introduced by Brauer [7] as vector space bases of certain associative algebras relevant in representation theory and thus became the historically first species of diagram monoids. We stress that even though the base set of $\mathcal{B}_n$ has been defined as a subset in the base set of $\mathcal{W}_n$, it is *not* true that $\mathcal{B}_n$ forms a submonoid of $\mathcal{W}_n$. On the other hand, it is easy to see that the 'forgetting' map $\varphi\colon \mathcal{W}_n \to \mathcal{B}_n$ defined by $\varphi(\pi; d) = \pi$ is a surjective homomorphism (the homomorphism just forgets the circles of its argument).

Clearly, both the reflection and the rotation respect $\mathcal{B}_n$ as a set and behave as anti-isomorphisms with respect to multiplication in $\mathcal{B}_n$. Thus, $\mathcal{B}_n$ forms an involution monoid under each of these unary operations; moreover, the homomorphism φ is compatible with both involutions * and $^\rho$. We summarize and augment the above information about the wire monoids and the Brauer monoids in the following lemma.

Lemma 2.1. *For each $n \geq 2$, the map $\varphi\colon (\pi; d) \mapsto \pi$ is a homomorphism from the monoid $\mathcal{W}_n$ onto the finite monoid $\mathcal{B}_n$; the homomorphism respects both involutions * and $^\rho$. For every idempotent in $\mathcal{B}_n$, its inverse image under φ is a commutative subsemigroup in $\mathcal{W}_n$.*

Proof. It remains to verify the last claim of the lemma. By the definition of φ, for each $\pi \in \mathcal{B}_n$, its inverse image under φ coincides with the set

$$\Pi := \{(\pi; d) \mid d = 0, 1, \dots\}.$$

If $\pi^2 = \pi$ in the Brauer monoid, then the product $(\pi; 0)(\pi; 0)$ in the wire monoid belongs to Π, whence $(\pi; 0)(\pi; 0) = (\pi; m)$ for some nonnegative integer m. Therefore, if we multiply two arbitrary elements $(\pi; k), (\pi; \ell) \in \Pi$, we get $(\pi; k + \ell + m)$ independently of the order of the factors. □

The *Jones monoid* $\mathcal{J}_n$ can be defined as the submonoid of the Brauer monoid $\mathcal{B}_n$ consisting of all elements of $\mathcal{B}_n$ that have a representation as a chip whose wires do not cross. (The name was suggested by Lau and FitzGerald [16] to honor the contribution of V. F. R. Jones to the theory, see, e.g., [14, Section 4].) Thus, $\mathcal{J}_n$ relates to $\mathcal{B}_n$ precisely as the Kauffman monoid $\mathcal{K}_n$ (in its incarnation as the connection monoid $\mathcal{C}_n$) relates to the wire monoid $\mathcal{W}_n$. Alternatively, one can define the Jones monoid as the image of the Kauffman monoid under the restriction of the 'forgetting' homomorphism φ to the latter. Clearly, $\mathcal{J}_n$ is closed under * and $^\rho$ and forms an involution monoid with respect to each of these operations. The following scheme summarizes the relations between the four species of diagram monoids introduced so far:

$$\begin{array}{ccc} \mathcal{W}_n & \xrightarrow{\ \varphi\ } & \mathcal{B}_n \\ \uparrow & & \uparrow \\ \mathcal{K}_n & \xrightarrow{\ \varphi\ } & \mathcal{J}_n \end{array}$$

The vertical arrows here stand for embeddings, the horizontal ones for surjections, and all maps respect multiplication and both involutions.

The following fact is just a specialization of Lemma 2.1.

Lemma 2.2. *For each $n \geq 2$, the map $\varphi\colon (\pi; d) \mapsto \pi$ is a homomorphism from the monoid $\mathcal{K}_n$ onto the finite monoid $\mathcal{J}_n$; the homomorphism respects both involutions * and $^\rho$. For every idempotent in $\mathcal{J}_n$, its inverse image under φ is a commutative subsemigroup in $\mathcal{K}_n$.*

As promised at the beginning of this section, we conclude with showing how one may bypass geometric considerations and define the Jones monoid in terms of generators and relations. Since the monoid $\mathcal{J}_n$ is the image of $\mathcal{K}_n$ under φ, it is generated by the hooks $h_1, \dots, h_{n-1}$ and the following relations hold in $\mathcal{J}_n$:

$$\begin{aligned} h_ih_j &= h_jh_i && \text{if } |i-j| \geq 2,\ i,j = 1,\dots,n-1; \\ h_ih_jh_i &= h_i && \text{if } |i-j| = 1,\ i,j = 1,\dots,n-1; \\ h_ih_i &= h_i && \text{for each } i = 1,\dots,n-1. \end{aligned} \tag{2.1}$$

In fact, it can be verified [6] that the monoid generated by $h_1, \dots, h_{n-1}$ subject to the relations (2.1), i.e., the monoid that spans the Temperley–Lieb algebra $\mathcal{TL}_n(\delta)$ with $\delta = 1$, is isomorphic to $\mathcal{J}_n$. Thus, one can define $\mathcal{J}_n$ by this presentation. Lemma 2.2 can be then recovered as follows. The homomorphism $\varphi\colon \mathcal{K}_n \twoheadrightarrow \mathcal{J}_n$ arises in this setting as the unique homomorphic extension of the map that sends the generators $h_1, \dots, h_{n-1}$ of $\mathcal{K}_n$ to the generators of $\mathcal{J}_n$ with the same names and 'erases' the generator c by sending it to 1; the fact that such an extension exists and enjoys all properties registered in Lemma 2.2 readily follows from the close similarity between the relations (1.1), (1.2), (1.4) on the one hand and the relations (2.1) on the other hand. The only claim in Lemma 2.2 which is not that apparent with this definition of $\mathcal{J}_n$ is the finiteness of the monoid. This indeed requires some work, see [6] for details. From the diagrammatic representation, it can be easily calculated that the cardinality of $\mathcal{J}_n$ is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$. For further interesting results concerning the monoids $\mathcal{K}_n$, $\mathcal{J}_n$ and similarly defined ones, the reader may consult [12].

3. A sufficient condition for the non-existence of a finite basis

We assume the reader's familiarity with the basic concepts of the theory of varieties [10, Chapter II] and of semigroup theory [11, Chapter 1].

We aim to establish a condition for the nonfinite basis property that would apply to both 'plain' semigroups and semigroups with involution as algebras of type (2,1). The two cases have much in common, and we use square brackets to indicate adjustments to be made in the involution case. First, let us formally introduce involution semigroups.

An algebra $\mathcal{S} = \langle S, \cdot, {}^\star \rangle$ of type (2,1) is called an *involution semigroup* if $\langle S, \cdot \rangle$ is a semigroup (referred to as the *semigroup reduct* of $\mathcal{S}$) and the identities

$$(xy)^\star \simeq y^\star x^\star \text{ and } (x^\star)^\star \simeq x$$

hold, in other words, if the unary operation $x \mapsto x^\star$ is an involutory anti-automorphism of $\langle S, \cdot \rangle$.

The *free involution semigroup* $\mathcal{FI}(X)$ on a given alphabet X can be constructed as follows. Let $\overline{X} := \{x^\star \mid x \in X\}$ be a disjoint copy of X. Define

$(x^\star)^\star := x$ for all $x^\star \in \overline{X}$. Then $\mathcal{FI}(X)$ is the free semigroup $(X \cup \overline{X})^+$ endowed with the involution defined by

$$(x_1 \cdots x_m)^\star := x_m^\star \cdots x_1^\star$$

for all $x_1, \dots, x_m \in X \cup \overline{X}$. We refer to elements of $\mathcal{FI}(X)$ as *involutory words over* X while elements of X^+ will be referred to as *plain words over* X.

If an involution semigroup $\mathcal{T} = \langle T, \cdot, ^\star \rangle$ is generated by a set $Y \subseteq T$, then every element in $\mathcal{T}$ can be represented by an involutory word over Y and thus by a plain word over $Y \cup \overline{Y}$ where $\overline{Y} = \{y^\star \mid y \in Y\}$. Hence, the reduct $\langle T, \cdot \rangle$ is generated by the set $Y \cup \overline{Y}$; in particular, $\mathcal{T}$ is finitely generated if and only if so is $\langle T, \cdot \rangle$. An algebra is said to be *locally finite* if each of its finitely generated subalgebras is finite. From the above remark, it follows that an involution semigroup $\mathcal{S} = \langle S, \cdot, ^\star \rangle$ is locally finite if and only if so is $\langle S, \cdot \rangle$. We denote by $\mathbf{L}$ the class of all locally finite semigroups. A variety of [involution] semigroups is *locally finite* if all its members are locally finite. Given a class $\mathbf{K}$ of [involution] semigroups, we denote by $\mathsf{var}\,\mathbf{K}$ the variety of [involution] semigroups it generates; if $\mathbf{K} = \{\mathcal{S}\}$, we write $\mathsf{var}\,\mathcal{S}$ rather than $\mathsf{var}\{\mathcal{S}\}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two subclasses of a fixed class $\mathbf{C}$ of algebras. The *Mal'cev product* $\mathbf{A} \textcircled{m} \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ (within $\mathbf{C}$) is the class of all algebras $\mathcal{C} \in \mathbf{C}$ for which there exists a congruence θ such that the quotient algebra $\mathcal{C}/\theta$ lies in $\mathbf{B}$ while all θ-classes that are subalgebras in $\mathcal{C}$ belong to $\mathbf{A}$. Note that for a congruence θ on a semigroup $\mathcal{S}$, a congruence class $s\theta$ forms a subsemigroup of $\mathcal{S}$ if and only if the element $s\theta$ is an idempotent of the quotient $\mathcal{S}/\theta$. Of essential use will be a powerful result by Brown [8, 9] that can be stated in terms of the Mal'cev product as follows.

Proposition 3.1 ([8, 9]). $\mathbf{L} \textcircled{m} \mathbf{L} = \mathbf{L}$ *where the Mal'cev product is considered within the class of all semigroups.*

Let $x_1, x_2, \dots, x_n, \dots$ be a sequence of letters. The sequence $\{Z_n\}_{n=1,2,\dots}$ of *Zimin words* is defined inductively by $Z_1 := x_1$, $Z_{n+1} := Z_n x_{n+1} Z_n$. We say that a word v is an [*involutory*] *isoterm* for a class $\mathbf{C}$ of semigroups [with involution] if the only [involutory] word v' such that all members of $\mathbf{C}$ satisfy the [involution] semigroup identity $v \simeq v'$ is the word v itself.

If a semigroup $\mathcal{S}$ satisfies the identities $x^2 y \simeq x^2 \simeq y x^2$, then $\mathcal{S}$ has a zero and the value of the word x^2 in $\mathcal{S}$ under every evaluation of the letter x in $\mathcal{S}$ is equal to zero. Having this in mind, we use the expression $x^2 \simeq 0$ as an abbreviation for the identities $x^2 y \simeq x^2 \simeq y x^2$.

The last ingredient that we need comes from [19, Proposition 3] for the plain case and from [3, Corollary 2.6] for the involution case.

Proposition 3.2 ([19, 3]). *Let* $\mathbf{V}$ *be a variety of* [*involution*] *semigroups. If*

(i) *all members of* $\mathbf{V}$ *satisfying* $x^2 \simeq 0$ *are locally finite, and*

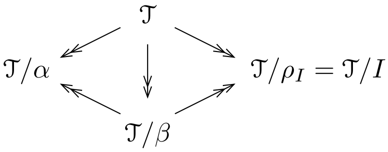
(ii) *each Zimin word is an* [*involutory*] *isoterm relative to* $\mathbf{V}$,

then $\mathbf{V}$ *is nonfinitely based.*

In the following, we shall present a specialization of Proposition 3.2 by presenting a sufficient condition for a variety $\mathbf{V}$ to satisfy condition (i). An essential step towards this result is the next lemma whose proof is a refinement of one of the crucial arguments in [20]. Here $\mathbf{Com}$ denotes the variety of all commutative semigroups.

Lemma 3.3. *Let $\mathcal{T}$ be a semigroup in $\mathbf{Com} \textcircled{m} \mathbf{L}$ and let I be the ideal of $\mathcal{T}$ generated by $\{t^2 \mid t \in \mathcal{T}\}$. Then the Rees quotient $\mathcal{T}/I$ is locally finite.*

Proof. Let α be a congruence on $\mathcal{T}$ such that $\mathcal{T}/\alpha$ is locally finite and the idempotent α-classes are commutative subsemigroups of $\mathcal{T}$. Let ρ_I be the Rees congruence of $\mathcal{T}$ corresponding to the ideal I and $\beta = \alpha \cap \rho_I$. We have the following commutative diagram in which all homomorphisms are canonical projections.



Recall that a semigroup is said to be *periodic* if each of its one-generated subsemigroups is finite. The semigroup $\mathcal{T}/\alpha$ is locally finite and thus periodic. Moreover, since the restrictions of α and β to the ideal I coincide, we have $I/\alpha = I/\beta$ whence I/β is periodic, as well. Since for each element of $\mathcal{T}/\beta$, its square belongs to I/β, it follows that $\mathcal{T}/\beta$ is also periodic, and so is each subsemigroup of $\mathcal{T}/\beta$.

Now let $A \in \mathcal{T}/\alpha$ be an idempotent α-class; by assumption, A is a commutative subsemigroup of $\mathcal{T}$. Then the inverse image of A (considered as an element of $\mathcal{T}/\alpha$) under the canonical projection $\mathcal{T}/\beta \twoheadrightarrow \mathcal{T}/\alpha$ is the subsemigroup A/β of $\mathcal{T}/\beta$, and this subsemigroup is at the same time commutative and periodic. It is well known (and easy to verify) that every commutative periodic semigroup is locally finite. We see that the congruence α/β on $\mathcal{T}/\beta$ satisfies the two conditions: (a) the quotient $(\mathcal{T}/\beta)\big/(\alpha/\beta) \cong \mathcal{T}/\alpha$ is locally finite and (b) the α/β-classes which are subsemigroups are locally finite. By Proposition 3.1, $\mathcal{T}/\beta$ is itself locally finite, and so is its quotient $\mathcal{T}/I$. □

For two semigroup varieties $\mathbf{V}$ and $\mathbf{W}$, their Mal'cev product $\mathbf{V} \textcircled{m} \mathbf{W}$ within the class of all semigroups may fail to be a variety, but it is always closed under forming subsemigroups and direct products, see [17, Theorems 1 and 2]. Therefore, the variety $\mathsf{var}(\mathbf{V} \textcircled{m} \mathbf{W})$ generated by $\mathbf{V} \textcircled{m} \mathbf{W}$ is comprised of all homomorphic images of the members of $\mathbf{V} \textcircled{m} \mathbf{W}$. We are now in a position to formulate and to prove our main result.

Theorem 3.4. *A variety $\mathbf{V}$ of [involution] semigroups is nonfinitely based if*

(i) *for some locally finite semigroup variety $\mathbf{W}$, [the class of all semigroup reducts of] $\mathbf{V}$ is contained in the variety $\mathsf{var}(\mathbf{Com} \textcircled{m} \mathbf{W})$, and*

(ii) *each Zimin word is an [involutory] isoterm relative to $\mathbf{V}$.*

Proof. By Proposition 3.2, it suffices to verify that all members of $\mathbf{V}$ satisfying $x^2 \simeq 0$ are locally finite. Since an involution semigroup is locally finite if and only if so is its semigroup reduct, it suffices to do so for the semigroup reducts of the members of $\mathbf{V}$. Let $\mathbf{W}$ be a locally finite semigroup variety as per condition (i). We need to check that each semigroup $\mathcal{S} \in \mathsf{var}(\mathbf{Com} \circledm \mathbf{W})$ which satisfies $x^2 \simeq 0$ is locally finite. As we observed prior to the formulation of the theorem, $\mathcal{S}$ is a homomorphic image of a semigroup $\mathcal{T} \in \mathbf{Com} \circledm \mathbf{W}$; let ψ stand for the corresponding homomorphism. Consider the ideal I in $\mathcal{T}$ generated by $\{t^2 \mid t \in \mathcal{T}\}$. Then $I \subseteq \psi^{-1}(0)$, and therefore, the homomorphism ψ factors through $\mathcal{T}/I$ which is locally finite by Lemma 3.3. Consequently, $\mathcal{S}$ is also locally finite. □

Remark 3.5. It follows immediately from the proof of Lemma 3.3 that Theorem 3.4 remains valid if we replace the variety $\mathbf{Com}$ of all commutative semigroups by an arbitrary semigroup variety all of whose periodic members are locally finite. For an example of a situation in which this extended version of Theorem 3.4 can be useful, we refer to [24].

Remark 3.6. For a locally finite [involution] semigroup variety $\mathbf{V}$, condition (i) is trivially satisfied with $\mathbf{W} = \mathbf{V}$. In this case, condition (ii) is sufficient for $\mathbf{V}$ to be nonfinitely based; moreover, $\mathbf{V}$ then is even *inherently nonfinitely based*, i.e., it is not contained in any finitely based locally finite variety. The corresponding result is captured by Sapir [19] for plain semigroups and by Auinger, Dolinka, and Volkov [3] for involution semigroups. It follows that the novelty in the present paper, though not always explicitly mentioned, is about *infinite* [involution] semigroups, or, to be more precise, [involution] semigroups which do not generate a locally finite variety.

Remark 3.7. Proposition 3.2 and therefore Theorem 3.4 formulate, in fact, sufficient conditions that the variety in question be not only nonfinitely based but even be *of infinite axiomatic rank*, that is, it has no identity basis that uses only finitely many variables. Consequently, in all our applications, the respective [involution] semigroups are also not only nonfinitely based but even of infinite axiomatic rank. This is worth registering because an infinite [involution] semigroup can be nonfinitely based but of finite axiomatic rank.

Remark 3.8. If two given varieties $\mathbf{X}$ and $\mathbf{Y}$ of [involution] semigroups satisfy $\mathbf{X} \subseteq \mathbf{Y}$, and $\mathbf{Y}$ satisfies condition (i) while $\mathbf{X}$ satisfies condition (ii), then all varieties $\mathbf{V}$ such that $\mathbf{X} \subseteq \mathbf{V} \subseteq \mathbf{Y}$ satisfy both conditions, and therefore, are nonfinitely based. Stated this way, Theorem 3.4 may be used to produce intervals consisting entirely of nonfinitely based varieties in the lattice of [involution] semigroup varieties. We conclude this section with an example of such an application.

For two varieties $\mathbf{V}$ and $\mathbf{W}$, we denote by $\mathbf{V} \vee \mathbf{W}$ their *join*, i.e., the least variety containing both $\mathbf{V}$ and $\mathbf{W}$. Sapir and Volkov [20] proved that for each locally finite semigroup variety $\mathbf{W}$ which contains the variety $\mathbf{B}$ of all *bands* (idempotent semigroups), the join $\mathbf{Com} \vee \mathbf{W}$ is nonfinitely based. More precisely, in [20] it is shown that each Zimin word is an isoterm relative to $\mathbf{Com} \vee \mathbf{B}$ and each member of $\mathbf{Com} \vee \mathbf{W}$ which satisfies $x^2 \simeq 0$ is locally finite (the latter by an argument that has been refined in the proof of Lemma 3.3). By Theorem 3.4, it follows that each variety $\mathbf{V}$ for which $\mathbf{Com} \vee \mathbf{B} \subseteq \mathbf{V} \subseteq \mathsf{var}(\mathbf{Com} \circledcirc \mathbf{W})$ is nonfinitely based. Notice that $\mathbf{Com} \vee \mathbf{W} \subseteq \mathsf{var}(\mathbf{Com} \circledcirc \mathbf{W})$ so that the quoted result from [20] appears as a special case.

One can obtain an analogous result for involution semigroups if $\mathbf{B}$ is replaced by the variety $\mathbf{B}^\star$ of all bands with involution and commutative semigroups are considered to be equipped with trivial involution (for the verification that all Zimin words are involutory isoterms relative to $\mathbf{Com} \vee \mathbf{B}^\star$, one can use Lemma 4.2 formulated in the next section).

4. Applications

For every n, there is an injective semigroup homomorphism $\mathcal{K}_n \hookrightarrow \mathcal{K}_{n+1}$ (induced by the map $c \mapsto c$, $h_i \mapsto h_i$ for $i = 1, \dots, n-1$) which is compatible with the reflection. Consequently, for every n, we have the inclusion $\mathsf{var}\,\mathcal{K}_n \subseteq \mathsf{var}\,\mathcal{K}_{n+1}$. As mentioned earlier, $\mathcal{K}_n$ is a submonoid of $\mathcal{W}_n$, whence $\mathsf{var}\,\mathcal{K}_n \subseteq \mathsf{var}\,\mathcal{W}_n$ for every n. These inclusions are true if the respective structures are considered either as semigroups or as involution semigroups with respect to the reflection. We start by applying Theorem 3.4 to the Kauffman monoids $\mathcal{K}_n$ and the wire monoids $\mathcal{W}_n$ with $n \geq 3$.

Theorem 4.1. *Let $n \geq 3$ and consider $\mathcal{K}_3$ and $\mathcal{W}_n$, either as semigroups or as involution semigroups with respect to reflection. Then every [involution] semigroup variety $\mathbf{V}$ such that $\mathsf{var}\,\mathcal{K}_3 \subseteq \mathbf{V} \subseteq \mathsf{var}\,\mathcal{W}_n$ is nonfinitely based.*

Proof. We invoke Theorem 3.4 in the form of Remark 3.8 and show that $\mathsf{var}\,\mathcal{W}_n$ satisfies (i) and $\mathsf{var}\,\mathcal{K}_3$ satisfies (ii). Thus, we are to check that the semigroup $\mathcal{W}_n$ belongs to the Mal'cev product of $\mathbf{Com}$ with a locally finite semigroup variety and that each Zimin word is an [involutory] isoterm relative to $\mathcal{K}_3$.

The first claim readily follows from Lemma 2.1. Indeed, by this lemma, there is a homomorphism $\varphi\colon \mathcal{W}_n \twoheadrightarrow \mathcal{B}_n$ with the property that for every idempotent in $\mathcal{B}_n$, its inverse image under φ is a commutative subsemigroup in $\mathcal{W}_n$. This immediately yields that $\mathcal{W}_n$ belongs to the Mal'cev product $\mathbf{Com} \circledcirc \mathsf{var}\,\mathcal{B}_n$, and $\mathsf{var}\,\mathcal{B}_n$ is locally finite as a variety generated by a finite algebra [10, Theorem 10.16].

In order to show that the Zimin words are isoterms relative to $\mathcal{K}_3$, consider the ideal C of $\mathcal{K}_3$ generated by c. Clearly, $\mathcal{K}_3 \setminus C = \{1, h_1, h_2, h_1h_2, h_2h_1\}$. If we denote the images of h_1 and h_2 in the Rees quotient $\mathcal{K}_3/C$ by a and b, respectively, then the relations of $\mathcal{K}_3$ translate into the following relations for a and b: $a^2 = 0,\ b^2 = 0,\ aba = a,\ bab = b$. These relations define the 6-element Brandt monoid B_2^1 (in the class of all monoids with 0). Thus, $\mathcal{K}_3/C$ satisfies the relations of B_2^1, and the Rees quotient also consists of 6 elements, so that $\mathcal{K}_3/C \cong B_2^1$. It is well known [19, Lemma 3.7] that each Zimin word is an isoterm relative to B_2^1. This completes the proof in the plain semigroup case.

If we consider $\mathcal{K}_3$ as an involution semigroup under reflection, we can employ the approach of Auinger et al. [2]. Recall that the 3-element *twisted semilattice* is the involution semigroup $\mathcal{TSL} = \langle \{e, f, 0\}, \cdot, ^\star \rangle$ in which $e^2 = e$, $f^2 = f$, and all other products are equal to 0, while the unary operation is defined by $e^\star = f$, $f^\star = e$, and $0^\star = 0$. The following observation has been made in the proof of [2, Theorem 3.1].

Lemma 4.2. *Let $\mathcal{T} = \langle T, \cdot, ^\star \rangle$ be an involution semigroup such that each Zimin word is an isoterm relative to its semigroup reduct $\langle T, \cdot \rangle$. If the* 3*-element twisted semilattice $\mathcal{TSL}$ belongs to the variety $\mathsf{var}\,\mathcal{T}$, then each Zimin word is also an involution isoterm relative to $\mathcal{T}$.*

Clearly, the ideal C of $\mathcal{K}_3$ is closed under reflection, which therefore induces an involution on $\mathcal{K}_3/C \cong B_2^1$. The latter involution swaps the idempotents ab and ba and fixes all other elements of B_2^1, whence the subset $\{ab, ba, 0\}$ of B_2^1 constitutes an involution subsemigroup isomorphic to $\mathcal{TSL}$. Hence, $\mathcal{TSL}$ belongs to the variety generated by $\mathcal{K}_3$ as an involution semigroup under reflection and Lemma 4.2 applies. □

The situation is somewhat more delicate if we consider $\mathcal{K}_n$ and $\mathcal{W}_n$ as involution semigroups under rotation; we denote these involution semigroups by $\mathcal{K}_n^\rho$ and $\mathcal{W}_n^\rho$, respectively. For every n we have the following embeddings.

- $\mathcal{K}_n^\rho \hookrightarrow \mathcal{K}_{n+2}^\rho$ and $\mathcal{W}_n^\rho \hookrightarrow \mathcal{W}_{n+2}^\rho$. These embeddings are obtained by adding one t-wire on top and one on bottom of each chip; for the case of Kauffman monoids, the embedding can be alternatively defined in terms of generators: it is induced by the map $c \mapsto c$, $h_i \mapsto h_{i+1}$ for $i = 1, \dots, n-1$.
- $\mathcal{K}_n^\rho \hookrightarrow \mathcal{K}_{2n}^\rho$ and $\mathcal{W}_n^\rho \hookrightarrow \mathcal{W}_{2n}^\rho$. These embeddings are obtained by 'doubling' each chip; in terms of generators for $\mathcal{K}_n^\rho$, the embedding is induced by the map $c \mapsto c^2$, $h_i \mapsto h_i h_{n+i}$ for $i = 1, \dots, n-1$.
- $\mathcal{W}_{2n}^\rho \hookrightarrow \mathcal{W}_{2n+1}^\rho$. The embedding is obtained by inserting a t-wire just into the middle of each chip.
- $\mathcal{K}_n^\rho \hookrightarrow \mathcal{W}_n^\rho$. This is the canonical embedding.

It follows that $\mathsf{var}\,\mathcal{K}_3^\rho \subseteq \mathsf{var}\,\mathcal{W}_n^\rho$ for $n = 3$ and each $n \geq 5$, and $\mathsf{var}\,\mathcal{K}_4^\rho \subseteq \mathsf{var}\,\mathcal{W}_n^\rho$ for each $n \geq 4$. We do not know whether $\mathsf{var}\,\mathcal{K}_3^\rho \subseteq \mathsf{var}\,\mathcal{W}_4^\rho$ or $\mathsf{var}\,\mathcal{K}_3^\rho \subseteq \mathsf{var}\,\mathcal{K}_4^\rho$. In any case, we have a version of Theorem 4.1 that is sufficient for our purposes.

Theorem 4.3. *Let $m \geq 4$; each variety $\mathbf{V}$ of involution semigroups satisfying $\mathsf{var}\,\mathcal{K}_3^\rho \subseteq \mathbf{V} \subseteq \mathsf{var}\,\mathcal{W}_{m+1}^\rho$ or $\mathsf{var}\,\mathcal{K}_4^\rho \subseteq \mathbf{V} \subseteq \mathsf{var}\,\mathcal{W}_m^\rho$ is nonfinitely based.*

Proof. We have already shown in the proof of Theorem 4.1 that the semigroup reducts of all members of $\mathsf{var}\,\mathcal{W}_m^\rho$ satisfying $x^2 \simeq 0$ are locally finite. In order to apply Theorem 3.4 (in the form of Remark 3.8), it remains to show that each Zimin word is an involutory isoterm relative to $\mathsf{var}\,\mathcal{K}_\ell^\rho$ for $\ell = 3$ and $\ell = 4$.

For $\ell = 4$, this follows from the analogous fact for the monoid $\mathcal{J}_4$ considered as an involution semigroup under rotation (this fact has been shown in [4, Theorem 2.13]); by Lemma 2.2, the latter monoid is a quotient of $\mathcal{K}_4^\rho$.

It remains to consider the case $\ell = 3$. We do not know whether or not $\mathcal{TSL}$ belongs to the variety $\mathsf{var}\,\mathcal{K}_3^\rho$; hence, we do not know if we can proceed as in the proof of Theorem 4.1. Nevertheless, we will show that each Zimin word is an involution isoterm relative to $\mathcal{K}_3^\rho$.

Arguing by contradiction, assume that for some n and some involutory word $w \neq Z_n$, the identity $Z_n \simeq w$ holds in $\mathcal{K}_3^\rho$. First we observe that each letter x_i, for $i = 1, 2, \dots, n$, occurs the same number of times in Z_n and w. For this, we substitute c for x_i and 1 for all other letters. The value of the word Z_n under this substitution is $c^{2^{n-i}}$ since it is easy to see that x_i occurs 2^{n-i} times in Z_n. Similarly, since $c^\rho = c$, the value of w is c^k, where k is the number of occurrences of x_i in w. As $Z_n \simeq w$ holds in $\mathcal{K}_3^\rho$, the two values should coincide whence $k = 2^{n-i}$. In a similar manner, one can verify that the only letters occurring in w are $x_1, x_2, \dots, x_n$.

We have already shown that Z_n is an isoterm relative to $\mathcal{K}_3$ considered as a plain semigroup. Hence, w must be a proper involutory word, that is, it has at least one occurrence of a 'starred' letter. We fix an $i \in \{1, 2, \dots, n\}$ such that $x_i^\star$ occurs in w and substitute h_1 for x_i and 1 for all other letters. It is easy to calculate that the value of the word Z_n under this substitution is $c^{2^{n-i}-1}h_1$. Since $h_1^\rho = h_2$ in $\mathcal{K}_3^\rho$ and x_i occurs 2^{n-i} times in w, the word w evaluates to a product p of 2^{n-i} factors, each of which is either h_1 or h_2 and at least one of which is h_2. As $Z_n \simeq w$ holds in $\mathcal{K}_3^\rho$, the value of p must coincide with $c^{2^{n-i}-1}h_1$, which is only possible when the first and the last factors of p are h_1. Then the relations (1.2) and (1.4) ensure that the value of p is $c^k h_1$, where k is the total number of occurrences of the factors h_1h_1 and h_2h_2 in p. However, p has at least one occurrence of h_1h_2 and at least one occurrence of h_2h_1, and therefore $k \leq 2^{n-i} - 3$, a contradiction. □

Remark 4.4. To get a version of Theorem 4.1 that could be stated and justified without any appeal to geometric considerations, one should change $\mathcal{W}_n$ to $\mathcal{K}_n$ in the formulation of Theorem 4.1 and refer to Lemma 2.2 instead of Lemma 2.1 in its proof. (Recall that we outlined a 'picture-free' proof of Lemma 2.2 at the end of Section 2.) This reduced version of Theorem 4.1 still suffices to solve the finite basis problem for the identities holding in the Kauffman monoids. The same observation applies to Theorem 4.3.

Remark 4.5. Theorems 4.1 and 4.3 imply that each of the monoids $\mathcal{W}_n$ and $\mathcal{K}_n$ with $n \geq 3$ is nonfinitely based as both a plain semigroup and an involution semigroup with either reflection or rotation. For the sake of completeness, we mention that the monoids $\mathcal{W}_2$ and $\mathcal{K}_2$ are easily seen to be commutative, and hence they are finitely based by a classical result of Perkins [18]. Moreover, both reflection and rotation act trivially in $\mathcal{W}_2$, and therefore, $\mathcal{W}_2$ and $\mathcal{K}_2$ are also finitely based as involution semigroups.

In a similar manner, Theorem 3.4 allows one to solve the finite basis problem for many other species of infinite diagram monoids in the setting of both plain and involution semigroups. These applications of Theorem 3.4 will be published in a separate paper, while here we restrict ourselves to demonstrating another application of rather a different flavor.

Recall the classical Rees matrix construction (see [11, Chapter 3] for details and for the explanation of the role played by this construction in the structure theory of semigroups). Let $\mathcal{G} = \langle G, \cdot\rangle$ be a semigroup, 0 a symbol not in G, and I, Λ non-empty sets. Given a $\Lambda \times I$-matrix $P = (p_{\lambda i})$ over $G \cup \{0\}$, we define a multiplication $\cdot$ on the set $(I \times G \times \Lambda) \cup \{0\}$ by the following rules:

$$a \cdot 0 = 0 \cdot a := 0 \quad \text{for all } a \in (I \times G \times \Lambda) \cup \{0\},$$

$$(i, g, \lambda) \cdot (j, h, \mu) := \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0. \end{cases}$$

Then $\langle (I \times G \times \Lambda) \cup \{0\}, \cdot\rangle$ becomes a semigroup denoted by $\mathcal{M}^0(I, \mathcal{G}, \Lambda; P)$ and is called the *Rees matrix semigroup over* $\mathcal{G}$ *with the sandwich matrix* P. For a semigroup $\mathcal{S}$, we let $\mathcal{S}^1$ stand for the monoid obtained from $\mathcal{S}$ by adjoining a new identity element.

Theorem 4.6. *Let $\mathcal{G} = \langle G, \cdot\rangle$ be an abelian group and $\mathcal{S} = \mathcal{M}^0(I, \mathcal{G}, \Lambda; P)$ a Rees matrix semigroup over $\mathcal{G}$. If the matrix P has a submatrix of one of the forms $\left(\begin{smallmatrix} a & b \\ c & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & b \\ c & 0\end{smallmatrix}\right)$ where $a, b, c \in G$, or $\left(\begin{smallmatrix} e & e \\ e & d\end{smallmatrix}\right)$ where e is the identity of $\mathcal{G}$ and $d \in G$ has infinite order, then the monoid $\mathcal{S}^1$ is nonfinitely based.*

Proof. Let $\mathcal{E} = \langle \{e\}, \cdot\rangle$ be the trivial group and $\overline{P} = (\bar{p}_{\lambda i})$ the $\Lambda \times I$-matrix over $\{e, 0\}$ obtained when each non-zero entry of P gets substituted by e. Consider the Rees matrix semigroup $\mathcal{T} = \mathcal{M}^0(I, \mathcal{E}, \Lambda; \overline{P})$. It is easy to see that the map ψ defined by

$$1 \mapsto 1,\ 0 \mapsto 0,\ (i, g, \lambda) \mapsto (i, e, \lambda)$$

is a homomorphism from $\mathcal{S}^1$ onto $\mathcal{T}^1$. It is known (see, e.g., the proof of [13, Theorem 3.3]) that every Rees matrix semigroup over $\mathcal{E}$ belongs to the variety generated by the 5-element semigroup A_2 that can be defined as the Rees matrix semigroup over $\mathcal{E}$ with the sandwich matrix $\left(\begin{smallmatrix} e & e \\ e & 0\end{smallmatrix}\right)$. Therefore, $\mathcal{T}^1$ lies in the variety $\mathsf{var}\, A_2^1$. The inverse image of an arbitrary element $(i, e, \lambda) \in \mathcal{T}$ under ψ consists of all triples of the form (i, g, λ) where g runs over G. If for some $j \in I$, $\mu \in \Lambda$, the triple (j, e, μ) is an idempotent in $\mathcal{T}$, then $\bar{p}_{\mu j} \neq 0$, whence $p_{\mu j} \neq 0$ as well. Therefore, the product of any two triples $(j, g, \mu), (j, h, \mu) \in \psi^{-1}(j, e, \mu)$ is equal to $(j, gp_{\mu j}h, \mu)$ and this result does not depend on the order of the factors since the group $\mathcal{G}$ is abelian. Taking into account that $\psi^{-1}(0) = \{0\}$ and $\psi^{-1}(1) = \{1\}$, we see that the inverse image under φ of every idempotent in $\mathcal{T}^1$ is a commutative subsemigroup in $\mathcal{S}^1$. Thus, $\mathcal{S}^1$ belongs to the Mal'cev product $\mathbf{Com} \circledcirc \mathsf{var}\, A_2^1$, and $\mathsf{var}\, A_2^1$ is locally finite as a variety generated by a finite algebra [10, Theorem 10.16].

In view of Theorem 3.4, it remains to verify that each Zimin word is an isoterm relative to $\mathcal{S}^1$. Here we invoke the premise that the matrix P has a 2×2-submatrix of a specific form. We fix such a submatrix P' of one of the given forms and let $\Lambda' = \{\lambda, \mu\} \subseteq \Lambda$ and $I' = \{i, j\} \subseteq I$ be such that P' occurs at the intersection of the rows whose indices are in Λ' with the columns whose indices are in I'.

First consider the case when P' is either $\left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. Clearly, the Rees matrix semigroup $\mathcal{U} = \mathcal{M}^0(I', \mathcal{G}, \Lambda'; P')$ is a subsemigroup of $\mathcal{S}$, whence $\mathcal{U}^1$ is a subsemigroup of $\mathcal{S}^1$. Then the image of $\mathcal{U}^1$ under the homomorphism φ is a subsemigroup $\mathcal{V}^1$ of $\mathcal{T}^1$ where $\mathcal{V}$ can be identified with the Rees matrix semigroup over $\mathcal{E}$ whose sandwich matrix is either $\left(\begin{smallmatrix} 0 & e \\ e & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} e & e \\ e & 0 \end{smallmatrix}\right)$. In the latter case, the semigroup $\mathcal{V}$ is isomorphic to the semigroup A_2. We have already used the fact that every Rees matrix semigroup over $\mathcal{E}$ belongs to the variety $\mathsf{var}\, A_2$; this implies that in any case, the Rees matrix semigroup $B = \mathcal{M}^0(I', \mathcal{E}, \Lambda'; \left(\begin{smallmatrix} 0 & e \\ e & 0 \end{smallmatrix}\right))$ belongs to the variety $\mathsf{var}\, \mathcal{V}$. Hence, $B^1 \in \mathsf{var}\, \mathcal{V}^1$, and it is easy to verify that the bijection

$$1 \mapsto 1,\ 0 \mapsto 0,\ (i, e, \lambda) \mapsto a,\ (j, e, \mu) \mapsto b,\ (i, e, \mu) \mapsto b,\ (j, e, \lambda) \mapsto ba$$

is an isomorphism between B^1 and the 6-element Brandt monoid B_2^1 defined in the proof of Theorem 4.1. Thus, B_2^1 lies in the variety $\mathsf{var}\, \mathcal{S}^1$, and each Zimin word is an isoterm relative to B_2^1 [19, Lemma 3.7].

Now suppose that $P' = \left(\begin{smallmatrix} e & e \\ e & d \end{smallmatrix}\right)$ with $d \in G$ being an element of infinite order. One readily verifies that the set

$$R = \big\{(k, d^n, \nu) \mid k \in I',\ \nu \in \Lambda',\ n = 0, 1, 2, \dots\big\}$$

forms a subsemigroup in $\mathcal{S}$ while the set

$$J = \big\{(k, d^n, \nu) \mid k \in I',\ \nu \in \Lambda',\ n = 1, 2, \dots\big\}$$

forms an ideal in R. It is easy to calculate that the Rees quotient R/J is isomorphic to the semigroup A_2, and we again conclude that B_2^1 lies in the variety $\mathsf{var}\, \mathcal{S}^1$. □

Remark 4.7. Suppose that $\mathcal{G} = \langle G, \cdot \rangle$ is an abelian group, I is a non-empty set, 0 is a symbol not in G, and $P = (p_{ij})$ is a symmetric $I \times I$-matrix over $G \cup \{0\}$. Then one can equip the Rees matrix semigroup $\mathcal{M}^0(I, \mathcal{G}, I; P)$ with an involution by letting $0^\star := 0, \quad (i, g, j)^\star := (j, g, i)$. A version of Theorem 4.6 holds also for involution monoids that are obtained from such involution semigroups by adjoining a new identity element.

Remark 4.8. Theorem 4.6 remains valid if we replace the abelian group $\mathcal{G}$ by an arbitrary semigroup $\mathcal{H}$ from a variety $\mathbf{U}$ all of whose periodic members are locally finite. In the matrix $\left(\begin{smallmatrix} e & e \\ e & d \end{smallmatrix}\right)$, the elements $e, d \in \mathcal{H}$ have to be chosen such that $e^2 = e$, $ed = d = de$, and $d^n \neq e$ for all positive integers n.

Remark 4.9. Readers familiar with the role of Rees matrix semigroups in the structure theory of semigroups will notice that Theorem 4.6 shows that for each *completely simple* semigroup $\mathcal{S}$ which admits two idempotents whose product has infinite order and whose maximal subgroups are abelian, the monoid $\mathcal{S}^1$ is nonfinitely based. Indeed, $\mathcal{S}$ admits a Rees matrix representation $\mathcal{M}(I, \mathcal{G}, \Lambda; P)$ (the construction mentioned above but without 0) such that P has a submatrix of the form $\left(\begin{smallmatrix} e & e \\ e & d \end{smallmatrix}\right)$ and d has infinite order in $\mathcal{G}$. The proof of Theorem 4.6 then shows that $\mathcal{S}^1 \in \mathsf{var}(\mathbf{Com} \circledcirc \mathbf{B})$ and $A_2^1 \in \mathsf{var}\,\mathcal{S}^1$; hence, each Zimin word is an isoterm relative to $\mathcal{S}^1$.

Remark 4.10. The results presented in this paper may create the impression that an involution semigroup and its semigroup reduct are always (non)finitely based at the same time. This is not true in general even in the case when one deals with groups (with the group inversion playing the role of the involution): there exists a nonfinitely based group $\langle G_1, \cdot\,, ^{-1}\rangle$ such that the semigroup $\langle G_1, \cdot\,\rangle$ is finitely based and on the other hand, there exists a finitely based group $\langle G_2, \cdot\,, ^{-1}\rangle$ such that the semigroup $\langle G_2, \cdot\,\rangle$ is nonfinitely based, see [23, Section 2] for references and a discussion.

Acknowledgment. The authors thank the anonymous referee for a useful suggestion incorporated in the present version of the paper.

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K. Auinger

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria

e-mail: karl.auinger@univie.ac.at

Yuzhu Chen, Xun Hu, Yanfeng Luo

(Yuzhu Chen, Xun Hu, Yanfeng Luo) Department of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, China; Key Laboratory of Applied Mathematics and Complex Systems, Gansu Province, China

e-mail: luoyf@lzu.edu.cn

(Xun Hu) Department of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing, 400033, China

M. V. Volkov

Institute of Mathematics and Computer Science, Ural Federal University, Lenina 51, 620000 Ekaterinburg, Russia

e-mail: mikhail.volkov@usu.ru

URL: http://csseminar.imkn.urfu.ru/volkov/