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On order types of linear basic algebras

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Abstract. Basic algebras form a common generalization of MV-algebras and of orthomodular lattices, the algebraic tool for axiomatization of many-valued Łukasiewicz logic and the logic of quantum mechanics. Hence, they are included among the so-called quantum structures. An important role is played by linearly ordered basic algebras because every subdirectly irreducible MV-algebra and every subdirectly irreducible commutative basic algebra is linearly ordered. Since subdirectly irreducible linearly ordered basic algebras exist of any infinite cardinality, the natural question is to describe all possible order types of these algebras. This problem is solved in the paper.

1. Introduction

The concept of a basic algebra was introduced by the second author with J. Kühr; see e.g. [6]–[10] for details and the motivation. Recall that an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ is called a *basic algebra* if it satisfies the axioms

(BA1): $x \oplus 0 = x$,
(BA2): $\neg\neg x = x$,
(BA3): $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
(BA4): $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

As usual, we denote $\neg 0$ by 1. It is elementary to prove that the following identities hold in basic algebras:

$$0 \oplus x = x, \quad x \oplus \neg x = 1 = \neg x \oplus x.$$

In a basic algebra, the order relation can be introduced as follows:

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1. \tag{1.1}$$

Then a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ can be considered as a bounded poset and, moreover, it is a lattice where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y).$$

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It is known that a basic algebra becomes an MV-algebra if and only if the operation $\oplus$ is associative. Moreover, every finite commutative basic algebra is an MV-algebra but there exist infinite commutative basic algebras which are not MV-algebras, see [2]. Commutative basic algebras [4] play an important role because their underlying lattices are distributive. It was shown by M. Botur [1] that every subdirectly irreducible commutative basic algebra is linearly ordered. Moreover, for any infinite cardinality, there exists a linearly ordered (commutative) subdirectly irreducible basic algebra, see [3]. Hence, it is a natural task to search for order types of linearly ordered basic algebras.

Before starting our task, we recall another description of basic algebras. Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $\leq$ its induced order defined by (1.1). As mentioned above, $L(\mathcal{A}) = (A; \vee, \wedge, 0, 1)$ for $1 = \neg 0$ is a bounded lattice with the property that for each element $a \in A$ there exists an antitone involution $x \mapsto x^a$ $(=\neg x \oplus a)$ in the interval $[a, 1]$, i.e., for $x, y \in [a, 1]$ we have

$$(x^a)^a = x \quad \text{and} \quad x \leq y \Rightarrow y^a \leq x^a.$$

Also conversely, having a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ such that for each $a \in L$ there exists an antitone involution $x \mapsto x^a$ in the interval $[a, 1]$, then it can be organized into a basic algebra as follows:

$$\neg x = x^0 \quad \text{and} \quad x \oplus y = (x^0 \vee y)^y.$$

Of course, $\oplus$ is a total operation because $x^0 \vee y \in [y, 1]$ for all $x, y \in L$. As shown e.g. in [2, 7], the above assignments form a one-to-one correspondence between the variety of basic algebras and the class of bounded lattices with antitone involutions in every interval $[a, 1]$. If this lattice is linearly ordered, i.e., if it is a chain, then for any x, y we have either $x^0 \leq y$ or $y \leq x^0$. In the first case we have $x \oplus y = 1$ and in the second one $x \oplus y = (x^0)^y$.

In what follows, we will often use this representation and we will deal mainly with linearly ordered sets with antitone involutions in every final interval. It is evident that for each natural number n, there is a unique linearly ordered n-element basic algebra on the chain $0 < 1 < \cdots < n - 1$ which is associative and hence an MV-algebra (the so-called *MV-chain*). This algebra is simple and consequently subdirectly irreducible.

2. Order types

Now we recall necessary concepts of the theory of linearly ordered sets. By an order type of $\langle L; \leq \rangle$ we mean the isomorphism type of this linearly ordered set. For the sake of brevity, we will not distinguish between a linearly ordered set and its isomorphism type whenever there is no danger of confusion. By ω (respectively, ω^*, η, n) is denoted the order type of the linearly ordered set of natural numbers with natural ordering (reverse ordering, natural ordering of rational numbers, natural ordering of the set $\{1, 2, \ldots, n\}$, respectively). For every linearly ordered set $\langle L; \leq \rangle$, let $\langle L; \leq \rangle^*$ denote $\langle L; \leq^{-1} \rangle$ where $\leq^{-1}$ is the inverse of $\leq$ (i.e., $a \leq^{-1} b$ iff $b \leq a$).

If α is an order type of a linearly ordered set $\langle L; \leq \rangle$, then α^* is the order type of $\langle L; \leq \rangle^*$. We call $\langle L; \leq \rangle$ *self-dual* if $\langle L; \leq \rangle$ is order isomorphic to $\langle L; \leq \rangle^*$, i.e., if $\alpha = \alpha^*$. For linearly ordered sets $\langle L_1; \leq_1 \rangle$ and $\langle L_2; \leq_2 \rangle$ where L_1, L_2 are disjoint, we denote by $\langle L_1; \leq_1 \rangle + \langle L_2; \leq \rangle$ the linearly ordered set $\langle L_1 \cup L_2; \leq \rangle$ where $\leq$ coincides with $\leq_i$ on L_i and $a \leq b$ for all $a \in L_1$, $b \in L_2$; it is the *ordinal sum* of $\langle L_1; \leq_1 \rangle$ and $\langle L_2; \leq_2 \rangle$. Analogously, we define $\sum_{i \in (I, \leq)} \langle L_i; \leq_i \rangle$ for every family of disjoint linearly ordered sets and a linearly ordered index set $\langle I, \leq \rangle$. This involves the sum of order types, $\sum_{i \in \beta} \alpha_i$, for order types α_i with $i \in \beta$.

The ordering on a subset B of $\langle L; \leq \rangle$ is defined as a restriction of $\leq$ to the set B. A subset B of $\langle L; \leq \rangle$ is called an *initial* (*final*) *interval* whenever for all $b \in B$ and $c \in L$ the inequality $c \leq b$ (respectively, $b \leq c$) yields $c \in B$. Denote by $(a]$ (or $[a)$) the initial (or final) interval $\{c \in L \mid c \leq a\}$ (or $\{c \in L \mid a \leq c\}$). An element b *covers* an element a in $\langle L; \leq \rangle$ if $a < b$ and there does not exist $c \in L$ with $a < c < b$. The cover of a will be denoted by a' if it exists. We define the order type α to be *finally self-dual* if for any $a \in \alpha$, the order type of the interval $[a)$ is self-dual. A linearly ordered set is called *scattered* if it does not contain some dense ordered subset.

By induction on the ordinal δ, we define the order types $(\omega^* + \omega)^\delta$ as follows:

$$(\omega^* + \omega)^0 = 1 \text{ and } (\omega^* + \omega)^{\delta+1} \text{ is defined as the sum } \sum_{i \in \omega^* + \omega} \alpha_i$$

where $\alpha_i = (\omega^* + \omega)^\delta$ for all $i \in \omega^* + \omega$.

Finally, for a limit ordinal δ, we consider the order spectrum

$$(\{(\omega^* + \omega)^\gamma; \gamma < \delta\}; \varphi_{\gamma_1,\gamma_2} \text{ for } \gamma_1 \leq \gamma_2 < \delta)$$

where $\varphi_{\gamma_1,\gamma_2}$ is some fixed embedding of $(\omega^* + \omega)^{\gamma_1}$ as a convex interval into $(\omega^* + \omega)^{\gamma_2}$. Clearly, this order type of the direct limit does not depend on the choice of embedding $\varphi_{\gamma_1,\gamma_2}$. This direct limit is $(\omega^* + \omega)^\delta$. It is plain that for every ordinal δ, the order type $(\omega^* + \omega)^\delta$ is self-dual and for each $a \in L$ (where L is a linearly ordered set of the order type $(\omega^* + \omega)^\delta$), its initial (final) intervals $(a]$ ($[a)$, respectively) have the same order type. To express the fact that it is an initial or final interval, we use the notation

$$(\omega^* + \omega)^\delta_i \quad \text{or} \quad (\omega^* + \omega)^\delta_f.$$

For any regular cardinal $\aleph_\alpha$, we denote by η_α the order type of any $\aleph_\alpha$-universal homogeneous linearly ordered set of cardinality $\aleph_\alpha$ (for its definition, construction, and existence under Generalized Continuum Hypothesis (GCH) see, for example, [12]). The following lemma can be found in [12].

Lemma 2.1. *Let $\langle L; \leq \rangle$ be a linearly ordered set of order type η_α. For any $a, b, c \in L$ with $a < b$, $a < c$, there exists an antitone involution* (*not necessarily unique*) *of the interval $\{d \in L \mid a < d\}$ onto itself which maps b to c.*

3. Subdirectly irreducible linear basic algebras

Using Lemma 2.1 and the GCH, we are able to prove the following result.

Theorem 3.1. *Assume GCH. For any regular cardinal $\aleph_\alpha$, there exist dense subdirectly irreducible non-commutative linear basic algebras of cardinality $\aleph_\alpha$. Their order type is $1 + \eta_\alpha + 1$.*

Proof. Let $\aleph_\alpha$ be a regular cardinal and consider a linearly ordered set $\langle L; \leq \rangle$ of the order type $1 + \eta_\alpha + 1$. Let $\neg$ be an antitone involution on $\langle L; \leq \rangle$ such that for some $c \in \eta_\alpha$, we have $\neg c = c$. According to Lemma 2.1, there are antitone involutions on every final interval $[a)$ of $\langle L; \leq \rangle$; thus, we obtain dense linear basic algebras of order types $1 + \eta_\alpha + 1$.

It is shown in [13] that there are dense linear basic algebras (MV-algebras) which are subdirectly irreducible. It is well known (see [8]) that basic algebras are congruence regular, i.e., their congruences θ are completely determined by any of the congruence classes a/θ of θ, in particular by the class $0/\theta$, the so-called ideal of L. It is also straightforward that the classes $0/\theta$ are the initial intervals of order types of linear basic algebras. However, not every initial interval of a linear basic algebra which is closed under $\oplus$ will be necessarily a congruence class $0/\theta$ of some congruence θ. This will be the case if and only if the initial interval is closed under three additional ideal terms, see [11]. Consequently, for the subdirect reducibility of the above constructed linear basic algebra, it is sufficient that there does not exist a least ideal.

To obtain subdirectly irreducible algebras, we have to define the antitone involutions on final sections of $\langle L; \leq \rangle$ in an appropriate way. Clearly, for $a < b < c$, we have $a < b < c = \neg c < \neg b < \neg a$. Denote by b_a the element $a \oplus b = (\neg a)^b$. We fix some element $e \in (0, c)$ and define the antitone involutions b_a in such a way that for all $0 < b \in (0, e)$ there exists $0 < a \in (0, b)$ with $e < b_a$. Due to Lemma 2.1, this choice is possible, and thus an arbitrary initial interval which is closed under $\oplus$ contains also $(e]$.

Hence, if we consider the set $\mathcal{M}$ of all proper ideals of L (i.e., non-zero ones), then their intersection $I = \cap \mathcal{M}$ contains $(e]$. This shows that I is the least non-zero ideal of L and, consequently, L is subdirectly irreducible. □

An open question remains whether there exist commutative basic algebras with the above properties.

In fact, an analogous statement holds also for any not necessarily regular cardinal. It is well known that the property of algebra to be or not to be subdirectly irreducible is expressible by means of an $L_{\omega_1\omega}$-formula. Similarly, the properties to be or not to be commutative, linear, or of dense order type are expressible by $L_{\omega\omega}$-formulas.

Also, the well-known Löwenheim-Skolem theorem for the language $L_{\omega_1\omega}$ states that for any formula Φ having a model of an arbitrary infinite cardinality $\aleph$, there are models of Φ of an arbitrary infinite cardinality less than $\aleph$.

Since for any cardinal there is its preceeding regular cardinal, we conclude that Theorem 4.1 and the Löwenheim-Skolem theorem for $L_{\omega_1\omega}$ lead to the following statement.

Theorem 3.2. *Assume GCH. For any infinite cardinal $\aleph_\alpha$, there exist dense subdirectly irreducible linear basic algebras of cardinality $\aleph_\alpha$.*

Let us note that the necessity of GCH in Theorem 3.1 is an open question.

Since from definition of self-dual linearly ordered sets we conclude the equality of order types of their initial intervals, we immediately obtain the following lemma.

Lemma 3.3. *Order types of initial intervals of an arbitrary finally self-dual linearly ordered set $\langle L;\leq\rangle$ are dual to order types of its final intervals, and for all $a < b$ from L, there is $d \in L$ such that the intervals $(0, d)$ and (a, b) are isomorphic.*

4. Order types of linearly ordered basic algebras

Now we are ready to state our theorem characterizing order types of linearly ordered basic algebras.

Theorem 4.1. *The order type δ corresponding to a linear basic algebra is equal to*

$$\delta = (\omega^* + \omega)_f^{(\alpha)} + \sum_{i\in\gamma}(\omega^* + \omega)_i^{(\alpha)} + (\omega^* + \omega)_i^{(\alpha)}$$

for some ordinal α and some finite (possible empty) or finally self-dual dense order type γ without least and greatest elements. Moreover, all the order types $(\omega^* + \omega)_i^{(\alpha)}$ are equal to $(\omega^* + \omega)^{(\alpha)}$.

Conversely, any order type δ described above is the order type of $\mathcal{L}(\mathcal{A})$ for some (not necessarily unique) linear basic algebra $\mathcal{A}$.

Proof. Recall that 0 ($\neg 0$) denotes the least (greatest) element of an arbitrary scattered finally self-dual linearly ordered set. Now let $\langle L;\leq\rangle$ be a finally self-dual linearly ordered set which is not dense. In this case, there are $a, a' \in L$ and, by Lemma 3.3, elements $0 < 0' = 1 < 0'' = 2 < \cdots < 0^{(n)} = n < \cdots$ where either $0^{(n)} = n = \neg 0$ for some natural n (in this case, we have a finite MV-algebra), or $n \neq \neg 0$ for all natural n. In the second case, for any final interval $[b)$ in $\langle L;\leq\rangle$, there exists an initial interval $\{b, b', \ldots, b^{(n)}, \ldots \mid n \in \omega\}$ (where $b^{(n+1)} = (b^{(n)})'$); hence, there is a final interval $\{-n \mid n \in \omega\}$ of $\langle L;\leq\rangle$ (where $'(-(n+1)) = -n$) of order type ω^*.

We denote the corresponding intervals of $\langle L;\leq\rangle$ by b^ω and ${}^{\omega^*}(\neg 0)$.

Apriori two cases can occur:

1. $\langle L;\leq\rangle = 0^\omega + {}^{\omega^*}(\neg 0)$; thus, the order type of $\langle L;\leq\rangle$ is $\omega + \omega^*$;
2. $\langle L;\leq\rangle \neq 0^\omega + {}^{\omega^*}(\neg 0)$, and hence there exists $c \in L$ not contained in $0^\omega \cup {}^{\omega^*}(\neg 0)$; thus, $n < c < (-n)$ for any natural n.

Consider further Case 2. We define the relation $\sim_f$ on L as follows: for $a < b$ in L, $a \sim_f b$ if and only if (a, b) is finite. Evidently, $\sim_f$ is an equivalence on L. Equivalence classes of $\sim_f$ are convex intervals in $\langle L; \leq\rangle$ and their order types are either n, ω, ω^*, or $\omega^* + \omega$.

Due to Lemma 3.3 and to the non-finiteness of L, we conclude that these classes are infinite. Moreover, for any isomorphism of $\langle L; \leq\rangle$ onto its final interval and for any antitone involution of this interval onto itself, they are mapped one to the other, and consequently, have to be isomorphic and self-dual. Thus, the order type of the class $0/_{\sim_f}$ $(\neg 0/_{\sim_f})$ is ω (ω^*), and for any $c \notin 0^\omega \cup {}^{\omega^*}(\neg 0)$ the order type of the class $c/_{\sim_f}$ is equal to $\omega^* + \omega$.

We denote by $\langle L; \leq\rangle'$ the factor set $L/_{\sim_f}$ with the order induced from $\langle L; \leq\rangle$. We immediately conclude that $\langle L; \leq\rangle'$ remains a finally self-dual linearly ordered set with the least element. Thus, $\langle L; \leq\rangle'$ is either finite, or $\langle L; \leq\rangle'$ is dense, or the considerations mentioned above for $\langle L; \leq\rangle$ can be repeated for $\langle L; \leq\rangle'$. In conclusion, we obtain the poset $\langle L; \leq\rangle'' = (\langle L; \leq\rangle')'$ of the type $\langle L/_{\sim^1}; \leq\rangle$, where $\sim^1$ is a preimage on $\langle L; \leq\rangle$ of the relation $\sim_f$ with respect to a natural homomorphism of a poset $\langle L; \leq\rangle$ onto a poset $\langle L; \leq\rangle' = \langle L/_{\sim_f}; \leq\rangle$. We iterate this process and for an arbitrary ordinal of type $\delta + 1$, we define the relation $\sim_f^{\delta+1}$ on $\langle L; \leq\rangle$ to be a preimage of $\sim_f$ onto a poset $\langle L; \leq\rangle^\delta = \langle L/_{\sim_f^\delta}; \leq\rangle$ with respect to the natural homomorphism of $\langle L; \leq\rangle$ onto $\langle L; \leq\rangle^\delta$.

For a limit ordinal δ, we consider the relation $\sim_f^\delta$ to be a union of the relations $\sim_f^\gamma$ on $\langle L; \leq\rangle$ for $\gamma < \delta$.

Since the sequence of intervals of the type $0/_{\sim_f^\alpha}$ is a strongly increasing chain, there exists an ordinal α of cardinality less than or equal to the cardinality of L such that the finally self-dual poset $\langle L; \leq\rangle^\alpha$ with the least element will be either finite or dense. Moreover, we have

$$\langle L; \leq\rangle = 0/_{\sim_f^\alpha} + \sum\{a/_{\sim_f^\alpha} \mid a/_{\sim_f^\alpha} \in \langle L/_{\sim_f^\alpha} \setminus \{0/_{\sim_f^\alpha}, \neg 0/_{\sim_f^\alpha}\}; \leq\rangle\} + \neg 0/_{\sim_f^\alpha}.$$

Clearly, order types of the intervals $a/_{\sim_f^\alpha}$ (distinct from $0/_{\sim_f^\alpha}$ and $\neg 0/_{\sim_f^\alpha}$) are equal to $(\omega^* + \omega)^{(\alpha)}$ and the intervals $0/_{\sim_f^\alpha}$ and $\neg 0/_{\sim_f^\alpha}$ have the types $(\omega^* + \omega)_f^{(\alpha)}$ and $(\omega^* + \omega)_i^{(\alpha)}$. □

Remark 4.2. If the order type mentioned in Theorem 4.1 is neither dense nor finite ($\alpha \neq 0$), then δ has an initial interval of the order type ω. Due to the congruence regularity of basic algebras, the order on their congruences is determined by set-inclusion $\subseteq$ on congruence classes with the least element 0.

So immediately verifying that the initial interval of the order type ω for an infinite but non-dense δ is closed under $\oplus$, we obtain that for each non-trivial congruence θ, $0/\theta$ contains this interval. This shows that there is a least non-trivial congruence θ_0 and thus the algebra is subdirectly irreducible.

Altogether, we obtain the following class of subdirectly irreducible linear basic algebras.

Corollary 4.3. *If $\mathcal{A}$ is a linear basic algebra whose underlying lattice is not dense then $\mathcal{A}$ is subdirectly irreducible.*

Further, choosing for ordinal types δ from Theorem 4.1 the ordinals α as arbitrary ordinals of a fixed infinite cardinality $\aleph$, and as γ an empty ordered type, we obtain the following statement.

Corollary 4.4. *For an arbitrary infinite cardinality $\aleph$, there are at least $\aleph^+$ subdirectly irreducible pairwise non-isomorphic linear basic algebras of cardinality $\aleph$.*

There is an interesting question whether there are MV-algebras corresponding to order types δ from Theorem 4.1 (as mentioned before, this is equivalent to associativity of $\oplus$). Also, commutativity of these algebras is of interest.

Recall that it is shown in [1] that subdirectly irreducible commutative basic algebras are linear. We show that the converse is not true. First, we observe that on an infinite linear basic algebra of order type $\omega + \omega^*$, the operations $\oplus$ and $\neg$ are defined uniquely (due to the uniqueness of antitone involutions on their final intervals). Denoting further the elements of an initial interval of order type ω (for an algebra of order type $\omega + \omega^*$) by the natural numbers n, and for final interval of order type ω^* by $-n$, we obtain the following relations for the operations $\neg$ and $\oplus$:

$$\neg n = -n, \text{ and } \neg(-n) = n;$$
$$n_1 \oplus n_2 = n_1 + n_2, \text{ and } (-n_1) \oplus (-n_2) = -0;$$
$$n_1 \oplus (-n_2) = -(n_2) \oplus n_1 = \begin{cases} -(n_2 - n_1) & \text{if } n_1 \leq n_2, \\ -0 & \text{if } n_2 < n_1. \end{cases}$$

This means that an algebra of order type $\omega + \omega^*$ is commutative. Moreover, it is an MV-algebra.

We prove by induction on ordinal α that the operations $\neg$ and $\oplus$ can be defined on the order types $(\omega^* + \omega)_f^{(\alpha)} + (\omega^* + \omega)_i^{(\alpha)}$ such that the corresponding basic algebras are commutative and associative. By definition of the operations on δ, we require the following additional condition (4.1) on the antitone bijections of final intervals of δ:

For any $a, b \in \delta$ and any $\gamma < \alpha$, the antitone bijections of intervals $[a)$ and $[b)$ on themselves restricted to the convex intervals of the order types $(\omega^* + \omega)_f^{(\gamma)}$ do not depend on the choice of a and b. (4.1)

It is obvious that the condition (4.1) can be fulfilled for the order type $\delta = (\omega^* + \omega)_f^{(\alpha)} + (\omega^* + \omega)_i^{(\alpha)}$.

Further, let us suppose that for the ordinals $\mu < \alpha$, on the order types $(\omega^* + \omega)_f^{(\mu)} + (\omega^* + \omega)_i^{(\mu)}$ we can define operations $\neg$ and $\oplus$ such that the corresponding basic algebras are commutative and associative. We assume that the condition (4.1) for the operations $\neg$ and $\oplus$ on the ordered type $\delta = (\omega^* + \omega)_f^{(\alpha)} + (\omega^* + \omega)_i^{(\alpha)}$ is fulfilled. Let α be some ordinal of type $\beta + 1$. Let θ

be a congruence of $\mathcal{A}$ of the order type δ such that its congruence class $0/\theta$ has the order type $(\omega^*+\omega)_f^{(\beta)}$. Then $\mathcal{A}/\theta$ is a linear basic algebra of the order type $\omega^*+\omega$ which is commutative and associative. From (4.1) for $\mathcal{A}$, it follows that the algebra $\mathcal{A}$ is commutative and associative as well. If α is a limit ordinal, then the algebra $\mathcal{A}$ is a union of increasing chains of its commutative and associative subalgebras of order types $(\omega^*+\omega)_f^{(\gamma)}+(\omega^*+\omega)_i^{(\gamma)}$ composed of its initial and final intervals of the type $(\omega^*+\omega)_f^{(\gamma)}$ and $(\omega^*+\omega)_i^{(\gamma)}$, respectively (for $\gamma<\alpha$). Consequently, the algebra $\mathcal{A}$ is commutative, associative, and its order type is $(\omega^*+\omega)_f^{(\gamma)}+(\omega^*+\omega)_i^{(\gamma)}$.

This gives the lower bound for the cardinality of the class of subdirectly irreducible MV-algebras.

Corollary 4.5. *For each infinite cardinal $\aleph$, there are at least $\aleph^+$ pairwise non-isomorphic linear scattered subdirectly irreducible MV-algebras of cardinality $\aleph$.*

The situation is different for linear basic algebras of the order type $\delta=\omega+\omega^*+\omega+\omega^*$. Such a basic algebra can be commutative or non-commutative. First observe that the operations $\oplus$ and $\neg$ of such an algebra are not defined uniquely: there is a countable set of antitone bijections of type δ onto itself (due to the possible choice of bijections of the interval $\omega^*+\omega$). Thus, there are countably many choices of different operations $\neg$ on the order type δ. Further, by a we denote the elements of $\mathcal{A}$ from the initial interval of the type ω, and by b those from the interval $\omega^*+\omega$. Due to the different antitone involutions of the interval $\omega^*+\omega$ onto itself, for any fixed choice of $\neg$, there are countably many operations $\oplus$ on δ. Then $b\oplus a$ can be chosen as an arbitrary element of the interval $\omega^*+\omega$. The situation is different for $a\oplus b$ which is defined for this type uniquely.

Hence, there is a possibility of the choice of $\oplus$ for a linear basic algebra $\mathcal{A}$ of the type $\omega+\omega^*+\omega+\omega^*$ such that $a\oplus b\neq b\oplus a$ (for some a,b or even for every a,b). The corresponding basic algebra will not be commutative. The same considerations are valid for the order types $(\omega^*+\omega)_f^{(\alpha)}+(\omega^*+\omega)^{(\alpha)}+(\omega^*+\omega)_i^{(\alpha)}$ for any ordinal α.

As a consequence, we obtain the following statement.

Corollary 4.6. *For any infinite cardinal $\aleph$, there are at least $\aleph^+$ pairwise non-isomorphic linear scattered non-commutative subdirectly irreducible basic algebras of cardinality $\aleph$.*

Let us mention some papers devoted to subdirectly irreducible basic algebras. It is proved in [3] that for every infinite cardinality, there are subdirectly irreducible commutative basic algebras which are not MV-algebras. We also mention the papers [2, 5] devoted to similar problems.

References

[1] Botur, M.: Subdirectly irreducible commutative basic algebras are linearly ordered. PhD thesis, Palacký University Olomouc (2009)
[2] Botur, M.: An example of a commutative basic algebra which is not an MV-algebra. Math. Slovaca **60**, 171–178 (2010).
[3] Botur, M., Chajda, I.: There exist subdirectly irreducible commutative basic algebras of an arbitrary infinite cardinality which are not MV-algebras. Algebra Universalis **65**, 185–192 (2011)
[4] Botur, M., Halaš, R.: Complete commutative basic algebras. Order **24**, 89–105 (2007)
[5] Botur, M., Halaš, R.: Finite commutative basic algebras are MV-effect algebras. J. Mult.-Valued Logic Soft Comput. **14**, 69–80 (2008).
[6] Botur, M., Kühr, J.: On (finite) distributive lattices with antitone involutions. Soft Computing **18**, 1033–1040 (2014)
[7] Chajda, I.: Basic algebras and their applications, an overview. Contributions to General Algebra **20**, Johannes Heyn, Klagenfurt (2011)
[8] Chajda, I., Halaš, R., Kühr, J.: Semilattice Structures. Heldermann, Lemgo (2007)
[9] Chajda, I., Halaš, R., Kühr, J.: Many-valued quantum algebras. Algebra Universalis **60**, 63–90 (2009)
[10] Chajda, I., Kolařík, M.: Independence of axiom system of basic algebras. Soft Computing **13**, 41–43 (2009)
[11] Chajda, I., Kühr, J.: Ideals and congruences of basic algebras. Soft Computing **17**, 401–410 (2013)
[12] Comfort, W.W., Negrepontis, S.: The theory of ultrafilters. Die Grundlehren der mathematischen Wissenschaften, vol. 211, Springer, New York (1974)
[13] Gaitán, H.: Subdirectly irreducible MV-algebras. Czechoslovak Math. J. **53**, 631–639 (2003)
[14] Keisler, H. J.: Model theory for infinitary logic. North-Holland, Amsterdam (1971)

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