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# Testing for edge terms is decidable

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Abstract. This paper defines a computational problem, the edge-like problem, and proves that the problem is a decidable one when the input sets are finite. The edge-like problem is relevant to the field of universal algebra as it is a common generalization of several problems currently of interest in that field, and this paper proves that several of these problems are decidable.

## 1. Introduction

In this paper, we examine a particular class of conditions that individually may or may not hold true for a given algebra, and we prove that satisfaction of such a condition is a decidable proposition. This paper generalizes the work of Maróti in [9] and the proof of this paper's main result proceeds along very similar lines to those of that paper.

In Section 2, we introduce some notation needed to simplify the proofs in this paper, and define the class of conditions that we will be examining. In Section 3, we define the characteristic triple of an operation, a partial evaluation of that operation which is compatible with the condition under examination. We then define what it means to compose a function with characteristic triples, and prove that this notion of composition is compatible with ordinary functional composition. In Section 4, we define weak near unanimity operations and examine the properties of their characteristic triples. Weak near unanimity operations form the starting point for our ultimate decision procedure for solving our original condition, and we prove that they can serve this purpose. In Section 5, we introduce a partial order on characteristic triples and prove that order filters constructed in a specific way are computable. We then prove that an operation satisfying our original condition will have a minimal characteristic triple with respect to this partial order, and that we can search for such an operation in finite time. In Section 6, we examine three corollaries of the main result that have consequences for the field of Universal Algebra, one of which is in fact Maróti's result on which this paper is based, namely the main result of [9].

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## 2. Preliminaries

Let us begin with some simplifying notation and terminology that will be in use throughout this paper.

**Definition 2.1.** • Let $\mathcal{O}_A$ be the *set of all operations on set* $A$.

- Given any set $\mathcal{F}$ of operations, let $\mathcal{F}^{(n)}$ denote the *set of* $n$*-ary operations in* $\mathcal{F}$.
- If $M$ is a matrix, let $M_i$ denote the *ith row of* $M$ and let $M^i$ denote the *ith column of* $M$. Let $M_i^j$ denote the *entry of* $M$ *in row* $i$, *column* $j$.
- Let $\omega^+$ be the *set that contains all finite ordinals and* $\omega$, *the smallest infinite ordinal.*
- Given $g \in \mathcal{O}_A^{(k)}$, $g' \in \mathcal{O}_A^{(\ell)}$ and $n \geq 0$, say that $g'$ is an $n$*-extension of* $g$ if there is an injection $\sigma \colon \{0, \ldots, k-1\} \to \{0, \ldots, \ell - 1\}$ such that
  - $g'(a_0, \ldots, a_{\ell-1}) = g(a_{\sigma(0)}, \ldots, a_{\sigma(k-1)})$, and
  - $\sigma$ restricted to $\{0, \ldots, n-1\}$ is the identity function.

Now we can define the type of problem we will be examining.

**Definition 2.2.** An *instance of the edge-like problem* is a tuple of the form $P = (A, \mathcal{F}, M, S)$ where

(1) $A$ is a finite set,
(2) $\mathcal{F}$ is a finite set of operations on $A$,
(3) $S \subseteq A$,
(4) $M$ is an $m \times n$ matrix with elements in $\{\mathbf{x}, \mathbf{y}\}$ (consider the rows of $M$ to be indexed by $n - m \leq i < n$ and the columns indexed by $0 \leq j < n$),
(5) no two rows of $M$ are equal,
(6) $M$ does not contain a row of all $\mathbf{x}$'s or a row of all $\mathbf{y}$'s, and
(7) $M$ does not contain a column of all $\mathbf{x}$'s.

**Definition 2.3.** • Given an instance of the edge-like problem, let $\mathcal{B}$ denote the *set of all binary functions* $f \colon S \times A \to A$.

- Given an instance of the edge-like problem and an operation $f \in \mathcal{O}_A^{(k)}$ for some $k \geq n$ and $i \geq n - m$ (notice that $n - m$ may be negative) define the *ith polymer of* $f$, $f|_i \in \mathcal{B}$ to be

$$f|_i(\mathbf{x}, \mathbf{y}) = \begin{cases} f(M_i, \mathbf{x}^{k-n}) & \text{if } n - m \leq i < n, \\ f(\mathbf{x}^i \mathbf{y} \mathbf{x}^{k-i-1}) & \text{if } n \leq i < k, \\ f(\mathbf{x}^k) & \text{otherwise.} \end{cases}$$

- For simplicity of notation, define $\nu = \{i : n - m \leq i < n\}$.
- *The class EL* (*edge-like*) is the class of all instances of the edge-like problem for which there is an idempotent $f \in \langle \mathcal{F} \rangle^{(k)}$ (the $k$-ary operations in the clone generated by $\mathcal{F}$) for some $k \geq n$ with $f|_i(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for all $n - m \leq i$, all $\mathbf{x} \in S$, and all $\mathbf{y} \in A$. In this case, we will say that $f$ *witnesses* $P$*'s membership in EL.*

For example, a finite algebra $\langle A, \mathcal{F}\rangle$ will support a near unanimity operation if and only if $(A, \mathcal{F}, \emptyset, A) \in EL$ (where $\emptyset$ denotes the $0 \times 0$ empty matrix), and it will support an edge operation (defined in Definition 6.3) if and only if $(A, \mathcal{F}, \left[\begin{smallmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} \\ \mathbf{y} & \mathbf{x} & \mathbf{y} \end{smallmatrix}\right], A) \in EL$.

We will show in Theorem 5.6 that, under certain conditions (namely if the algebra generates a variety that omits type 1), membership in the class $EL$ can be decided. Section 6 shows why this is relevant to the field of universal algebra.

## 3. Characteristic triples

For the remainder of this paper (excepting Section 6), fix an instance $P = (A, \mathcal{F}, M, S)$ (with $M$ an $m \times n$ matrix) of the edge-like problem for consideration.

We will now separate $\mathcal{O}_A$ into equivalence classes dependent on $P$ in such a way as to characterize those operations that will witness $P$'s membership in $EL$.

**Definition 3.1.** (1) Define the *characteristic triple of* $f \in \mathcal{O}_A$ to be $\mathrm{T}(f) = (\rho_f, \alpha_f, \chi_f)$ where $\rho_f \colon A \to A$, $\alpha_f \colon \nu \to \mathcal{B}$, and $\chi_f \colon \mathcal{B} \to \omega^+$ are defined as

$$\rho_f(a) = f(a, a, \dots, a), \quad \alpha_f(i) = f|_i, \quad \chi_f(c) = \big|\{i \geq n : f|_i = c\}\big|,$$

for all $a \in A$ and $c \in \mathcal{B}$. Call $\alpha_f$ the *characteristic sequence of* $f$ and $\chi_f$ the *characteristic function of* $f$, and call $\rho_f$ the *idempotence test of* $f$.

(2) Given an instance of the edge-like problem $P$, let $\mathrm{T}(P) = (\rho_P, \alpha_P, \chi_P)$ be *the characteristic triple of any operation witnessing the membership of* $P$ *in* $EL$, namely $\rho_P$ is the identity function on $A$, $\alpha_P = \mathbf{x}^m$ and

$$\chi_P(c) = \begin{cases} \omega & \text{if } c = \mathbf{x}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that a function $f$ witnesses $P$'s membership in $EL$ if and only if $\mathrm{T}(f) = \mathrm{T}(P)$. Now we need to consider which characteristic triples are characteristic triples of operations on $A$.

**Lemma 3.2.** *A triple* $(\rho, \alpha, \chi) \in A^A \times \mathcal{B}^\nu \times (\omega^+)^{\mathcal{B}}$ *is the characteristic triple of some operation* $f$ *if and only if the following hold:*

(1) *there is a unique* $b \in \mathcal{B}$ *with* $\chi(b) = \omega$*;*
(2) *for all* $c \in \mathcal{B}$ *with* $\chi(c) > 0$*, we have* $c(\mathbf{x}, \mathbf{x}) = b(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{x})$*;*
(3) *for all* $i \in \nu$*, we have* $\alpha(i)(\mathbf{x}, \mathbf{x}) = \rho(\mathbf{x})$*.*

*Proof.* It is trivial to see that for any $f \in \mathcal{O}_A$ the stated conditions hold.

Given $(\rho, \alpha, \chi)$ satisfying the stated conditions, we wish to construct a function $f \in \mathcal{O}_A$ such that $\rho_f = \rho$, $\alpha_f = \alpha$, and $\chi_f = \chi$. By our earlier assumption, $\sum_{c \in A, c \neq b} \chi(c) = k$ is finite, so we can choose a sequence $\zeta_i \in \mathcal{B}$ for $i \geq n$ such that $\chi(c) = |\{i : \zeta_i = c\}|$ for each $c \in \mathcal{B}$. For each $i \in \nu$, set $\zeta_i = \alpha(i)$.

Then there is a function $f \in \mathcal{O}_A^{(n+k)}$ such that

$$\begin{aligned} f(M_i\mathbf{x}^k) &= \zeta_i && \text{if } i \in \nu, \\ f(\mathbf{x}^i\mathbf{y}\mathbf{x}^{n+k-i-1}) &= \zeta_i && \text{if } n \leq i < n+k, \\ f(\mathbf{x}^{n+k}) &= b(\mathbf{x},\mathbf{y}) && \text{and} \\ f(a^{n+k}) &= \rho(a) && \text{for all } a \in A. \end{aligned}$$

Clearly then $\mathrm{T}(f) = (\rho, \alpha, \chi)$. □

**Definition 3.3.** (1) Given $\mathcal{G} \subseteq \mathcal{O}_A$, let $\mathrm{T}(\mathcal{G}) = \{\mathrm{T}(f) : f \in \mathcal{G}\}$ be the *set of characteristic triples of functions in* $\mathcal{G}$, and let $\mathrm{T}_A = \mathrm{T}(\mathcal{O}_A)$.

(2) Given $\mathcal{U} \subseteq \mathrm{T}_A$, let $X(\mathcal{U})$ be the *projection of* $\mathcal{U}$ *onto the third coordinate*, in particular $X(\mathrm{T}(\mathcal{G})) = \{\chi_f : f \in \mathcal{G}\}$, and let $\mathcal{X}_A = X(\mathrm{T}_A)$.

In order to search for the appropriate characteristic triple in $\langle \mathcal{F} \rangle$, we need some method of composing functions with characteristic triples. Unfortunately, this composition is not itself a function, but it is a computable relation under the right conditions (see Lemma 5.3).

**Definition 3.4.** (1) By *a composition of* $f \in \mathcal{O}_A^{(k)}$ *with* $n$*-extensions of* $g_0, \ldots, g_{k-1} \in \mathcal{O}_A$, we mean an operation of the form $f(g_0', \ldots, g_{k-1}') \in \mathcal{O}_A^{(\ell)}$ where $g_i' \in \mathcal{O}_A^{(\ell)}$ is an $n$-extension of $g_i$.

(2) Say that $\mathcal{T} = (\rho, \alpha, \chi) \in \mathrm{T}_A$ *is a composition of* $f \in \mathcal{O}_A^{(k)}$ *with* $\mathrm{T}_0, \ldots, \mathrm{T}_{k-1} \in \mathrm{T}_A$ (where $\mathrm{T}_i = (\rho_i, \alpha_i, \chi_i)$) if

$$\rho(a) = f(\rho_0(a), \ldots, \rho_{k-1}(a)), \quad \text{and} \quad \alpha(i) = f(\alpha_0(i), \ldots, \alpha_{k-1}(i)),$$

for all $i \in \nu$ and $a \in A$, and there is a $\mu\colon \mathcal{B}^k \to \omega^+$ such that

$$\chi(c) = \sum_{\bar{b} \in \mathcal{B}^k,\, f(\bar{b}) = c} \mu(\bar{b}) \quad \text{and} \quad \chi_i(c) = \sum_{\bar{b} \in \mathcal{B}^k,\, b_i = c} \mu(\bar{b}),$$

for every $c \in \mathcal{B}$ and every $i < k$.

(3) Given $\mathcal{G}, \mathcal{H} \subseteq \mathcal{O}_A$, let $\mathrm{C}_{\mathcal{G}}(\mathcal{H})$ be the *set of all possible compositions of operations in* $\mathcal{G}$ *with* $n$*-extensions of operations in* $\mathcal{H}$. Given $\mathcal{G} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq \mathrm{T}_A$, let $\mathrm{C}_{\mathcal{G}}(\mathcal{U})$ be the *set of all possible compositions of operations from* $\mathcal{G}$ *with characteristic triples from* $\mathcal{U}$.

(4) Also, inductively define $\mathrm{C}_{\mathcal{G}}^{i+1}(\cdot) = \mathrm{C}_{\mathcal{G}}(\mathrm{C}_{\mathcal{G}}^{i}(\cdot))$ where $\mathrm{C}_{\mathcal{G}}^{0}$ is the identity map.

The obvious question following from the preceding definition is whether or not composition with characteristic triples behaves like composition with operations. The succeeding Lemma shows that T and $\mathrm{C}_{\mathcal{F}}$ commute as functions, demonstrating the relevance of our definition of composition with characteristic triples.

**Lemma 3.5.** $\mathrm{T}\,\mathrm{C}_{\mathcal{G}}(\mathcal{H}) = \mathrm{C}_{\mathcal{G}}\,\mathrm{T}(\mathcal{H})$ *for all* $\mathcal{G}, \mathcal{H} \subseteq \mathcal{O}_A$.

*Proof.* To prove that $\mathrm{T}\,\mathrm{C}_{\mathcal{G}}(\mathcal{H}) \subseteq \mathrm{C}_{\mathcal{G}}\,\mathrm{T}(\mathcal{H})$, take $f \in \mathcal{G}^{(k)}$ and $g_0, \dots, g_{k-1} \in \mathcal{H}$ with $h = f(g_0', \dots, g_{k-1}')$, where $g_i' \in \mathcal{O}_A^{(\ell)}$ is an $n$-extension of $g_i$. Then we need to prove that $\mathrm{T}(h)$ is a composition of $f$ with $\mathrm{T}(g_0), \dots, \mathrm{T}(g_{k-1})$. For any $a \in A$, notice that

$$\begin{aligned}\rho_h(a) &= h(a, \dots, a) = f(g_0'(a, \dots, a), \dots, g_{k-1}'(a, \dots, a)) \\ &= f(g_0(a, \dots, a), \dots, g_{k-1}(a, \dots, a)) = f(\rho_{g_0}(a), \dots, \rho_{g_{k-1}}(a)).\end{aligned}$$

For any $i \in \nu$, notice that

$$\begin{aligned}\alpha_h(i) &= h|_i = f(g_0'|_i, \dots, g_{k-1}'|_i) \\ &= f(g_0|_i, \dots, g_{k-1}|_i) = f(\alpha_{g_0}(i), \dots, \alpha_{g_{k-1}}(i)).\end{aligned}$$

Now define $\mu\colon \mathcal{B}^k \to \omega^+$ as $\mu(\bar{b}) = \bigl|\{j' \geq n : (g_0'|_j, \dots, g_{k-1}'|_j) = \bar{b}\}\bigr|$. Then for each $c \in \mathcal{B}$,

$$\begin{aligned}\sum_{\bar{b} \in \mathcal{B}^k,\, f(\bar{b}) = c} \mu(\bar{b}) &= \bigl|\{j \geq n : f(g_0'|_j, \dots, g_{k-1}'|_j) = c\}\bigr| \\ &= \bigl|\{j \geq n : h|_j = c\}\bigr| = \chi_h(c),\end{aligned}$$

completing the proof that $\mathrm{T}(h)$ is a composition of $f$ with $\mathrm{T}(g_0), \dots, \mathrm{T}(g_{k-1})$.

To prove that $\mathrm{C}_{\mathcal{G}}\,\mathrm{T}(\mathcal{H}) \subseteq \mathrm{T}\,\mathrm{C}_{\mathcal{G}}(\mathcal{H})$, take $\mathcal{T} = (\rho, \alpha, \chi)$, a composition of $f \in \mathcal{G}^{(k)}$ with $\mathrm{T}(g_0), \dots, \mathrm{T}(g_{k-1})$, where $g_i \in \mathcal{H}^{(\ell_i)}$; let $\mu\colon \mathcal{B}^k \to \omega^+$ witness this composition. Specifically,

$$\begin{aligned}&\rho(a) = f(\rho_{g_0}(a), \dots, \rho_{g_{k-1}}(a)) \text{ for all } a \in A, \\ &\alpha(i) = f(\alpha_{g_0}(i), \dots, \alpha_{g_{k-1}}(i)) \text{ for all } i \in \nu, \\ &\chi(c) = \sum_{\bar{b} \in \mathcal{B}^k,\, f(\bar{b}) = c} \mu(\bar{b}) \quad \text{and} \quad \chi_{g_i}(c) = \sum_{\bar{b} \in \mathcal{B}^k,\, b_i = c} \mu(\bar{b}),\end{aligned}$$

for all $c \in \mathcal{B}$ and $i < k$. To complete the proof, we must find $g_0', \dots, g_{k-1}' \in \mathcal{O}_A^{(\ell)}$, for some $\ell$, such that $g_i'$ is an $n$-extension of $g_i$ and such that $\mathrm{T}(h) = \mathcal{T}$, where $h = f(g_0', \dots, g_{k-1}')$.

Let $\zeta\colon \{j : j \geq n\} \to \mathcal{B}^k$ be a mapping such that $\mu(\bar{b}) = |\{j \geq n : \zeta(j) = \bar{b}\}|$ for all $\bar{b} \in \mathcal{B}^k$. Then we have that

$$\bigl|\{j \geq n : g_i|_j = c\}\bigr| = \chi_{g_i}(c) = \sum_{\bar{b} \in \mathcal{B}^k,\, b_i = c} \mu(\bar{b}) = |\{j \geq n : \zeta(j)_i = c\}|,$$

for each $i < k$ and each $c \in \mathcal{B}$. So for each $i < k$, we can choose a permutation $\sigma_i'\colon \{j : j \geq n\} \to \{j : j \geq n\}$ such that $g_i|_j = \zeta(\sigma_i'(j))_i$ for all $j \geq n$. For each $i < k$, define $\sigma_i\colon \omega \to \omega$ as

$$\sigma_i(j) = \begin{cases} j & \text{if } j < n, \\ \sigma_i'(j) & \text{otherwise.}\end{cases}$$

Letting $\ell = \max\{\sigma_i(j) : i < k, j < \ell_i\}$, then for each $i < k$ the restriction of $\sigma_i$ to $\{j : j < \ell_i\}$ is an injection into the set $\{j : j < \ell\}$. Define the operations $g_0', \dots, g_{k-1}' \in \mathcal{O}_A^{(\ell)}$ as $g_i'(x_0, \dots, x_{\ell-1}) = g_i(x_{\sigma_i(0)}, \dots, x_{\sigma_i(\ell_i - 1)})$. Clearly, each

$g_i'$ is an $n$-extension of $g_i$ and $g_i'|_j = g_i|_{\sigma_i^{-1}(j)}$, so let $h = f(g_0', \ldots, g_{k-1}')$. Then for each $x \in A$,

$$\begin{aligned}\chi_h(c) &= \left|\{j \geq n : h|_j = c\}\right| = \left|\{j \geq n : f(g_0'|_j, \ldots, g_{k-1}'|_j) = c\}\right| \\ &= \left|\{j \geq n : f(g_0|_{\sigma_0^{-1}(j)}, \ldots, g_{k-1}|_{\sigma_{k-1}^{-1}(j)}) = c\}\right| \\ &= \left|\{j \geq n : f(\zeta\sigma_0\sigma_0^{-1}(j)_0, \ldots, \zeta\sigma_{k-1}\sigma_{k-1}^{-1}(j)_{k-1}) = c\}\right| \\ &= |\{j \geq n : f(\zeta(j)) = c\}| = \sum_{\bar{b} \in \mathcal{B}^k,\, f(\bar{b}) = c} \mu(\bar{b}) = \chi(c).\end{aligned}$$

Also, for each $i \in \nu$, it is clear that

$$\alpha_h(i) = h|_i = f(g_0'|_i, \ldots, g_{k-1}'|_i) = f(g_0|_i, \ldots, g_{k-1}|_i) = \alpha(i).$$

The fact that $\rho_h = \rho$ is left as an exercise for the reader; therefore clearly, $\mathrm{T}(h) = \mathcal{T}$, completing the proof. $\square$

## 4. Weak near unanimity operations

In order to begin searching $\langle \mathcal{F} \rangle$ for a function to witness $P$'s membership in $EL$, we need a set of characteristic triples in $\mathrm{T}(\langle \mathcal{F} \rangle)$ that will contain such a function if it exists and which can be computed. To this end, we will consider weak near unanimity operations since their characteristic triples are easily calculated (see Lemma 5.5) and they can be used as a "basis" of sorts for any witnessing operation (see Lemma 4.4).

**Definition 4.1.** [10] An operation $g \in \mathcal{O}_A^{(k)}$ is called a *weak near unanimity operation* (*weak NU operation*) if it is idempotent and for all $i, j < k$,

$$g(\mathbf{x}^i \mathbf{y} \mathbf{x}^{k-i-1}) = g(\mathbf{x}^j \mathbf{y} \mathbf{x}^{k-j-1}).$$

In the remaining sections of this paper, we will frequently need to compose operations in particular ways and to introduce many "dummy" variables, so the next definition provides notation to easily allow for such procedures.

**Definition 4.2.** (1) For $f \in \mathcal{O}_A^{(k)}$ and $i < n$, define $\delta_i(f) \in \mathcal{O}_A^{(n+k-1)}$ as

$$\delta_i(f)(x_0, \ldots, x_{n+k-2}) = f(x_i, x_n, x_{n+1}, \ldots, x_{n+k-2}),$$

and for each $i > 0$, inductively define $\gamma_i(f) \in \mathcal{O}_A^{(ki-i+1)}$ as

$$\gamma_1(f)(x_0, \ldots, x_{k-1}) = f(x_0, \ldots, x_{k-1}) \text{ and}$$
$$\gamma_{i+1}(f)(x_0, \ldots, x_{ki+k-i-1}) = f(\gamma_i(f)(x_0, \ldots, x_{ki-i}), x_{ki-i+1}, \ldots, x_{ki+k-i-1}).$$

(2) For $f \in \mathcal{O}_A$, define $\Gamma(f) = \{\gamma_i(f) : i > 0\}$.
(3) For $\mathcal{G} \subseteq \mathcal{O}_A$, define $\Delta(\mathcal{G}) = \{\delta_i(f) : f \in \mathcal{G}, i < n\}$.

For our purposes, we require a weak NU operation with a particularly useful property. The next lemma shows that such a weak NU operation is always present whenever there is any weak NU operation.

**Lemma 4.3.** *Let $\mathcal{G} \subseteq \mathcal{O}_A$ and let $g \in \langle \mathcal{G} \rangle^{(k)}$ be a weak near unanimity operation. Then there is weak near unanimity operation $g' \in \langle \mathcal{G} \rangle^{(\ell)}$ for some $\ell$ such that $\gamma_2(g')(\mathbf{y}\mathbf{x}^{2\ell-2}) = g'(\mathbf{y}\mathbf{x}^{\ell-1})$.*

*Proof.* Construct a sequence of functions $g_i \in \langle \mathcal{G} \rangle^{(k^i)}$ as follows: $g_1 = g$ and

$$g_{i+1}(x_0, \ldots, x_{k^{i+1}-1}) = g(g_i(x_0, \ldots, x_{k^i-1}), \ldots, g_i(x_{(k-1)k^i}, \ldots, x_{k^{i+1}-1})).$$

Clearly, the associated binary functions $h_i(\mathbf{x}, \mathbf{y}) = g_i(\mathbf{y}\mathbf{x}^{k^i-1})$ have the property that $h_{i+1}(\mathbf{x}, \mathbf{y}) = h_1(\mathbf{x}, h_i(\mathbf{x}, \mathbf{y}))$, and so we also know that $h_{i+j}(\mathbf{x}, \mathbf{y}) = h_i(\mathbf{x}, h_j(\mathbf{x}, \mathbf{y}))$.

By construction, we know that $h_{|A|!} = h_{2|A|!}$, and so we can choose $g' = g_{|A|!}$ with $\ell = k^{|A|!}$, completing the proof. $\square$

All that remains in this section is to prove that we now have a subset of the clone that will always contain an operation which witnesses $P$'s membership in $EL$, if such an operation exists in $\langle \mathcal{F} \rangle$.

**Lemma 4.4.** *Suppose that $g \in \langle \mathcal{F} \rangle^{(k)}$ is a weak near unanimity operation with $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$. Then $\langle \mathcal{F} \rangle$ contains an operation that witnesses $P$'s membership in the class $EL$ if and only if $\mathrm{T}(P) \in \bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}} \, \mathrm{T}(\Delta\Gamma(g))$.*

*Proof.* Since $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}} \, \Delta\Gamma(g) \subseteq \langle \mathcal{F} \rangle$ the reverse implication trivially follows from Lemma 3.5.

If $f \in \langle \mathcal{F} \rangle^{(\ell)}$ witnesses $P$'s membership in $EL$ (necessitating $\ell \geq n$), then we will construct an operation in $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}} \, \Delta\Gamma(g)$ that also witnesses that membership. This is sufficient to prove the lemma since $\mathrm{T}(\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}} \, \Delta\Gamma(g)) = \bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}} \, \mathrm{T}(\Delta\Gamma(g))$ (by Lemma 3.5).

- For each $n - m < j < \ell$, let $\eta_j$ be the $(k-1)$-length sequence of variables $x_{\ell+(j-n+m)(k-1)}, \ldots, x_{\ell+(j-n+m)(k-1)+k-2}$. What we require of these $\eta_j$'s is that they be sequences of variables with index at least $\ell$ such that no two $\eta_j$'s have any variables in common.
- For each $i < n$, define $e_i$ to be the sequence of variables obtained by concatenating each $\eta_j$ for which the $(j - m + n, i)$ entry of $M$ is $\mathbf{y}$, where $j \in \nu$. In other words, for each $i < n$, we are going to construct $e_i$ such that if the $i$th column of $M$ has a $\mathbf{y}$ in row $j + m - n$, the corresponding $\eta_j$ will be included in $e_i$.
- For each $n \leq i < \ell$, let $e_i = \eta_i$.
- For each $i < n$, let $m_i$ be the number of $\mathbf{y}$'s in the $i$th column of $M$.
- For each $i < \ell$, define a function $\theta_i$ of arity $k\ell - (n-m)(k-1)$ as follows:

$$\theta_i(\overline{x}) = \begin{cases} \gamma_{m_i}(g)(x_i, e_i) & \text{if } i < n, \\ g(x_i, e_i) & \text{if } n \leq i < \ell. \end{cases}$$

- Define a function $h \in \bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}} \, \Delta\Gamma(g)$ of arity $k\ell - (n-m)(k-1)$ as follows: $h(\overline{x}) = f(\theta_0(\overline{x}), \ldots, \theta_{\ell-1}(\overline{x}))$.

Notice that $\gamma_{m_i}(g)(x_i, e_i)$ is a function of arity $m_i(k-1)+1$ such that the only variable of $\{x_0, \ldots, x_{n-1}\}$ not discarded is $x_i$. This tells us that $\gamma_{m_i}(g)(x_i, e_i)$ is an $n$-extension of $\delta_i \gamma_{m_i}(g)$ for each $i$ and so $h$ is in the relevant set.

Let $p$ be the binary function defined by $p(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}, \mathbf{x}^{k-1})$. We will now prove that $h|_i(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for every $i$ and every $\mathbf{x} \in S$.

**Case 1:** If $i \in \nu$, then $h|_i(\mathbf{x}, \mathbf{y}) = f(M_{i+m-n}(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}) = \mathbf{x}$.

*Proof.* By definition,

$$h|_i(\mathbf{x}, \mathbf{y}) = h(M_{i+m-n}(\mathbf{x}, \mathbf{y}), \mathbf{x}^{k\ell-(n-m)(k-1)-n}) = f(a_0, \ldots, a_{\ell-1})$$

for some $a_i \in \mathcal{B}$. Notice that every input variable later than $x_{n-1}$ receives an input of $\mathbf{x}$, so $a_j(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for each $j \geq n$. This also tells us that

$$a_j(\mathbf{x}, \mathbf{y}) = \gamma_{m_i}(g)(M^j_{i+m-n}, \mathbf{x}^{m_j(k-1)}) = p(\mathbf{x}, M^j_{i+m-n})$$

for each $j < n$, and so,

$$h|_i(\mathbf{x}, \mathbf{y}) = f(a_0, \ldots, a_{\ell-1}) = f(M_{i+m-n}(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}). \qquad \square$$

**Case 2:** If $\ell \leq i < \ell + m(k-1)$, then $\ell + j(k-1) \leq i < \ell + (j+1)(k-1)$ for some $j < m$, and so $h|_i(\mathbf{x}, \mathbf{y}) = f(M_j(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}) = \mathbf{x}$.

*Proof.* Notice first that the only input variable receiving a value of $\mathbf{y}$ is $x_i$; all others receive a value of $\mathbf{x}$. Also notice that the only $\eta$ in which $x_i$ appears is $\eta_j$, and so for each $b < n$, $x_i$ will appear in $e_b$ exactly if $M^j_{b+m-n} = \mathbf{y}$. Now as in case 1 (above),

$$h|_i(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}^i, \mathbf{y}, \mathbf{x}^{k\ell-(n-m)(k-1)-i-1}) = f(a_0, \ldots, a_{\ell-1})$$

for some $a_i \in \mathcal{B}$. We know that for each $b < n$, $a_b(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, M^j_{b+m-n})$ and for each $b \geq n$, $a_b(\mathbf{x}, \mathbf{y}) = \mathbf{x}$; therefore,

$$h|_i(\mathbf{x}, \mathbf{y}) = f(a_0, \ldots, a_{\ell-1}) = f(M^j_{i+m-n}(\mathbf{x}, p(\mathbf{x}, \mathbf{y})), \mathbf{x}^{\ell-n}). \qquad \square$$

The remaining cases are left as an exercise for the reader. Therefore, $h$ witnesses $P$'s membership in $EL$. $\square$

## 5. Comparisons, computability, and the main result

Now that we have proven (in Lemma 4.4) that if $P$ is in $EL$, then there will be a witnessing operation in $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}}(\Delta\Gamma(g))$, we must prove that we can effectively compute the characteristic triples of all operations in this set. To that end we introduce a partial order on characteristic triples that, when applied in the correct way, will allow us to calculate the minimal elements of the set in question, which will turn out to be an order filter.

**Definition 5.1.** (1) For each $k > 1$, define a *partial order* $\sqsubseteq_k$ on $\omega^+$ such that 0 and $\omega$ are each comparable only with themselves, and for all positive $a, b$, we have $a \sqsubseteq_k b$ if and only if $a \leq b$ and $k \mid (b - a)$.

(2) Acting coordinate-wise, we can extend this partial order to one on $(\omega^+)^{\mathcal{B}}$, and we can extend this to a partial order on $\mathrm{T}_A$ by saying that $(\rho, \alpha, \chi) \sqsubseteq_k (\rho', \alpha', \chi')$ if and only if $\rho = \rho'$, $\alpha = \alpha'$ and $\chi \sqsubseteq_k \chi'$.

(3) For $\mathcal{U} \subseteq \mathrm{T}_A$, let $\mathrm{F}_k(\mathcal{U})$ denote the *order filter* (*upward closed set*) *with respect to* $\sqsubseteq_k$ *generated by* $\mathcal{U}$.

Now we will demonstrate that we can meaningfully compose functions with order filters of characteristic triples.

**Lemma 5.2.** *Let* $k > 1$ *and* $\mathcal{U} \subseteq \mathrm{T}_A$. *Then* $\mathrm{F}_k \mathrm{C}_{\mathcal{F}}(\mathcal{U}) \subseteq \mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U})$ *and* $\mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U})$ *is an order filter.*

*Proof.* Take $(\rho, \alpha, \chi) \in \mathrm{C}_{\mathcal{F}}(\mathcal{U})$ with $\chi \sqsubseteq_k \chi'$. Then $(\rho, \alpha, \chi)$ is a composition of some $f \in \mathcal{F}^{(\ell)}$ with characteristic triples $(\rho_i, \alpha_i, \chi_i) \in \mathcal{U}$ for $i < \ell$, and so there is a $\mu \colon \mathcal{B}^\ell \to \omega^+$ such that

$$\chi(c) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, f(\bar{b}) = c} \mu(\bar{b}) \quad \text{and} \quad \chi_i(c) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, b_i = c} \mu(\bar{b}).$$

We will show that $(\rho, \alpha, \chi')$ is a composition of $f$ with characteristic triples in $\mathrm{F}_k(\mathcal{U})$.

Let $D$ be the set of all $d \in \mathcal{B}$ such that $\chi(d) \neq \chi'(d)$. By the definition of $\sqsubseteq_k$, we know that $0 < \chi(d) < \chi'(d) < \omega$ and $k | (\chi'(d) - \chi(d))$ for all $d \in D$. For each $d \in D$, pick $\bar{b}_d \in \mathcal{B}^\ell$ such that $f(\bar{b}_d) = d$ and $0 < \mu(\bar{b}_d) < \omega$. Define $\mu' \colon \mathcal{B}^\ell \to \omega^+$ as follows:

$$\mu'(\bar{b}) = \begin{cases} \mu(\bar{b}) + \chi'(d) - \chi(d) & \text{if } \bar{b} = \bar{b}_d \text{ for some } d \in D, \\ \mu(\bar{b}) & \text{otherwise.} \end{cases}$$

Clearly, we can see that $\chi'(c) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, f(\bar{b}) = c} \mu'(\bar{b})$, and so we can use $\mu'$ to define $\chi_i' \colon \mathcal{B} \to \omega^+$ for each $i < \ell$ as $\chi_i'(c) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, b_i = c} \mu'(\bar{b})$.

It immediately follows that $\chi_i \sqsubseteq_k \chi_i'$ for each $i < \ell$, and so we have that $(\rho_i, \alpha_i, \chi_i) \sqsubseteq_k (\rho_i, \alpha_i, \chi_i')$, demonstrating that $(\rho_i, \alpha_i, \chi_i') \in \mathrm{F}_k(\mathcal{U})$. To complete this part of the proof, we need only notice that $\mu'$ was constructed to witness the fact that $(\rho, \alpha, \chi')$ is a composition of $f$ with $(\rho_i, \alpha_i, \chi_i')$, $i < \ell$.

To show that $\mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U})$ is an order filter, notice that

$$\mathrm{F}_k \mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U}) \subseteq \mathrm{C}_{\mathcal{F}} \mathrm{F}_k \mathrm{F}_k(\mathcal{U}) = \mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U}) \subseteq \mathrm{F}_k \mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U}). \qquad \square$$

**Lemma 5.3.** *Let* $k > 1$ *and let* $\mathcal{U} \subseteq \mathrm{T}_A$ *be finite. Then the* $\sqsubseteq_k$*-minimal elements of* $\mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U})$ *can be effectively computed.*

*Proof.* Let $(\rho, \alpha, \chi)$ be an arbitrary minimal element of $\mathrm{C}_{\mathcal{F}} \mathrm{F}_k(\mathcal{U})$. Then $(\rho, \alpha, \chi)$ is a composition of $f \in \mathcal{F}^{(\ell)}$ with characteristic triples

$$(\rho_0, \alpha_0, \chi_0), \ldots, (\rho_{\ell-1}, \alpha_{\ell-1}, \chi_{\ell-1}) \in \mathrm{F}_k(\mathcal{U})$$

witnessed by a mapping $\mu\colon \mathcal{B}^\ell \to \omega^+$. Notice that $f$ and $\mu$ uniquely determine $\chi$ and $\chi_0, \ldots, \chi_{\ell-1}$, and similarly $f$, $\mu$, $\rho_i \in A^A$, and $\alpha_i \in \mathcal{B}^\nu$ for $i < \ell$ uniquely determine $(\rho, \alpha, \chi)$ and $(\rho_i, \alpha_i, \chi_i)$ for $i < \ell$.

Since $\mathcal{B}^\ell$ is finite, $(\omega^+)^{\mathcal{B}^\ell}$ is well-founded under $\sqsubseteq_k$, and so we may assume that $\mu$ is minimal among mappings that witness the fact that $(\rho, \alpha, \chi)$ is a composition of $f$ with elements of $\mathrm{F}_k(\mathcal{U})$.

Define $p = \max(\{k\} \cup \{\chi'(b) : \chi' \in X(\mathcal{U}), b \in \mathcal{B}, \chi'(b) \neq \omega\})$, which is a natural number dependent only on $k$ and $\mathcal{U}$.

**Claim:** For all $\bar{b} \in \mathcal{B}^\ell$, if $\mu(\bar{b}) > p$, then $\mu(\bar{b}) = \omega$.

*Proof.* To get a contradiction, assume that $p < \mu(\bar{c}) < \omega$ for some $\bar{c} \in \mathcal{B}^\ell$. Define $\mu'\colon \mathcal{B}^\ell \to \omega^+$ as

$$\mu'(\bar{b}) = \begin{cases} \mu(\bar{b}) - k & \text{if } \bar{b} = \bar{c}, \\ \mu(\bar{b}) & \text{otherwise,} \end{cases}$$

and define $\chi'$ and $\chi'_0, \ldots, \chi'_{\ell-1}$ as

$$\chi'(d) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, f(\bar{b}) = d} \mu'(\bar{b}) \quad \text{and} \quad \chi'_i(d) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, b_i = d} \mu'(\bar{b}).$$

Observe that $\mu'(\bar{c}) = \mu(\bar{c}) - k > p - k \geq 0$.

We will now argue that $\mu' \sqsubseteq \mu$ and that $\mu'$ also witnesses that $(\rho, \alpha, \chi)$ is a composition of $f$ with elements of $\mathrm{F}_k(\mathcal{U})$, contradicting the minimality of $\mu$.

First, we must argue that $(\rho_i, \alpha_i, \chi'_i) \in \mathrm{F}_k(\mathcal{U})$ for each $i < \ell$. Clearly, we have $\chi'_i(b) = \chi_i(b)$ for all $b \neq c_i$ and all $i < \ell$; now consider the value of $\chi'_i(c_i)$.

**Case 1:** $\chi'_i(c_i) = \omega$, in which case $\chi_i(c_i) = \omega$ as well, hence

$$(\rho_i, \alpha_i, \chi'_i) = (\rho_i, \alpha_i, \chi_i) \in \mathrm{F}_k(\mathcal{U}).$$

**Case 2:** $\chi'_i(c_i) = \chi_i(c_i) - k$ and so $\chi'_i(c_i) \geq \mu(\bar{c}) - k > p - k \geq 0$. Since $(\rho_i, \alpha_i, \chi_i) \in \mathrm{F}_k(\mathcal{U})$ there is a characteristic triple $(\rho_i, \alpha_i, \chi''_i) \in \mathcal{U}$ such that $\chi''_i \sqsubseteq_k \chi_i$. By choice of $p$, we have that $\chi''_i(c_i) \leq p < \mu(\bar{c}) \leq \chi_i(c_i)$, and so $\chi''_i \leq \chi_i(c_i) - k$. Therefore, $\chi''_i \sqsubseteq_k \chi'_i$, and so $(\rho_i, \alpha_i, \chi'_i) \in \mathrm{F}_k(\mathcal{U})$.

Analogously, $\chi'(b) = \chi(b)$ for all $b \neq f(\bar{c})$, and either $\chi'(f(\bar{c})) = \omega = \chi(f(\bar{c}))$ or $\chi'(f(\bar{c})) = \chi(f(\bar{c})) - k > p - k \geq 0$; hence, $\chi' \sqsubseteq_k \chi$. Since $(\rho_i, \alpha_i, \chi'_i) \in \mathrm{F}_k(\mathcal{U})$, we get that $(\rho, \alpha, \chi') \in \mathrm{C}_{\mathcal{F}}\,\mathrm{F}_k(\mathcal{U})$. From the minimality of $\chi$, we get that $\chi' = \chi$, and so $\mu'$ contradicts the minimality of $\mu$ among representations of $\chi$. □

Therefore, the following algorithm will calculate all the $\sqsubseteq_k$-minimal elements of $\mathrm{C}_{\mathcal{F}}\,\mathrm{F}_k(\mathcal{U})$.

**Algorithm.** Input: Natural number $k > 1$ and finite sets $A$, $\mathcal{F} \subseteq \mathcal{O}_A$ and $\mathcal{U} \subseteq \mathrm{T}_A$.

Output: All the $\sqsubseteq_k$-minimal elements of $\mathrm{C}_{\mathcal{F}}\,\mathrm{F}_k(\mathcal{U})$.

(1) Set $R = \emptyset$.

(2) For each $f \in \mathcal{F}$ (say $f \in \mathcal{F}^{(\ell)}$), do:
  (a) Set $p = \max(\{k\} \cup \{\chi'(b) : \chi' \in X(\mathcal{U}), b \in \mathcal{B}, \chi'(b) \neq \omega\})$.
  (b) For each $\mu \colon \mathcal{B}^\ell \to \{0, 1, \ldots, p, \omega\}$, do:
    (i) For each $c \in \mathcal{B}$, calculate $\chi(c) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, f(\bar{b}) = c} \mu(\bar{b})$.
    (ii) For each $c \in \mathcal{B}$ and each $i < \ell$, calculate $\chi_i(c) = \sum_{\bar{b} \in \mathcal{B}^\ell,\, b_i = c} \mu(\bar{b})$.
    (iii) For each $\overline{\rho} \in (A^A)^\ell$ and each $\overline{\alpha} \in (\mathcal{B}^\nu)^\ell$, do:
      If all $(\rho_i, \alpha_i, \chi_i) \in \mathrm{F}_k(\mathcal{U})$, then $R = R \cup \{(f(\overline{\rho}), f(\overline{\alpha}), \chi)\}$.

(3) The minimal elements of $R$ are the minimal elements of $\mathrm{C}_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$. □

**Lemma 5.4.** *Let $k > 1$ and let $\mathcal{U} \subseteq \mathrm{T}_A$ be a finite set. Then $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$ is an order filter with respect to $\sqsubseteq_k$ and its minimal elements can be effectively computed.*

*Proof.* By Lemmas 5.2 and 5.3, $\mathrm{C}^i_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$ is an order filter for each $i > 0$ and its minimal elements can be effectively computed. If we let $\mathcal{U}_i$ for $i < \omega$ be defined as the set of minimal elements of $\mathrm{C}^i_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$ then the minimal elements of $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$ will be the minimal elements of $\bigcup_{i<\omega} \mathcal{U}_i$.

Since $\mathrm{T}_A$ is well-founded under $\sqsubseteq_k$, the increasing (under inclusion) sequence of filters $\bigcup_{j<i} \mathrm{C}^i_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$ must eventually stabilize and so the sequence of $\mathcal{U}_i$'s must also eventually stabilize. Since we can calculate each $\mathcal{U}_i$, we simply continue to do so until we reach some $\ell$ for which $\mathcal{U}_\ell = \mathcal{U}_{\ell-1}$, then the minimal elements of $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}}\, \mathrm{F}_k(\mathcal{U})$ will be the minimal elements of $\bigcup_{i<\ell} \mathcal{U}_i$, a finite set whose elements we will already have computed. □

Now that we have shown that we can effectively compute the minimal elements of arbitrary composition of functions with characteristic triples forming an order filter, we must actually have an order filter of characteristic triples with which to compose said functions.

**Lemma 5.5.** *Let $g \in \mathcal{O}^{(k)}_A$ be a weak near unanimity operation such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$. Then $\mathrm{T}\,\Delta\Gamma(g) = \mathrm{F}_{k-1}\,\mathrm{T}\,\Delta(\{g\})$.*

*Proof.* We will simply calculate the characteristic triples of every $\delta_i(\gamma_j(g))$ and notice that they form an order filter with respect to $\sqsubseteq_{k-1}$ whose minimal elements are exactly $\mathrm{T}\,\Delta(\{g\})$.

First, let us pick $p, q \in \mathcal{B}$ such that $p(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}\mathbf{x}^{k-1})$ and $q(\mathbf{x}, \mathbf{y}) = \mathbf{x}$. Then for every $i < n$, $j > 0$, and $\ell \in \nu$, let $g' = \delta_i(\gamma_j(g))$; it is clear that

$$\begin{aligned}
\rho_{g'}(\mathbf{y}) &= \mathbf{y}, \\
\rho_{\delta_n(\gamma_j(g))}(\mathbf{y}) &= \mathbf{y}, \\
\alpha_{g'}(\ell) &= \begin{cases} p & \text{if } M^i_\ell = \mathbf{y}, \\ q & \text{otherwise}, \end{cases} \\
\alpha_{\delta_n(\gamma_j(g))}(\ell) &= q,
\end{aligned}$$

and, with regard to characteristic functions, for every $c \in \mathcal{B}$ it is clear that

$$\chi_{g'}(c) = \begin{cases} \omega & \text{if } c = q, \\ j(k-1) & \text{if } c = p, \\ 0 & \text{otherwise;} \end{cases}$$

$$\chi_{\delta_n(\gamma_j(g))}(c) = \begin{cases} \omega & \text{if } c = q, \\ j(k-1)+1 & \text{if } c = p, \\ 0 & \text{otherwise.} \end{cases}$$

These characteristic triples satisfy the requirements, completing the proof. □

And now we can proceed to the main result of this paper.

**Theorem 5.6.** *Given an instance of the edge-like problem $P = (A, \mathcal{F}, M, S)$ such that $\langle A, \mathcal{F}\rangle$ generates a variety that omits type* 1*, it is decidable whether or not $P \in EL$.*

*Proof.* Since $\langle A, \mathcal{F}\rangle$ generates a variety that omits type 1, there must be a weak near unanimity term in $\langle \mathcal{F}\rangle$ (see [5]), and so by Lemma 4.3, there must be a weak near unanimity operation $g \in \langle \mathcal{F}\rangle^{(k)}$ such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$. Clearly, we can calculate the minimal elements of $\mathrm{F}_{k-1}\,\mathrm{T}\,\Delta(\{g\})$ (given in the proof of Lemma 5.5), and so by Lemma 5.4, we can compute the minimal elements of $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}}\,\mathrm{F}_{k-1}\,\mathrm{T}\,\Delta(\{g\})$, calling this set $\mathcal{U}$. Since $\mathrm{T}(P)$ is minimal with respect to $\sqsubseteq_{k-1}$, Lemma 4.4 tells us that $\langle \mathcal{F}\rangle$ will contain a term that witnesses the membership of $P$ in $EL$ if an only if $\mathrm{T}(P) \in \mathcal{U}$. □

**Algorithm.** Input: Finite sets $A$, $\mathcal{F} \subseteq \mathcal{O}_A$ and $S \subseteq A$, and matrix $M \in M_{m\times n}(\{\mathbf{x}, \mathbf{y}\})$.

Condition: This algorithm is only guaranteed if $\langle A, \mathcal{F}\rangle$ generates a variety that omits type 1.

Output: Whether or not $P = (A, \mathcal{F}, M, S)$ is in $EL$.

(1) Search for a weak near unanimity term $g \in \langle \mathcal{F}\rangle^{(k)}$ such that $\gamma_2(g)(\mathbf{y}\mathbf{x}^{2k-2}) = g(\mathbf{y}\mathbf{x}^{k-1})$. Since $\langle A, \mathcal{F}\rangle$ generates a variety that omits type 1, we are guaranteed to find such a term.
(2) Calculate $\mathcal{U}$, the set of minimal elements of $\bigcup_{i<\omega} \mathrm{C}^i_{\mathcal{F}}\,\mathrm{F}_{k-1}\,\mathrm{T}\,\Delta(\{g\})$ (note that Lemma 5.4 explains how to do this).
(3) $P \in EL$ if and only if $\mathrm{T}(P) \in \mathcal{U}$ (this is given by Lemmas 4.4 and 5.5).

## 6. Consequences

**Definition 6.1.** Say that $f \in \mathcal{O}^{(k)}_A$ is a *near unanimity operation* if for all $i < k$, $f(\mathbf{x}^i\mathbf{y}\mathbf{x}^{k-i-1}) = \mathbf{x}$.

An algebra with a near unanimity operation has many useful properties, including but not limited to having few subpowers [2] and being congruence distributive. That the presence of a near unanimity operation is a decidable proposition was first proven by Maróti in [9], and forms the foundation on which this paper builds.

**Corollary 6.2.** *It is decidable whether or not a finite algebra supports a near unanimity term.*

*Proof.* First notice that if an algebra generates a variety that admits type 1, then it cannot support a near unanimity term.

Given an algebra $\langle A, \mathcal{F}\rangle$, we can test whether or not it generates a variety that omits type 1 (see [6] or [7]) and if it does not, it cannot contain a near unanimity operation. Otherwise, we can construct an instance of the edge-like problem, namely $P = (A, \mathcal{F}, \emptyset, A)$ (where $\emptyset$ represents the $0 \times 0$ empty matrix), such that $\langle A, \mathcal{F}\rangle$ supports a near unanimity term if and only if $P \in EL$. By Theorem 5.6, this is a decidable proposition. $\square$

**Definition 6.3.** [2] Say that $f \in \mathcal{O}_A^{(k)}$ is an *edge operation* if

$$f(\mathbf{yyx}^{k-2}) = f(\mathbf{yxyx}^{k-3}) = \mathbf{x}$$

and for all $3 \leq i < k$

$$f(\mathbf{x}^i\mathbf{yx}^{k-i-1}) = \mathbf{x}.$$

Those algebras with edge operations are exactly those algebras that have few subpowers (see [2]). Additionally, their associated relational structures produce constraint satisfaction problems that can be decided in polynomial time [4].

**Corollary 6.4.** *It is decidable whether or not a finite algebra supports an edge term.*

*Proof.* As in the preceding corollary, given an algebra $\langle A, \mathcal{F}\rangle$, we can test whether or not it generates a variety that omits type 1, and if it does not, then it cannot contain an edge operation. Otherwise, we can construct an instance of the edge-like problem, namely

$$P = (A, \mathcal{F}, \left[\begin{smallmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} \\ \mathbf{y} & \mathbf{x} & \mathbf{y} \end{smallmatrix}\right], A)$$

such that $\langle A, \mathcal{F}\rangle$ supports an edge term if and only if $P \in EL$. By Theorem 5.6, this is a decidable proposition. $\square$

**Definition 6.5.** [1] Given an algebra $\mathbf{A}$, say that $S \subseteq A$ *is an absorbing set* if there is an idempotent term operation $f$ on $\mathbf{A}$ (say it is $k$-ary) such that for all $i < k$, all $\overline{a} \in S^k$ and all $b_i \in A$,

$$f(a_0, \ldots, a_{i-1}, b_i, a_{a+1}, \ldots, a_{k-1}) \in S.$$

**Corollary 6.6** (Originally observed by Matt Valeriote in [11])**.** *If an algebra $\langle A, \mathcal{F}\rangle$ generates a variety that omits type 1, then it is decidable whether or not $\{a\}$ is an absorbing set for any $a \in A$.*

*Proof.* Given an algebra $\langle A, \mathcal{F}\rangle$ generatng a variety that omits type 1, we can construct an instance of the edge-like problem, namely

$$P = (A, \mathcal{F}, \emptyset, \{a\})$$

(where $\emptyset$ represents the $0 \times 0$ empty matrix) such that $\{a\}$ is an absorbing set in $\langle A, \mathcal{F}\rangle$ if and only if $P \in EL$. By Theorem 5.6 this is a decidable proposition. □

**Question.** Is it decidable whether or not $S \subseteq A$ is an absorbing subset of $\langle A, \mathcal{F}\rangle$ in the case when $|S| > 1$?

Note that Bulín proved in [3] that for relational structures with bounded width, it is indeed decidable whether or not a particular subset is absorbing.

**Question.** Can testing for membership in $EL$ be done in time bounded by a primitive recursive function?

**Question.** What is the computational complexity of testing for a near unanimity term? An edge term?

It is worth noting that in [8], it is proven that detecting an edge term in an idempotent algebra can be accomplished using an algorithm in co-NP.

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