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Idempotent generated algebras and Boolean powers of commutative rings

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Abstract. A Boolean power S of a commutative ring R has the structure of a commutative R-algebra, and with respect to this structure, each element of S can be written uniquely as an R-linear combination of orthogonal idempotents so that the sum of the idempotents is 1 and their coefficients are distinct. In order to formalize this decomposition property, we introduce the concept of a Specker R-algebra, and we prove that the Boolean powers of R are up to isomorphism precisely the Specker R-algebras. We also show that these algebras are characterized in terms of a functorial construction having roots in the work of Bergman and Rota. When R is indecomposable, we prove that S is a Specker R-algebra iff S is a projective R-module, thus strengthening a theorem of Bergman, and when R is a domain, we show that S is a Specker R-algebra iff S is a torsion-free R-module.

For indecomposable R, we prove that the category of Specker R-algebras is equivalent to the category of Boolean algebras, and hence is dually equivalent to the category of Stone spaces. In addition, when R is a domain, we show that the category of Baer Specker R-algebras is equivalent to the category of complete Boolean algebras, and hence is dually equivalent to the category of extremally disconnected compact Hausdorff spaces.

For totally ordered R, we prove that there is a unique partial order on a Specker R-algebra S for which it is an f-algebra over R, and show that S is isomorphic to the R-algebra of piecewise constant continuous functions from a Stone space X to R equipped with the interval topology.

1. Introduction

For a commutative ring R and a Boolean algebra B, the Boolean power of R by B is the R-algebra $C(X, R_{\text{disc}})$ of continuous functions from the Stone space X of B to the discrete space R (see, e.g., [2] or [7, Ch. IV, §5]). Each element of a Boolean power of R can be written uniquely as an R-linear combination of orthogonal idempotents so that the sum of the idempotents is 1 and their coefficients are distinct. In this note we formalize this decomposition property by introducing the class of Specker R-algebras. We prove that an R-algebra S is isomorphic to a Boolean power of R iff S is a Specker R-algebra, and we characterize Specker R-algebras (hence Boolean powers of R) in several

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other ways for various choices of the commutative ring R, such as when R is indecomposable, an integral domain, or totally ordered.

Our terminology is motivated by Conrad's concept of a Specker ℓ-group. We recall [8, Sec. 4.7] that an element $g > 0$ of an ℓ-group G is *singular* if $h \wedge (g - h) = 0$ for all $h \in G$ with $0 \leq h \leq g$, and that G is a Specker ℓ-group if it is generated by its singular elements. Conrad proved in [8, Sec. 4.7] that a Specker ℓ-group admits a unique multiplication such that $gh = g \wedge h$ for all singular elements g, h. Under this multiplication, the singular elements become idempotents, and hence a Specker ℓ-group with strong order unit, when viewed as a ring, is generated as a $\mathbb{Z}$-algebra by its idempotents. Moreover, it is a torsion-free $\mathbb{Z}$-algebra, and hence its elements admit a unique orthogonal decomposition. Our definition of a Specker R-algebra extracts these key features of Specker ℓ-groups.

For a commutative ring R, we give several equivalent characterizations for a commutative R-algebra to be a Specker R-algebra. One of these characterizations produces a functor from the category $\mathbf{BA}$ of Boolean algebras to the category $\mathbf{Sp}_R$ of Specker R-algebras. This functor has its roots in the work of Bergman [3] and Rota [16]. We show this functor is left adjoint to the functor that sends a Specker R-algebra to its Boolean algebra of idempotents. We prove that the ring R is indecomposable iff these functors establish an equivalence of $\mathbf{Sp}_R$ and $\mathbf{BA}$. It follows then from Stone duality that when R is indecomposable, $\mathbf{Sp}_R$ is dually equivalent to the category **Stone** of Stone spaces (zero-dimensional compact Hausdorff spaces). Hence, when R is indecomposable, Specker R-algebras are algebraic counterparts of Stone spaces in the category of commutative R-algebras.

It follows from the work of Bergman [3] that every Specker R-algebra is a free R-module. For indecomposable R, we show that the converse is also true. In fact, we prove a stronger result: An idempotent generated commutative R-algebra S (with R indecomposable) is a Specker R-algebra iff S is a projective R-module. A simple example shows that the assumption of indecomposability is necessary here. When R is a domain, an even stronger result is true: S is a Specker R-algebra iff S is a torsion-free R-module. Thus, the case when R is a domain provides the most direct generalization of the ℓ-group case.

For a domain R, we prove that the Stone space of the idempotents of a Specker R-algebra S can be described as the space of minimal prime ideals of S, and that a Specker R-algebra S is an injective object in $\mathbf{Sp}_R$ iff S is a Baer ring. This yields an equivalence between the category $\mathbf{BSp}_R$ of Baer Specker R-algebras and the category **cBA** of complete Boolean algebras, and hence a dual equivalence between $\mathbf{BSp}_R$ and the category **ED** of extremally disconnected compact Hausdorff spaces.

We conclude the article by considering the case when R is a totally ordered ring. Such a ring is then automatically indecomposable. We prove that there is a unique partial order on a Specker R-algebra S for which it is an f-algebra over R, and show that S is isomorphic to the R-algebra of piecewise constant continuous functions from a Stone space X to R, where R is given the interval topology. These results give a more general point of view on similar results obtained for $R = \mathbb{Z}$ by Ribenboim [15] and Conrad [8], and for $R = \mathbb{R}$ as considered in [5].

2. Specker algebras and Boolean powers of a commutative ring

All algebras considered in this article are commutative and unital, and all algebra homomorphisms are unital. Throughout, R will be a commutative ring with 1. In this section, we introduce Specker R-algebras and use them to characterize Boolean powers of R. A key property of Specker R-algebras is that their elements can be decomposed uniquely into R-linear combinations of idempotents so that the sum of the idempotents is 1 and their coefficients are distinct. We begin the section by formalizing the terminology needed to make precise this decomposition property.

Let S be a commutative R-algebra. As S is a commutative ring with 1, it is well known that the set $\mathrm{Id}(S)$ of idempotents of S is a Boolean algebra via the operations: $e \vee f = e + f - ef$, $e \wedge f = ef$, and $\neg e = 1 - e$.

We call an R-algebra S *idempotent generated* if S is generated as an R-algebra by a set of idempotents. If the idempotents belong to some Boolean subalgebra B of $\mathrm{Id}(S)$, we say that B *generates* S. Because we are assuming S is commutative, each monomial of idempotents is equal to $e_1 \cdots e_r$ for some $e_i \in \mathrm{Id}(S)$. Therefore, each element is, in fact, an R-linear combination of idempotents. Thus, an idempotent generated R-algebra S is generated as an R-module by its idempotents, and if B generates S, then B generates S both as an R-algebra and as an R-module.

We call a set E of nonzero idempotents of S *orthogonal* if $e \wedge f = 0$ for all $e \neq f$ in E, and we say that $s \in S$ has an *orthogonal decomposition* or that s is in *orthogonal form* if $s = \sum_{i=1}^n a_i e_i$ with the $e_i \in \mathrm{Id}(S)$ orthogonal. If, in addition, $\bigvee e_i = 1$, we call the decomposition a *full orthogonal decomposition* and the set $\{e_1, \dots, e_n\}$ a *full orthogonal set*. By possibly adding a term with a 0 coefficient, we can turn any orthogonal decomposition into a full orthogonal decomposition.

We call a nonzero idempotent e of S *faithful* if for each $a \in R$, whenever $ae = 0$, then $a = 0$. Let B be a Boolean subalgebra of $\mathrm{Id}(S)$ that generates S. We say that B is a *generating algebra of faithful idempotents of S* if each nonzero $e \in B$ is faithful.

Lemma 2.1. *Let S be a commutative R-algebra and let B be a Boolean subalgebra of* $\mathrm{Id}(S)$ *that generates S. Then each $s \in S$ can be written in full orthogonal form $s = \sum_{i=1}^n a_i e_i$, where the $a_i \in R$ are distinct and $e_i \in B$. Moreover, such a decomposition is unique iff B is a generating algebra of faithful idempotents of S.*

Proof. The proof that each $s \in S$ can be written in full orthogonal form is a standard argument. Write $s = \sum_{i=1}^n a_i e_i$ with $a_i \in R$ and $e_i \in B$. Each e_i can then be refined into a sum of idempotents, each of which is a meet of a set of idempotents in $\{e_1, \ldots, e_n, 1 - e_1, \ldots, 1 - e_n\}$, in such a way that the resulting refinements of the e_i are orthogonal. By combining terms with the same coefficient, s can be written in orthogonal form with distinct coefficients. If the decomposition is not in full orthogonal form, adding the term $0f$, where f is the negation of the join of the idempotents in the decomposition, turns it into a full orthogonal decomposition.

Suppose that each element has a unique full orthogonal decomposition and suppose that $ae = 0$ for some $a \in R$ and nonzero $e \in B$. Then since $ae = 0e$, uniqueness implies that $a = 0$, and hence e is faithful. Conversely, suppose that B is a generating algebra of faithful idempotents of S. Let $s \in S$ and write $s = \sum_i a_i e_i = \sum_j b_j f_j$ with each sum a full orthogonal decomposition with distinct coefficients. First consider i with $a_i \neq 0$. Multiplying both sides by e_i yields $a_i e_i = \sum_j b_j(e_i f_j)$. Since e_i is faithful and $a_i \neq 0$, there is j with $e_i f_j \neq 0$. Multiplying by f_j yields $a_i e_i f_j = b_j e_i f_j$. Therefore, since $e_i f_j$ is faithful, $a_i = b_j$. Because the b_j are distinct, there is a unique j with $e_i f_j \neq 0$. Since $a_i = b_j$, we then have $a_i e_i = b_j e_i f_j = a_i e_i f_j$. Thus, $0 = a_i(e_i - e_i f_j) = a_i e_i (1 - f_j) = a_i(e_i \wedge \neg f_j)$, so by faithfulness, $e_i \wedge \neg f_j = 0$, hence $e_i \leq f_j$. Reversing the roles of i and j yields $f_j \leq e_i$, so $e_i = f_j$. This implies that, after suitable renumbering, $e_i = f_i$ and $a_i = b_i$ for each i with $a_i \neq 0$. If the decomposition $\sum a_i e_i$ has a zero coefficient, say $0 = a_k$, then as the decomposition is full and the coefficients are distinct, $e_k = \neg(\bigvee_{i \neq k} e_i)$, which implies that the idempotent corresponding to a zero coefficient is uniquely determined. Consequently, s has a unique full orthogonal decomposition. □

Remark 2.2. (1) Orthogonal and full orthogonal decompositions will be our main technical tool. As we already pointed out, any orthogonal decomposition can be turned into a full orthogonal decomposition by possibly adding a term with a 0 coefficient, so depending on our need, we will freely work with either orthogonal or full orthogonal decompositions. If B is a generating algebra of faithful idempotents of S and $s \in S$ is nonzero, then by possibly dropping a term with a 0 coefficient, the same argument as in the proof of Lemma 2.1 produces a unique orthogonal decomposition $s = \sum_{i=1}^n a_i e_i$, where the $a_i \in R$ are distinct and nonzero.

(2) If $e_1, \ldots, e_n$ is an orthogonal set of faithful idempotents and $\sum a_i e_i = \sum b_i e_i$ for $a_i, b_i \in R$, then $a_i = b_i$ for each i; for, if we multiply by e_j, we get $a_j e_j = b_j e_j$, so $(a_j - b_j)e_j = 0$, and thus $a_j - b_j = 0$ since e_j is faithful. This holds regardless of whether the coefficients in either expression are distinct. We will use this fact several times.

Definition 2.3. We call an R-algebra S a *Specker R-algebra* if S is a commutative R-algebra that has a generating algebra of faithful idempotents.

Obviously, each Specker R-algebra is idempotent generated. Moreover, if S is a Specker R-algebra, then $1 \in \mathrm{Id}(S)$ is faithful, which means the natural map $R \to S$ sending $a \in R$ to $a \cdot 1 \in S$ is 1-1. Thus, R is isomorphic to an R-subalgebra of S. Throughout, we will freely identify R with an R-subalgebra of S.

To characterize Specker R-algebras among idempotent generated commutative R-algebras, we introduce a construction that associates with each Boolean algebra B an idempotent generated commutative R-algebra $R[B]$. This construction has its roots in the work of Bergman [3] and Rota [16].

Definition 2.4. Let B be a Boolean algebra. We denote by $R[B]$ the quotient ring $R[\{x_e : e \in B\}]/I_B$ of the polynomial ring over R in variables indexed by the elements of B modulo the ideal I_B generated by the following elements, as e, f range over B:

$$x_{e\wedge f} - x_e x_f, \quad x_{e\vee f} - (x_e + x_f - x_e x_f), \quad x_{\neg e} - (1 - x_e), \quad x_0.$$

For $e \in B$, we set $y_e = x_e + I_B \in R[B]$. Considering the generators of I_B, we see that for all $e, f \in B$:

$$y_{e\wedge f} = y_e y_f, \quad y_{e\vee f} = y_e + y_f - y_e y_f, \quad y_{\neg e} = 1 - y_e, \quad y_0 = 0.$$

It is obvious that $R[B]$ is a commutative R-algebra. From the relations above, it is also clear that y_e is an idempotent of $R[B]$ for each $e \in B$. Therefore, each $s \in R[B]$ can be written as $s = \sum a_i y_{e_i}$ with $a_i \in R$ and $e_i \in B$. Thus, $R[B]$ is idempotent generated. Moreover, $i_B \colon B \to \mathrm{Id}(R[B])$, given by $i_B(e) = y_e$, is a well-defined Boolean homomorphism. The following universal mapping property is an easy consequence of the definition of $R[B]$.

Lemma 2.5. *Let S be a commutative R-algebra. If B is a Boolean algebra and $\sigma \colon B \to \mathrm{Id}(S)$ is a Boolean homomorphism, then there is a unique R-algebra homomorphism $\alpha \colon R[B] \to S$ satisfying $\alpha \circ i_B = \sigma$.*

Proof. There is an R-algebra homomorphism $\gamma \colon R[\{x_e : e \in B\}] \to S$ such that $\gamma(x_e) = \sigma(e)$ for each $e \in B$. Since σ is a Boolean homomorphism, each generator of I_B lies in the kernel of γ. Therefore, we get an induced R-algebra homomorphism $\alpha \colon R[B] \to S$ with $\alpha(x_e + I_B) = \sigma(e)$. Thus, $\alpha \circ i_B = \sigma$. Clearly, α is the unique R-algebra homomorphism satisfying this equation since $R[B]$ is generated by the y_e. □

Lemma 2.6. *Let B be a Boolean algebra.*

(1) *If $e \in B$ is nonzero, then $y_e \in R[B]$ is faithful.*
(2) *i_B is a Boolean isomorphism from B onto the generating algebra of faithful idempotents $\{y_e : e \in B\}$ of $\mathrm{Id}(R[B])$.*

Proof. (1): Suppose $e \neq 0$. Then there is a Boolean homomorphism σ from B onto the two-element Boolean algebra $\mathbf{2}$ with $\sigma(e) = 1$. Viewing $\mathbf{2}$ as a subalgebra of $\mathrm{Id}(R)$, we can view σ as a Boolean homomorphism from B to $\mathrm{Id}(R)$. Then by Lemma 2.5, there is an R-algebra homomorphism $\alpha \colon R[B] \to R$ that sends y_e to $\sigma(e) = 1$. Consequently, if $ay_e = 0$, then $0 = \alpha(ay_e) = a$. This shows that y_e is faithful.

(2): It is obvious that $\{y_e : e \in B\}$ is a generating algebra of idempotents of $\mathrm{Id}(R[B])$ and that $i_B \colon B \to \{y_e : e \in B\}$ is an onto Boolean homomorphism. That $\{y_e : e \in B\}$ is faithful, and so i_B is 1-1, follows from (1). □

We are ready to prove the main result of this section, which gives several characterizations of Specker R-algebras, one of which is as Boolean powers of R.

Theorem 2.7. *Let S be a commutative R-algebra. The following are equivalent.*

(1) *S is a Specker R-algebra.*
(2) *S is isomorphic to $R[B]$ for some Boolean algebra B.*
(3) *S is isomorphic to a Boolean power of R.*
(4) *There is a Boolean subalgebra B of $\mathrm{Id}(S)$ such that S is generated by B and every Boolean homomorphism $B \to \mathbf{2}$ lifts to an R-algebra homomorphism $S \to R$.*

Proof. (1) ⇒ (2): Let B be a generating algebra of faithful idempotents of S. By Lemma 2.5, the identity map $B \to B$ lifts to an R-algebra homomorphism $\alpha \colon R[B] \to S$. By assumption, B generates S, so α is onto. To see that α is 1–1, suppose that $s \in R[B]$ with $\alpha(s) = 0$. Since $R[B]$ is idempotent generated, by Lemma 2.1, we can write $s = \sum a_i y_{e_i}$, where the $a_i \in R$ are distinct and the $e_i \in B$ are orthogonal. Therefore, $0 = \alpha(s) = \sum a_i e_i$. Multiplying by e_i gives $a_i e_i = 0$, which since the nonzero idempotents in B are faithful, implies that $a_i = 0$. This yields $s = 0$; hence, α is an isomorphism.

(2) ⇒ (3): We show that $R[B]$ is isomorphic to $C(X, R_{\mathrm{disc}})$, where X is the Stone space of B. By Stone duality, we identify B with the Boolean algebra of clopen subsets of X. For e a clopen subset of X, let χ_e be the characteristic function of e, and define $\sigma \colon B \to C(X, R_{\mathrm{disc}})$ by $e \mapsto \chi_e$. It is easy to see that this is a Boolean homomorphism from B to the idempotents of $C(X, R_{\mathrm{disc}})$. Thus, by Lemma 2.5, there is an R-algebra homomorphism $\alpha \colon R[B] \to C(X, R_{\mathrm{disc}})$ which sends y_e to $\sigma(e)$ for each $e \in B$. By Lemma 2.1, each $s \in R[B]$ can be written in the form $s = \sum a_i y_{e_i}$ with the $a_i \in R$ distinct and the $e_i \in B$ orthogonal. Then $\alpha(s)$ is the continuous function $X \to R$ such that

$$\alpha(s)(x) = \begin{cases} a_i & \text{if } x \in e_i, \\ 0 & \text{otherwise.} \end{cases}$$

If $s \neq 0$, then there is i with $e_i \neq \varnothing$ and $a_i \neq 0$. So, $\alpha(s) \neq 0$ in $C(X, R_{\mathrm{disc}})$. Thus, α is 1–1. To see α is onto, let $f \in C(X, R_{\mathrm{disc}})$. For each $a \in R$, we see that $f^{-1}(a)$ is clopen in X, and the various $f^{-1}(a)$ cover X. By compactness, there are finitely many distinct a_i such that $X = f^{-1}(a_1) \cup \cdots \cup f^{-1}(a_n)$. If $e_i = f^{-1}(a_i)$, then $f = \sum a_i \chi_{e_i}$, so $f = \alpha\left(\sum a_i y_{e_i}\right)$. Thus, α is onto. Consequently, α is an R-algebra isomorphism between $R[B]$ and $C(X, R_{\mathrm{disc}})$.

(3) $\Rightarrow$ (1): Let X be a Stone space and set $S = C(X, R_{\text{disc}})$. For each clopen subset U of X, the characteristic function χ_U of U is an idempotent of S. Let $B = \{\chi_U : U \text{ is clopen in } X\}$. Then B is a Boolean subalgebra of $\text{Id}(S)$. Moreover, each nonzero $\chi_U \in B$ is faithful since if $a \in R$ with $a\chi_U = 0$, then $a\chi_U(x) = 0$ for all $x \in X$. As χ_U is nonzero, U is nonempty. Let $x \in U$. Then $0 = a\chi_U(x) = a$. Thus, χ_U is faithful. Finally, we show that B generates S. Take $s \in S$. For each $a \in R$, the pullback $s^{-1}(a)$ is a clopen subset of X. Moreover, X is covered by the various $s^{-1}(a)$. Since X is compact, there are distinct $a_1, \dots, a_n \in R$ with $X = s^{-1}(a_1) \cup \cdots \cup s^{-1}(a_n)$. If $U_i = s^{-1}(a_i)$, then $s = \sum a_i \chi_{U_i}$. Thus, B generates S. Consequently, S is a Specker R-algebra.

(2) $\Rightarrow$ (4): Suppose that $S \cong R[C]$ for some Boolean algebra C. Let $B = \{y_c : c \in C\}$. By Lemma 2.6, B is isomorphic to C, so $R[B]$ is isomorphic to $R[C]$. We identify S with $R[B]$. Let $\sigma \colon B \to \mathbf{2}$ be a Boolean homomorphism. By viewing $\mathbf{2}$ as a Boolean subalgebra of $\text{Id}(R)$, we can view σ as a Boolean homomorphism from B to $\text{Id}(R)$. Then Lemma 2.5 yields an R-algebra homomorphism $S \to R$ lifting σ.

(4) $\Rightarrow$ (1): It suffices to show that every nonzero idempotent in B is faithful. Let $0 \neq e \in B$, and let $a \in R$ with $ae = 0$. Since $0 \neq e$, there is a Boolean homomorphism $\sigma \colon B \to \mathbf{2}$ such that $\sigma(e) = 1$. By (4), σ lifts to an R-algebra homomorphism $\alpha \colon S \to R$. Thus, $0 = \alpha(ae) = a\sigma(e) = a$, so that e is faithful. $\square$

Remark 2.8. (1) Let S be a Specker R-algebra. We will see in Section 3 that a generating algebra of faithful idempotents of S need not be unique, but that it is unique up to isomorphism.

(2) The proof of (1) $\Rightarrow$ (2) of Theorem 2.7 shows that if B is a generating algebra of faithful idempotents of S, then $S \cong R[B]$. We will make use of this fact later on.

(3) In the statement of Theorem 2.7(4), the requirement that S is generated by B is not redundant. For let R be an integral domain and let $S = R[x]/(x^2)$. As R has no zerodivisors, the only R-algebra homomorphism from S to R sends the coset of x to 0. Therefore, each Boolean homomorphism $\mathbf{2} \to \mathbf{2}$ lifts uniquely to an R-algebra homomorphism $S \to R$. By the definition of S, each element of S can be written uniquely as the coset of some linear polynomial $a + bx$ for $a, b \in R$. If $s = a + bx + (x^2)$ is idempotent, then $s^2 = s$ yields $a^2 + 2abx + (x^2) = a + bx + (x^2)$. Uniqueness then yields $a^2 = a$ and $2ab = b$. Therefore, $a \in \text{Id}(R)$, and as R is a domain, this forces $a \in \{0, 1\}$, so $b = 0$. Thus, $s \in \{0 + (x^2), 1 + (x^2)\}$, and so $\text{Id}(S) = \{0, 1\}$. It follows that S is not generated over R by idempotents.

Remark 2.9. While in this article we focus on viewing Boolean powers as $C(X, R_{\text{disc}})$, Foster's original conception of a Boolean power [9, 10, 11] also has an interesting interpretation in our setting. Let R be a commutative ring and let B be a Boolean algebra. Consider the set $R[B]^{\perp}$ of all functions $f \colon R \to B$ such that

(1) $f(a) = 0$ for all but finitely many $a \in R$,
(2) $f(a) \wedge f(b) = 0$ for all $a \neq b$ in R,
(3) $\bigvee \mathrm{Im} f = 1$.

Then $R[B]^{\perp}$ has an R-algebra structure given by

(4) $(f + g)(a) = \bigvee\{f(b) \wedge g(c) : b + c = a\}$,
(5) $(fg)(a) = \bigvee\{f(b) \wedge g(c) : bc = a\}$,
(6) $(bf)(a) = \bigvee\{f(c) : bc = a\}$.

As noticed by Jónsson in the review of [11] and further elaborated by Banaschewski and Nelson [2], $R[B]^{\perp}$ is isomorphic to the Boolean power $C(X, R_{\mathrm{disc}})$, where X is the Stone space of B.

As the notation $(-)^{\perp}$ suggests, $R[B]^{\perp}$ encodes full orthogonal decompositions of elements of $R[B]$ into an algebra of functions from R to B. Indeed, for a Specker R-algebra S with a generating algebra of faithful idempotents B, define $(-)^{\perp} : S \to R[B]^{\perp}$ as follows. For $s \in S$, write $s = \sum_{i=1}^{n} a_i e_i$ in full orthogonal form, and define $s^{\perp} : R \to B$ by

$$s^{\perp}(a) = \begin{cases} e_i & \text{if } a = a_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

One can show that $(-)^{\perp} : S \to R[B]^{\perp}$ is an R-algebra isomorphism. Thus, the interpretation of a Specker R-algebra S in terms of $R[B]^{\perp}$ is convenient for applications where the decomposition data for elements in S needs to be tracked under the algebraic operations of S.

3. Specker algebras over an indecomposable ring

In this section, we show that if R is an indecomposable ring (that is, $\mathrm{Id}(R) = \{0, 1\}$), then the results of the previous section can be strengthened considerably. Namely, we show that for indecomposable R, the category $\mathbf{Sp}_R$ of Specker R-algebras is equivalent to the category **BA** of Boolean algebras, and hence, by Stone duality, is dually equivalent to the category **Stone** of Stone spaces (zero-dimensional compact Hausdorff spaces). We also show that for indecomposable R, Specker R-algebras are exactly the idempotent generated R-algebras which are projective as an R-module.

We start by pointing out that for indecomposable R, the representation of Theorem 2.7 of Specker R-algebras as Boolean powers of R yields another representation of Specker R-algebras as idempotent generated subalgebras of R^I for some set I.

Proposition 3.1. *Let R be indecomposable. A commutative R-algebra S is a Specker R-algebra iff S is isomorphic to an idempotent generated R-subalgebra of R^I for some set I.*

Proof. Let S be a Specker R-algebra. By Theorem 2.7, $S \cong C(X, R_{\mathrm{disc}})$ for some Stone space X, so S is isomorphic to an idempotent generated subalgebra of R^X. Conversely, suppose that S is an idempotent generated subalgebra of R^I for some set I. Since R is indecomposable, it is easy to see that $\mathrm{Id}(R^I) = \{f \in R^I : f(i) \in \{0,1\}\ \forall i \in I\}$. From this description, it is clear that each nonzero idempotent of R^I is faithful. Therefore, S has a generating algebra of faithful idempotents, hence is a Specker R-algebra. □

In the next lemma, we characterize the idempotents of $R[B]$. For this, we view the Boolean algebra $\mathrm{Id}(R)$ as a Boolean ring. Then $\mathrm{Id}(R)[B]$ is an $\mathrm{Id}(R)$-algebra, which is a Boolean ring, and hence can be viewed as a Boolean algebra.

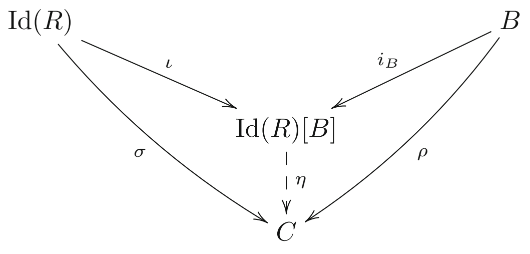
Lemma 3.2. *Let B be a Boolean algebra.*

(1) $s \in \mathrm{Id}(R[B])$ *iff* $s = \sum a_i y_{e_i}$ *with the* $a_i \in \mathrm{Id}(R)$ *distinct and the* $e_i \in B$ *a full orthogonal set.*
(2) $\mathrm{Id}(R[B]) \cong \mathrm{Id}(R)[B]$ *as Boolean algebras.*
(3) $\mathrm{Id}(R[B])$ *is isomorphic to the coproduct of* $\mathrm{Id}(R)$ *and* B.
(4) *If* R *is indecomposable, then* $i_B \colon B \to \mathrm{Id}(R[B])$ *is a Boolean isomorphism.*

Proof. (1): Let $s = \sum a_i y_{e_i}$ with $a_i \in \mathrm{Id}(R)$ and the $e_i \in B$ orthogonal. Then $y_{e_i} y_{e_j} = 0$ for $i \neq j$, and so $s^2 = \sum a_i^2 (y_{e_i})^2 = \sum a_i y_{e_i} = s$. Thus, s is idempotent. For the converse, let $s \in R[B]$ be idempotent. By Lemma 2.6, the y_e form a generating algebra of faithful idempotents of $R[B]$. Thus, by Lemma 2.1, we can write $s = \sum a_i y_{e_i}$ with the $a_i \in R$ distinct and the $e_i \in B$ a full orthogonal set. If $i \neq j$, then $y_{e_i} y_{e_j} = y_{e_i \wedge e_j} = y_0 = 0$. Therefore, $s^2 = \sum a_i^2 y_{e_i}$. By Remark 2.2(2), the equation $s^2 = s$ implies $a_i^2 = a_i$ for each i, so $a_i \in \mathrm{Id}(R)$.

(2): The inclusion map $\iota\colon \mathrm{Id}(R) \to \mathrm{Id}(R[B])$ is a Boolean homomorphism, and so a ring homomorphism of Boolean rings. For $\mathrm{Id}(R[B])$ as an $\mathrm{Id}(R)$-algebra, the Boolean homomorphism $i_B \colon B \to \mathrm{Id}(R[B])$ sending e to y_e extends to an $\mathrm{Id}(R)$-algebra homomorphism $\alpha\colon \mathrm{Id}(R)[B] \to \mathrm{Id}(R[B])$, according to Lemma 2.5. By (1) and Lemma 2.1, α is an isomorphism of Boolean rings, hence an isomorphism of Boolean algebras.

(3): Let C be a Boolean algebra and suppose $\sigma\colon \mathrm{Id}(R) \to C$ and $\rho\colon B \to C$ are Boolean homomorphisms. Then we see that C, viewed as a Boolean ring, is an $\mathrm{Id}(R)$-algebra, and Lemma 2.5 gives an $\mathrm{Id}(R)$-algebra homomorphism $\eta\colon \mathrm{Id}(R)[B] \to C$, sending ay_e to $\sigma(a)\rho(e) = \sigma(a) \wedge \rho(e)$. Clearly, η is the unique Boolean homomorphism making the following diagram commute. Thus, by (2), $\mathrm{Id}(R[B]) \cong \mathrm{Id}(R)[B]$ is the coproduct of $\mathrm{Id}(R)$ and B.



(4): Since R is indecomposable, (1) implies that $\mathrm{Id}(R[B]) = \{y_e : e \in B\}$. Now apply Lemma 2.6. □

Remark 3.3. As follows from the proofs of Lemma 3.2 and Theorem 2.7, for Boolean algebras B and C, the coproduct of B and C in **BA** can be described as the Boolean power of B by C. In particular, we see that $B[C] \cong C[B]$. This isomorphism is contained in [7, Exercise IV.5.1].

As promised in Remark 2.8, we next show that a generating algebra of faithful idempotents of a Specker R-algebra is not unique.

Example 3.4. Suppose that the ring R is not indecomposable and let $B = \{0, 1, e, \neg e\}$ be the four-element Boolean algebra. By Lemma 2.6, $\{y_b : b \in B\}$ is a generating algebra of faithful idempotents of $R[B]$. Let $a \in R$ be an idempotent with $a \neq 0, 1$ and set $g = ay_e + (1 - a)y_{\neg e}$. By Lemma 3.2(1), g is an idempotent in $R[B]$. Also,

$$1 - g = y_e + y_{\neg e} - (ay_e + (1 - a)y_{\neg e}) = (1 - a)y_e + ay_{\neg e}.$$

Now, g is faithful since if $c \in R$ with $cg = 0$, then $cay_e + c(1 - a)y_{\neg e} = 0$. By Remark 2.2(2), $ca = c(1 - a) = 0$, which forces $c = 0$. A similar argument shows $1 - g$ is faithful. Let $C = \{0, 1, g, 1 - g\}$. Since $y_e = ag + (1 - a)(1 - g)$, we see that C is a generating algebra of faithful idempotents of $R[B]$ different than $\{y_b : b \in B\}$.

On the other hand, we next prove that a generating algebra of faithful idempotents of a Specker R-algebra is unique up to isomorphism.

Theorem 3.5. *Let S be a Specker R-algebra. If B and C are both generating algebras of faithful idempotents of S, then B is isomorphic to C.*

Proof. We identify S with $R[B]$. Let P be a prime ideal of R and let PS be the ideal of S generated by P. Thus, PS consists of the sums of elements of the form ps with $p \in P$ and $s \in S$. We show that $S/PS \cong (R/P)[B]$. We note that $(R/P)[B]$ is an R-algebra, where scalar multiplication is given by $a \cdot (\sum (b_i + P)y_{e_i}) = \sum (ab_i + P)y_{e_i}$ for $a, b_i \in R$ and $e_i \in B$. By Lemma 2.5, the identity homomorphism $B \to B$ lifts to an R-algebra homomorphism $\alpha \colon R[B] \to (R/P)[B]$. It is clear that α is onto, and $\ker(\alpha)$ contains PS. If $s \in \ker(\alpha)$, write $s = \sum a_i y_{e_i}$ in unique full orthogonal form. Then $0 = \alpha(s) = \sum (a_i + P)y_{e_i}$. By Remark 2.2(2), each $a_i \in P$, so $s \in PS$. Therefore, $\ker(\alpha) = PS$, and so $S/PS \cong (R/P)[B]$. Now, since R/P is a domain, it is indecomposable. Thus, by Lemma 3.2(4), $B \cong \mathrm{Id}((R/P)[B]) \cong \mathrm{Id}(S/PS)$. Applying the same argument for C, for any prime ideal P of R, we then get $B \cong \mathrm{Id}(S/PS) \cong C$, so $B \cong C$. □

Example 3.4 and Theorem 3.5 show that while a generating algebra B of faithful idempotents of a Specker R-algebra may not be unique, it is unique up to isomorphism. In the following theorem, we show that if R is indecomposable, then a Specker R-algebra S has a unique generating algebra of faithful idempotents, namely $\mathrm{Id}(S)$.

Theorem 3.6. *Let R be indecomposable. An idempotent generated commutative R-algebra S is a Specker R-algebra iff each nonzero idempotent in* $\mathrm{Id}(S)$ *is faithful. Consequently, if S is a Specker R-algebra, then* $\mathrm{Id}(S)$ *is the unique generating algebra of faithful idempotents of S.*

Proof. If each nonzero idempotent of S is faithful, then $\mathrm{Id}(S)$ is a generating algebra of faithful idempotents of S, and so S is a Specker R-algebra. Conversely, suppose that S is a Specker R-algebra. Then S has a generating algebra of faithful idempotents B, and we identify S with $R[B]$. Because R is indecomposable, Lemma 3.2(4) implies that $\mathrm{Id}(S) = \{y_e : e \in B\}$. Thus, by Lemma 2.6, each nonzero idempotent of S is faithful. □

The results of the previous section yield two functors, $\mathcal{I}\colon \mathbf{Sp}_R \to \mathbf{BA}$ and $\mathcal{S}\colon \mathbf{BA} \to \mathbf{Sp}_R$. The functor $\mathcal{I}$ associates with each $S \in \mathbf{Sp}_R$ the Boolean algebra $\mathrm{Id}(S)$ of idempotents of S, and with each R-algebra homomorphism $\alpha\colon S \to S'$ the restriction $\mathcal{I}(\alpha) = \alpha|_{\mathrm{Id}(S)}$ of α to $\mathrm{Id}(S)$. The functor $\mathcal{S}$ associates with each $B \in \mathbf{BA}$ the Specker R-algebra $R[B]$, and with each Boolean homomorphism $\sigma\colon B \to B'$ the induced R-algebra homomorphism $\alpha\colon R[B] \to R[B']$ that sends each y_e to $y_{\sigma(e)}$.

Lemma 3.7. *The functor $\mathcal{S}$ is left adjoint to the functor $\mathcal{I}$.*

Proof. By definition, $\mathcal{I}(\mathcal{S}(B)) = \mathrm{Id}(R[B])$. By [14, Ch. IV, Thm. 1.2], the universal mapping property established in Lemma 2.5 is equivalent to the fact that $\mathcal{S}$ is left adjoint to $\mathcal{I}$. □

We show that the functors $\mathcal{I}, \mathcal{S}$ form an equivalence of $\mathbf{Sp}_R$ and $\mathbf{BA}$ precisely when R is indecomposable.

Theorem 3.8. *The following are equivalent.*

(1) *R is indecomposable.*
(2) $\mathcal{I} \circ \mathcal{S} \cong 1_{\mathbf{BA}}$.
(3) $\mathcal{S} \circ \mathcal{I} \cong 1_{\mathbf{Sp}_R}$.
(4) *The functors $\mathcal{I}$ and $\mathcal{S}$ yield an equivalence of* $\mathbf{Sp}_R$ *and* $\mathbf{BA}$.

Proof. (1) ⇔ (2): Suppose that R is indecomposable. We have $\mathcal{I}(\mathcal{S}(B)) = \mathrm{Id}(R[B])$. By Lemma 3.2(4), $i_B\colon B \to \mathrm{Id}(R[B])$ is a Boolean isomorphism, and by Lemma 3.7, i_B is natural, so (2) follows. Conversely, if (2) holds, then $\mathrm{Id}(R[\mathbf{2}]) \cong \mathbf{2}$. Observe that $R[\mathbf{2}] \cong R$. To see this, let $\mathbf{2} = \{0, 1\}$ and recall from Definition 2.4 the ideal $I_{\mathbf{2}}$ defining $R[\mathbf{2}]$. By listing all the generators for $I_{\mathbf{2}}$, we see that $I_{\mathbf{2}}$ is generated by $x_0, x_1 - 1$. Therefore, we have $R[\mathbf{2}] \cong R[x_0, x_1]/(x_0, x_1 - 1) \cong R$. Thus, $\mathbf{2} \cong \mathrm{Id}(R[\mathbf{2}]) \cong \mathrm{Id}(R)$, which shows that R is indecomposable.

(1) ⇔ (3): Suppose that R is indecomposable. We have $\mathcal{S}(\mathcal{I}(S)) = R[\mathrm{Id}(S)]$ for each Specker R-algebra S. Furthermore, by Theorem 3.6, $\mathrm{Id}(S)$ is a generating algebra of faithful idempotents of S. Consequently, the R-algebra homomorphism $\alpha_S\colon R[\mathrm{Id}(S)] \to S$ sending y_e to e for each $e \in \mathrm{Id}(S)$ is an

isomorphism. By Lemma 3.7, α_S is natural, so (3) follows. Conversely, if (3) holds, then $R[\mathrm{Id}(R)] \cong R$ via α_R. If $e \neq 0, 1$ is an idempotent in R, then $\alpha_R(ey_{\neg e}) = e(\neg e) = 0$, a contradiction to Lemma 2.6. Thus, $\mathrm{Id}(R) = \{0, 1\}$, so R is indecomposable.

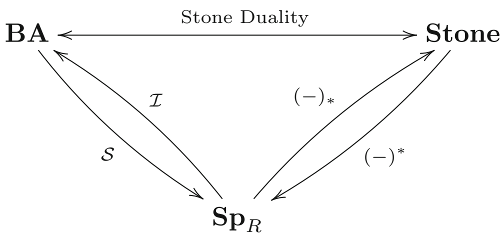
(1) $\Leftrightarrow$ (4): In view of Lemma 3.7, (2) and (3) together are equivalent to (4). Thus, by what we have proven already, (1) implies both (2) and (3), so implies (4). Conversely, if (4) holds, then (2) holds, so (1) holds as (1) is equivalent to (2). □

Corollary 3.9. *If R is indecomposable, then* $\mathbf{Sp}_R$ *is dually equivalent to* **Stone**.

Proof. By Theorem 3.8, $\mathbf{Sp}_R$ is equivalent to **BA**. By Stone duality, **BA** is dually equivalent to **Stone**. Combining these two results yields that $\mathbf{Sp}_R$ is dually equivalent to **Stone**. □

Remark 3.10. In [15, Sec. 7], Ribenboim defines the category of Boolean powers of $\mathbb{Z}$ and proves that this category is equivalent to **BA**. In view of Theorem 2.7 and Remark 2.9, Ribenboim's result is a particular case of Theorem 3.8. Similarly, it follows from [5, Thm. 6.8] that $\mathbf{Sp}_\mathbb{R}$ is equivalent to **BA**. Again, this result is a particular case of Theorem 3.8. Moreover, by Theorem 2.7, Specker $\mathbb{R}$-algebras are isomorphic to Boolean powers of $\mathbb{R}$.

As noted in the proof of Corollary 3.9, the functors $\mathcal{I}$ and $\mathcal{S}$ of Theorem 3.8 compose with the functors of Stone duality to give functors between $\mathbf{Sp}_R$ and **Stone**. The resulting contravariant functor from **Stone** to $\mathbf{Sp}_R$ is the Boolean power functor $(-)^*\colon \mathbf{Stone} \to \mathbf{Sp}_R$ that associates with each $X \in$ **Stone** the Boolean power $X^* = C(X, R_{\mathrm{disc}})$, and with each continuous map $\varphi\colon X \to Y$ the R-algebra homomorphism $\varphi^*\colon Y^* \to X^*$ given by $\varphi(f) = f \circ \varphi$. The functor $(-)_*\colon \mathbf{Sp}_R \to \mathbf{Stone}$ sends the Specker R-algebra S to the Stone space of $\mathrm{Id}(S)$ and associates with each R-algebra homomorphism $S \to T$ the continuous map from the Stone space of $\mathrm{Id}(S)$ to the Stone space of $\mathrm{Id}(T)$. By Corollary 3.9, these two functors yield a dual equivalence when R is indecomposable. In general, we have the following diagram.



We show in Proposition 3.11 that the functor $(-)_*\colon \mathbf{Sp}_R \to \mathbf{Stone}$ has a natural interpretation when R is indecomposable, one that does not require reference to $\mathrm{Id}(S)$. Let S be a Specker R-algebra and let $\mathrm{Hom}_R(S, R)$ be the set of R-algebra homomorphisms from S to R. We define a topology on $\mathrm{Hom}_R(S,R)$ by setting $U_s = \{\alpha \in \mathrm{Hom}_R(S,R) : \alpha(s) = 0\}$ and taking $\{U_s : s \in S\}$ as a subbasis. Recall that the Stone space of a Boolean algebra B can be described as the set $\mathrm{Hom}(B, \mathbf{2})$ of Boolean homomorphisms from B to $\mathbf{2}$, topologized by the basis $\{\widehat{e} : e \in B\}$, where $\widehat{e} = \{\sigma \in \mathrm{Hom}(B, \mathbf{2}) : \sigma(e) = 0\}$.

Proposition 3.11. *Let R be indecomposable, and let S be a Specker R-algebra. Then $\mathrm{Hom}_R(S,R)$ is homeomorphic to $\mathrm{Hom}(\mathrm{Id}(S), \mathbf{2})$.*

Proof. Set $B = \mathrm{Id}(S)$; define $\varphi\colon \mathrm{Hom}_R(S,R) \to \mathrm{Hom}(B, \mathbf{2})$ by $\varphi(\alpha) = \alpha|_B$. By Theorem 2.7, φ is onto. It is 1–1 because if $\alpha|_B = \beta|_B$, then α, β are R-algebra homomorphisms which agree on a generating set of S, so $\alpha = \beta$. We have

$$\varphi^{-1}(\widehat{e}) = \{\alpha \in \mathrm{Hom}_R(S,R) : \alpha(e) = 0\} = U_e,$$

which proves that φ is continuous. It also shows that $\varphi(U_e) = \widehat{e}$. Now, let $s \in S$. If $s = 0$, then $U_s = \mathrm{Hom}_R(S,R)$, so $\varphi(U_s) = \mathrm{Hom}(B, \mathbf{2})$ is open. Otherwise, we can write $s = \sum_i a_i e_i$ with the $a_i \in R$ nonzero and the $e_i \in B$ orthogonal. If $\alpha \in U_s$, then $s \in \ker(\alpha)$, so $a_i e_i = s e_i \in \ker(\alpha)$. Thus, $e_i \in \ker(\alpha)$ since otherwise $\alpha(e_i) = 1$, and this contradicts $a_i \neq 0$. Therefore, $\alpha \in U_{e_1} \cap \cdots \cap U_{e_n}$. The reverse inclusion is obvious. Thus, $U_s = U_{e_1} \cap \cdots \cap U_{e_n}$, and so $\varphi(U_s) = \widehat{e_1} \cap \cdots \cap \widehat{e_n}$. Since the U_s form a subbasis for $\mathrm{Hom}_R(S,R)$, this proves that φ^{-1} is continuous. Consequently, φ is a homeomorphism. □

It follows that when R is indecomposable, the space $\mathrm{Hom}_R(S,R)$ of a Specker R-algebra S is homeomorphic to the Stone space of $\mathrm{Id}(S)$. This allows us to describe the contravariant functor $(-)_*\colon \mathbf{Sp}_R \to \mathbf{Stone}$ as follows. Associate with each $S \in \mathbf{Sp}_R$ the Stone space $S_* = \mathrm{Hom}_R(S,R)$, and with each R-algebra homomorphism $\alpha\colon S \to T$, the continuous map $\alpha_*\colon T_* \to S_*$ given by $\alpha_*(\delta) = \delta \circ \alpha$ for each $\delta \in T_* = \mathrm{Hom}_R(T,R)$. Thus, we have a description of $(-)_*$ that does not require passing to idempotents.

We conclude this section by giving a module-theoretic characterization of Specker R-algebras for indecomposable R. Bergman [3, Cor. 3.5] has shown that a Boolean power $C(X, R_{\mathrm{disc}})$ of the ring R is a free R-module having a basis of idempotents. Thus, by Theorem 2.7, every Specker R-algebra is a free R-module having a basis of idempotents. We prove in the next theorem that the converse of the corollary is true when R is indecomposable, and that in this case, freeness is equivalent to projectivity. For an R-module M, we denote the annihilator of m in R by $\mathrm{ann}_R(m)$, so $\mathrm{ann}_R(m) := \{r \in R : rm = 0\}$.

Theorem 3.12. *Let R be indecomposable and let S be an idempotent generated commutative R-algebra. Then the following are equivalent.*

(1) *S is a Specker R-algebra.*
(2) *S is a free R-module.*
(3) *S is a projective R-module.*

Proof. (1) ⇒ (2): This follows from [3, Cor. 3.5] and Theorem 2.7.

(2) ⇒ (3): This is obvious.

(3) $\Rightarrow$ (1): Let $B = \mathrm{Id}(S)$. By Lemma 2.5, the inclusion $B \to S$ lifts to an R-algebra homomorphism $\alpha\colon R[B] \to S$. Since S is generated by B, we have that α is onto. In particular, for each idempotent $e \in S$, we have $\alpha(y_e) = e$. Now since S is a projective R-module, there exists an R-module homomorphism $\beta\colon S \to R[B]$ such that $\alpha(\beta(s)) = s$ for all $s \in S$. Let e be an idempotent in S. Write $\beta(e) = \sum a_i y_{e_i}$ with $a_i \in R$ and the $e_i \in \mathrm{Id}(S)$ orthogonal. Then $e = \alpha(\beta(e)) = \sum a_i \alpha(y_{e_i}) = \sum a_i e_i$.

First observe that for every $a \in \mathrm{ann}_R(e)$, we have $aa_1 = \cdots = aa_n = 0$. Indeed, for $a \in \mathrm{ann}_R(e)$, we have $0 = \beta(ae) = a\beta(e) = \sum aa_i y_{e_i}$, so that since by Lemma 2.6, each y_{e_i} is faithful, so $aa_i = 0$. This in turn implies that if $\mathrm{ann}_R(e) \neq 0$, then $\sum_{i=1}^n a_i R$ is a proper ideal of R, as every element in $\mathrm{ann}_R(e)$ annihilates $\sum a_i R$. We use these observations to show that every nonzero idempotent is faithful.

Suppose $\mathrm{ann}_R(e) \neq 0$. We show $e = 0$. First we claim $\mathrm{ann}_R(e) + \sum a_i R = R$. Since $e = \sum a_i e_i$ is an orthogonal decomposition of e and e is idempotent, it follows that $a_i e_i = a_i^2 e_i$, and hence $a_i(1 - a_i)e_i = 0$ for each i. Thus, $(1 - a_1) \cdots (1 - a_n) \in \mathrm{ann}_R(e)$. But $1 - (1 - a_1) \cdots (1 - a_n) \in \sum a_i R$, yielding $\mathrm{ann}_R(e) + \sum a_i R = R$, as desired. Therefore, there exist $a \in \mathrm{ann}_R(e)$ and $b_1, \ldots, b_n \in R$ such that $a + \sum a_i b_i = 1$. By assumption, $\mathrm{ann}_R(e) \neq 0$, so as established above, $\sum a_i R$ is a proper ideal of R. In particular, $1 \neq \sum a_i b_i$, so since $1 = a + \sum a_i b_i$, this forces $a \neq 0$. As noted above, $aa_1 = \cdots = aa_n = 0$. Thus, $a(\sum a_i b_i) = 0$, so that multiplying both sides of the equation $a + \sum a_i b_i = 1$ by a yields $a^2 = a$. Therefore, $a \in \mathrm{Id}(R)$, and since R is indecomposable and $a \neq 0$, this forces $a = 1$. But $ae = 0$, so we conclude that $e = 0$. This proves that every nonzero idempotent in S is faithful, and hence, as S is idempotent generated, S is a Specker R-algebra. ☐

Remark 3.13. The assumption of indecomposability in the theorem is necessary: If R is not indecomposable, then there exists an idempotent a in R distinct from $0, 1$, so that $R = aR \oplus (1-a)R$, and hence $S := R/aR$ is isomorphic as an R-module to $(1 - a)R$, a direct summand of the free R-module R, and thus is a projective R-module. The algebra S is generated as an R-algebra by the idempotent $1 + aR$. Yet $1 + aR$ is not faithful because it is annihilated by a, so S is not a Specker R-algebra.

4. Specker algebras over a domain

As follows from the previous section, having R indecomposable allows one to prove several strong results about Specker R-algebras. Some of these results can be strengthened further provided R is a domain, a case we consider in more detail in this section. We first show that among idempotent generated commutative algebras over a domain R, the Specker R-algebras are simply those that are torsion-free R-modules. We then give a necessary and sufficient condition for a Specker R-algebra S to be a weak Baer ring and a Baer ring.

For a domain R, the characterization of Specker R-algebras that are Baer rings yields a characterization of injective objects as well as the construction of injective hulls in $\mathbf{Sp}_R$. In addition, it provides a description of $S_* = \mathrm{Hom}_R(S, R)$ by means of minimal prime ideals of S.

Proposition 4.1. *Let R be a domain and let S be an idempotent generated commutative R-algebra. Then S is a Specker R-algebra iff S is a torsion-free R-module.*

Proof. As discussed before Theorem 3.12, a Specker R-algebra S is a free R-module, and hence with R a domain, S is torsion-free. Conversely, if S is an idempotent generated commutative R-algebra that is torsion-free, then nonzero idempotents are faithful, and hence by Theorem 3.6, S is a Specker R-algebra. □

Next, we recall the well-known definition of a Baer ring and a weak Baer ring in the case of a commutative ring.

Definition 4.2. A commutative ring R is a *Baer ring* if the annihilator ideal of each subset of R is a principal ideal generated by an idempotent, and R is a *weak Baer ring* if the annihilator ideal of each element of R is a principal ideal generated by an idempotent.

As we noted after Definition 2.3, we will view R as an R-subalgebra of each Specker R-algebra S.

Theorem 4.3. *Let S be a Specker R-algebra.*

(1) *S is weak Baer iff R is weak Baer.*
(2) *S is Baer iff S is weak Baer and* $\mathrm{Id}(S)$ *is a complete Boolean algebra.*

Proof. (1): Let B be a generating algebra of faithful idempotents of S. Suppose that S is weak Baer and let $a \in R$. Then there is $e \in \mathrm{Id}(S)$ with $\mathrm{ann}_S(a) = eS$. By Lemma 3.2(1) and Theorem 2.7, we can write $e = \sum b_i e_i$ in full orthogonal form with $b_i \in \mathrm{Id}(R)$ and $e_i \in B$. Since $0 = ae = \sum (ab_i)e_i$, by Remark 2.2(2), we see that $ab_i = 0$ for all i. Therefore, $b_j \in eS$ for each j, hence $b_j = es$ for some $s \in S$. Then $eb_j = e(es) = es = b_j$. The equation $eb_j = b_j$ yields $\sum (b_i b_j)e_i = b_j = \sum b_j e_i$ since $\sum e_i = \bigvee e_i = 1$. Remark 2.2(2) yields $b_i b_j = b_j$. Applying the same argument to $eb_i = b_i$ gives $b_i b_j = b_i$, so $b_i = b_j$ for each i, j. Thus, $e = \sum b_i e_i = b_1 \sum e_i = b_1$. Consequently, $e = b_1 \in R$. From this it follows that $\mathrm{ann}_R(a) = b_1 R$ is generated by the idempotent b_1, so R is weak Baer.

Conversely, suppose that R is weak Baer. Let $s \in S$ and write $s = \sum a_i e_i$ in full orthogonal form with the $a_i \in R$ distinct and $e_i \in B$. Since R is weak Baer, $\mathrm{ann}_R(a_i) = b_i R$ for some idempotent $b_i \in R$. Let $e = \sum b_i e_i$. By Lemma 3.2(1), e is an idempotent in S. We claim that $\mathrm{ann}_S(s) = eS$. We have $es = (\sum b_i e_i)(\sum a_i e_i) = \sum (b_i a_i) e_i = 0$ because the e_i are orthogonal and the b_i annihilate the a_i. So $eS \subseteq \mathrm{ann}_S(s)$. To prove the reverse inclusion, we first show that if $b \in R$ and $g \in \mathrm{Id}(S)$ with $bg \in \mathrm{ann}_S(s)$, then $bg \in eS$. If $bgs = 0$, then $\sum (ba_i)(e_i g) = 0$. Thus, by Remark 2.2(2), for each i with $e_i g \neq 0$, we have $ba_i = 0$. When this occurs, $b \in b_i R$, so $b = bb_i$. Consequently,

$$\begin{aligned} e(bg) &= \left(\sum b_i e_i\right) bg = \sum (b_i b)(e_i g) = \sum b(e_i g) = \left(\sum e_i\right) bg \\ &= 1 \cdot bg = bg. \end{aligned}$$

Thus, $bg \in eS$. In general, if $t \in \mathrm{ann}_S(s)$, write $t = \sum c_j f_j$ in orthogonal form. Then each $c_j f_j = tf_j \in \mathrm{ann}_S(s)$. By the previous argument, each $c_j f_j \in eS$, so $t \in eS$. This proves that $\mathrm{ann}_S(s) = eS$, so S is weak Baer.

(2): First suppose that S is weak Baer and $\mathrm{Id}(S)$ is complete, and let $I \subseteq S$. Then $\mathrm{ann}_S(I) = \bigcap_{s \in I} \mathrm{ann}_S(s)$. Since S is weak Baer, there is $e_s \in \mathrm{Id}(S)$ with $\mathrm{ann}_S(s) = e_s S$. Consequently, $\mathrm{ann}_S(I) = \bigcap_{s \in I} e_s S$. Let $e = \bigwedge e_s$. We show that $\mathrm{ann}_S(I) = eS$. Since $e \leq e_s$ for each s, we have $ee_s = e$, so $e \in \bigcap e_s S = \mathrm{ann}_S(I)$. Conversely, let $t \in \mathrm{ann}_S(I)$. Then $ts = 0$ for all $s \in I$, so $t \in e_s S$ for each s, which yields $te_s = t$. Let $t = \sum b_i f_i$ be the full orthogonal decomposition of t with the $b_i \in R$ distinct and $f_i \in B$. Then $te_s = t$ yields $\sum b_i f_i e_s = \sum b_i f_i$. By Remark 2.2(2), $f_i e_s = f_i$, so $f_i \leq e_s$ for each s. Therefore, $f_i \leq e$, so $f_i e = f_i$. Since this is true for all i, we have $te = t$. This yields $t \in eS$. Thus, $\mathrm{ann}_S(I) = eS$, and so S is Baer.

Next suppose that S is Baer. Then S is weak Baer. Let $\{e_i : i \in I\}$ be a family of idempotents of S. Set $K = \{1 - e_i : i \in I\}$. Then $\mathrm{ann}_S(1 - e_i) = e_i S$, so $\mathrm{ann}_S(K) = \bigcap e_i S$. Since S is Baer, $\mathrm{ann}_S(K) = eS$ for some $e \in \mathrm{Id}(S)$. We show that $e = \bigwedge e_i$. First, as $e \in \mathrm{ann}_S(K)$, we have $ee_i = e$, so $e \leq e_i$. Thus, e is a lower bound of the e_i. Next, let $f \in \mathrm{Id}(S)$ be a lower bound of the e_i. Then $fe_i = f$, so $(1 - e_i)f = 0$. Therefore, $f \in \mathrm{ann}_S(K) = eS$. This implies that $fe = f$, so $f \leq e$. Thus, $e = \bigwedge_i e_i$. Consequently, $\mathrm{Id}(S)$ is a complete Boolean algebra. □

Corollary 4.4. *Let S be a Specker R-algebra.*

1. *If R is indecomposable, then S is Baer iff R is a domain and* $\mathrm{Id}(S)$ *is a complete Boolean algebra.*
2. *If R is a domain, then S is weak Baer.*

Proof. (1): By Theorem 4.3(2), S is Baer iff S is weak Baer and $\mathrm{Id}(S)$ is complete. By Theorem 4.3(1), S is weak Baer iff R is weak Baer. Now, since R is indecomposable, the only idempotents are $0, 1$, so if R is weak Baer, then $\mathrm{ann}_R(a) = 0$ for each non-zero $a \in R$, which means each nonzero element is a nonzerodivisor, so R is a domain. Conversely, if R is a domain, then trivially R is Baer. Thus, (1) follows from Theorem 4.3.

(2) This follows from Theorem 4.3 since a domain is a Baer ring. □

Next we show that when R is a domain, then the Stone space S_* is also homeomorphic to the space $\mathrm{Min}(S)$ of minimal prime ideals of S with the subspace topology inherited from the Zariski topology on the prime spectrum of S. Therefore, the closed sets of $\mathrm{Min}(S)$ are the sets of the form $Z(I) =$

$\{P \in \mathrm{Min}(S) : I \subseteq P\}$ for some ideal I of S. For $s \in S$, we set $Z(s) = Z(sS)$. Let $s = \sum_{i=1}^{n} a_i e_i$ with the a_i nonzero and the e_i orthogonal. As $e_i s = a_i e_i$, it follows that $Z(s) = Z(e_1) \cap \cdots \cap Z(e_n)$, and so $Z(s)$ is a clopen subset of $\mathrm{Min}(S)$. This contrasts the case of the prime spectrum of S, where $Z(s)$ is clopen iff s is an idempotent.

Lemma 4.5. *When R is a domain, the following are equivalent for a prime ideal P of a Specker R-algebra S.*

(1) *P is a minimal prime ideal of S.*
(2) $P \cap R = 0$.
(3) *Every element of P is a zerodivisor.*

Proof. (1) ⇒ (2): Suppose P is a minimal prime ideal of S. Then every element of P is a zerodivisor in S (see, e.g., [13, Cor. 1.2]). Thus, if $a \in P \cap R$, then there exists $0 \neq s \in S$ such that $as = 0$. But by Proposition 4.1, S is a torsion-free R-module, so necessarily $a = 0$, and hence $P \cap R = 0$.

(2) ⇒ (3): Let $s \in P$ be nonzero and write $s = \sum a_i e_i$ in orthogonal form with the a_i distinct and nonzero. Then $a_i e_i = s e_i \in P$ for each i, so since $P \cap R = 0$, it must be that $e_i \in P$. Thus, $1 \neq \sum e_i$, and hence since $(1 - \sum e_i)s = 0$, we see that s is a zerodivisor in S.

(3) ⇒ (1): By Corollary 4.4, S is a weak Baer ring. Statement (1) now follows from [13, Lem. 3.8] □

Theorem 4.6. *If R is a domain and S is a Specker R-algebra, then S_* is homeomorphic to* $\mathrm{Min}(S)$.

Proof. By Proposition 3.11, we identify S_* with $\mathrm{Hom}_R(S, R)$. If $\alpha \in S_*$, then as R is a domain, $P := \ker(\alpha)$ is a prime ideal. Moreover, $P \cap R = 0$ since $a \in R$ implies $\alpha(a) = a$. Consequently, by Lemma 4.5, P is a minimal prime ideal of S. Conversely, if P is a minimal prime ideal of S, then consider the composition of the canonical R-algebra homomorphisms $R \to S \to S/P$. By Lemma 4.5, $R \cap P = 0$, so this composition is 1–1. To see that it is onto, observe that since S/P is a domain, $e + P \in \{0 + P, 1 + P\}$ for each idempotent $e \in S$. Therefore, S/P is generated over R by $1 + P$, and so the homomorphism $R \to S/P$ is onto. Thus, there is $\alpha \in S_*$ with $P = \ker(\alpha)$. This shows that there is a bijection $\psi \colon S_* \to \mathrm{Min}(S)$, given by $\psi(\alpha) = \ker(\alpha)$. We note that $\psi^{-1}(Z(s)) = \{\alpha \in S_* : \alpha(s) = 0\} = U_s$. Now, to see that ψ is continuous, if I is an ideal of S, then

$$\psi^{-1}(Z(I)) = \psi^{-1}\big(\bigcap_{s \in I} Z(s)\big) = \bigcap_{s \in I} \psi^{-1}(Z(s)) = \bigcap_{s \in I} U_s. \tag{4.1}$$

It follows from the proof of Proposition 3.11 that if $s = \sum a_i e_i$ is in orthogonal form, then $U_s = U_{e_1} \cap \cdots \cap U_{e_n}$. Because $U_e = S_* - U_{\neg e}$ for each $e \in \mathrm{Id}(S)$, we see that each U_e is clopen, and so U_s is clopen. Therefore, Equation (4.1) shows that ψ is continuous. In addition, because ψ is onto and $\psi^{-1}(Z(s)) = U_s$, we have $\psi(U_s) = Z(s)$. Thus, ψ^{-1} is continuous, so ψ is a homeomorphism. □

Let $\mathbf{BSp}_R$ be the full subcategory of $\mathbf{Sp}_R$ consisting of Baer Specker R-algebras; let $\mathbf{cBA}$ be the full subcategory of $\mathbf{BA}$ consisting of complete Boolean algebras, and let $\mathbf{ED}$ be the full subcategory of $\mathbf{Stone}$ consisting of extremally disconnected spaces.

Theorem 4.7. *Let R be a domain.*

(1) *The categories $\mathbf{BSp}_R$ and $\mathbf{cBA}$ are equivalent.*
(2) *The categories $\mathbf{BSp}_R$ and $\mathbf{ED}$ are dually equivalent.*

Proof. (1): By Corollary 4.4, when R is a domain, a Specker R-algebra is a Baer ring iff $\mathrm{Id}(S)$ is a complete Boolean algebra. Now apply Theorem 3.8 to obtain that the restrictions of the functors $\mathcal{I}$ and $\mathcal{S}$ yield an equivalence of $\mathbf{BSp}_R$ and $\mathbf{cBA}$.

(2): Stone duality yields that the restriction of $(-)_*$ to $\mathbf{BSp}_R$ lands in $\mathbf{ED}$. When R is a domain, $\mathrm{Id}(X^*)$ consists of the characteristic functions of clopen subsets of X. Stone duality and Corollary 4.4 then yield that the restriction of $(-)^*$ to $\mathbf{ED}$ lands in $\mathbf{BSp}_R$. Now apply Corollary 3.9 to conclude that the restrictions of $(-)_*$ and $(-)^*$ yield a dual equivalence of $\mathbf{BSp}_R$ and $\mathbf{ED}$. □

Since injectives in $\mathbf{BA}$ are exactly the complete Boolean algebras, as an immediate consequence of Theorem 4.7, we obtain the following corollary.

Corollary 4.8. *When R is a domain, the injective objects in $\mathbf{Sp}_R$ are the Baer Specker R-algebras.*

Remark 4.9. In fact, when R is a domain, each $S \in \mathbf{Sp}_R$ has its injective hull in $\mathbf{Sp}_R$, which can be constructed as follows. Let $\mathrm{DM}(\mathrm{Id}(S))$ be the Dedekind-MacNeille completion of the Boolean algebra $\mathrm{Id}(S)$. Then, by [1, Prop. 3] and Theorem 3.8, $R[\mathrm{DM}(\mathrm{Id}(S))]$ is the injective hull of S in $\mathbf{Sp}_R$.

5. Specker algebras over a totally ordered ring

Recall (see, e.g., [6, Ch. XVII]) that a ring R with a partial order $\leq$ is an ℓ-*ring* (lattice-ordered ring) if (i) $(R, \leq)$ is a lattice, (ii) $a \leq b$ implies $a+c \leq b+c$ for each c, and (iii) $0 \leq a, b$ implies $0 \leq ab$. An ℓ-ring R is *totally ordered* if the order on R is a total order, and it is an f-*ring* if it is a subdirect product of totally ordered rings. It is well known (see, e.g., [6, Ch. XVII, Corollary to Thm. 8]) that an ℓ-ring R is an f-ring iff for each $a, b, c \in R$ with $a \wedge b = 0$ and $c \geq 0$, we have $ac \wedge b = 0$.

In this final section we consider the case when R is a totally ordered ring. Our motivation for considering Specker algebras over totally ordered rings stems from the case when $R = \mathbb{Z}$ as treated by Ribenboim [15] and Conrad [8], and the case $R = \mathbb{R}$ studied in [5]. These approaches all have in common a lifting of the order from the totally ordered ring to what is a fortiori a Specker R-algebra, and in all three cases the lift produces the same order. We show in

Theorem 5.1 that if R is totally ordered, then there is a unique partial order on a Specker R-algebra that makes it into an f-algebra over R.

We start by noting that each totally ordered ring R is indecomposable. To see this, we first note that if $a \in R$, then $a^2 \geq 0$, since if $a \geq 0$, then $a^2 \geq 0$, and if $a \leq 0$, then $-a \geq 0$, so $a^2 = (-a)^2 \geq 0$. Let $e \in R$ be idempotent. Then $0 \leq e$ since $e = e^2$. Either $e \leq 1-e$ or vice versa. If $e \leq 1-e$, then multiplying by e yields $e^2 \leq 0$, which forces $e = 0$. On the other hand, if $1 - e \leq e$, then multiplying by $1 - e$, which is nonnegative since it is an idempotent, we get $1 - e \leq 0$. Like before this forces $1 - e = 0$, so $e = 1$. Thus, $\mathrm{Id}(R) = \{0, 1\}$.

Let $(S, \leq)$ be a partially ordered R-algebra, where R is totally ordered. We call S an *ℓ-algebra over R* if S is both an ℓ-ring and an R-algebra such that whenever $0 \leq s \in S$ and $0 \leq a \in R$, then $as \geq 0$. Furthermore, we call S an *f-algebra over R* if S is both an ℓ-algebra over R and an f-ring.

Theorem 5.1. *Let R be totally ordered and let S be a Specker R-algebra. Then there is a unique partial order on S for which $(S, \leq)$ is an f-algebra over R.*

Proof. By Theorem 2.7, we identify S with $C(X, R_{\mathrm{disc}})$ for some Stone space X. Since R is totally ordered, there is a partial order on S, defined by $f \leq g$ if $f(x) \leq g(x)$ for each $x \in X$. It is elementary to see that S with this partial order is an ℓ-algebra over R. Let $f, g \in S$ with $f \wedge g = 0$ and let $h \geq 0$. Then for each $x \in X$, either $f(x) = 0$ or $g(x) = 0$. Therefore, $fh(x) = 0$ or $g(x) = 0$, so $fh \wedge g = 0$. Thus, S is an f-ring, and so is an f-algebra.

To prove uniqueness, suppose we have a partial order $\leq'$ on S for which $(S, \leq')$ is an f-algebra over R. As squares in S are positive [6, Ch. XVII, Lem. 6.2], idempotents in S are positive. Let $f \in S$ be nonzero, and write $f = \sum a_i \chi_{U_i}$ for some nonzero $a_i \in R$ and U_i nonempty pairwise disjoint clopen subsets of X. Since the a_i are distinct nonzero values of f, we see that $0 \leq f$ iff each $a_i \geq 0$. Therefore, if $0 \leq f$, then $0 \leq' f$. Conversely, let $0 \leq' f$ and let $f = \sum a_i \chi_{U_i}$ as above. Note that $f\chi_{U_j} = a_j \chi_{U_j}$ for each j. Since $0 \leq' f, \chi_{U_j}$, we have $0 \leq' f\chi_{U_j}$, so $0 \leq' a_j \chi_{U_j}$. As $0 \neq \chi_{U_j}$, if $a_j < 0$, then $-a_j > 0$, and since S is an f-algebra, $0 \leq' (-a_j)\chi_{U_j}$. Therefore, $a_j \chi_{U_j} \leq' 0$. This implies $a_j \chi_{U_j} = 0$, which is impossible since the χ_{U_j} are faithful idempotents and $a_j \neq 0$. Thus, $a_j \geq 0$ for each j. Consequently, $0 \leq f$, and so $\leq'$ is equal to $\leq$. □

Remark 5.2. Ribenboim [15, Thm. 5] shows that when B is a Boolean algebra, the order on $\mathbb{Z}$ lifts to the Boolean power of $\mathbb{Z}$ by B in such a way that the resulting abelian group is an ℓ-group. His approach is through Foster's version of Boolean powers (see Remark 2.9), while Theorem 5.1 recovers his result via Jónsson's interpretation of Boolean powers. In this sense, our proof is similar in spirit to Conrad's point of view of Specker ℓ-groups, which emphasizes the fact that such an ℓ-group can be viewed as a subdirect product of copies of $\mathbb{Z}$, and hence inherits the order from this product; see [8, Sec. 4].

Let R be totally ordered and let S and T be ℓ-algebras over R. We recall that an *ℓ-algebra homomorphism* $\alpha\colon S \to T$ is an R-algebra homomorphism that is in addition a lattice homomorphism. The following corollary allows us to conclude that when R is totally ordered, an R-algebra homomorphism between Specker R-algebras is automatically an ℓ-algebra homomorphism. Thus, the category of Specker R-algebras and ℓ-algebra homomorphisms is a full subcategory of the category of commutative R-algebras and R-algebra homomorphisms. The corollary is motivated by a similar result for rings of real-valued continuous functions [12, Thm. 1.6], and its proof is a modification of the proof of that result.

Corollary 5.3. *If $S, T \in \mathbf{Sp}_R$, then each R-algebra homomorphism $\alpha\colon S \to T$ is an ℓ-algebra homomorphism.*

Proof. Identifying S with $C(X, R_{\text{disc}})$ and using $f \geq 0$ iff $f(x) \geq 0$ for all $x \in X$, we see that the unique partial order on S has positive cone

$$\left\{\sum a_i e_i : a_i \geq 0, e_i \in \operatorname{Id}(S)\right\}.$$

From the description of the positive cone it follows that α is order-preserving. Let $s \in S$. We recall that the ℓ-ring S has an absolute value. Since $S = C(X, R_{\text{disc}})$, we can define it explicitly as $|s|(x) = |s(x)|$ for each $x \in X$. Then $|s|^2 = s^2$ and $\alpha(|s|)^2 = \alpha(|s|^2) = \alpha(s^2) = \alpha(s)^2$. Therefore, as $\alpha(|s|) \geq 0$ and an element of an ℓ-ring has at most one positive square root, $\alpha(|s|) = |\alpha(s)|$ (see, e.g., [6, Ch. XVII]). Because of the ℓ-ring formula

$$2(a \vee b) = a + b + |a - b|,$$

we have $\alpha(2(a \vee b)) = \alpha(a) + \alpha(b) + |\alpha(a) - \alpha(b)|$. Consequently, $2\alpha(a \vee b) = 2(\alpha(a) \vee \alpha(b))$. Since each nonzero element of an ℓ-group has infinite order (see, e.g., [6, Ch. XIII, Cor. 3.1]), $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$. Thus, α is an ℓ-algebra homomorphism. $\square$

We conclude this article with a few comments on the interpretation of the Boolean power representation of a Specker R-algebra when R is a totally ordered ring. As follows from Theorem 2.7, Specker R-algebras are represented as $X^* = C(X, R_{\text{disc}})$, where X is a Stone space and R_{disc} is viewed as a discrete space. Since R is totally ordered, we can equip R with the interval topology. Sometimes this interval topology is discrete, e.g., when $R = \mathbb{Z}$, but often it is not, e.g., when $R = \mathbb{R}$. In this situation, there is another natural object to study, namely the algebra $C(X, R)$ of continuous functions from a Stone space X to R, where R has the interval topology. As the discrete topology is finer than the interval topology, we have that $C(X, R_{\text{disc}})$ is an R-subalgebra of $C(X, R)$. Often $C(X, R_{\text{disc}})$ is a proper R-subalgebra of $C(X, R)$. For example, if X is the one-point compactification of the positive integers, then $f\colon X \to \mathbb{R}$ given by $f(n) = 1/n$ and $f(\infty) = 0$ is in $C(X, \mathbb{R}) - C(X, \mathbb{R}_{\text{disc}})$. Let $FC(X, R)$ be the set of finitely-valued continuous functions from X to R. It is obvious that $FC(X, R)$ is an R-subalgebra of $C(X, R)$.

Proposition 5.4. $C(X, R_{\text{disc}}) = FC(X, R)$.

Proof. Let $f \in C(X, R_{\text{disc}})$. Since $C(X, R_{\text{disc}})$ is a Specker R-algebra, f is finitely valued. Let $\{a_1, \ldots, a_n\}$ be the values of f; $\{f^{-1}(a_1), \ldots, f^{-1}(a_n)\}$ is then a partition of R. As the interval topology is Hausdorff, points of X are closed, so each $f^{-1}(a_i)$ is closed. Thus, $\{f^{-1}(a_1), \ldots, f^{-1}(a_n)\}$ is a partition of R into finitely many closed sets. This implies that each $f^{-1}(a_i)$ is clopen. Therefore, $f \in FC(X, R)$. Conversely, if $f \in FC(X, R)$, then f is a finitely-valued function in $C(X, R)$. Using again that points of X are closed, we conclude that $f \in C(X, R_{\text{disc}})$. Thus, $FC(X, R) = C(X, R_{\text{disc}})$. □

Remark 5.5. One way to think about $FC(X, R)$ is as piecewise constant continuous functions from X to R. We recall (see, e.g., [5, Example 2.4(3)]) that a continuous function $f : X \to R$ is piecewise constant if there exist a clopen partition $\{P_1, \ldots, P_n\}$ of X and $a_i \in R$ such that $f(x) = a_i$ for each $x \in P_i$. Let $PC(X, R)$ be the subset of $C(X, R)$ consisting of piecewise constant functions. Then it is obvious that $PC(X, R)$ is an R-subalgebra of $C(X, R)$, and it follows from the definitions of $FC(X, R)$ and $PC(X, R)$ that $FC(X, R) = PC(X, R)$. By Proposition 5.4, $PC(X, R) = C(X, R_{\text{disc}})$. Thus, when R is totally ordered, another way to think about the Boolean power of R by B is as the R-algebra of piecewise constant continuous functions from the Stone space X of B to R, where R has the interval topology. Consequently, for a totally ordered ring R, we obtain the following two representations of a Specker R-algebra: as the R-algebra $C(X, R_{\text{disc}})$ or as the R-algebra $PC(X, R)$. In [5, Thm. 6.2], it is proved that a Specker $\mathbb{R}$-algebra is isomorphic to $PC(X, \mathbb{R})$. As follows from the discussion above, this result is a particular case of Theorem 2.7.

In the following work [4], we use the ideas developed in the present article to generalize the Boolean power construction to the setting of compact Hausdorff spaces.

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