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A note on strongly Jónsson binary relational structures

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Abstract. Let X be a set and let R be a binary relation on X. A subset L of X is said to be a *lower set of* $\mathbf{X} := (X, R)$ provided whenever $x \in L$ and $y \in X$ with yRx, then also $y \in L$. In this note, we study binary relational structures $\mathbf{X}$ with the property that distinct lower sets of $\mathbf{X}$ have distinct cardinalities.

1. Introduction

We begin by recalling the following famous problem, which has spawned a massive literature over the years:

Problem 1.1 (Jónsson's Problem)**.** For which infinite cardinals κ does there exist an algebra $\mathbf{A}$ of size κ, with but finitely many operations (of finite arity), for which every proper subuniverse of $\mathbf{A}$ has cardinality less than κ?

Infinite algebras satisfying the above condition are known as *Jónsson algebras*. In the modern era, the theory of Jónsson algebras has proved to be a useful tool in the investigation of large cardinals. We refer the reader to [1] and [2] for a survey of the classical results on these algebras.

The canonical interpretation of the Jónsson property for posets already appears in the literature. In [3], a poset $\mathbf{P} := (P, \preceq)$ is called a *Jónsson poset* if every proper order ideal of $\mathbf{P}$ has cardinality strictly less than $|P|$. Jónsson posets were utilized therein to obtain new results on unary Jónsson algebras.

Now consider strengthening the Jónsson property but dispensing with the assumption that the binary relation is a partial order. To wit, let $\mathbf{X} := (X, R)$ be a binary relational structure, and recall that a subset $L \subseteq X$ is a *lower set of* $\mathbf{X}$ or an *R-lower subset of* X provided for all $x \in L$ and $y \in X$: if yRx, then $y \in L$. Say that $\mathbf{X}$ is *strongly Jónsson* provided distinct lower sets of $\mathbf{X}$ have distinct cardinalities.

An analogous 'strongly Jónsson' property for modules over a commutative ring was recently studied by the author. In particular, a structure theorem for such modules was presented in [4, Theorem 1]. The purpose of this note is to port over Theorem 1 to the binary relational universe; Theorem 2.7 is our principle result.

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2. Main results

We begin with comments on notation and terminology that will be used throughout the paper. The symbol $\leq$ will exclusively refer to the usual inclusive epsilon order on the class *Ord* of ordinal numbers; we denote the subclass of cardinal numbers by *Card*. If R is a binary relation on a set X and $R^{\neq} := R - \{(x,x) : x \in X\}$, then $\mathbf{G} := (X, R^{\neq})$ is a directed loopless graph. Recall that a directed graph $\mathbf{G}$ is *acyclic* provided $\mathbf{G}$ has no cycles. We will make use of the following well-known fact:

Fact 2.1. Let R be a binary relation on a set X. Then R admits a linear extension if and only if $(X, R^{\neq})$ is acyclic.

Sketch of Proof. Necessity is obvious. To prove sufficiency, note that the reflexive transitive closure $\overline{R}$ of R is both reflexive and transitive. If $(X, R^{\neq})$ is acyclic, it is easy to see that $\overline{R}$ is also antisymmetric. Thus $(X, \overline{R})$ is a poset, and it is well-known that every poset has a linear extension. □

We will soon show that there is a large class $\mathcal{C}$ of binary structures such that for all $\mathbf{X} \in \mathcal{C}$, if $\mathbf{X}$ is strongly Jónsson, then $\mathbf{X}$ is isomorphic to a suborder of the inclusive epsilon order on a countable cardinal. First, we have a lemma.

Lemma 2.2. *Let $\alpha \leq \omega$ be an ordinal and suppose that $\leq$ is an extension of a binary relation $\preceq$ on α. If $\mathbf{X} := (\alpha, \preceq)$ is strongly Jónsson, then $n \preceq n+1$ for all $n \in \omega$ such that $n+1 < \alpha$.*

Proof. Assume that $\mathbf{X}$ is strongly Jónsson with $\leq$ an extension of $\preceq$, and let $n \in \omega$ be arbitrary. Further, suppose that $n+1 < \alpha$. If $n = 0$ and $0 \npreceq 1$, then $\{0\}$ and $\{1\}$ are distinct lower sets of $\mathbf{X}$ of the same size, a contradiction. Assume now that $n \neq 0$. Since $\mathbf{X}$ is strongly Jónsson and $\{0, 1, \ldots, n\}$ is a $\preceq$-lower set of $\mathbf{X}$ of size $n+1$, it follows that $\{0, \ldots, n-1, n+1\}$ is *not* a lower set of $\mathbf{X}$. Thus $n \preceq n+1$, and the proof of the lemma is complete. □

Proposition 2.3. *Let $\mathbf{X} := (X, R)$ be a structure such that $\mathbf{G} := (X, R^{\neq})$ is an acyclic digraph. Then $\mathbf{X}$ is strongly Jónsson if and only if there exists an ordinal $\alpha \leq \omega$ and a relation $\preceq$ on α such that $\mathbf{X} \cong (\alpha, \preceq)$, and both of the following hold:*

(1) *$\leq$ is an extension of $\preceq$ on α.*
(2) *$n \preceq n+1$ for all n such that $n+1 < \alpha$.*

Proof. We prove only the nontrivial direction. Thus, assume that $\mathbf{X}$ is strongly Jónsson and $\mathbf{G}$ is acyclic. Now let $\preceq^*$ be a linear extension of R (which exists by Fact 2.1). Then clearly, $(X, \preceq^*)$ is strongly Jónsson as well. We claim that $\preceq^*$ is a well-order on X. Otherwise, there exists an infinite strictly decreasing sequence $\cdots \prec^* x_3 \prec^* x_2 \prec^* x_1 \prec^* x_0$ in X. Let C be the complement of the $\preceq^*$–principal order filter $[x_0)$. Then C and $C \cup \{x_0\}$ are distinct $\preceq^*$-lower subsets of X of the same cardinality, a contradiction. Since $\preceq^*$ well orders X

and $(X, \preceq^*)$ is strongly Jónsson, it is easy to see that $(X, \preceq^*) \cong (\alpha, \leq)$ for some ordinal $\alpha \leq \omega$. Let $\varphi\colon X \to \alpha$ be an isomorphism, and set

$$\preceq := \{(\varphi(x), \varphi(y)) : (x, y) \in R\}.$$

Then clearly, (1) above holds; Lemma 2.2 implies (2). □

Corollary 2.4. *Let* $\mathbf{P} := (P, \preceq)$ *be a poset. Then* $\mathbf{P}$ *is strongly Jónsson if and only if* $\mathbf{P} \cong (\alpha, \leq)$ *for some ordinal* $\alpha \leq \omega$.

More generally, we now consider the problem of classifying the strongly Jónsson binary relational structures. Toward this end, we will require a characterization of the countable Jónsson posets. Recall from the introduction that a poset $\mathbf{P} := (P, \preceq)$ is Jónsson provided every proper lower set of $\mathbf{P}$ has cardinality less than $|P|$.

Lemma 2.5. *Let* $\mathbf{P} := (P, \preceq)$ *be a countable poset. Then* $\mathbf{P}$ *is Jónsson if and only if* $(P, \preceq^*) \cong (|P|, \leq)$ *for every linear extension* $\preceq^*$ *of* $\preceq$.

Proof. If $\mathbf{P}$ is Jónsson, then the proof that every linear extension is isomorphic to $(|P|, \leq)$ is analogous to the proof of Proposition 2.3. Conversely, suppose that $\mathbf{P}$ is not Jónsson. Then P is infinite; further, $[p_0)^c$ is infinite for some $p_0 \in P$. Now set $R := \preceq \cup \{(x, p_0) : x \in [p_0)^c\}$. We claim that $\mathbf{G} := (P, R^{\neq})$ is acyclic (this is known; we include the short proof). Suppose by way of contradiction that $x_0, \ldots, x_n$ are distinct and

$$C := \{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)\} \subseteq R^{\neq} \text{ for some } n > 0. \quad (2.1)$$

Since $\preceq$ is a partial order, it follows that $C \nsubseteq \preceq$. Without loss of generality, we may assume that $(x_0, x_1) = (x, p_0)$ for some $x \notin [p_0)$. One shows by induction that $p_0 \preceq x_i$ for all i with $1 \leq i \leq n$. In particular, $p_0 \preceq x_n$. But then $(x_n, x_0) \notin \{(x, p_0) : x \in [p_0)^c\}$. It follows that $x_n \preceq x_0$, and $x = x_0 \in [p_0)$, a contradiction. Fact 2.1 now implies that there exists a linear extension $\preceq^*$ of R (hence also of $\preceq$). Then the set $\{p \in P : p \preceq^* p_0\}$ is infinite, and thus $(P, \preceq^*)$ is not isomorphic to $(|P|, \leq)$. □

We now translate the classification problem for strongly Jónsson structures to an analogous problem for weighted posets. Note that to every structure $\mathbf{X} := (X, R)$, we may associate a weighted poset as follows. Let $\overline{R}$ be the reflexive transitive closure of R. Now define $\sim$ on X by $x \sim y$ if and only if $x\overline{R}y$ and $y\overline{R}x$. Setting $P := X/{\sim}$ and $\preceq := R/{\sim}$, it is well-known that $\mathbf{P} := (P, \preceq)$ is a poset. Define a weight function w on P by $w([p]) := |[p]|$.

Say that a weighted poset $\mathbf{P}_w := (P, \preceq, w)$ with $w\colon P \to \mathit{Card} - \{0\}$ is a *strongly Jónsson weighted poset* if distinct lower sets of $\mathbf{P}$ have distinct total weights (the *total weight* of a subset X of P is $w(X) := \sum_{x \in X} w(x)$). It is straightforward to verify that $\mathbf{X}$ is strongly Jónsson if and only if $\mathbf{P}_w$ is a strongly Jónsson weighted poset. Thus, the classification problem has been reduced to classifying the strongly Jónsson weighted posets. We give an example of such a structure and then we present the main result of the paper.

Example 2.6. Let a be an ordinal, $P := (\omega - \{0\}) \cup \{\aleph_{i+1} : i < a\}$, and w the identity on P. Then $(P, \leq, w)$ is a strongly Jónsson weighted poset.

Theorem 2.7. *Let* $\mathbf{P}_w := (P, \preceq, w)$ *be a weighted poset such that* $w(p) \in$ *Card* $- \{0\}$ *for all* $p \in P$. *Then* $\mathbf{P}_w$ *is a strongly Jónsson weighted poset if and only if* $\mathbf{P}_w = (\mathbf{P}_0)_w \oplus (\mathbf{P}_1)_w$ *for some weighted posets* $(\mathbf{P}_0)_w$ *and* $(\mathbf{P}_1)_w$ *such that the following hold:*

(1) $w(p) < \aleph_0$ *for every* $p \in P_0$.
(2) $w(p) \geq \aleph_0$ *for every* $p \in P_1$.
(3) $(\mathbf{P}_0)_w$ *is strongly Jónsson.*
(4) P_0 *is countable.*
(5) $\mathbf{P}_0$ *is Jónsson.*
(6) $(w(P_0))^+ + \aleph_0 \leq w(p)$ *for every* $p \in P_1$.
(7) $\mathbf{P}_1$ *is isomorphic to an ordinal. Moreover, for every* $p \in P_1$, *we have* $\sum\{w(x) : x \in P_1, x \prec p\} < w(p)$.

Proof. If $\mathbf{P}_w$ decomposes into an ordinal sum of weighted posets $(\mathbf{P}_0)_w$ and $(\mathbf{P}_1)_w$ that satisfy (1)–(7), then it is straightforward to check that $\mathbf{P}_w$ is a strongly Jónsson weighted poset.

Conversely, suppose that $\mathbf{P}_w$ is a strongly Jónsson weighted poset. We first establish that

$$\text{for all } p \in P : w([p)^c) < w(p) + \aleph_0. \tag{2.2}$$

If not, then for some $p \in P$, $[p)^c$ and $[p)^c \cup \{p\}$ are distinct lower sets of $\mathbf{P}$ of the same weight, a contradiction.

Now set $P_0 := \{p \in P : w(p) < \aleph_0\}$ and $P_1 := P_0^c$. Then (1) and (2) are clear. Moreover, if $x \in P_0$ and $y \in P_1$, then by (2.2), $y \notin [x)^c$. Thus, $x \leq y$, and we see that $\mathbf{P}_w = (\mathbf{P}_0)_w \oplus (\mathbf{P}_1)_w$. It follows that every lower set of $\mathbf{P}_0$ is also a lower set of $\mathbf{P}$, whence $\mathbf{P}_0 := (P_0, \preceq, w)$ is a strongly Jónsson weighted poset. This proves (3).

We now show that P_0 is countable. Toward this end, let $\preceq^*$ be a linear extension of $\preceq$. Then one shows by contradiction, as in the proof of Proposition 2.3, that $\preceq^*$ well orders P_0 (the same argument goes through, but the contradiction is the existence of distinct lower sets of $\mathbf{P}_0$ with the same total weight). This in turn implies via an analogous argument that $(P_0, \preceq^*)$ is isomorphic to $(\alpha, \leq)$ for some ordinal $\alpha \leq \omega$. We have proved (4). Invoking Lemma 2.5, (5) follows.

Lastly, we prove (6) and (7). For (6), it suffices to show that $w(P_0) < w(p)$ for all $p \in P_1$. Let $p \in P_1$ be arbitrary. Since $\mathbf{P} = \mathbf{P}_0 \dotplus \mathbf{P}_1$, we see that $P_0 \subseteq [p)^c$. Hence by (2.2), we have $w(P_0) \leq w([p)^c) < w(p) + \aleph_0 = w(p)$. As for (7), let $a, b \in P_1$ be arbitrary. If a and b are incomparable, then $a \in [b)^c$ and $b \in [a)^c$. Thus by (2.2), $w(a) < w(b) < w(a)$, which is absurd. We apply (2.2) yet again to conclude that $a \prec b$ if and only if $w(a) < w(b)$, whence $\preceq$ is a well-order on P_1. Now let $p \in P_1$. That $\sum\{w(x) : x \in P_1, x \prec p\} < w(p)$ follows immediately from (2.2). This concludes the proof. $\square$

Remark 2.8. Though $(\mathbf{P}_1)_w$ is a strongly Jónsson weighted poset, observe from Example 2.6 that $\mathbf{P}_1$ can be (isomorphic to) *any* ordinal. Thus. $\mathbf{P}_1$ need not be a Jónsson poset.

Several additional remarks are now in order. First, $\mathbf{P}_0$ is determined up to isomorphism by Lemma 2.5 and (5) of Theorem 2.7. Further, the weighted poset $(\mathbf{P}_1)_w$ is determined up to isomorphism. It remains to classify the weight function w on P_0. We address this problem with an example. Let $0 < k < \omega$, and suppose that $X := \{x_i : i < k\}$ is a set of k positive integers such that for all $Y, Z \in \mathcal{P}(X)$: if $Y \neq Z$, then $\sum_{y \in Y} Y \neq \sum_{z \in Z} Z$ (for example, $X := \{2^i : i < k\}$). Define w on $P_0 := k$ by $w(i) := x_i$ for all $i < k$. Then $(\mathbf{P}_0)_w := (P_0, =, w)$ is a countable strongly Jónsson weighted poset. Therefore, the problem of classifying the weight functions on countable strongly Jónsson weighted posets (with weights in $\omega - \{0\}$) is at least as hard as determining all k-element sets of positive integers for which distinct subsets have distinct sums. The latter is a well-known combinatorial problem (the so-called *distinct subset-sum problem*) for which no complete solution is known. Thus, it is likely that the classification given in Theorem 2.7 cannot be considerably strengthened.

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