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Lattice subordinations and Priestley duality

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Abstract. There is a well-known correspondence between Heyting algebras and **S4**-algebras. Our aim is to extend this correspondence to distributive lattices by defining analogues of **S4**-algebras for them. For this purpose, we introduce binary relations on Boolean algebras that resemble de Vries proximities. We term such binary relations lattice subordinations. We show that the correspondence between Heyting algebras and **S4**-algebras extends naturally to distributive lattices and Boolean algebras with a lattice subordination. We also introduce Heyting lattice subordinations and prove that the category of Boolean algebras with a Heyting lattice subordination is isomorphic to the category of **S4**-algebras, thus obtaining the correspondence between Heyting algebras and **S4**-algebras as a particular case of our approach.

In addition, we provide a uniform approach to dualities for these classes of algebras. Namely, we generalize Priestley spaces to quasi-ordered Priestley spaces and show that lattice subordinations on a Boolean algebra B correspond to Priestley quasi-orders on the Stone space of B. This results in a duality between the category of Boolean algebras with a lattice subordination and the category of quasi-ordered Priestley spaces that restricts to Priestley duality for distributive lattices. We also prove that Heyting lattice subordinations on B correspond to Esakia quasi-orders on the Stone space of B. This yields Esakia duality for **S4**-algebras, which restricts to Esakia duality for Heyting algebras.

1. Introduction

A Priestley space is a partially ordered Stone space $(X, \leq)$ in which, whenever $x \not\leq y$, there is a clopen up-set U containing x and missing y. The well-known Priestley duality establishes that the category of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order-preserving maps. An Esakia space is a Priestley space in which the down-set of each clopen set is clopen. The well-known Esakia duality provides a dual equivalence between the category of Heyting algebras with Heyting homomorphisms and the category of Esakia spaces with continuous bounded morphisms (order-preserving maps for which $f(x) \leq y$ implies that there exists z with $x \leq z$ and $f(z) = y$). These landmark theorems were established by Priestley [16, 17] and Esakia [9], respectively.

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A standard proof of Priestley duality exploits the prime spectrum functor and the clopen up-set functor. The prime spectrum functor associates with each bounded distributive lattice D the prime filters of D ordered by inclusion and topologized by the patch topology of the Stone topology on the prime filters of D. The clopen up-set functor associates with each Priestley space X the bounded distributive lattice of clopen up-sets of X. The restriction of these functors also yields Esakia duality. Namely, if D is a Heyting algebra, then the prime spectrum of D is an Esakia space, and if X is an Esakia space, then the clopen up-sets of X form a Heyting algebra. However, Esakia's original proof of his duality was different. He approached things from the point of view of modal logic.

The celebrated Stone duality [20] between Boolean algebras and Stone spaces was generalized by Halmos [13] to **S5**-algebras (monadic algebras in Halmos' terminology) and special equivalence relations on Stone spaces. **S4**-algebras (also known as interior algebras [5], closure algebras [14], or topological Boolean algebras [18]) generalize **S5**-algebras. It was known from the work of McKinsey and Tarski [15] (see also [18]) that there is a close correspondence between **S4**-algebras and Heyting algebras. This correspondence is at the heart of Gödel's translation [12] of intuitionistic logic into **S4**. Esakia generalized Halmos duality for **S5**-algebras to **S4**-algebras. The objects of the resulting dual category are special quasi-ordered Stone spaces. Esakia studied the **S4**-algebras that correspond to those quasi-ordered Stone spaces where the quasi-order is a partial order (stencil **S4**-algebras in Esakia's terminology), and showed that they are equivalent to the category of Heyting algebras. Esakia duality for Heyting algebras follows.

More precisely, **S4**-algebras $(B, \Box)$ can be thought of as pairs (B, D), where B is a Boolean algebra, D is a bounded sublattice of B (that is, containing the 0 and 1 of B), and the inclusion $D \hookrightarrow B$ has a right adjoint [6, 10, 11]. This right adjoint is responsible for the existence of $\Box\colon B \to B$ such that $(B, \Box)$ is an **S4**-algebra. It is also this right adjoint that turns the bounded distributive lattice D into a Heyting algebra. It follows that there is a functor from **S4**-algebras to Heyting algebras (that sends each **S4**-algebra (B, D) to D). This functor has a left adjoint (which sends each Heyting algebra to its free Boolean extension). This yields an equivalence between the category of Heyting algebras and the category of the **S4**-algebras that are generated by D (the stencil **S4**-algebras). The dual spaces of stencil **S4**-algebras turn out to be exactly the Esakia spaces, so Heyting algebras dually correspond to Esakia spaces, and Esakia duality follows.

If the inclusion $D \hookrightarrow B$ does not have a right adjoint, then D is not a Heyting algebra. This indicates an alternate route to Priestley duality for bounded distributive lattices [11]. Consider the category consisting of the pairs (B, D) where B is a Boolean algebra and D is a bounded sublattice of B. Its full subcategory consisting of the pairs (B, D) where B is generated by D is equivalent to the category of bounded distributive lattices. Moreover, the dual space of (B, D) is a Priestley space iff B is generated by D, and Priestley duality follows.

Of course, the pairs (B, D) where D happens to be a Heyting algebra can be described by means of **S4**-algebras $(B, \Box)$: D consists of the fixed points of $\Box$. This can no longer be done if D is merely a bounded distributive lattice. The aim of this paper is to show that the pairs (B, D) where D is a bounded sublattice of B can nevertheless be described by means of special binary relations on B that resemble de Vries proximities on B. We term these binary relations lattice subordinations. We prove that the correspondence between **S4**-algebras and Heyting algebras extends naturally to the correspondence between Boolean algebras with a lattice subordination and bounded distributive lattices. We also introduce Heyting lattice subordinations and Boolean lattice subordinations. These are lattice subordinations that satisfy additional conditions. We prove that the category of Boolean algebras with a Heyting lattice subordination is isomorphic to the category of **S4**-algebras and that if the subordination is in addition Boolean, then the resulting category is isomorphic to the category of **S5**-algebras. Thus, Boolean algebras with a lattice subordination can be viewed as analogues of **S4**-algebras for bounded distributive lattices.

We also provide a uniform framework for presenting dualities for these classes of algebras. We generalize Priestley spaces to quasi-ordered Priestley spaces and show that quasi-ordered Priestley spaces dually correspond to Boolean algebras with a lattice subordination. We also describe the Boolean algebras with a lattice subordination that correspond to Priestley spaces. Priestley duality follows. We prove that quasi-ordered Esakia spaces dually correspond to Boolean algebras with a Heyting lattice subordination, and we describe the Boolean algebras with a Heyting lattice subordination that correspond to Esakia spaces. Esakia duality for both **S4**-algebras and Heyting algebras as well as Halmos duality for **S5**-algebras follow.

The paper is organized as follows. In Section 2 we introduce lattice subordinations on Boolean algebras and establish their basic properties. We also compare lattice subordinations to de Vries proximities on Boolean algebras. In Section 3 we introduce Boolean lattice subordinations and establish their basic properties. In Section 4 we introduce Heyting lattice subordinations and establish their basic properties. We prove that the category of Boolean algebras with a Heyting lattice subordination is isomorphic to the category of **S4**-algebras, and that the category of Boolean algebras with a Heyting lattice subordination that in addition is a Boolean lattice subordination is isomorphic to the category of **S5**-algebras. In Section 5 we introduce quasi-ordered Priestley spaces and quasi-ordered Esakia spaces, and prove our main duality results. Esakia duality for **S4**-algebras and Halmos duality for **S5**-algebras follow. Finally, in Section 6 we show how our results produce Priestley duality for bounded distributive lattices and Esakia duality for Heyting algebras.

2. Lattice subordinations

We begin by introducing the central concept of the paper.

Definition 2.1. Let B be a Boolean algebra. We call a binary relation $\prec$ on B a *lattice subordination* if $\prec$ satisfies the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$.
(S2) $a \prec b, c$ implies $a \prec b \wedge c$.
(S3) $a, b \prec c$ implies $a \vee b \prec c$.
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.
(S5) $a \prec b$ implies that there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$.

We next collect some basic properties of lattice subordinations.

Lemma 2.2. *Let $\prec$ be a lattice subordination on a Boolean algebra B.*

(1) *$a \prec b$ implies $a \leq b$.*
(2) *$a \prec b \leq c$ implies $a \prec c$, and $a \leq b \prec c$ implies $a \prec c$.*
(3) *$0 \prec a$ and $a \prec 1$ for each $a \in B$.*
(4) *$a \prec b$ and $c \prec d$ imply $a \wedge c \prec b \wedge d$ and $a \vee c \prec b \vee d$.*
(5) *$a \prec b$ implies that there exists $c \in B$ with $a \prec c \prec b$.*
(6) *$a \prec c \prec b$ implies $a \prec b$.*
(7) *$a \prec b$ iff there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$.*

Proof. (1): If $a \prec b$, then by (S5), there exists $c \prec c$ with $a \leq c \leq b$. Therefore, $a \leq b$.

(2): This follows from (S4).

(3): Let $a \in B$. By (S1), $0 \prec 0 \leq a$ and $a \leq 1 \prec 1$. Therefore, by (2), $0 \prec a \prec 1$.

(4): We have $a \wedge c \leq a \prec b$. So, by (2), $a \wedge c \prec b$. Similarly, $a \wedge c \prec d$. Therefore, by (S2), $a \wedge c \prec b \wedge d$. A similar argument (that uses (S3) instead) gives $a \vee c \prec b \vee d$.

(5): If $a \prec b$, then by (S5), there exists $c \prec c$ with $a \leq c \leq b$. As $a \leq c \prec c$, by (2) we have $a \prec c$. That $c \prec b$ is similar.

(6): Let $a \prec c \prec b$. By (1), $a \leq c \prec b$, and by (2), $a \prec b$.

(7): One implication is (S5). For the other implication, if we have $c \prec c$ and $a \leq c \leq b$, then by (2), $a \prec c \prec b$. Now apply (6). □

Definition 2.3. For a lattice subordination $\prec$ on a Boolean algebra B, let $D_{\prec} = \{a \in B : a \prec a\}$ be the set of *reflexive* elements of $\prec$.

Lemma 2.4. *Let $\prec$ be a lattice subordination on a Boolean algebra B. Then $D_{\prec}$ is a bounded sublattice of B.*

Proof. By (S1), $0, 1 \in D_{\prec}$. Let $a, b \in D_{\prec}$. Then $a \prec a$ and $b \prec b$. By Lemma 2.2(4), $a \wedge b \prec a \wedge b$ and $a \vee b \prec a \vee b$. Therefore, $a \wedge b, a \vee b \in D_{\prec}$, and so $D_{\prec}$ is a bounded sublattice of B. □

Definition 2.5. For a bounded sublattice D of a Boolean algebra B, define $\prec_D$ on B by setting $a \prec_D b$ iff there exists $c \in D$ with $a \le c \le b$.

Lemma 2.6. *Let D be a bounded sublattice of a Boolean algebra B. Then $\prec_D$ is a lattice subordination on B.*

Proof. As $0, 1 \in D$, we have that $\prec_D$ satisfies (S1). Let $a \prec_D b, c$. Then there exist $x, y \in D$ such that $a \le x \le b$ and $a \le y \le c$. Therefore, $a \le x \wedge y \le b \wedge c$. As $x \wedge y \in D$, we conclude that $a \prec_D b \wedge c$, so $\prec_D$ satisfies (S2). That $\prec_D$ satisfies (S3) is similar and uses the fact that $x, y \in D$ imply $x \vee y \in D$. That $\prec_D$ satisfies (S4) is obvious. As D is the set of reflexive elements of $\prec_D$, it is immediate that $\prec_D$ satisfies (S5). □

Lemma 2.7. *Let B be a Boolean algebra.*

(1) *If $\prec$ is a lattice subordination on B, then $\prec\, = \,\prec_{D_\prec}$.*
(2) *If D is a bounded sublattice of B, then $D = D_{\prec_D}$.*

Proof. (1) For $a, b \in B$, we have:

$$\begin{aligned} a \prec_{D_\prec} b &\iff \exists c \in D_\prec : a \le c \le b \\ &\iff \exists c \in B : c \prec c \ \&\ a \le c \le b \iff a \prec b. \end{aligned}$$

Here the last equivalence follows from Lemma 2.2(7). Thus, $\prec\, = \,\prec_{D_\prec}$.

(2) Let $a \in B$. Then

$$a \in D_{\prec_D} \iff a \prec_D a \iff \exists c \in D : a \le c \le a \iff a \in D.$$

Thus, $D = D_{\prec_D}$. □

This establishes a 1–1 correspondence between lattice subordinations on B and bounded sublattices of B. We extend this to an isomorphism of appropriate categories.

Definition 2.8. (1) Let BLS be the category whose objects are pairs $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a lattice subordination on B, and whose morphisms are Boolean homomorphisms $h\colon B_1 \to B_2$ that satisfy $a \prec_1 b \Rightarrow h(a) \prec_2 h(b)$; that is, Boolean homomorphisms preserving lattice subordination. It is straightforward that BLS is a category where composition of two morphisms is the usual function composition.

(2) Let BDA be the category whose objects are pairs (B, D), where B is a Boolean algebra and D is a bounded sublattice of B, and whose morphisms are Boolean homomorphisms $h\colon B_1 \to B_2$ satisfying $a \in D_1 \Rightarrow h(a) \in D_2$. It is straightforward that BDA is a category where composition of two morphisms is the usual function composition.

Remark 2.9. If $h\colon B_1 \to B_2$ is a morphism in BDA, then the restriction of h to D_1 is a bounded lattice homomorphism from D_1 to D_2.

Theorem 2.10. BLS *is isomorphic to* BDA.

Proof. First, we define a functor $\Phi\colon \mathsf{BLS} \to \mathsf{BDA}$ as follows. For $(B, \prec) \in \mathsf{BLS}$, let $\Phi(B, \prec) = (B, D_\prec)$ and for $h \in \hom_{\mathsf{BLS}}((B_1, \prec_1), (B_2, \prec_2))$, let $\Phi(h) = h$. By Lemma 2.4, $\Phi(B, \prec) \in \mathsf{BDA}$. Suppose $h \in \hom_{\mathsf{BLS}}((B_1, \prec_1), (B_2, \prec_2))$. Then h is a Boolean homomorphism. Let $a \in D_{\prec_1}$. This implies $a \prec_1 a$. As h preserves lattice subordination, $h(a) \prec_2 h(a)$. Thus, $h(a) \in D_{\prec_2}$, and so $\Phi(h) = h \in \hom_{\mathsf{BDA}}((B_1, D_{\prec_1}), (B_2, D_{\prec_2}))$. It follows that Φ is a well-defined functor.

Next, we define a functor $\Psi\colon \mathsf{BDA} \to \mathsf{BLS}$ as follows. For $(B, D) \in \mathsf{BDA}$, let $\Psi(B, D) = (B, \prec_D)$ and for $h \in \hom_{\mathsf{BDA}}((B_1, D_1), (B_2, D_2))$, let $\Psi(h) = h$. By Lemma 2.6, $\Psi(B, D) \in \mathsf{BLS}$. Suppose that $h \in \hom_{\mathsf{BDA}}((B_1, D_1), (B_2, D_2))$. Then h is a Boolean homomorphism. Let $a, b \in B_1$ with $a \prec_{D_1} b$. This implies that there exists $c \in D_1$ with $a \le_1 c \le_1 b$. Thus, $h(a) \le_2 h(c) \le_2 h(b)$, and as $h(c) \in D_2$, this yields that $h(a) \prec_{D_2} h(b)$. From this we conclude that $\Psi(h) = h \in \hom_{\mathsf{BLS}}((B_1, \prec_{D_1}), (B_2, \prec_{D_2}))$, so Ψ is also a well-defined functor.

Let $(B, \prec) \in \mathsf{BLS}$. By Lemma 2.7(1), $\Psi\Phi(B, \prec) = \Psi(B, D_\prec) = (B, \prec_{D_\prec}) = (B, \prec)$. Let $(B, D) \in \mathsf{BDA}$. By Lemma 2.7(2), $\Phi\Psi(B, D) = \Phi(B, \prec_D) = (B, D_{\prec_D}) = (B, D)$. Thus, BLS is isomorphic to BDA. □

Remark 2.11. We conclude this section by comparing lattice subordinations to de Vries proximities. We recall [8] that a *de Vries proximity* on a Boolean algebra B is a binary relation $\prec$ on B satisfying the following axioms:

(DV1) $1 \prec 1$.
(DV2) $a \prec b$ implies $a \le b$.
(DV3) $a \le b \prec c \le d$ implies $a \prec d$.
(DV4) $a \prec b, c$ implies $a \prec b \wedge c$.
(DV5) $a \prec b$ implies $\neg b \prec \neg a$.
(DV6) $a \prec b$ implies that there exists $c \in B$ with $a \prec c \prec b$.
(DV7) $a \ne 0$ implies that there exists $b \ne 0$ with $b \prec a$.

A de Vries proximity $\prec$ is *zero-dimensional* [2] if in addition it satisfies the following strong form of (DV6):

(SDV6) $a \prec b$ implies that there exists $c \in B$ with $c \prec c$ and $a \prec c \prec b$.

It follows from Definition 2.1 and Lemma 2.2 that if $\prec$ is a lattice subordination on B, then $\prec$ satisfies all the axioms of a zero-dimensional de Vries proximity except (DV5) and (DV7). The following simple example shows that lattice subordinations do not always satisfy these two axioms. Let $B = \{0, a, \neg a, 1\}$ be the four-element Boolean algebra, let $D = \{0, a, 1\}$ be its bounded sublattice, and let $\prec_D$ be the corresponding lattice subordination. Then $a \prec_D a$, but $\neg a \not\prec_D \neg a$, so $\prec_D$ does not satisfy (DV5). Also, $\neg a \ne 0$, but there is no $b \ne 0$ in B with $b \prec_D \neg a$. So $\prec_D$ does not satisfy (DV7).

3. Boolean lattice subordinations

In this section, we introduce Boolean lattice subordinations and show that the category of Boolean algebras with a Boolean lattice subordination is isomorphic to the category of pairs (B, D), where D is a Boolean subalgebra of B.

Definition 3.1. Let $\prec$ be a lattice subordination on a Boolean algebra B. We call $\prec$ a *Boolean lattice subordination* if $a \prec b$ implies $\neg b \prec \neg a$.

Lemma 3.2. *Let $\prec$ be a lattice subordination on B. Then the following conditions are equivalent:*

(1) *$\prec$ is a Boolean lattice subordination.*
(2) *$a \in D_\prec$ implies $\neg a \in D_\prec$.*
(3) *$D_\prec$ is a Boolean subalgebra of B.*

Proof. (1)⇒(2): If $a \in D_\prec$, then $a \prec a$. As $\prec$ is a Boolean lattice subordination, this implies $\neg a \prec \neg a$. Therefore, $\neg a \in D_\prec$.

(2)⇒(3): $D_\prec$ is a bounded sublattice of B that is closed under $\neg$, so $D_\prec$ is a Boolean subalgebra of B.

(3)⇒(1): Let $a \prec b$. Then there exists $c \in D_\prec$ with $a \le c \le b$. Therefore, $\neg b \le \neg c \le \neg a$, and as $\neg c \in D_\prec$, we conclude that $\neg b \prec \neg a$. □

Lemma 3.3. *Let D be a bounded sublattice of B. Then D is a Boolean subalgebra of B iff $\prec_D$ is a Boolean lattice subordination on B.*

Proof. First suppose that D is a Boolean subalgebra of B. Let $a \prec_D b$. Then there exists $c \in D$ with $a \le c \le b$. Therefore, $\neg b \le \neg c \le \neg a$. As $\neg c \in D$, we obtain that $\neg b \prec_D \neg a$. Thus, $\prec_D$ is a Boolean lattice subordination on B. Next suppose that $\prec_D$ is a Boolean lattice subordination on B and $a \in D$. Then $a \prec_D a$. Therefore, $\neg a \prec_D \neg a$, which implies that $\neg a \in D$. Thus, D is a Boolean subalgebra of B. □

Definition 3.4.

(1) Let BLB be the full subcategory of BLS consisting of Boolean algebras with a Boolean lattice subordination.
(2) Let BBA be the full subcategory of BDA consisting of the pairs (B, D), where D is a Boolean subalgebra of B.

Remark 3.5. If $h\colon B_1 \to B_2$ is a morphism in BBA, then the restriction of h to D_1 is a Boolean algebra homomorphism from D_1 to D_2.

Theorem 3.6. BLB *is isomorphic to* BBA.

Proof. Apply Theorem 2.10, Lemma 3.2, and Lemma 3.3. □

Remark 3.7. Boolean lattice subordinations obviously satisfy (DV5). However, they still do not have to satisfy (DV7). Let $B = \{0, a, \neg a, 1\}$ be the four-element Boolean algebra, let $D = \{0, 1\}$ be its two-element Boolean subalgebra, and let $\prec_D$ be the corresponding Boolean lattice subordination. Then $a \neq 0$, but there is no $b \neq 0$ in B such that $b \prec_D a$. So $\prec_D$ does not satisfy (DV7).

4. Heyting lattice subordinations

In this section, we introduce Heyting lattice subordinations. We prove that the category of Boolean algebras with a Heyting lattice subordination is isomorphic to the category of **S4**-algebras, and that its full subcategory, where the subordination is also Boolean, is isomorphic to the category of **S5**-algebras.

Definition 4.1. Let $\prec$ be a lattice subordination on a Boolean algebra B. We call $\prec$ a *Heyting lattice subordination* if for each $a \in B$ the set $\{b \in B \colon b \prec a\}$ has a largest element, which we denote by $\Box a$.

Lemma 4.2. *Let $\prec$ be a Heyting lattice subordination on B. Then for each $a \in B$, we have $\Box a \prec \Box a$. Consequently, $a \prec b$ implies $\Box a \prec \Box b$.*

Proof. We clearly have $\Box a \prec a$. Therefore, there exists $c \in B$ with $c \prec c$ and $\Box a \leq c \leq a$. Thus, $\Box a \leq c$ and $c \prec a$, which implies $\Box a = c$. Consequently, $\Box a \prec \Box a$. Now suppose $a \prec b$. Then there exists $c \in B$ with $c \prec c$ and $a \leq c \leq b$. Therefore, $\Box c \prec \Box c$ and $\Box a \leq \Box c \leq \Box b$. Thus, $\Box a \prec \Box b$. $\Box$

Lemma 4.3. *Let $\prec$ be a Heyting lattice subordination on B; then $\Box \colon B \to D_\prec$ is a right adjoint to the inclusion $D_\prec \hookrightarrow B$.*

Proof. That $\Box \colon B \to D_\prec$ is well defined follows from Lemma 4.2. Let $x \in D_\prec$ and $y \in B$. As $\Box y \leq y$, if $x \leq \Box y$, then $x \leq y$. Conversely, if $x \leq y$, then $x \prec x \leq y$, so $x \prec y$, yielding $x \leq \Box y$. Thus, $x \leq y$ iff $x \leq \Box y$, which means that $\Box \colon B \to D_\prec$ is a right adjoint to the inclusion $D_\prec \hookrightarrow B$. $\Box$

Definition 4.4. ([10, Sec. II.5]) Let D be a bounded sublattice of B. We call D *relatively complete* (in B) if the inclusion $D \hookrightarrow B$ has a right adjoint, which we denote by $\Box \colon B \to D$. In other words, D is relatively complete (in B) iff for each $a \in B$, the set $\{d \in D \colon d \leq a\}$ has a largest element.

Lemma 4.5. *Let D be a bounded sublattice of a Boolean algebra B. Then D is relatively complete iff $\prec_D$ is a Heyting lattice subordination.*

Proof. Let $a \in B$, and consider $\{d \in D : d \leq a\}$ and $\{b \in B : b \prec_D a\}$. Clearly, $\{d \in D : d \leq a\}$ is a cofinal subset of $\{b \in B : b \prec_D a\}$. Therefore, $\{d \in D : d \leq a\}$ has a largest element iff $\{b \in B : b \prec_D a\}$ has a largest element. Thus, D is a relatively complete sublattice of B iff $\prec_D$ is a Heyting lattice subordination on B. $\Box$

As follows from Theorem 2.10 and Lemma 4.5, the full subcategory of BLS consisting of Boolean algebras with a Heyting lattice subordination is isomorphic to the full subcategory of BDA consisting of the pairs (B, D), where D

is a relatively complete sublattice of B. However, it is more interesting to consider the non-full subcategory of BLS whose morphisms also preserve the right adjoint $\Box$.

Lemma 4.6. *Let $(B_1, \prec_1)$ and $(B_2, \prec_2)$ be Boolean algebras with a Heyting lattice subordination and let $h\colon B_1 \to B_2$ be a morphism in* BLS*.*

(1) $h(\Box_1 a) \leq_2 \Box_2 h(a)$.
(2) $a \prec_1 b$ *implies* $h(\Box_1 a) \prec_2 \Box_2 h(b)$ *and* $\Box_2 h(a) \prec_2 h(\Box_1 b)$.

Proof. (1): Since $\Box_1 a \leq_1 a$, we have $h(\Box_1 a) \leq_2 h(a)$. Because $\Box_1 a \in D_{\prec_1}$ and $h\colon B_1 \to B_2$ is a morphism in BLS, $h(\Box_1 a) \in D_{\prec_2}$. So, $h(\Box_1 a) \leq_2 \Box_2 h(a)$.

(2) Suppose $a \prec_1 b$. Then $h(a) \prec_2 h(b)$. By Lemma 4.2, $\Box_2 h(a) \prec_2 \Box_2 h(b)$. Thus, by (1), $h(\Box_1 a) \prec_2 \Box_2 h(b)$. Also, $a \prec_1 b$ implies $a \leq_1 \Box_1 b$. Because $\Box_1 b \in D_{\prec_1}$, so $a \prec_1 \Box_1 b$. Therefore, $h(a) \prec_2 h(\Box_1 b)$. Since $\Box_2 h(a) \leq_2 h(a)$, we conclude that $\Box_2 h(a) \prec_2 h(\Box_1 b)$. $\Box$

Definition 4.7. (1) Let BLH be the category whose objects are Boolean algebras with a Heyting lattice subordination and whose morphisms are Boolean homomorphisms $h\colon B_1 \to B_2$ satisfying both $a \prec_1 b$ implies $h(a) \prec_2 h(b)$ and $e \prec_2 \Box_2 h(a)$ implies that there exists b with $b \prec_1 a$ and $e \prec_2 h(b)$. It is easy to check that BLS is a category where composition of two morphisms is the usual function composition.

(2) Let BHA be the category whose objects are pairs (B, D), where B is a Boolean algebra and D is a relatively complete sublattice of B, and whose morphisms are Boolean homomorphisms $h\colon B_1 \to B_2$ satisfying $a \in D_1$ implies $h(a) \in D_2$ and $h(\Box_1 a) = \Box_2 h(a)$. It is straightforward that BHA is a category where composition of two morphisms is the usual function composition.

Theorem 4.8. BLH *is isomorphic to* BHA*.*

Proof. By Theorem 2.10 and Lemma 4.5, it is sufficient to show that for a morphism $h\colon B_1 \to B_2$ in BLS (or equivalently in BDA), $h(\Box_1 a) = \Box_2 h(a)$ iff $e \prec_2 \Box_2 h(a)$ implies that there exists b with $b \prec_1 a$ and $e \prec_2 h(b)$. First, suppose that $h(\Box_1 a) = \Box_2 h(a)$. Let $e \prec_2 \Box_2 h(a)$. Set $b = \Box_1 a$. Because $\Box_1 a \in D_{\prec_1}$, so $b = \Box_1 a \prec_1 \Box_1 a \leq_1 a$. Therefore, $b \prec_1 a$. Moreover, $h(b) = h(\Box_1 a) = \Box_2 h(a)$, so $e \prec_2 h(b)$. Next, suppose that $e \prec_2 \Box_2 h(a)$ implies that there exists b with $b \prec_1 a$ and $e \prec_2 h(b)$. Let $a \in B_1$. By Lemma 4.6(1), $h(\Box_1 a) \leq_2 \Box_2 h(a)$. Let $e \prec_2 \Box_2 h(a)$. Then there exists b such that $b \prec_1 a$ and $e \prec_2 h(b)$. From $b \prec_1 a$, it follows that $b \leq_1 \Box_1 a$. So, $h(b) \leq_2 h(\Box_1 a)$. Thus, $e \prec_2 h(\Box_1 a)$, which yields $e \leq_2 h(\Box_1 a)$. In particular, since $\Box_2 h(a) \in D_{\prec_2}$, we have $\Box_2 h(a) \prec_2 \Box_2 h(a)$, so $\Box_2 h(a) \leq_2 h(\Box_1 a)$. Thus, $h(\Box_1 a) = \Box_2 h(a)$. $\Box$

We recall that an **S4**-algebra (also known as an interior algebra [5], closure algebra [14], or topological Boolean algebra [18]) is a pair $(B, \Box)$ where B is a Boolean algebra and $\Box\colon B \to B$ satisfies (i) $\Box a \leq a$, (ii) $\Box a \leq \Box\Box a$, (iii) $\Box(a \wedge b) = \Box a \wedge \Box b$, and (iv) $\Box 1 = 1$. Let $\mathsf{S4}$ be the category of **S4**-algebras and **S4**-algebra homomorphisms.

Corollary 4.9. BLH *is isomorphic to* S4.

Proof. It is known (see, e.g., [10, Sec. II.5] or [11, Sec. 4]) that S4 is isomorphic to BHA. By Theorem 4.8, BHA is isomorphic to BLH. The result follows. ☐

We recall that an **S5**-algebra (also known as a monadic algebra [13]) is an **S4**-algebra satisfying $\neg\Box a \leq \Box\neg\Box a$. Let S5 be the full subcategory of S4 consisting of **S5**-algebras. Let BLHB be the full subcategory of BLH consisting of those objects of BLH where the Heyting lattice subordination is Boolean, and let BHBA be the full subcategory of BHA consisting of those pairs (B, D) in BHA where D is a Boolean subalgebra of B.

Corollary 4.10. *The categories* S5*,* BLHB*, and* BHBA *are isomorphic.*

Proof. It follows from [13, pp. 44–46] that S5 is isomorphic to BHBA. That BLHB is isomorphic to BHBA follows from Theorem 4.8 and Theorem 3.6. The result follows. ☐

5. Duality

In this section, we show that lattice subordinations and Heyting lattice subordinations on a Boolean algebra B can be described by means of Priestley quasi-orders and Esakia quasi-orders on the Stone space X of B. We also show that Boolean subordinations on B can be described by means of Priestley equivalence relations on X. Esakia duality for **S4**-algebras and Halmos duality for **S5**-algebras follow.

Definition 5.1. Let X be a Stone space. We call a quasi-order $\leq$ on X a *Priestley quasi-order* if $x \not\leq y$ implies that there exists a clopen up-set U of X with $x \in U$ and $y \notin U$. We call a pair $(X, \leq)$ a *quasi-ordered Priestley space* if X is a Stone space and $\leq$ is a Priestley quasi-order on X. If $\leq$ is a partial order, then $(X, \leq)$ is a Priestley space. Let QPS be the category of quasi-ordered Priestley spaces and continuous order-preserving maps, and let PS be the full subcategory of QPS consisting of all Priestley spaces.

Let B be a Boolean algebra and let X be its Stone space. It follows from the dual characterization of bounded sublattices of a bounded distributive lattice [1, 7, 19, 3, 4] that bounded sublattices of B are in 1–1 correspondence with Priestley quasi-orders on X. The correspondence is obtained as follows: If D is a bounded sublattice of B, then the corresponding Priestley quasi-order $\leq_D$ on X is defined by setting $x \leq_D y$ iff $x \cap D \subseteq y$; and if $\leq$ is a Priestley quasi-order on X, then the corresponding bounded sublattice of B is $D_{\leq} = \{a \in B : \varphi(a)$ is an up-set of $X\}$, where $\varphi(a) = \{x \in X : a \in x\}$. This yields a 1–1 correspondence between quasi-ordered Priestley spaces and the pairs (B, D), where B is a Boolean algebra and D is a bounded sublattice of D. This 1–1 correspondence extends naturally to a dual equivalence between QPS and BDA.

Theorem 5.2. BDA *is dually equivalent to* QPS.

Proof. First define a contravariant functor $(-)_*\colon \mathsf{BDA} \to \mathsf{QPS}$ as follows. For $(B, D) \in \mathsf{BDA}$, let $(B, D)_* = (X, \leq)$, with X the Stone space of B and $x \leq y$ iff $x \cap D \subseteq y$. From the paragraph above, $\leq$ is a Priestley quasi-order on X. For $h \in \hom_{\mathsf{BDA}}((B_1, D_1), (B_2, D_2))$, let $h_*\colon (B_2, D_2)_* \to (B_1, D_1)_*$ be given by $h_*(x) = h^{-1}(x)$. That h_* is a well-defined continuous map follows from Stone duality. To see that h_* is order-preserving, let $x \leq y$. Then $x \cap D_2 \subseteq y$. Therefore, $h^{-1}(x) \cap D_1 \subseteq h^{-1}(y)$. Thus, h_* is order-preserving, and it is straightforward to see that $(-)_*$ is a well-defined contravariant functor.

Next, define a contravariant functor $(-)^*\colon \mathsf{QPS} \to \mathsf{BDA}$ as follows. For $(X, \leq) \in \mathsf{QPS}$, let $(X, \leq)^* = (B_X, D_X)$, where B_X is the Boolean algebra of clopen subsets of X and D_X is the bounded sublattice of B_X consisting of all clopen up-sets of $(X, \leq)$. For $f \in \hom_{\mathsf{QPS}}((X_1, \leq_1), (X_2, \leq_2))$, let $f^*\colon (X_2, \leq_2)^* \to (X_1, \leq_1)^*$ be given by $f^*(U) = f^{-1}(U)$. That f^* is a well-defined Boolean homomorphism follows from Stone duality. Moreover, as f is continuous and order-preserving, the f-inverse image of each clopen up-set of X_2 is a clopen up-set of X_1. Therefore, $f^* \in \hom_{\mathsf{BDA}}((X_2, \leq_2)^*, (X_1, \leq_1)^*)$, and it is straightforward to see that $(-)^*$ is a well-defined contravariant functor.

Finally, the 1–1 correspondence between bounded sublattices of Boolean algebras and Priestley quasi-orders on Stone spaces yields that BDA is dually equivalent to QPS. □

As an immediate consequence of Theorem 2.10 and Theorem 5.2 we obtain:

Corollary 5.3. BLS *is dually equivalent to* QPS.

For the reader's convenience, we give a direct description of the contravariant functors that establish this dual equivalence. Let $(B, \prec) \in \mathsf{BLS}$. For $S \subseteq B$, let $\Uparrow S = \{a \in B : \exists b \in S \text{ with } b \prec a\}$. Define $(-)_*\colon \mathsf{BLS} \to \mathsf{QPS}$ as follows. For $(B, \prec) \in \mathsf{BLS}$, let $(B, \prec)_* = (X, \leq)$, where X is the Stone space of B and $x \leq y$ iff $\Uparrow x \subseteq y$. For $h \in \hom_{\mathsf{BLS}}((B_1, \prec_1), (B_2, \prec_2))$, let $h_*\colon (B_2, \prec_2)_* \to (B_1, \prec_1)_*$ be given by $h_*(x) = h^{-1}(x)$. That $(-)_*$ is well defined follows from the following lemma.

Lemma 5.4.

(1) *Let* $(B, \prec) \in \mathsf{BLS}$, $D_\prec = \{a \in B : a \prec a\}$, *and* $x, y \in (B, \prec)_*$. *Then* $\Uparrow x \subseteq y$ *iff* $x \cap D_\prec \subseteq y$.

(2) *Let* $h \in \hom_{\mathsf{BLS}}((B_1, \prec_1), (B_2, \prec_2))$ *and* $x, y \in (B_2, \prec_2)_*$. *If* $\Uparrow x \subseteq y$, *then* $\Uparrow h^{-1}(x) \subseteq h^{-1}(y)$.

Proof. (1): First suppose that $\Uparrow x \subseteq y$ and $a \in x \cap D_\prec$. Then $a \in \Uparrow x$, so $a \in y$, yielding $x \cap D_\prec \subseteq y$. Next, suppose that $x \cap D_\prec \subseteq y$ and $a \in \Uparrow x$. Then there exists $b \in x$ with $b \prec a$. Therefore, there exists $c \in D_\prec$ with $b \leq c \leq a$. As $c \in x \cap D_\prec$, we have $c \in y$, so $a \in y$, yielding $\Uparrow x \subseteq y$.

(2): Let $a \in {\uparrow\!\!\uparrow} h^{-1}(x)$. Then there exists $b \in h^{-1}(x)$ with $b \prec_1 a$. Therefore, $h(b) \prec_2 h(a)$, so $h(a) \in {\uparrow\!\!\uparrow} x$. Thus, $h(a) \in y$, yielding $a \in h^{-1}(y)$. □

Define $(-)^* \colon \mathsf{QPS} \to \mathsf{BLS}$ as follows. For $(X, \leq) \in \mathsf{QPS}$, let $(X, \leq)^* = (B_X, \prec)$, where B_X is the Boolean algebra of clopen subsets of X and $U \prec V$ iff there exists a clopen up-set W of X such that $U \subseteq W \subseteq V$. As $\emptyset, X$ are clopen up-sets and finite intersections and unions of clopen up-sets are clopen up-sets, $\prec$ satisfies axioms (S1)–(S3). It follows from the definition that $\prec$ satisfies axiom (S4). Finally, as clopen up-sets are the reflexive elements of $\prec$, it is immediate that $\prec$ satisfies axiom (S5). For $f \in \mathrm{hom}_{\mathsf{QPS}}((X_1, \leq_1), (X_2, \leq_2))$, let $f^* \colon (X_2, \leq_2)^* \to (X_1, \leq_1)^*$ be given by $f^*(U) = f^{-1}(U)$. As the f-inverse image of a clopen up-set of X_2 is a clopen up-set of X_1, it is obvious that $U \prec_2 V$ implies $f^{-1}(U) \prec_1 f^{-1}(V)$. Thus, $(-)^*$ is a well-defined contravariant functor. The functors $(-)_* \colon \mathsf{BLS} \to \mathsf{QPS}$ and $(-)^* \colon \mathsf{QPS} \to \mathsf{BLS}$ provide the dual equivalence of Corollary 5.3.

Definition 5.5. Let X be a Stone space and let $\leq$ be a Priestley quasi-order on X.

(1) We call $\leq$ an *Esakia quasi-order* if ${\downarrow}U$ is clopen for each clopen U of X, where ${\downarrow}U = \{x \in X : \exists y \in U \text{ with } x \leq y\}$.
(2) We call a pair $(X, \leq)$ a *quasi-ordered Esakia space* if X is a Stone space and $\leq$ is an Esakia quasi-order on X.
(3) Let QES be the category of quasi-ordered Esakia spaces and continuous bounded morphisms between them, where $f \colon (X_1, \leq_1) \to (X_2, \leq_2)$ is a bounded morphism if f is order-preserving and $f(x) \leq_2 y$ implies that there exists z with $x \leq_1 z$ and $f(z) = y$.

Lemma 5.6.

(1) *Suppose that* $(B, \prec) \in \mathsf{BLS}$ *and* $(X, \leq) = (B, \prec)_*$. *If* $(B, \prec) \in \mathsf{BLH}$, *then* $(X, \leq) \in \mathsf{QES}$.
(2) *Suppose that* $(X, \leq) \in \mathsf{QPS}$ *and* $(B, \prec) = (X, \leq)^*$. *If* $(X, \leq) \in \mathsf{QES}$, *then* $(B, \prec) \in \mathsf{BLH}$.

Proof. (1): For $a \in B$, let $\Box\varphi(a) = -{\downarrow}-\varphi(a)$, where $-$ is set-theoretic complement. We show $\varphi(\Box a) = \Box\varphi(a)$. For $x \in X$, let ${\uparrow}x = \{y \in X : x \leq y\}$. Then $x \in \varphi(\Box a)$ iff $\Box a \in x$, and $x \in \Box\varphi(a)$ iff ${\uparrow}x \subseteq \varphi(a)$. Observe that ${\uparrow}x \subseteq \varphi(a)$ iff $(\forall y \in X)(x \leq y \Rightarrow a \in y)$ iff $(\forall y \in X)({\uparrow\!\!\uparrow} x \subseteq y \Rightarrow a \in y)$. Since ${\uparrow\!\!\uparrow} x$ is a filter, by the ultrafilter theorem for Boolean algebras, the last condition is equivalent to $a \in {\uparrow\!\!\uparrow} x$. As $b \prec a$ is equivalent to $b \prec \Box a$, we obtain $a \in {\uparrow\!\!\uparrow} x$ iff $\Box a \in x$. Therefore, $\varphi(\Box a) = \Box\varphi(a)$. Let U be a clopen subset of X. Then $-U$ is also clopen, so there exists $a \in B$ with $-U = \varphi(a)$. Thus, ${\downarrow}U = --{\downarrow}--U = -(-{\downarrow}-\varphi(a)) = -\Box\varphi(a) = -\varphi(\Box a)$, and as $\varphi(\Box a)$ is clopen, so is ${\downarrow}U = -\varphi(\Box a)$. Consequently, $(X, \leq) \in \mathsf{QES}$.

(2): Let B_X be the Boolean algebra of clopen subsets of X. It is sufficient to show that for each $U \in B_X$, the largest element of $\{V \in B_X : V \prec U\}$ is $-{\downarrow}-U$. Clearly, $-{\downarrow}-U \in \{V \in B_X : V \prec U\}$. Let $V \prec U$ and let D_X be the bounded sublattice of B_X consisting of clopen up-sets of X. Then there exists $W \in D_X$ with $V \subseteq W \subseteq U$. As $W \in D_X$, we have $W \subseteq -{\downarrow}-U$, so $V \subseteq -{\downarrow}-U$, and so $-{\downarrow}-U$ is the largest element of $\{V \in B_X : V \prec U\}$. □

Lemma 5.7.
(1) $h \in \hom_{\mathsf{BLH}}((B_1, \prec_1), (B_2, \prec_2)) \Rightarrow h_* \in \hom_{\mathsf{QES}}((B_2, \prec_2)_*, (B_1, \prec_1)_*)$.
(2) $f \in \hom_{\mathsf{QES}}((X_1, \leq_1), (X_2, \leq_2)) \Rightarrow f^* \in \hom_{\mathsf{BLH}}((X_2, \leq_2)^*, (X_1, \leq_1)^*)$.

Proof. (1): By Lemma 5.4(2), h_* is order-preserving. Let $h_*(x) \leq_2 y$. Then ${\Uparrow}h^{-1}(x) \subseteq y$. We claim that the filter F generated by ${\Uparrow}x \cup h[y]$ has the empty intersection with the ideal I generated by $h[B_1 - y]$. If not, then there exist $a \in {\Uparrow}x$, $b \in y$ and $c \notin y$ such that $a \wedge h(b) \leq_2 h(c)$. Thus, $a \leq_2 h(\neg b \vee c)$. Since $a \in {\Uparrow}x$, there exists $e \in x$ with $e \prec_2 a$, so $e \prec_2 h(\neg b \vee c)$, and so $e \prec_2 \Box_2 h(\neg b \vee c)$. Since h is a morphism in BLH, there exists d with $d \prec_1 \neg b \vee c$ and $e \prec_2 h(d)$. Thus, $h(d) \in x$, so $d \in h^{-1}(x)$, and so $\neg b \vee c \in {\Uparrow}h^{-1}(x)$. This yields that $\neg b \vee c \in y$, a contradiction. Consequently, there exists $z \in (B_2, \prec_2)_*$ such that $F \subseteq z$ and $I \cap z = \emptyset$. From $F \subseteq z$, it follows that ${\Uparrow}x \subseteq z$; from $h[y] \subseteq z$, it follows that $y \subseteq h^{-1}(z)$, and from $h[B_1 - y] \cap z = \emptyset$, it follows that $h^{-1}(z) \subseteq y$. Therefore, there exists z with $x \leq_1 z$ and $h_*(z) = y$, which then implies that $h_* \in \hom_{\mathsf{QES}}((B_2, \prec_2)_*, (B_1, \prec_1)_*)$.

(2): Let $U, V \in B_{X_2}$. Now, $U \prec_2 V$ implies $f^{-1}(U) \prec_1 f^{-1}(V)$, as we already saw. Let $E \in B_{X_1}$, $U \in B_{X_2}$, and $E \prec_1 \Box_1 f^{-1}(U)$. Then there exists $W \in D_{X_1}$ with $E \subseteq W \subseteq \Box_1 f^{-1}(U)$. From $W \subseteq \Box_1 f^{-1}(U)$, it follows that $W \subseteq f^{-1}(U)$, so $f(W) \subseteq U$. Since W is a clopen up-set and f is a continuous bounded morphism, $f(W)$ is a closed up-set of X_2. As $(X_2, \leq_2)$ is a quasi-ordered Priestley space, there exists $V \in D_{X_2}$ such that $f(W) \subseteq V \subseteq U$. From $V \in D_{X_2}$, it follows that $V \prec_2 U$, and from $E \subseteq W$ and $f(W) \subseteq V$, it follows that $f(E) \subseteq V$, so $E \subseteq f^{-1}(V)$. Since $f^{-1}(V) \in D_{X_1}$, this yields $E \prec_1 f^{-1}(V)$. Therefore, there exists $V \in B_{X_2}$ such that $V \prec_2 U$ and $E \prec_1 f^{-1}(V)$. Thus, $f^* \in \hom_{\mathsf{BLH}}((X_2, \leq_2)^*, (X_1, \leq_1)^*)$. □

Theorem 5.8. BLH *is dually equivalent to* QES.

Proof. By Lemmas 5.6 and 5.7, $(-)_* : \mathsf{BLH} \to \mathsf{QES}$ and $(-)^* : \mathsf{QES} \to \mathsf{BLH}$ are well-defined contravariant functors. Now apply Corollary 5.3. □

Corollary 5.9. BHA *is dually equivalent to* QES.

Proof. Apply Theorems 4.8 and 5.8. □

Corollary 5.10 (Esakia Duality for **S4**-algebras)**.** $\mathsf{S4}$ *is dually equivalent to* QES.

Proof. Apply Corollary 4.9 and Theorem 5.8. □

Definition 5.11. (1) We call an equivalence relation $\sim$ on a Stone space X a *Priestley equivalence relation* if $(X, \sim)$ is a quasi-ordered Priestley space. Let EPS be the category whose objects are pairs $(X, \sim)$, where X is a Stone space and $\sim$ is a Priestley equivalence relation on X, and whose morphisms are continuous maps that preserve $\sim$ (that is, maps $f\colon X_1 \to X_2$ such that $x \sim_1 y$ implies $f(x) \sim_2 f(y)$).

(2) We call an equivalence relation $\sim$ on a Stone space X an *Esakia equivalence relation* if $(X, \sim)$ is a quasi-ordered Esakia space, and we call such a pair $(X, \sim)$ a *Halmos space* because these spaces were first considered by Halmos [13]. Let HS be the category of Halmos spaces and continuous bounded morphisms between them.

Clearly EPS is a full subcategory of QPS and HS is a full subcategory of QES.

Theorem 5.12.
(1) BBA *is dually equivalent to* EPS.
(2) BLB *is dually equivalent to* EPS.
(3) BLHB *is dually equivalent to* HS.
(4) BHBA *is dually equivalent to* HS.
(5) (Halmos Duality for **S5**-algebras) $\mathsf{S5}$ *is dually equivalent to* HS.

Proof. (1): Let $(B, D) \in \mathsf{BBA}$. Then D is a Boolean subalgebra of B, so $\leq$ given by $x \leq y$ iff $x \cap D \subseteq y$ is an equivalence relation on the Stone space X of B. Theorem 5.2 now yields $(B, \prec)_* \in \mathsf{EPS}$. Let $(X, \sim) \in \mathsf{EPS}$. As $\sim$ is an equivalence relation, $\sim$-up-sets are closed under set-theoretic complement, so $U \prec V$ implies $-V \prec -U$, and hence applying Theorem 5.2 again yields $(X, \sim)^* \in \mathsf{BBA}$. One more application of Theorem 5.2 gives that the restrictions of $(-)_*$ and $(-)^*$ are well-defined contravariant functors $(-)_*\colon \mathsf{BBA} \to \mathsf{EPS}$ and $(-)^*\colon \mathsf{EPS} \to \mathsf{BBA}$ that establish a dual equivalence of BBA and EPS.

(2): This follows from (1) and Theorem 3.6.

(3): Let $(B, \prec) \in \mathsf{BLHB}$. By (2) and Lemma 5.6(1), $(B, \prec)_*$ is a Halmos space. If $h \in \hom_{\mathsf{BLHB}}((B_1, \prec_1), (B_2, \prec_2))$, then by (2) and Lemma 5.7(1), $h_* \in \hom_{\mathsf{HS}}((B_2, \prec_2)_*, (B_1, \prec_1)_*)$. This implies that the restriction of $(-)_*$ to BLHB is a well-defined contravariant functor to HS. Let $(X, \sim) \in \mathsf{HS}$. By (2) and Lemma 5.6(2), $(X, \sim)^* \in \mathsf{BLHB}$. If $f \in \hom_{\mathsf{HS}}((X_1, \sim_1), (X_2, \sim_2))$, then by (2) and Lemma 5.7(2), $f^* \in \hom_{\mathsf{BLHB}}((X_2, \sim_2)^*, (X_1, \sim_1)^*)$. This implies that the restriction of $(-)^*$ to HS is a well-defined contravariant functor to BLHB. Now apply Theorem 5.8.

(4) and (5): These follow from (3) and Corollary 4.10. □

6. Boolean envelopes

In this final section, we introduce full subcategories of BLS and BLH consisting of the objects that are generated by the reflexive elements of the subordination. We prove that such objects $(B, \prec)$ can dually be described by Priestley orders and Esakia orders on the Stone space of B. Priestley duality for bounded distributive lattices and Esakia duality for Heyting algebras follow.

Definition 6.1. (1) Let D be a bounded sublattice of a Boolean algebra B. We call B the *Boolean envelope* of D if B is generated by D; that is, B is the smallest Boolean subalgebra of B containing D.

(2) Let $(B, D) \in \mathsf{BDA}$. We call (B, D) *D-generated* if B is the Boolean envelope of D. Let GBDA be the full subcategory of BDA consisting of the D-generated objects of BDA. Also, let GBHA be the full subcategory of BHA consisting of D-generated objects of BHA.

(3) Let $(B, \prec) \in \mathsf{BLS}$. We call $(B, \prec)$ *D-generated* if B is the Boolean envelope of $D_\prec$. Let GBLS be the full subcategory of BLS consisting of the D-generated objects of BLS. Also, let GBLH be the full subcategory of BLH consisting of D-generated objects of BLH.

The next theorem is an immediate consequence of Theorem 2.10, Theorem 4.8, and Definition 6.1.

Theorem 6.2. GBLS *is isomorphic to* GBDA *and* GBLH *is isomorphic to* GBHA.

Let DL be the category of bounded distributive lattices and bounded lattice homomorphisms. There is a natural functor $\Gamma\colon \mathsf{BDA} \to \mathsf{DL}$ that sends each (B, D) to D and each $h \in \hom_{\mathsf{BDA}}((B_1, D_1), (B_2, D_2))$ to the restriction of h to D_1. Equivalently there is a functor $\Gamma\colon \mathsf{BLS} \to \mathsf{DL}$ that sends each $(B, \prec)$ to $D_\prec$ and each $h \in \hom_{\mathsf{BLS}}((B_1, \prec_1), (B_2, \prec_2))$ to the restriction of h to $D_{\prec_1}$.

Let HA be the category of Heyting algebras and Heyting homomorphisms. Then HA is a non-full subcategory of DL. If $(B, D) \in \mathsf{BHA}$, then $D \in \mathsf{HA}$; also, if $h \in \hom_{\mathsf{BHA}}((B_1, D_1), (B_2, D_2))$, then the restriction of h to D_1 is a morphism in HA (see, e.g., [10, Sec. II.2]). Therefore, Γ restricts to a functor from BHA to HA. Equivalently, if $(B, \prec) \in \mathsf{BLH}$, then $D_\prec \in \mathsf{HA}$; and if $h \in \hom_{\mathsf{BLH}}((B_1, \prec_1), (B_2, \prec_2))$, then the restriction of h to $D_{\prec_1}$ is a morphism in HA. Therefore, Γ restricts to a functor from BLH to HA.

The functor $\Gamma\colon \mathsf{BDA} \to \mathsf{DL}$ has a left adjoint $\Delta\colon \mathsf{DL} \to \mathsf{BDA}$ sending each D to the pair $\Delta(D) = (B(D), D)$, where $B(D)$ is the free Boolean extension of D, and each $h \in \hom_{\mathsf{DL}}(D_1, D_2)$ to its unique extension $\Delta(h)\colon B(D_1) \to B(D_2)$ (see, e.g., [6, 10, 11]). Equivalently, the left adjoint from $\Delta\colon \mathsf{DL} \to \mathsf{BLS}$ to $\Gamma\colon \mathsf{BLS} \to \mathsf{DL}$ sends each D to the pair $\Delta(D) = (B(D), \prec_D)$ and each $h \in \hom_{\mathsf{DL}}(D_1, D_2)$ to $\Delta(h)\colon B(D_1) \to B(D_2)$.

Similarly, $\Gamma\colon \mathsf{BHA} \to \mathsf{HA}$ has a left adjoint $\Delta\colon \mathsf{HA} \to \mathsf{BHA}$, or equivalently, $\Gamma\colon \mathsf{BLH} \to \mathsf{HA}$ has a left adjoint $\Delta\colon \mathsf{HA} \to \mathsf{BLH}$. If $D \in \mathsf{DL}$, then $(B(D), D) \in \mathsf{GBDA}$, and if $D \in \mathsf{HA}$, then $(B(D), D) \in \mathsf{GBHA}$. Equivalently, if $D \in \mathsf{DL}$, then $(B(D), \prec_D) \in \mathsf{GBLS}$, and if $D \in \mathsf{HA}$, then $(B(D), \prec_D) \in \mathsf{GBLH}$. Therefore, we arrive at the following theorem.

Theorem 6.3.

(1) (see [6, 10, 11]) DL *is equivalent to* GBDA *and* HA *is equivalent to* GBHA.
(2) DL *is equivalent to* GBLS *and* HA *is equivalent to* GBLH.

Lemma 6.4.

(1) *Let* $(B, \prec) \in \mathsf{BLS}$ *and let* $(X, \leq) = (B, \prec)_*$. *Then* $(B, \prec) \in \mathsf{GBLS}$ *iff* $\leq$ *is a partial order.*

(2) *Let* $(B, D) \in \mathsf{BDA}$ *and let* $(X, \leq) = (B, D)_*$. *Then* $(B, D) \in \mathsf{GBDA}$ *iff* $\leq$ *is a partial order.*

(3) *Let* $(B, \prec) \in \mathsf{BLH}$ *and let* $(X, \leq) = (B, \prec)_*$. *Then* $(B, \prec) \in \mathsf{GBLH}$ *iff* $\leq$ *is a partial order.*

(4) *Let* $(B, D) \in \mathsf{BHA}$ *and let* $(X, \leq) = (B, D)_*$. *Then* $(B, D) \in \mathsf{GBHA}$ *iff* $\leq$ *is a partial order.*

Proof. (1): First suppose that $(B, \prec) \in \mathsf{GBLS}$. To see that $\leq$ is a partial order, let $x, y \in X$ with $x \leq y$ and $y \leq x$. Then $\Uparrow x \subseteq y$ and $\Uparrow y \subseteq x$. We must show that $x = y$. If not, then as x, y are ultrafilters of B, we may assume that $x \nsubseteq y$. Therefore, there exists $a \in x - y$. Since B is the Boolean envelope of $D_\prec$, there exist $a_i, b_i \in D_\prec$ such that $a = \bigvee_{i=1}^{n}(a_i \wedge \neg b_i)$. As $a \in x$ and x is an ultrafilter, there exists k such that $a_k \wedge \neg b_k \in x$, so $a_k \in x$ and $b_k \notin x$. Since $a \notin y$, for each i we have $a_i \wedge \neg b_i \notin y$, so $a_k \notin y$ or $b_k \in y$. If $a_k \notin y$, then $a_k \notin \Uparrow x$. But $a_k \prec a_k$, so $a_k \notin \Uparrow x$ implies $a_k \notin x$, a contradiction. If $b_k \in y$, then as $b_k \prec b_k$, we have $b_k \in \Uparrow y$. Therefore, $b_k \in x$, which is again a contradiction. Thus, such an a does not exist, so $x = y$, and so $\leq$ is a partial order.

Let $\leq$ be a partial order. As B is isomorphic to the Boolean algebra of clopen subsets of X and $D_\prec$ is isomorphic to the bounded sublattice of clopen up-sets of X, it is sufficient to show that each clopen of X is a Boolean combination of clopen up-sets of X. Let U be a clopen subset of X and let $x \in U$. As $\leq$ is a partial order, for each $y \notin U$, either $x \nleq y$ or $y \nleq x$. If $x \nleq y$, then there exists a clopen up-set V_y of X such that $x \in V_y$ and $y \notin V_y$; and if $y \nleq x$, then there exists a clopen up-set W_y of X such that $y \in W_y$ and $x \notin W_y$. Therefore, $x \in \bigcap\{V_y : x \nleq y\} \cap \bigcap\{-W_y : y \nleq x\} \subseteq U$, which by compactness implies that there are finitely many $V_1, \dots, V_n$ and $W_1, \dots, W_m$ such that $x \in V_1 \cap \dots \cap V_n \cap -W_1 \cap \dots \cap -W_m \subseteq U$. Thus, for each $x \in U$, there is a neighborhood of x that is a Boolean combination of clopen up-sets of X and is contained in U. Applying compactness again yields that U is a finite union of Boolean combinations of clopen up-sets of X, hence is a Boolean combination of clopen up-sets of X. Consequently, $(B, \prec) \in \mathsf{GBLS}$.

(2): This follows from (1).

(3): This is a particular case of (1).

(4): This follows from (3). □

Definition 6.5. (1) Let $(X, \leq)$ be a quasi-ordered Priestley space. We call $(X, \leq)$ a *(partially ordered) Priestley space* if $\leq$ is a partial order. Let PS be the full subcategory of QPS consisting of Priestley spaces.

(2) Let $(X, \leq)$ be a quasi-ordered Esakia space. We call $(X, \leq)$ a *(partially ordered) Esakia space* if $\leq$ is a partial order. Let ES be the full subcategory of QES consisting of Esakia spaces.

Theorem 6.6.
(1) GBLS *is dually equivalent to* PS.
(2) GBDA *is dually equivalent to* PS.
(3) GBLH *is dually equivalent to* ES.
(4) GBHA *is dually equivalent to* ES.

Proof. (1): Apply Corollary 5.3 and Lemma 6.4(1).
(2): This follows from (1) and Theorem 6.2.
(3): This follows from Theorem 5.8 and Lemma 6.4(3).
(4): This follows from (3) and Theorem 6.2. □

Corollary 6.7.
(1) (Priestley Duality for Bounded Distributive Lattices) DL *is dually equivalent to* PS.
(2) (Esakia Duality for Heyting Algebras) HA *is dually equivalent to* ES.

Proof. Apply Theorem 6.3 and Theorem 6.6. □

Remark 6.8. If $(B, D) \in$ BBA, then D is a Boolean subalgebra of B, so (B, D) is D-generated iff $B = D$. Therefore, the full subcategory GBBA of BBA consisting of D-generated objects of BBA is isomorphic to the category BA of Boolean algebras and Boolean homomorphisms. Applying Theorem 3.6, we obtain that the full subcategory GBLB of BLB consisting of D-generated objects of BLB is also isomorphic to BA. This implies that each of GBLB and GBBA is dually equivalent to the category Stone of Stone spaces and continuous maps.

We conclude the paper with five tables. In the first four tables we list the categories considered in this paper. For readability, we only list the objects of the categories. In the fifth table, we describe the obtained isomorphisms, equivalences, and dual equivalences, together with relevant theorem numbers. For two categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathcal{C} \cong \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are isomorphic, $\mathcal{C} \sim \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent, and $\mathcal{C} \overset{\mathrm{d}}{\sim} \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are dually equivalent.

Table 1. Categories of Boolean algebras with subordination

Category	Objects
BLS	Boolean algebras with a lattice subordination
GBLS	D-generated objects of BLS
BLB	objects of BLS where the lattice subordination is Boolean
GBLB	D-generated objects of BLB
BLH	Boolean algebras with a Heyting lattice subordination
GBLH	D-generated objects of BLH
BLHB	objects of BLH where the Heyting lattice subordination is Boolean

Table 2. Categories of pairs (B, D) with D subordinate to B

Category	Objects
BDA	pairs (B, D) where B is a Boolean algebra and D is a bounded sublattice of B
GBDA	D-generated objects of BDA
BBA	objects of BDA where D is a Boolean subalgebra of B
GBBA	D-generated objects of BBA
BHA	pairs (B, D) where B is a Boolean algebra and D is a relatively complete sublattice of B
GBHA	D-generated objects of BHA
BHBA	objects of BHA where D is a Boolean subalgebra of B

Table 3. Categories of algebras

Category	Objects
S4	**S4**-algebras
S5	**S5**-algebras
DL	bounded distributive lattices
HA	Heyting algebras
BA	Boolean algebras

Table 4. Categories of spaces

Category	Objects
QPS	quasi-ordered Priestley spaces
EPS	objects of QPS where the quasi-order is an equivalence relation
PS	Priestley spaces
QES	quasi-ordered Esakia spaces
HS	Halmos spaces
ES	Esakia spaces
Stone	Stone spaces

Table 5. Isomorphisms, equivalences, and dual equivalences

BLS	$\cong$	BDA	$\overset{d}{\sim}$	QPS				Thm. 2.10, 5.2, Cor. 5.3
BLB	$\cong$	BBA	$\overset{d}{\sim}$	EPS				Thm. 3.6, 5.12
BLH	$\cong$	BHA	$\cong$	S4	$\overset{d}{\sim}$	QES		Thm. 4.8, 5.8, Cor. 4.9, 5.9, 5.10
BLHB	$\cong$	BHBA	$\cong$	S5	$\overset{d}{\sim}$	HS		Cor. 4.10, Thm. 5.12
GBLS	$\cong$	GBDA	$\sim$	DL	$\overset{d}{\sim}$	PS		Thm. 6.3, 6.6, Cor. 6.7
GBLH	$\cong$	GBHA	$\sim$	HA	$\overset{d}{\sim}$	ES		Thm. 6.3, 6.6, Cor. 6.7
GBLB	$\cong$	GBBA	$\cong$	BA	$\overset{d}{\sim}$	Stone		Rem. 6.8

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