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Many-sorted and single-sorted algebras

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Abstract. This paper specifies a detailed, fully type-based general method for translating the class of all pure, many-sorted algebras of a given constant-free type into an equivalent variety of single-sorted algebras of defined, constant-free type. The complexity of the identities defining the variety is a linear function of the number of sorts and the arity of the fundamental operations.

1. Introduction

Many-sorted or heterogeneous algebras were initially studied as generalizations of classical single-sorted or homogeneous algebras [3, 10, 11]. Categorical methods based on monads were used subsequently, often in connection with applications to computer science [2, 7]. There is also a categorical approach using algebraic theories, including explicit discussion of heterogeneous algebraic formulations of stacks, directed graphs, and sequential automata [1]. Certain many-sorted algebras were also employed in model theory [6, 12]. Recent applications include quasigroup homotopies and web geometry [8, 9, 22], fibred automata and continued fractions [13, 14, 15, 16, 21], and conformal and vertex algebras [23].

Despite their utility, many-sorted algebras have a tendency towards notational awkwardness, and their theory does not always extend without caveats from the theory of single-sorted algebras [5, 7]. Particularly troublesome are the many-sorted algebras that are not "pure," mixing empty and non-empty sorts. Against this background, a monadicity observation of Barr [2, Thm 5], refined by Goguen and Meseguer [7, p. 331], shows that the class of all pure heterogeneous algebras of a given type, or of all pure heterogeneous algebras in a certain variety, is equivalent to a variety of single-sorted algebras. It is important to note that the types or defining identities for the single-sorted varieties are not given explicitly by this general monadic approach, although they have been determined in specific cases [22, 23]. The current paper is concerned

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with the general problem of determining good sets of explicit defining identities and quasi-identities for the variety of single-sorted algebras equivalent to a class of all pure heterogeneous algebras of a given type.

Translation from certain many-sorted to single-sorted algebras has been discussed in the model-theoretic literature with specific types and identities (compare [6, 12]). These discussions have taken place within restricted contexts (a universally pseudorecursive class in [6], or a strongly Abelian class in [12]), always excluding any consideration of the empty set, or even insisting that the sets underlying different sorts be disjoint [12, Defn 11.1(2)]. Nevertheless, admission of the empty set is crucial in many applications (to fibred automata, for instance), as is the possibility of a common underlying set for the different sorts (in the treatment of quasigroup homotopies, for example).

Motivated by the increasing range of applications of many-sorted algebras, the philosophy of the current paper seeks to combine the general applicability of the categorical approach with the specificity of the model-theoretic approach. The goal is to give a detailed, fully type-based treatment of the equivalence between the class of all pure heterogeneous algebras of a given type and a variety of single-sorted algebras. At such a level of detail, the actual equivalence depends on properties of the types involved, since they influence the specific way in which many-sorted operations may be encoded as single-sorted operations. Here, we concentrate on the case where there are no constants in the many-sorted or single-sorted types (the case considered in [6, §2.3]). This case corresponds to the quasigroup-homotopy example [22], although that example exploited specific features to encode three many-sorted operations into one single-sorted operation, and is thus not subsumed by the current work. The case with constants is also amenable to a general treatment, along somewhat different lines from those adopted in this paper, instead tracking the methods used for conformal and vertex algebras in [23].

The plan of the paper is as follows. Section 2 presents the notation used for handling many-sorted sets and functions between them, including the key concept of a pure many-sorted set (Definition 2.1). Section 3 treats the diagonal algebras (generalizations of rectangular bands) that encode product sets as single-sorted algebras. Our conventions for many-sorted algebras are introduced in Section 4. Section 5 gives a new and concise treatment of words, absolutely free algebras, and derived types for many-sorted algebras along the lines of the comparable treatment of single-sorted algebras in [24, IV.1.3]. This treatment avoids tiresome recursive definitions by means of an algebraic technique. Section 6 covers a limited notion of satisfaction of identities within pure many-sorted algebras, including fibred and reversible automata as examples. For the constant-free case, Section 7 deals with the process of homogenization that is used to convert from many-sorted or heterogeneous algebras to single-sorted or homogeneous algebras. Section 8 offers a brief digression on the transfer of identities during the process of homogenization. The converse process of heterogenization is the topic of Section 9. In order to create a many-sorted type from a single-sorted type, combinatorial constructs known as input and output functions are required (Definitions 9.1, 9.4). The single-sorted algebras corresponding to pure many-sorted algebras are described in Definition 9.1 as being heterogenizable, characterized by simple new identities and quasi-identities involving only a small number of variables that is a linear function of the number of sorts and the arity of a fundamental operation. Although Proposition 9.8 shows that these quasi-identities are equivalent to identities (as the monadicity results [2, 7] predict), we consider the quasi-identities to be more fundamental in our general context, because of their simpler form. (This by no means precludes the existence of nice identities in particular cases such as [22].) The main equivalence results (Theorem 10.1 going from pure many-sorted algebras of a given type to a variety of single-sorted algebras, and Corollary 10.3 reversing the process) are summarized in Section 10.

For algebraic and categorical concepts not otherwise given explicitly within the paper, readers are referred to [20, 24].

2. Many-sorted sets

Let n be a positive integer. Consider the set

$$\underline{n} = \{i \in \mathbb{N} \mid i < n\} \tag{2.1}$$

of natural numbers less than n, so that $|\underline{n}| = n$. The set (2.1) will also be considered as the object set of a small discrete category $\underline{n}$. The *category of n-sorted sets* is the functor category $\mathbf{Set}^{\underline{n}}$. Thus an object A of $\mathbf{Set}^{\underline{n}}$ involves a set A_i (a *sort*) as the image of i, for each $i < n$. Each such functor A has a limit

$$\varprojlim A = \prod_{i<n} A_i\,, \tag{2.2}$$

the product of the sets A_i, and a colimit

$$\varinjlim A = \sum_{i<n} A_i\,, \tag{2.3}$$

the disjoint union of the sets A_i. (Readers who are unfamiliar with limits may simply regard (2.2) and (2.3) as convenient notations.) If S is a set, then the *constant* n-sorted set S is defined by $S_i = S$ for $i < n$.

Definition 2.1. An n-sorted set A is *pure* if there is a function $\varinjlim A \to \varprojlim A$. In other words, the condition $(\exists\ i < n\,.\ A_i = \varnothing) \Rightarrow (\forall\ j < n\,,\ A_j = \varnothing)$ holds.

A morphism $f\colon A \to B$ of $\mathbf{Set}^{\underline{n}}$ is a natural transformation. Since $\underline{n}$ is discrete, the naturality conditions are trivial, and one just has the component functions $f_i\colon A_i \to B_i$ for each $i < n$. If these components are subset inclusions, A is said to be a (*many-sorted*) *subset* of B. The (*n-sorted*) *power set* 2^B of an n-sorted set B is defined by $(2^B)_i = 2^{B_i}$ for $i < n$. Given n-sorted sets X and Y, their *product* $X \times Y$ is the componentwise product functor with

$(X \times Y)_i = X_i \times Y_i$ for $i < n$. The *graph* of a morphism $f\colon A \to B$ is the subset $\operatorname{gr} f$ of $A \times B$ with $(\operatorname{gr} f)_i = \{(a, af_i) \mid a \in A_i\}$ for $i < n$.

3. Diagonal algebras

This section provides a summary of the relationship between products of sets and diagonal algebras (compare [17, 18, 19], [20, Example 5.2.2]). Let n be a positive integer.

Definition 3.1. A *diagonal algebra* (of *degree* n) is an algebra (D, d) with an idempotent n-ary operation d satisfying

$$(x_{0,0} \cdots x_{0,n-1} d) \cdots (x_{n-1,0} \cdots x_{n-1,n-1} d) d = x_{0,0} \cdots x_{n-1,n-1} d\,,$$

an identity known as the *diagonal identity.*

Proposition 3.2. *Let A be an n-sorted set. Define an operation d on the product $\varprojlim A$—compare* (2.2)*—by*

$$(x_0^0, \dots, x_0^{n-1}) \cdots (x_{n-1}^0, \dots, x_{n-1}^{n-1}) d = (x_0^0, \dots, x_{n-1}^{n-1})\,. \tag{3.1}$$

Then $\big(\varprojlim A, d\big)$ is a diagonal algebra of degree n.

Proof. It is straightforward to verify the idempotence and diagonal identity for the operation (3.1). □

Proposition 3.3. *Let (D, d) be a diagonal algebra of degree n. For $i < n$, define a relation θ_i on D by*

$$\begin{aligned}(x, y) \in \theta_i \quad \Leftrightarrow \quad & \forall\ x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1} \in D\,, \\ & x_0 \cdots x_{i-1} x x_{i+1} \cdots x_{n-1} d = x_0 \cdots x_{i-1} y x_{i+1} \cdots x_{n-1} d\,.\end{aligned} \tag{3.2}$$

(a) *For each $i < n$, the relation θ_i is an equivalence relation on D.*
(b) *There is a pure n-sorted set*

$$D^\theta\colon i \mapsto D^{\theta_i}\,.$$

(c) *For $x_j \in D$ and $i < n$, one has*

$$x_i\ \theta_i\ x_0 \cdots x_i \cdots x_{n-1} d\,.$$

(d) *There is an isomorphism*

$$p\colon (D, d) \to \big(\varprojlim D^\theta, d\big); \quad x \mapsto (x^{\theta_0}, \dots, x^{\theta_{n-1}}) \tag{3.3}$$

of diagonal algebras.

Proof. (a): This is immediate from the definition of θ_i.

(b): If D is (non-)empty, so is each quotient D^{θ_i}.

(c): For elements y_k of D, idempotence and the diagonal identity yield

$$\begin{aligned} & y_0 \cdots (x_0 \cdots x_i \cdots x_{n-1} d) \cdots y_{n-1} d \\ & \quad = (y_0 \cdots y_0 d) \cdots (x_0 \cdots x_i \cdots x_{n-1} d) \cdots (y_{n-1} \cdots y_{n-1} d) d \\ & \quad = y_0 \cdots x_i \cdots y_{n-1} d\,, \end{aligned}$$

so $x_0 \cdots x_i \cdots x_{n-1}d \ \theta_i \ x_i$.

(d): For elements x_i of D with $i < n$, the diagonal identity gives

$$\begin{aligned} x_0p \cdots x_{n-1}pd &= (x_0^{\theta_0}, \dots, x_0^{\theta_{n-1}}) \cdots (x_{n-1}^{\theta_0}, \dots, x_{n-1}^{\theta_{n-1}})d \\ &= (x_0^{\theta_0}, \dots, x_{n-1}^{\theta_{n-1}}) \\ &= \big((x_0 \cdots x_{n-1}d)^{\theta_0}, \dots, (x_0 \cdots x_{n-1}d)^{\theta_{n-1}}\big) \\ &= (x_0 \cdots x_{n-1}d)p\,, \end{aligned}$$

the penultimate equality holding by (c). Thus, p is a homomorphism of diagonal algebras. Now suppose that for elements x and y of D, one has $xp = yp$. Then

$$x = xx \cdots xd = yx \cdots xd = yy \cdots xd = \cdots = yy \cdots yd = y$$

by successive application of the relationships $(x, y) \in \theta_0, \theta_1, \dots, \theta_{n-1}$. This means that p injects. Finally, consider the problem of showing that p surjects. Consider an element $(x_0^{\theta_0}, \dots, x_{n-1}^{\theta_{n-1}})$ of $\varprojlim D^\theta$. It will be shown by induction that for each $j < n$, there is an element y_j of D such that

$$(y_j^{\theta_0}, \dots, y_j^{\theta_j}, x_{j+1}^{\theta_{j+1}}, \dots, x_{n-1}^{\theta_{n-1}}) = (x_0^{\theta_0}, \dots, x_j^{\theta_j}, x_{j+1}^{\theta_{j+1}}, \dots, x_{n-1}^{\theta_{n-1}})\,. \tag{3.4}$$

For $j = 0$, take $y_0 = x_0$. Now suppose that (3.4) holds. Define

$$y_{j+1} = y_j \cdots y_j x_{j+1} \cdots x_{n-1}d\,.$$

By (c), one then has

$$x_i \ \theta_i \ y_j \ \theta_i \ y_j \cdots \overbrace{y_j}^{\text{slot } i} \cdots y_j x_{j+1} \cdots x_{n-1}d = y_{j+1}$$

for $i \le j$ and $x_{j+1} \ \theta_{j+1} \ y_j \cdots y_j x_{j+1} \cdots x_i \cdots x_{n-1}d = y_{j+1}$. This completes the induction step. $\square$

Theorem 3.4. *Let $\mathcal{P}_n$ be the full subcategory of $\mathbf{Set}^{\underline{n}}$ consisting of pure n-sorted sets. Let $\mathcal{D}_n$ be the variety of diagonal algebras of degree n, considered as a category with homomorphisms as morphisms. Then the categories $\mathcal{P}_n$ and $\mathcal{D}_n$ are equivalent.*

Proof. In view of Proposition 3.3(d), it remains to be shown that for a pure n-sorted set A, there is an isomorphism $A \cong (\varprojlim A)^\theta$, or in other words a set isomorphism $A_i \cong (\varprojlim A)^{\theta_i}$ for each $i < n$. This set isomorphism is achieved through the well-defined and mutually inverse maps

$$x^i \mapsto (x^1, \dots, x^i, \dots, x^{n-1})^{\theta_i}$$

and

$$(x^1, \dots, x^i, \dots, x^{n-1})^{\theta_i} \mapsto x^i\,.$$

Indeed, (3.1) and the definition of θ_i imply

$$(x^0, \dots, x^i, \dots, x^{n-1}) \ \theta_i \ (y^0, \dots, x^i, \dots, y^{n-1})$$

for elements x^j and y^j of A_j, with $j < n$. $\square$

4. Many-sorted algebras

Definition 4.1. Let n be a positive integer. An *n-sorted type* τ is a function

$$\tau\colon \Omega \to \mathbb{N}^{\underline{n}} \times \underline{n}\,; \quad \omega \mapsto (i \mapsto i\omega\tau, \iota_\omega)\,. \tag{4.1}$$

The set Ω is known as the *operator domain*, and its elements ω are known as *operators*. The type is said to be *constant-free* if for all $\omega \in \Omega$, there exists an $i < n$ such that $i\omega\tau > 0$.

Definition 4.2. A *τ-algebra* or *n-sorted algebra* (A, Ω) of (n-*sorted*) *type* τ is an n-sorted set A equipped with an *operation*

$$\begin{aligned} \omega\colon \prod_{i<n} A_i^{i\omega\tau} &\to A_{\iota_\omega}; \\ (a_1^0, \dots, a_{0\omega\tau}^0, \dots, a_1^{n-1}, \dots, a_{(n-1)\omega\tau}^{n-1}) & \\ \mapsto a_1^0 \cdots a_{0\omega\tau}^0 \cdots a_1^{n-1} \cdots a_{(n-1)\omega\tau}^{n-1}\omega & \end{aligned} \tag{4.2}$$

for each operator ω in the operator domain Ω.

Remark 4.3. The notation of (4.2) is to be understood with the convention that when $k\omega\tau = 0$ for some $0 \le k < n$, then no arguments $a_1^k, a_2^k, \dots$ appear in the operation ω.

Example 4.4. Consider the n-sorted set $\underline{\underline{n}}$ with $\underline{\underline{n}}_i = \{i\}$ for $i < n$. The *trivial* τ-algebra is defined to be $\big(\underline{\underline{n}}, \Omega\big)$ in which the image of an operation ω is $\{\iota_\omega\}$.

A subset B of an n-sorted algebra (A, Ω) is said to be a (*many-sorted*) *subalgebra* if B is closed under the operations (4.2).

Example 4.5. Let X be an n-sorted set. Consider $(\varprojlim 2^X) \times \underline{\underline{n}}$, the n-sorted product of the constant n-sorted set $\varprojlim 2^X$ with the n-sorted set $\underline{\underline{n}}$ of Example 4.4. Define the n-sorted set $2^X n$ by $(2^X n)_i = (\varprojlim 2^X) \times \{i\}$ for $i < n$. Then a τ-algebra $(2^X n, \Omega)$ is defined by the operation

$$\begin{aligned} \omega\colon \big((S_{0,0,1}, \dots, S_{n-1,0,1}, 0), \dots, (S_{0,0,0\omega\tau}, \dots, S_{n-1,0,0\omega\tau}, 0), \\ \dots, (S_{0,n-1,(n-1)\omega\tau}, \dots, S_{n-1,n-1,(n-1)\omega\tau}, n-1)\big) \\ \mapsto \big(S_{0,0,1} \cup \dots \cup S_{0,0,0\omega\tau} \cup \dots \cup S_{0,n-1,(n-1)\omega\tau}, \dots, \\ S_{n-1,0,1} \cup \dots \cup S_{n-1,0,0\omega\tau} \cup \dots \cup S_{n-1,n-1,(n-1)\omega\tau}, \iota_\omega\big) \end{aligned}$$

for each operator ω. This algebra is known as the (n-*sorted*) *power* (τ-)*algebra* of the n-sorted set X. Now let $2^X_{<\infty} n$ be the subset of $2^X n$ determined by the finite subsets of X. Then $(2^X_{<\infty} n, \Omega)$ forms a subalgebra of $(2^X n, \Omega)$.

Intersections of subalgebras are subalgebras (compare, e.g., [4, Prop. 1(2)0]). For a subset S of (the underlying n-sorted set A of) an n-sorted algebra (A, Ω), the *subalgebra* $\langle S \rangle$ *generated* by S is the intersection of all the subalgebras containing S. Now consider n-sorted algebras (A,Ω) and (B,Ω). The *product* (*algebra*) $(A\times B,\Omega)$ is defined by the componentwise structure

$$\begin{aligned}\prod_{i<n}(A_i^{i\omega\tau},B_i^{i\omega\tau}) &\to (A_{\iota_\omega},B_{\iota_\omega});\\ \big((a_1^0,b_1^0),\dots,(a^{n-1}_{(n-1)\omega\tau},b^{n-1}_{(n-1)\omega\tau})\big)&\\ \mapsto (a_1^0\cdots a^{n-1}_{(n-1)\omega\tau}\omega,\, b_1^0\cdots b^{n-1}_{(n-1)\omega\tau}\omega)&\end{aligned}$$

for each operator ω. A $(\mathbf{Set}^{\underline{n}})$-morphism $f\colon A\to B$ is a (*many-sorted*) (*algebra*) *homomorphism* $f\colon(A,\Omega)\to(B,\Omega)$ if its graph $\operatorname{gr} f$ is a subalgebra of $(A\times B,\Omega)$. The *image* $\operatorname{im} f$ or Af of f is the projection of $\operatorname{gr} f$ onto B. It forms a subalgebra of (B,Ω).

5. Derived types

Fix a positive integer n, and let X be an n-sorted set. Consider the free monoid $(\Omega+\varinjlim X)^*$ over the disjoint union of the sets Ω and X_i for $i<n$ (compare (2.3)). For $i<n$, define a subset

$$A_i = X_i\cup\bigcup\{(\Omega+\varinjlim X)^*\omega \mid \iota_\omega = i\}$$

of $(\Omega+\varinjlim X)^*$. For each operator ω, (4.2) defines an operation yielding an n-sorted algebra (A,Ω). Note that X is a subset of A. Then the *τ-word algebra* or *absolutely free τ-algebra* over X is defined to be the subalgebra $(X\Omega,\Omega)$ of (A,Ω) generated by X. Elements of the disjoint union $\varinjlim X\Omega$ are known as *Ω-words* over X. In particular, elements of $X\Omega_i$ are known as *i-flavored* Ω-words over X.

Example 5.1. Consider the constant-free two-sorted type

$$\tau=\big\{\big(f,(\{0\mapsto 1,1\mapsto 0\},1)\big),\big(g,(\{0\mapsto 0,1\mapsto 1\},0)\big)\big\}\ .$$

Thus, an algebra (B,Ω) of type τ has operations $f\colon B_0\to B_1$ and $g\colon B_1\to B_0$. Consider the two-sorted set X with $X_0=\{x\}$ and $X_1=\{y\}$. Let M be the free monoid over the alphabet $\{f,g,x,y\}$. Then $A_0=\{x\}\cup\{wg\mid w\in M\}$ and $A_1=\{y\}\cup\{wf\mid w\in M\}$. Finally, $X\Omega_0=\{x,yg,xfg,ygfg,\dots\}$ and $X\Omega_1=\{y,xf,ygf,xfgf,\dots\}$.

The absolutely free τ-algebra $(X\Omega,\Omega)$ over an n-sorted set X has the following universality property.

Proposition 5.2. *Let (B,Ω) be a τ-algebra. For each given $(\mathbf{Set}^{\underline{n}})$-morphism $f\colon X\to B$, there is a unique algebra homomorphism $\overline{f}\colon(X\Omega,\Omega)\to(B,\Omega)$ that extends $f\colon X\to B$.*

Proof. The graph of $\overline{f}$ is the subalgebra of $(X\Omega\times B,\Omega)$ generated by the graph of $f\colon X\to B$. □

The next definition makes immediate use of Proposition 5.2. For each $i < n$, write the ordered set of positive integers as

$$P_i = \{x_1^i < x_2^i < x_3^i < \cdots\} \tag{5.1}$$

and define a corresponding n-sorted set P. Using say x_1^i instead of the numeral 1 will have typographical advantages in what follows.

Definition 5.3. Let $s\colon P \to 2^P n$ be the singleton embedding with component

$$s_i\colon P_i \to (2^P n)_i; \quad x_k^i \mapsto (\varnothing, \dots, \overbrace{\{x_k^i\}}^{\text{slot } i}, \dots, \varnothing, i) \quad \text{for each } i < n.$$

Proposition 5.2 gives a homomorphic extension $\overline{s}\colon (P\Omega, \Omega) \to (2^P n, \Omega)$ of s into the power set algebra (compare Example 4.5), and the co-restriction $\widetilde{s}\colon (P\Omega, \Omega) \to (2^P_{<\infty} n, \Omega)$ of $\overline{s}$ to the n-sorted subalgebra $2^P_{<\infty} n$. Then the *argument map* is defined to be the disjoint union $\arg\colon \varinjlim P\Omega \to \varinjlim 2^P_{<\infty} n$ of the components $\widetilde{s}_i$ of $\widetilde{s}$ for $i < n$.

Example 5.4. Consider the two-sorted type

$$\tau = \big\{\big(f, (\{0 \mapsto 1, 1 \mapsto 1\}, 0)\big), \big(g, (\{0 \mapsto 1, 1 \mapsto 1\}, 1)\big)\big\}\ .$$

The argument of the word $x_1^0 x_1^1 f x_2^1 g$ is $(\{x_1^0\}, \{x_1^1, x_2^1\}, 1)$.

Definition 5.5. Consider the function $\max\colon \varinjlim 2^P_{<\infty} n \to \mathbb{N}^{\underline{n}} \times \underline{n}$ restricting to

$$(2^P_{<\infty} n)_i \to \mathbb{N}^{\underline{n}} \times \underline{n}\,; \quad (S_0, \dots, S_{n-1}, i) \mapsto (\max S_0, \dots, \max S_{n-1}, i) \tag{5.2}$$

for each $i < n$. Note that the sets S_i (for $i < n$) appearing in (5.2) are finite (possibly empty) subsets of the set (5.1) of positive integers. As such, the maximum $\max S_i$ is a natural number, 0 if S_i is empty. Then given an n-sorted type (4.1), the composite

$$\varinjlim P\Omega \xrightarrow{\arg} \varinjlim 2^P_{<\infty} n \xrightarrow{\max} \mathbb{N}^{\underline{n}} \times \underline{n} \tag{5.3}$$

is called the *derived type* $\tau'\colon \varinjlim P\Omega \to \mathbb{N}^{\underline{n}} \times \underline{n}$.

Note that a type τ', derived from a constant-free type τ, is itself constant-free.

Example 5.6. In the context of Example 5.4, the word $x_{01} x_{11} f x_{12} g$ has derived type $(1, 2, 1)$.

The codomain $\mathbb{N}^{\underline{n}} \times \underline{n}$ of the derived type (5.3) carries a componentwise order, as the product of copies of the ordered set of natural numbers and the antichain $\underline{n}$. For a function $f\colon A \to B$ to an ordered codomain $(B, \leq)$, the *epigraph* $\operatorname{epi} f$ of f is the subset $\{(a, b) \mid a \in A\,,\ af \leq b\}$ of $A \times B$.

Definition 5.7. Let $\tau\colon \Omega \to \mathbb{N}^{\underline{n}} \times \underline{n}$ be an n-sorted type. Define $\overline{\Omega} = \operatorname{epi} \tau'$, the epigraph of the derived type (5.3). Then the n-sorted type

$$\overline{\tau}\colon \overline{\Omega} \to \mathbb{N}^{\underline{n}} \times \underline{n}\,; \quad \big(w, (r_0, \dots, r_{n-1}, i)\big) \mapsto (r_0, \dots, r_{n-1}, i)$$

is called the *closure* of τ.

Again, note that the closure $\overline{\tau}$ of a constant-free type τ is itself constant-free.

6. Identities

Let (B,Ω) be a τ-algebra. Let $\big(w,(r_0,\dots,r_{n-1},i)\big)$ be an *i-flavored $\overline{\Omega}$-word*, an element of the operator domain $\overline{\Omega}$ of the closure of τ. Let X be the subset of P with $X_j=\{x_k^j \mid 1\le k\le r_j\}$ for each $j<n$. Note that $w\in(X\Omega)_i$. Now define an operation

$$\big(w,(r_0,\dots,r_{n-1},i)\big)\colon \prod_{j<n} B_j^{r_j}\to B_i \tag{6.1}$$

as follows. For an element

$$(b_1^0,\dots,b_{r_0}^0,\dots,b_1^{n-1},\dots,b_{r_{n-1}}^{n-1}) \tag{6.2}$$

of $\prod_{j<n} B_j^{r_j}$, consider the $(\mathbf{Set}^{\underline{n}})$-morphism $f\colon X\to B$ that has components $f_j\colon X_j\to B_j;\ x_k^j\mapsto b_k^j$ for $j<n$. Let $\overline{f}\colon (X\Omega,\Omega)\to(B,\Omega)$ be the homomorphic extension of f given by Proposition 5.2. Then the effect of the operation (6.1) on (6.2) is defined to be $w\overline{f}$. With this definition, the following proposition is obtained.

Proposition 6.1. *An n-sorted algebra (B,Ω) of type τ augments to an n-sorted algebra $(B,\overline{\Omega})$ of type $\overline{\tau}$.*

Definition 6.2. Let $\tau\colon\Omega\to\mathbb{N}^{\underline{n}}\times\underline{n}$ be an n-sorted type. Then for $i<n$, an *(i-flavored) identity* of type τ is a pair (u,v) of i-flavored $\overline{\Omega}$-words over a set X.

Remark 6.3. An identity (u,v) as in Definition 6.2 is usually written in the form $u=v$. Furthermore, i-flavored words $\big(w,(r_0,\dots,r_{n-1},i)\big)$ appearing in identities are often abbreviated as w, with the corresponding elements $(r_0,\dots,r_{n-1},i)$ from the epigraph of the derived type (5.3) being understood implicitly.

Definition 6.4. Let $u=v$ be an i-flavored identity of type τ over a set X. Then a pure τ-algebra (B,Ω) is said to *purely satisfy* the identity $u=v$ if the operations u and v coincide on the augmented algebra $(B,\overline{\Omega})$.

Remark 6.5. Fuller discussions of the satisfaction of identities in many-sorted algebras, including the critical role of quantifiers, may be found in references such as [5, 7]. Definition 6.4 is introduced for the limited purposes of the current paper, chiefly the presentation of the following examples, and Proposition 8.2 below.

Example 6.6 ([13, 14, 21])**.** A *fibred automaton* is a 2-sorted algebra (A,Ω) with a *maternal operation* $\delta\colon A_0\to A_0$, a *paternal operation* $\varepsilon\colon A_0\to A_1$, and an *action* $\mu\colon A_0\times A_1\to A_0$. A pure fibred automaton purely satisfies the 0-flavored identities $x_1^0\delta x_1^0\varepsilon\mu = x_1^0 = x_1^0 x_1^1\mu\delta$ and the 1-flavored identity $x_1^0 x_1^1\mu\varepsilon = x_1^1$. Note that the latter identity is formally written as the pair

$$\big((x_1^0 x_1^1\mu\varepsilon,(1,1,1)),(x_1^1,(1,1,1))\big)$$

if the abbreviation of Remark 6.3 is not used. Since $\tau'\colon x_1^1 \mapsto (0,1,1)$, one has $\big(x_1^1,(1,1,1)\big)$ in the epigraph $\overline{\Omega}$ of τ'.

Example 6.7 ([8, 9, 22])**.** A *reversible automaton* (*of quasigroup type*) is a 3-sorted algebra (A,Ω) equipped with a *multiplication* $\mu\colon A_1\times A_2 \to A_0$, a *right division* $\rho\colon A_0\times A_2\to A_1$, and *left division* $\lambda\colon A_1\times A_0\to A_2$. A pure reversible automaton purely satisfies the 0-flavored identities $x_1^0x_1^2\rho x_1^2\mu = x_1^0 = x_1^1x_1^1x_1^0\lambda\mu$, the 1-flavored identity $x_1^1x_1^2\mu x_1^2\rho = x_1^1$, and the 2-flavored identity $x_1^1x_1^1x_1^2\mu\lambda = x_1^2$.

7. Homogenization

Let n be a positive integer. The topic of this section is the passage from n-sorted (heterogeneous) algebras to single-sorted (homogeneous) algebras, a process known as *homogenization*.

Definition 7.1. Let $\tau\colon \Omega\to\mathbb{N}^{\underline{n}}\times\underline{n}\,;\ \omega\mapsto(i\mapsto i\omega\tau,\iota_\omega)$ be an n-sorted type. Then its *pre-homogenization* or *pre-homogenized type* is

$$\tau^{\flat'}\colon\Omega\to\mathbb{N};\quad \omega\mapsto\sum_{i<n} i\omega\tau\,, \tag{7.1}$$

a single-sorted type.

Definition 7.2. Let τ be a constant-free n-sorted type, with operator domain Ω. For an operator ω from Ω, define the *socle*

$$s_\omega = \min\{i \mid 0 < i\omega\tau\}$$

and the *top*

$$t_\omega = \max\{i \mid 0 < i\omega\tau\}\,.$$

Note that these numbers are well defined, since τ is constant-free.

Definition 7.3. Let τ be a constant-free n-sorted type, with operator domain Ω. Let (A,Ω) be a τ-algebra. Its *pre-homogenization* is the $\tau^{\flat'}$-algebra $(\varprojlim A,\Omega)$ equipped with an operation

$$\begin{aligned}\omega^\flat\colon(\varprojlim A)^{\omega\tau^{\flat'}} &\to \varprojlim A; \\ \big((a^{s_\omega}_{0,1},\dots,a^{s_\omega}_{n-1,1}),\dots,&(a^{s_\omega}_{0,s_\omega\omega\tau},\dots,a^{s_\omega}_{n-1,s_\omega\omega\tau}),\dots \\ \dots,(a^{t_\omega}_{0,1},\dots,a^{t_\omega}_{n-1,1}),&\dots,(a^{t_\omega}_{0,t_\omega\omega\tau},\dots,a^{t_\omega}_{n-1,t_\omega\omega\tau})\big) \\ \mapsto \big(a^{s_\omega}_{0,1},\dots,&\overbrace{a^{s_\omega}_{s_\omega,1}\cdots a^{s_\omega}_{s_\omega,s_\omega\omega\tau}\cdots a^{t_\omega}_{t_\omega,1}\cdots a^{t_\omega}_{t_\omega,t_\omega\omega\tau}\omega}^{\text{slot }\iota_\omega},\dots,a^{s_\omega}_{n-1,1}\big)\end{aligned} \tag{7.2}$$

for each operator ω.

Remark 7.4. Note that in (7.2), the superscripts placed on the arguments of the homogeneous operation $\omega^\flat$ track the sorts that actually appear in the domain of the heterogeneous operation ω.

Definition 7.5. Let $\tau\colon \Omega \to \mathbb{N}^{\underline{n}} \times \underline{n}\,;\ \omega \mapsto (i \mapsto i\omega\tau, \iota_\omega)$ be an n-sorted type. Then its *homogenization* or *homogenized type* is the disjoint union

$$\tau^\flat = \tau^{\flat'} + \{(d,n)\} \tag{7.3}$$

of the pre-homogenized type (7.1) with the type (d,n) of a diagonal algebra of degree n.

Definition 7.6. Let τ be a constant-free n-sorted type, with operator domain Ω. Let (A,Ω) be a τ-algebra. Its *homogenization* is the $\tau^\flat$-algebra $(\varprojlim A, \Omega, d)$, equipped with the operation (7.2) for each operator ω, such that the reduct $(\varprojlim A, d)$ is a diagonal algebra.

Proposition 7.7. *Let τ be a constant-free n-sorted type, with operator domain Ω. Let (A,Ω) be a τ-algebra. Then the homogenization $(\varprojlim A, \Omega, d)$ satisfies the quasi-identity*

$$\forall\ i < n\,,\ \forall\ 1 \leq j \leq i\omega\tau\,,\ z^0 \cdots x^i_j \cdots z^{n-1} d = z^0 \cdots y^i_j \cdots z^{n-1} d$$

$$\Rightarrow z^0 \cdots \overbrace{\left(x_1^{s_\omega} \cdots x^{t_\omega}_{t_\omega\omega\tau}\omega^\flat\right)}^{\textit{slot } \iota_\omega} \cdots z^{n-1} d = z^0 \cdots \overbrace{\left(y_1^{s_\omega} \cdots y^{t_\omega}_{t_\omega\omega\tau}\omega^\flat\right)}^{\textit{slot } \iota_\omega} \cdots z^{n-1} d$$

for each operator ω in Ω.

Proof. Let ω be an operator in Ω. In (7.2), the ι_ω slot of the value of the operation $\omega^\flat$ only depends on the component $a^i_{i,j}$ of the argument

$$(a^i_{0,j}, \ldots, a^i_{i,j}, \ldots, a^i_{n-1,j})\,,$$

for each $i < n$ and $1 \leq j \leq i\omega\tau$. Thus,

$$\forall\ i < n\,,\ \forall\ 1 \leq j \leq i\omega\tau\,,$$
$$(x^i_j, y^i_j) \in \theta_i \ \Rightarrow\ \left(x_1^{s_\omega} \cdots x^{t_\omega}_{t_\omega\omega\tau}\omega^\flat, y_1^{s_\omega} \cdots y^{t_\omega}_{t_\omega\omega\tau}\omega^\flat\right) \in \theta_{\iota_\omega}$$

in the homogenization $(\varprojlim A, \Omega, d)$. By (3.2), this implication translates to the quasi-identity of the proposition. □

Proposition 7.8. *Let τ be a constant-free n-sorted type, with operator domain Ω. Let (A,Ω) be a τ-algebra. Then the homogenization $(\varprojlim A, \Omega, d)$ satisfies the identity*

$$z^0 \cdots \overbrace{\left(x_1^{s_\omega} \cdots x^{t_\omega}_{t_\omega\omega\tau}\omega^\flat\right)}^{\textit{slot } i} \cdots z^{n-1} d = z^0 \cdots \overbrace{x^i_1}^{\textit{slot } i} \cdots z^{n-1} d$$

for each operator ω in Ω and index $\iota_\omega \neq i < n$.

Proof. Let ω be an operator in Ω, and suppose $\iota_\omega \neq i < n$. In (7.2), the i-slot of the value of the operation $\omega^\flat$ is $a^{s_\omega}_{i,1}$, which is the i-slot of the first argument. Thus, $\left(x_1^{s_\omega} \cdots x^{t_\omega}_{t_\omega\omega\tau}\omega^\flat, x^i_1\right) \in \theta_i$ in the homogenization $(\varprojlim A, \Omega, d)$. By (3.2), this implication translates to the identity of the proposition. □

8. Homogenized identities

Consider homogenization of a pure many-sorted algebra having constant-free type. Proposition 7.7 gave quasi-identities satisfied by the homogenization. Proposition 7.8 gave identities satisfied by the homogenization. These quasi-identities and identities, which are the main concern of the current paper, depend only on the type of the many-sorted algebra. However, the original pure many-sorted algebra may purely satisfy various identities itself. This brief section (which is incidental to the remainder of the paper, and may therefore be skipped if desired) illustrates one approach to the transfer of such identities from the many-sorted algebra to its homogenization. An examination of alternative approaches, and the relationships between them, is deferred to a later paper (cf. [6, Prop. 2.8]).

Lemma 8.1. *As in Definition* 7.6*, the augmentation* $(A, \overline{\Omega})$ *of* (A, Ω) *given by Proposition* 6.1 *will homogenize to a* $\overline{\tau}^{\flat}$*-algebra* $(\varprojlim A, \overline{\Omega}, d)$*. Each operator* $\big(w, (r_0, \dots, r_{n-1}, i)\big)$ *from* $\overline{\Omega}$ *thus yields an operation* $\big(w, (r_0, \dots, r_{n-1}, i)\big)^{\flat}$ *on* $(\varprojlim A, \overline{\Omega}, d)$*.*

Proposition 8.2. *Let* τ *be a constant-free* n*-sorted type, with operator domain* Ω*. Let* (A, Ω) *be a pure* τ*-algebra. Suppose that for* $i < n$*, the* n*-sorted algebra* (A, Ω) *purely satisfies an* i*-flavored identity* $u = v$*. Then the identity*

$$x_0 \cdots \overbrace{\ u^{\flat}\ }^{\textit{slot } i} \cdots x_{n-1} d = x_0 \cdots \overbrace{\ v^{\flat}\ }^{\textit{slot } i} \cdots x_{n-1} d$$

is satisfied by the homogenization $(\varprojlim A, \Omega, d)$*.*

Proof. Apply Proposition 7.7 to the $\overline{\tau}$-algebra $(A, \overline{\Omega})$. □

Example 8.3. Let $((A_0, A_1), \{\delta, \varepsilon, \mu\})$ be a pure fibred automaton, as defined in Example 6.6. The homogenization of the automaton $((A_0, A_1), \{\delta, \varepsilon, \mu\})$ is the algebra $(\varprojlim A = A_0 \times A_1, \{\delta^{\flat}, \varepsilon^{\flat}, \mu^{\flat}, d\})$ with

$$\begin{aligned}
\delta^{\flat} &\colon \varprojlim A \to \varprojlim A; && (a^0_{0,1}, a^0_{1,1}) \mapsto (a^0_{0,1}\delta, a^0_{1,1})\,,\\
\varepsilon^{\flat} &\colon \varprojlim A \to \varprojlim A; && (a^0_{0,1}, a^0_{1,1}) \mapsto (a^0_{0,1}, a^0_{0,1}\varepsilon)\,,\\
\mu^{\flat} &\colon \big(\varprojlim A\big)^2 \to \varprojlim A; && ((a^0_{0,1}, a^0_{1,1}), (a^1_{0,1}, a^1_{1,1})) \mapsto (a^0_{0,1}a^1_{1,1}\mu, a^0_{1,1})
\end{aligned}$$

and d as a binary diagonal operation. Now the augmentation of the automaton has operations

$$\big(x^0_1 x^1_1 \mu\varepsilon, (1,1,1)\big) \colon (x_0, x_1) \mapsto x_0 x_1 \mu\varepsilon$$

and

$$\big(x^1_1, (1,1,1)\big) \colon (x_0, x_1) \mapsto x_1\,.$$

Operations of the homogenized augmentation are thus defined by

$$\begin{aligned}
\big(x^0_1 x^1_1 \mu\varepsilon, (1,1,1)\big)^{\flat} &\colon \big(\varprojlim A\big)^2 \to \varprojlim A;\\
&((a^0_{0,1}, a^0_{1,1}), (a^1_{0,1}, a^1_{1,1})) \mapsto (a^0_{0,1}, a^0_{0,1}a^1_{1,1}\mu\varepsilon)
\end{aligned}$$

and

$$\left(x_1^1,(1,1,1)\right)^\flat\colon \left(\varprojlim A\right)^2 \to \varprojlim A;$$
$$\left((a_{0,1}^0,a_{1,1}^0),(a_{0,1}^1,a_{1,1}^1)\right)\mapsto (a_{0,1}^0,a_{1,1}^1)\,.$$

Since the pure fibred automaton $((A_0,A_1),\{\delta,\varepsilon,\mu\})$ purely satisfies the identity $x_{01}x_{11}\mu\varepsilon = x_{11}$, it follows by Proposition 8.2 that the identity

$$x_1\Big(x_2x_3\big(x_1^0x_1^1\mu\varepsilon,(1,1,1)\big)^\flat\Big)d = x_1\Big(x_2x_3\big(x_1^1,(1,1,1)\big)^\flat\Big)d$$

is satisfied by the homogenization of the fibred automaton.

9. Heterogenization

Let n be a positive integer. The topic of this section is the passage from single-sorted (homogeneous) algebras to n-sorted (heterogeneous) algebras, a process known as *heterogenization.*

Definition 9.1. Let $\tau\colon\Omega\to\mathbb{N}$ be a single-sorted type. Then an algebra (A,Ω,d) is said to be (*monotonically*) *heterogenizable* if the following hold:

(a) The type τ is constant-free, i.e., $0\notin\Omega\tau$.
(b) The reduct (A,d) is a diagonal algebra of degree n.
(c) For each ω in Ω, there is a *selection function* $f_\omega\colon\{1,\dots,\omega\tau\}\to\underline{n}$ and a natural number $g_\omega<n$ such that the identities

$$z^0\cdots\overbrace{(x_1\cdots x_{\omega\tau}\omega)}^{\text{slot } i}\cdots z^{n-1}d = z^0\cdots\overbrace{x_1}^{\text{slot } i}\cdots z^{n-1}d$$

for each $g_\omega\neq i<n$, and the quasi-identity

$$\forall\ i<n\,,\ \forall\ j\in f_\omega^{-1}\{i\}\,,$$
$$z^0\cdots\overbrace{x_j}^{\text{slot } i}\cdots z^{n-1}d = z^0\cdots\overbrace{y_j}^{\text{slot } i}\cdots z^{n-1}d$$
$$\Rightarrow z^0\cdots\overbrace{(x_1\cdots x_{\omega\tau}\omega)}^{\text{slot } g_\omega}\cdots z^{n-1}d = z^0\cdots\overbrace{(y_1\cdots y_{\omega\tau}\omega)}^{\text{slot } g_\omega}\cdots z^{n-1}d$$

are satisfied.
(d) The selection function $f_\omega\colon\{1<\cdots<\omega\tau\}\to\{0<\cdots<n-1\}$ is monotone for each ω in Ω.

For a monotonically heterogenizable algebra (A,Ω,d) as above, the *input function* is defined to be the function

$$k\colon\Omega\to\mathbb{N}^n;\quad \omega\mapsto\big(|f_\omega^{-1}\{0\}|,\dots,|f_\omega^{-1}\{n-1\}|\big)\,.$$

The *output function* is defined to be the function $g\colon\Omega\to\underline{n};\quad \omega\mapsto g_\omega\,.$

Example 9.2. For a positive integer n, note that diagonal algebras of degree n are heterogenizable, since the conditions (c) and (d) of Definition 9.1 are satisfied vacuously.

Using Propositions 7.7 and 7.8, one easily checks the following.

Proposition 9.3. *Suppose $\sigma\colon \Omega \to \mathbb{N}^{\underline{n}} \times \underline{n};\ \omega \mapsto (i \mapsto i\omega\sigma, \iota_\omega)$ is an n-sorted type. Let (B, Ω) be a σ-algebra. Then the homogenization $(\varinjlim B, \Omega, d)$ is monotonically heterogenizable, with input function*

$$k\colon \omega \mapsto \big(0\omega\sigma, \dots, (n-1)\omega\sigma\big)$$

and output function $g\colon \omega \mapsto \iota_\omega$.

Definition 9.4. Let $\tau\colon \Omega \to \mathbb{N}$ be a single-sorted type.

(a) An *output function* (of *degree* n) is a function $h\colon \Omega \to \underline{n}\,;\quad \omega \mapsto h_\omega\,.$
(b) Define the sum function $\Sigma\colon \mathbb{N}^n \to \mathbb{N};\quad (l_0, \dots, l_{n-1}) \mapsto l_0 + \dots + l_{n-1}\,.$ Then an *input function* (of *degree* n) is a function

$$l\colon \Omega \to \mathbb{N}^n;\quad \omega \mapsto (l_0^\omega, \dots, l_{n-1}^\omega)$$

such that $l\Sigma = \tau$. (In other words, $l_0^\omega + \dots + l_{n-1}^\omega = \omega\tau$ for each ω.)

Definition 9.5. Let $\tau\colon \Omega \to \mathbb{N}$ be a single-sorted type, with given input function l and output function h of degree n. Then the function

$$\tau_{lh}^\sharp\colon \Omega \to \mathbb{N}^{\underline{n}} \times \underline{n}\,;\quad \omega \mapsto (i \mapsto l_i^\omega, h_\omega)$$

is the corresponding *heterogenization* or *heterogenized type.* If the input and output functions are clear from the context, the heterogenized type will be denoted simply by $\tau^\sharp$.

Lemma 9.6. *Let $\tau\colon \Omega \to \mathbb{N}$ be a constant-free single-sorted type, with given input function l and output function h of degree n. Then the heterogenized type $\tau_{lh}^\sharp$ is also constant-free.*

Proof. If $\tau_{lh}^\sharp$ were not constant-free, there would be an operator ω from Ω with $l_i^\omega = 0$ for each $i < n$. Then $0 = \sum_{i<n} l_i^\omega = \omega\tau$ would contradict the constant-freedom of τ. $\square$

In the context of Lemma 9.6, the socle and top of an operator ω within the heterogenized type $\tau_{lh}^\sharp$ will be written as $s_\omega^\sharp$ and $t_\omega^\sharp$, respectively. Thus,

$$s_\omega^\sharp = \min\{i < n \mid l_i^\omega \neq 0\} \tag{9.1}$$

and

$$t_\omega^\sharp = \max\{i < n \mid l_i^\omega \neq 0\} \tag{9.2}$$

for each operator ω.

Definition 9.7. Let $\tau\colon \Omega \to \mathbb{N}$ be a single-sorted type, with given input function l and output function h of degree n. A monotonically heterogenizable algebra (A, Ω, d) is *compatible* if l is its input function and h is its output function.

Proposition 9.8. *Let $\tau\colon \Omega \to \mathbb{N}$ be a single-sorted type, with given input function l and output function h of degree n. Then the class of monotonically heterogenizable, compatible algebras (A, Ω, d) forms a variety.*

Proof. An algebra (A, Ω, d) is monotonically heterogenizable and compatible if and only if:

(a) The following identities are satisfied for each $\omega \in \Omega$ and $h_\omega \neq i < n$:

$$z^0 \cdots \overbrace{(x_1 \cdots x_{\omega\tau}\omega)}^{\text{slot } i} \cdots z^{n-1} d = z^0 \cdots \overbrace{x_1}^{\text{slot } i} \cdots z^{n-1} d \tag{9.3}$$

(b) The following quasi-identities are satisfied for each $\omega \in \Omega$:

$$\begin{aligned} &\forall\ i < n\,,\ \forall\ l_0^\omega + \cdots + l_{i-1}^\omega < j \leq l_0^\omega + \cdots + l_i^\omega\,,\\ &z^0 \cdots \overbrace{x_j}^{\text{slot } i} \cdots z^{n-1} d = z^0 \cdots \overbrace{y_j}^{\text{slot } i} \cdots z_n d \\ &\Rightarrow z^0 \cdots \overbrace{(x_1 \cdots x_{\omega\tau}\omega)}^{\text{slot } h_\omega} \cdots z^{n-1} d = z^0 \cdots \overbrace{(y_1 \cdots y_{\omega\tau}\omega)}^{\text{slot } h_\omega} \cdots z^{n-1} d \end{aligned} \tag{9.4}$$

(c) The reduct (A, d) is a diagonal algebra.

It will be shown below that (b) is equivalent to the following:

(b′) The following identities are satisfied for each $\omega \in \Omega$:

$$\begin{aligned} u^0 \cdots u^{h_\omega - 1} \Bigg(\Big(z^0 \cdots \overbrace{x_1}^{\text{slot } s_\omega^\sharp} \cdots z^{n-1} d \Big) \cdots \\ \cdots \Big(z^0 \cdots \overbrace{x_{\omega\tau}}^{\text{slot } t_\omega^\sharp} \cdots z^{n-1} d \Big) \omega \Bigg) u^{h_\omega+1} \cdots u^{n-1} d \\ = u^0 \cdots u^{h_\omega - 1} \big(z^0 \cdots z^{h_\omega - 1}\, (x_1 \cdots x_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1} d \big) u^{h_\omega+1} \cdots u^{n-1} d \end{aligned}$$

Note the use of the notation (9.1) and (9.2) in (b′). The conditions (a), (b′), and (c) will then specify the class of monotonically heterogenizable, compatible algebras as a variety.

(b) ⇒ (b′): If the quasi-identity (9.4) holds for a certain operator ω, then the congruences

$$\forall\ i < n\,,\ \forall\ l_0^\omega + \cdots + l_{i-1}^\omega < j \leq l_0^\omega + \cdots + l_i^\omega\,,\ x_j\ \theta_i\ y_j$$

imply the congruence $x_1 \cdots x_{\omega\tau}\omega\ \theta_{h_\omega}\ y_1 \cdots y_{\omega\tau}\omega$. Now for $i < n$ and

$$l_0^\omega + \cdots + l_{i-1}^\omega < j \leq l_0^\omega + \cdots + l_i^\omega\,,$$

take $y_j = z_1 \cdots z_{i-1} x_j z_{i+1} \cdots z_{n-1} d$, so that $x_j\ \theta_i\ y_j$ by Proposition 3.3(c). The quasi-identity (9.4) then yields the congruence

$$x_1 \cdots x_{\omega\tau}\omega\ \theta_{h_\omega}\ \Big(z^0 \cdots \overbrace{x_1}^{\text{slot } s_\omega^\sharp} \cdots z^{n-1} d \Big) \cdots \Big(z^0 \cdots \overbrace{x_{\omega\tau}}^{\text{slot } t_\omega^\sharp} \cdots z^{n-1} d \Big) \omega\,.$$

Proposition 3.3(c) in turn gives the congruence

$$z^0 \cdots z^{h_\omega - 1}\, (x_1 \cdots x_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1} d\ \theta_{h_\omega}\ x_1 \cdots x_{\omega\tau}\omega\,,$$

so

$$z^0 \cdots z^{h_\omega - 1}\,(x_1 \cdots x_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d \quad \theta_{h_\omega}$$
$$\Big(z^0 \cdots \overbrace{x_1}^{\text{slot } s^\sharp_\omega} \cdots z^{n-1}d\Big) \cdots \Big(z^0 \cdots \overbrace{x_{\omega\tau}}^{\text{slot } t^\sharp_\omega} \cdots z^{n-1}d\Big)\omega$$

follows by transitivity. By (3.2), the identity of (b′) holds for the operator ω.

(b′) ⇒ (b): Suppose the identity of (b′) holds for a certain operator ω, and

$$\forall\ i < n\,,\ \forall\ l^\omega_0 + \cdots + l^\omega_{i-1} < j \leq l^\omega_0 + \cdots + l^\omega_i\,,$$
$$z^0 \cdots \overbrace{x_j}^{\text{slot } i} \cdots z^{n-1}d = z^0 \cdots \overbrace{y_j}^{\text{slot } i} \cdots z_n d\,.$$

Then

$$
\begin{aligned}
&u^0 \cdots u^{h_\omega-1}\big(z^0 \cdots z^{h_\omega-1}\,(x_1 \cdots x_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d\big)u^{h_\omega+1} \cdots u^{n-1}d\\
&= u^0 \cdots\\
&\quad u^{h_\omega-1}\bigg(\Big(z^0 \cdots \overbrace{x_1}^{\text{slot } s^\sharp_\omega} \cdots z^{n-1}d\Big) \cdots \Big(z^0 \cdots \overbrace{x_{\omega\tau}}^{\text{slot } t^\sharp_\omega} \cdots z^{n-1}d\Big)\omega\bigg)u^{h_\omega+1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots u^{n-1}d\\
&= u^0 \cdots\\
&\quad u^{h_\omega-1}\bigg(\Big(z^0 \cdots \overbrace{y_1}^{\text{slot } s^\sharp_\omega} \cdots z^{n-1}d\Big) \cdots \Big(z^0 \cdots \overbrace{y_{\omega\tau}}^{\text{slot } t^\sharp_\omega} \cdots z^{n-1}d\Big)\omega\bigg)u^{h_\omega+1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots u^{n-1}d\\
&= u^0 \cdots u^{h_\omega-1}\big(z^0 \cdots z^{h_\omega-1}\,(y_1 \cdots y_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d\big)u^{h_\omega+1} \cdots u^{n-1}d\,,
\end{aligned}
$$

so that

$$z^0 \cdots z^{h_\omega-1}\,(x_1 \cdots x_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d$$
$$\theta_{h_\omega} \quad z^0 \cdots z^{h_\omega-1}\,(y_1 \cdots y_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d$$

by (3.2). Now by Proposition 3.3(c),

$$x_1 \cdots x_{\omega\tau}\omega \quad \theta_{h_\omega} \quad z^0 \cdots z^{h_\omega-1}\,(x_1 \cdots x_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d,$$
$$z^0 \cdots z^{h_\omega-1}\,(y_1 \cdots y_{\omega\tau}\omega)\, z^{h_\omega+1} \cdots z^{n-1}d \quad \theta_{h_\omega} \quad y_1 \cdots y_{\omega\tau}\omega\,.$$

Transitivity of θ_{h_ω} then yields $x_1 \cdots x_{\omega\tau}\omega \quad \theta_{h_\omega} \quad y_1 \cdots y_{\omega\tau}\omega$, which by (3.2) translates to the desired conclusion

$$z^0 \cdots \overbrace{(x_1 \cdots x_{\omega\tau}\omega)}^{\text{slot } h_\omega} \cdots z^{n-1}d = z^0 \cdots \overbrace{(y_1 \cdots y_{\omega\tau}\omega)}^{\text{slot } h_\omega} \cdots z^{n-1}. \qquad \square$$

Proposition 9.9. *Let $\tau\colon \Omega \to \mathbb{N}$ be a single-sorted type, with given input function l and output function h of degree n. Let (A, Ω, d) be a monotonically heterogenizable, compatible algebra. Then there is an n-sorted $\tau^\sharp_{lh}$-algebra*

(A^θ, Ω) with well-defined operation

$$\begin{aligned}\omega^\sharp \colon & \prod_{i<n} (A^{\theta_i})^{i\omega\tau^\sharp} \to A^{\theta_{h_\omega}}; \\ & \big(a^{\theta_0}_{0,1}, \dots, a^{\theta_0}_{0,0\omega\tau^\sharp}, \dots, a^{\theta_{n-1}}_{n-1,1}, \dots, a^{\theta_{n-1}}_{n-1,(n-1)\omega\tau^\sharp}\big) \mapsto a_{s^\sharp_\omega,1} \cdots a_{t^\sharp_\omega, t^\sharp_\omega \omega\tau^\sharp} \omega^{\theta_{h_\omega}}\end{aligned} \tag{9.5}$$

for each operator ω in Ω.

Proof. Suppose that for all $i < n$ and all $1 \le j \le i\omega\tau^\sharp$, $(a_{i,j}, b_{i,j}) \in \theta_i$ for elements $a_{i,j}, b_{i,j}$ of A. Then for $i < n$ and $1 \le j \le i\omega\tau^\sharp$, the identities

$$z^0 \cdots \overbrace{a_{i,j}}^{\text{slot } i} \cdots z^{n-1} d = z^0 \cdots \overbrace{b_{i,j}}^{\text{slot } i} \cdots z^{n-1} d$$

are satisfied. Since (A, Ω, d) is a monotonically heterogenizable, compatible algebra, the identity

$$z^0 \cdots \overbrace{(a_{s^\sharp_\omega,1} \cdots a_{t^\sharp_\omega, t^\sharp_\omega \omega\tau^\sharp} \omega)}^{\text{slot } h_\omega} \cdots z^{n-1} = z^0 \cdots \overbrace{(b_{s^\sharp_\omega,1} \cdots b_{t^\sharp_\omega, t^\sharp_\omega \omega\tau^\sharp} \omega)}^{\text{slot } h_\omega} \cdots z^{n-1}$$

is satisfied. Thus, $a_{s^\sharp_\omega,1} \cdots a_{t^\sharp_\omega, t^\sharp_\omega \omega\tau^\sharp}\omega$ and $b_{s^\sharp_\omega,1} \cdots b_{t^\sharp_\omega, t^\sharp_\omega \omega\tau^\sharp}\omega$ are related by θ_{h_ω}, as required for the operation (9.5) to be well defined. □

Definition 9.10. The n-sorted $\tau^\sharp$-algebra (A^θ, Ω) of Proposition 9.9 is called the *heterogenization* of the single-sorted algebra (A, Ω, d).

10. Equivalence

Let n be a positive integer. The results of this section describe the equivalence between classes of pure n-sorted algebras and classes of single-sorted algebras.

Theorem 10.1. *Let $\tau \colon \Omega \to \mathbb{N}^{\underline{n}} \times \underline{n};\ \omega \mapsto (i \mapsto i\omega\tau, \iota_\omega)$ be a constant-free n-sorted type, with homogenization $\tau^\flat$. Consider the input function*

$$k \colon \omega \mapsto \big(0\omega\tau, \dots, (n-1)\omega\tau\big)$$

and output function $g \colon \omega \mapsto \iota_\omega$. Then the following classes are equivalent:

(a) *the class of pure n-sorted τ-algebras,*
(b) *the variety of single-sorted $\tau^\flat$-algebras (D, Ω, d) that are monotonically heterogenizable and compatible with the input function k and the output function g.*

Proof. Let (A, Ω) be a pure τ-algebra. By Proposition 9.3, the homogenization $(\varprojlim A, \Omega, d)$ is monotonically heterogenizable, with input function k and output function g. Proposition 9.9 then furnishes an n-sorted $(\tau^\flat)^\sharp_{kg}$-algebra $\big((\varprojlim A)^\theta, \Omega\big)$. Let ω be an operator from Ω, giving an operation of (A, Ω) that may be written as follows:

$$\omega : (a^{s_\omega}_{s_\omega,1}, \dots, a^{t_\omega}_{t_\omega, t_\omega\omega\tau}) \mapsto a^{s_\omega}_{s_\omega,1} \cdots a^{t_\omega}_{t_\omega, t_\omega\omega\tau}\omega \tag{10.1}$$

—compare with (4.2). (The apparently redundant indexing prepares for subsequent padding.) The corresponding operation (7.2) of the homogenization $(\varprojlim A, \Omega, d)$ may be summarized as

$$\begin{aligned}\omega^\flat :&\big((a^0_{s_\omega,1},\dots,a^{s_\omega}_{s_\omega,1},\dots,a^{n-1}_{s_\omega,1}),\dots,(\dots,a^{t_\omega}_{t_\omega,t_\omega\omega\tau},\dots,a^{n-1}_{t_\omega,t_\omega\omega\tau})\big)\\ &\mapsto (a^0_{s_\omega,1},\dots,\overbrace{a^{s_\omega}_{s_\omega,1}\cdots a^{t_\omega}_{t_\omega,t_\omega\omega\tau}\omega}^{\text{slot }\iota_\omega},\dots,a^{n-1}_{s_\omega,1})\,.\end{aligned}$$

In turn, (9.5) provides an operation $(\omega^\flat)^\sharp$ of $\big((\varprojlim A)^\theta, \Omega\big)$ that may be written briefly as:

$$\begin{aligned}\omega^{\flat\sharp}\colon &\big((a^0_{s_\omega,1},\dots,a^{s_\omega}_{s_\omega,1},\dots,a^{n-1}_{s_\omega,1})^{\theta_{s_\omega}},\dots,(\dots,a^{t_\omega}_{t_\omega,t_\omega\omega\tau},\dots,a^{n-1}_{t_\omega,t_\omega\omega\tau})^{\theta_{t_\omega}}\big)\\ &\mapsto (a^0_{s_\omega,1},\dots,\overbrace{a^{s_\omega}_{s_\omega,1}\cdots a^{t_\omega}_{t_\omega,t_\omega\omega\tau}\omega}^{\text{slot }\iota_\omega},\dots,a^{n-1}_{s_\omega,1})^{\theta_{\iota_\omega}}\,.\end{aligned}\tag{10.2}$$

Comparing (10.1) with (10.2), it becomes apparent that the n-set isomorphism $A \cong (\varprojlim A)^\theta$ of Theorem 3.4 serves to yield an n-sorted algebra isomorphism $(A, \Omega) \cong \big((\varprojlim A)^\theta, \Omega\big)$.

Conversely, let (D, Ω, d) be a $\tau^\flat$-algebra that is monotonically heterogenizable and compatible with the input function k and output function g. Proposition 9.9 gives an n-sorted $(\tau^\flat)^\sharp_{kg}$-algebra (D^θ, Ω). Let ω be an operator from Ω, giving an operation

$$\omega\colon (d_1,\dots,d_{\omega\tau}) \mapsto d_1\cdots d_{\omega\tau}\omega \tag{10.3}$$

of (D, Ω, d). The operation (9.5) of the heterogenization (D^θ, Ω) may be written as

$$\omega^\sharp\colon \big((d_1)^{\theta_{s_\omega}},\dots,(d_{\omega\tau})^{\theta_{t_\omega}}\big) \mapsto (d_1\cdots d_{\omega\tau}\omega)^{\theta_{g_\omega}}\,.$$

In turn, (7.2) provides an operation

$$\begin{aligned}\omega^{\sharp\flat}\colon &\big((d_1^{\theta_0},\dots,d_1^{\theta_{n-1}}),\dots,(d_{\omega\tau}^{\theta_0},\dots,d_{\omega\tau}^{\theta_{n-1}})\big)\\ &\mapsto \big(d_1^{\theta_0},\dots,(d_1\cdots d_{\omega\tau}\omega)^{\theta_{g_\omega}},\dots,d_1^{\theta_{n-1}}\big)\end{aligned}\tag{10.4}$$

of $\big(\varprojlim D^\theta, \Omega\big)$. Comparing (10.3) with (10.4), and recalling that (D, Ω, d) satisfies the identities and quasi-identities of Definition 9.1(c), it then becomes apparent that the diagonal algebra isomorphism (3.3) yields a $\tau^\flat$-algebra isomorphism $(D, \Omega, d) \cong \big(\varprojlim D^\theta, \Omega, d\big)$. $\square$

The form of (10.4) suggests identities that are equivalent to the identities and quasi-identities of Definition 9.1(c) (cf. [6, (2.5)]).

Corollary 10.2. *Let n be a positive integer, and let d be a diagonal operation of degree n. Suppose that $\tau\colon \omega \to \mathbb{N}$ is a single-sorted type, with input function $k\colon \Omega \to \mathbb{N}^n$ and output function $g\colon \Omega \to \underline{n}$. Then the identities*

$$x_1^0\cdots\overbrace{(x_1\cdots x_{\omega\tau}\omega)}^{\textit{slot } g_\omega}\cdots x_1^{n-1}d = (x_1^0\cdots x_1^{n-1}d)\cdots(x_{\omega\tau}^0\cdots x_{\omega\tau}^{n-1}d)\omega \tag{10.5}$$

are equivalent to the identities and quasi-identities of Definition 9.1(c) *for each operator* $\omega \in \Omega$.

Note that the complexity of the identities and quasi-identities of Definition 9.1(c), as determined by the number of variables, is linear, namely $\omega\tau + n - 1$ for arity $\omega\tau$ and degree n. (Even the equivalent identities of (b′) in the proof of Proposition 9.8 still only have linear complexity $\omega\tau + 2n - 2$.) By contrast, the superficially elegant identities (10.5) have quadratic complexity $n \times \omega\tau$.

Theorem 10.1 was stated in terms of a many-sorted type τ and its homogenization $\tau^\sharp$. It may be reformulated in terms of a single-sorted type σ (with input function l and output function h) and its heterogenization $\sigma^\sharp_{lh}$.

Corollary 10.3. *Let* $\sigma\colon \Omega \to \mathbb{N}$ *be a constant-free single-sorted type, with given input function* l *and output function* h *of degree* n. *Then the following classes are equivalent:*

(a) *the variety of* $\sigma + \{(d, n)\}$*-algebras* (B, Ω, d), *monotonically heterogenizable and compatible with the input function* l *and output function* h,

(b) *the class of pure* $\sigma^\sharp_{lh}$*-algebras.*

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