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Hilbert algebras with supremum

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Abstract. In this paper, we will study the class of Hilbert algebras with supremum, i.e., Hilbert algebras where the associated order is a join-semilattice. First, we will give a simplified topological duality for Hilbert algebras using sober topological spaces with a basis of open-compact sets satisfying an additional condition. Next, we will extend this duality to Hilbert algebras with supremum. We shall prove that the ordered set of all ideals of a Hilbert algebra with supremum has a lattice structure. We will also see that in this lattice, it is possible to define an implication, but the resulting structure is neither a Heyting algebra nor an implicative semilattice. Finally, we will give a dual description of the lattice of ideals of a Hilbert algebra with supremum.

1. Introduction

Hilbert algebras represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. There are many examples of Hilbert algebras, but there exists a particular example that is very important. It is well known that every poset $\langle A, \leq \rangle$ with greatest element 1 induces a structure of Hilbert algebra defining an implication $\to$ on A as follows: $a \to b = 1$ when $a \leq b$, and $a \to b = b$ when $a \nleq b$. This example allows us to define Hilbert algebras on semilattices or lattices which are not implicative semilattices or Heyting algebras. For instance, the Boolean lattice with four elements and the implication given by the order is a Hilbert algebra which has as a reduct a bounded distributive lattice that is not a Heyting algebra. These examples motivate the study of Hilbert algebras with lattice operations. These classes of enriched Hilbert algebras are subclasses of BCK-algebras with lattices operations considered by P. M. Idziak in [9]. We note that the class of Hilbert algebras where the induced order is a meet-semilattice is considered in [8] under the name of Hilbert algebras with infimum. In this paper we will consider mainly Hilbert algebras where the induced order is a join-semilattice. This class is a variety, as will be shown later.

A duality theory for Hilbert algebras was developed recently in [5]. The dual space of a Hilbert algebra is defined in [5] as an ordered topological space $\langle X, \leq, \mathcal{T}_{\mathcal{K}} \rangle$ called an *ordered Hilbert space*, where $\mathcal{K}$ is a basis of open-compact and decreasing subsets for the topology $\mathcal{T}_{\mathcal{K}}$ satisfying additional conditions.

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It is not hard to see that the order $\leq$ of an ordered Hilbert space is the dual order of the topological specialization order. In other words, $\leq$ can be defined as: $x \leq y$ iff y belongs to the topological closure of $\{x\}$ for each pair $x, y \in X$. This simple observation allows us to consider a simplified duality for Hilbert algebras. The new dual spaces associated with Hilbert algebras are sober topological spaces with a basis of open-compact sets satisfying an additional condition. This new definition simplifies the representation and duality theory for Hilbert algebras developed in [5].

This paper has two main objectives. First, we shall give a simplified representation and duality for Hilbert algebras, and second, we will apply these results to study the representation and duality of Hilbert algebras with supremum.

The paper is organized as follows. In Section 3, we shall give the new representation and duality for Hilbert algebras by means of sober spaces. In Section 4, we will define Hilbert algebras with supremum or $H^{\vee}$-algebras. We shall see that this class is equational. In Section 5, we will develop the topological representation for $H^{\vee}$-algebras using the simplified representation. We will introduce irreducible H-relations and irreducible H-functional relations, and we will show that they give the dual description of semi-homomorphisms of Hilbert algebras preserving $\vee$ and homomorphisms of Hilbert algebras preserving $\vee$, respectively. In Section 6, we will prove that the irreducible H-functional relations between $H^{\vee}$-spaces can be characterized by means of special partial functions between $H^{\vee}$-spaces. In Section 7, we will study the ideals in $H^{\vee}$-algebras. We will prove that the ordered set $\mathrm{Id}(A)$ of all ideals of an $H^{\vee}$-algebra $\langle A, \rightarrow, \vee, 1\rangle$ has a lattice structure, but is not a distributive lattice. Also, we will prove that it is possible to define an implication $\twoheadrightarrow$ in $\mathrm{Id}(A)$, which is not an implicative semilattice either. Finally, in Section 8, we shall introduce the notion of open directed subsets and we will prove that these sets give the dual description of ideals in $H^{\vee}$-algebras.

2. Preliminaries

Definition 2.1. A *Hilbert algebra* is an algebra $A = \langle A, \rightarrow, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in A:

(1) $a \rightarrow (b \rightarrow a) = 1$.
(2) $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$.
(3) $a \rightarrow b = 1 = b \rightarrow a$ implies $a = b$.

In [7], Diego proves that the class of Hilbert algebras forms a variety which is denoted by $\mathbb{H}$. It is easy to see that the binary relation $\leq$ defined in a Hilbert algebra A by $a \leq b$ if and only if $a \rightarrow b = 1$ is a partial order on A with greatest element 1. This order is called the *natural ordering* on A. A *bounded Hilbert algebra* is a Hilbert algebra A with an element $0 \in A$ such that $0 \leq a$ for all $a \in A$.

Let us recall that the class of Hilbert algebras is a subclass of the BCK-algebras, because a Hilbert algebra A is a BCK-algebra that satisfies the equation $x \to (y \to z) = (x \to y) \to (x \to z)$ (see, for instance, [6] page 165).

Lemma 2.2. *Let A be a Hilbert algebra and $a, b, c \in A$. Then the following equalities are satisfied:*

(1) $a \to a = 1$,
(2) $1 \to a = a$,
(3) $a \to (b \to c) = b \to (a \to c)$,
(4) $a \to (b \to c) = (a \to b) \to (a \to c)$,
(5) $a \to ((a \to b) \to b) = 1$,
(6) $a \to (a \to b) = a \to b$,
(7) $((a \to b) \to b) \to b = a \to b$,
(8) $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

Let us consider a poset $\langle X, \leq \rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x, y \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$). The set of all subsets of X is denoted by $\mathcal{P}(X)$, and the set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. The set complement of a subset $Y \subseteq X$ will be denoted by Y^c or $X - Y$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X \mid \exists y \in Y\ (y \leq x)\}$ ($(Y] = \{x \in X \mid \exists y \in Y\ (x \leq y)\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. A subset $K \subseteq X$ is called *dually directed* if for any $x, y \in K$ there exists $z \in K$ such that $z \leq x$ and $z \leq y$.

In [2], it was proved that if $\langle X, \leq \rangle$ is a poset, then $\langle \mathcal{P}_i(X), \Rightarrow, X \rangle$ is a Hilbert algebra where the implication $\Rightarrow$ is defined by

$$U \Rightarrow V = (U \cap V^c]^c = \{x \mid [x) \cap U \subseteq V\}$$

for $U, V \in \mathcal{P}_i(X)$.

Example 2.3. An H-set is a triple $\langle X, \leq, \mathcal{K} \rangle$ where $\langle X, \leq \rangle$ is a poset and $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}(X)$. Every H-set defines a structure $H_{\mathcal{K}}(X)$ as follows:

$$H_{\mathcal{K}}(X) = \{U \in \mathcal{P}(X) \mid \exists W \in \mathcal{K} \text{ and } \exists V \subseteq W\ (U = W \Rightarrow V)\}.$$

In [3], it was proved that $\langle H_{\mathcal{K}}(X), \Rightarrow, X \rangle$ is a Hilbert algebra and a subalgebra of $\langle \mathcal{P}_i(X), \Rightarrow, X \rangle$.

Let A be a Hilbert algebra. A subset $D \subseteq A$ is a *deductive system* of A if $1 \in D$ and if $a, a \to b \in D$, then $b \in D$. The set of all deductive systems of a Hilbert algebra A is denoted by $\mathrm{Ds}(A)$. It is easy to prove that $\mathrm{Ds}(A)$ is closed under arbitrary intersections. The deductive system generated by a set X is $\langle X \rangle = \bigcap\{D \in \mathrm{Ds}(A) \mid X \subseteq D\}$. If $X = \{a\}$, then we have $\langle a \rangle = \{b \in A \mid a \leq b\} = [a)$. We shall say that a proper deductive system D is *irreducible* if for any $D_1, D_2 \in \mathrm{Ds}(A)$ such that $D = D_1 \cap D_2$, it follows that $D = D_1$ or $D = D_2$. The set of all irreducible deductive systems of a Hilbert algebra A is denoted by $X(A)$. Let us recall that a deductive system is *irreducible* iff for every $a, b \in A$ such that $a, b \notin D$, there exists $c \notin D$ such that $a, b \leq c$. A subset I of A is called an *order-ideal* of A if $b \in I$ and $a \leq b$, then $a \in I$, and for each $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideals of A will be denoted by $\mathrm{Id}(A)$. For more details on this topic see [7], [10], or [6].

The following is a Hilbert algebra analogue of Birkhoff's Prime Filter Lemma, and it is proved in [2].

Theorem 2.4. *Let A be a Hilbert algebra. Let $D \in \mathrm{Ds}(A)$ and let $I \in \mathrm{Id}(A)$ such that $D \cap I = \emptyset$. Then there exists $P \in X(A)$ such that $D \subseteq P$ and $P \cap I = \emptyset$.*

Let A be a Hilbert algebra. Let us consider the poset $\langle X(A), \subseteq \rangle$ and the mapping $\varphi \colon A \to \mathcal{P}_i(X(A))$ defined by $\varphi(a) = \{P \in X(A) \mid a \in P\}$. For a proof of the next theorem, see [7] or [2].

Theorem 2.5. *Let A be Hilbert algebra. Then A is isomorphic to the subalgebra $\varphi[A] = \{\varphi(a) \mid a \in A\}$ of $\langle \mathcal{P}_i(X(A)), \Rightarrow, X(A) \rangle$.*

3. A simplified representation for Hilbert algebras

In this section, we will simplify the topological representation for Hilbert algebras given in [5]. First, we will recall some topological notions.

Let $X = \langle X, \mathcal{T} \rangle$ be a topological space. The *closure* of a set $Y \subseteq X$ is denoted by $\mathrm{cl}(Y)$. The interior of a set Y is denoted by $\mathrm{int}(Y)$. We recall that the *specialization order* of X is defined by $x \preceq y$ if $x \in \mathrm{cl}(\{y\}) = \mathrm{cl}(y)$. The relation $\preceq$ is reflexive and transitive, and it is a partial order if X is T_0. The dual order of $\preceq$ is denoted by $\leq$, i.e., $x \leq y$ if $y \in \mathrm{cl}(x)$. We note that $\mathrm{cl}(x) = [x)$, and that an open (resp. closed) subset is a decreasing (resp. increasing) subset with respect to $\leq$.

An arbitrary non-empty subset Y of a topological space X is *irreducible* if $Y \subseteq Z \cup W$ for closed subsets Z and W implies $Y \subseteq Z$ or $Y \subseteq W$. A topological space X is *sober* if for every irreducible closed set Y, there exists a unique $x \in X$ such that $\mathrm{cl}(x) = Y$. Notice that a sober space is automatically T_0. From now on, for every sober space X, we will consider the order $\leq$. A subset $Y \subseteq X$ is *saturated* if it is an intersection of open sets, or equivalently, if Y is a decreasing set in the order $\leq$. The saturation $\mathrm{sat}(Y)$ of a subset Y is the smallest saturated set containing Y. We note that $\mathrm{sat}(Y) = (Y]$.

In [5], we introduced the dual space of a Hilbert algebra as an ordered topological space satisfying certain additional conditions as stated below.

Definition 3.1. An *ordered Hilbert space* is a structure $\langle X, \leq, \mathcal{T}_{\mathcal{K}} \rangle$ such that:

(1) $\mathcal{K}$ is a basis of open, compact, and decreasing subsets for the topology $\mathcal{T}_{\mathcal{K}}$ defined on X.
(2) For every $A, B \in \mathcal{K}$, $\mathrm{sat}(A \cap B^c) \in \mathcal{K}$.

(3) For every $x, y \in X$, if $x \nleq y$, then there exists $U \in \mathcal{K}$ such that $x \notin U$ and $y \in U$.
(4) If Y is a closed subset and $L \subseteq \mathcal{K}$ is a dually directed set such that $Y \cap U \neq \emptyset$ for every $U \in L$, then $\bigcap\{U \mid U \in L\} \cap Y \neq \emptyset$.

Remark 3.2. Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be a topological space with a basis $\mathcal{K}$ of open and compact subsets such that $(A \cap B^c] \in \mathcal{K}$ for every $A, B \in \mathcal{K}$. Consider the set $D(X) = \{U \subseteq X \mid U^c \in \mathcal{K}\}$. It is easy to prove that $\langle D(X), \Rightarrow, X \rangle$ is a Hilbert algebra where the implication $\Rightarrow$ is defined by

$$U \Rightarrow V = (\mathrm{sat}(U \cap V^c))^c = (U \cap V^c]^c$$

for each $U, V \in D(X)$. We also note that $\langle D(X), \Rightarrow, X \rangle$ is a subalgebra of the Hilbert algebra $\langle H_{\mathcal{K}}(X), \Rightarrow, X \rangle$ defined as in Example 2.3. In this case, the H-set is the triple $\langle X, \leq, \mathcal{K} \rangle$ where the order $\leq$ is the dual of the specialization order.

We will give an equivalent definition of ordered Hilbert spaces without using the order. This allows us to give a simplified topological duality. Before hand, we will give some necessary concepts and show the equivalence between sober spaces and the last two conditions of the definition given above.

Theorem 3.3. *Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be a topological space with a base $\mathcal{K}$ of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$ on X. Suppose that for every $A, B \in \mathcal{K}$, $\mathrm{sat}(A \cap B^c) \in \mathcal{K}$. Then the following conditions are equivalent:*

(1) *X is T_0, and for each closed subset Y and for each subset $L \subseteq \mathcal{K}$ dually directed such that $Y \cap U \neq \emptyset$ for all $U \in L$, one has $\bigcap\{U \mid U \in L\} \cap Y \neq \emptyset$.*
(2) *X is T_0, and $\varepsilon_X : X \to X(D(X))$, where $\varepsilon_X(x) = \{U \in D(X) \mid x \in U\}$ for each $x \in X$, is onto.*
(3) *$\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is sober.*

Proof. (1) $\Rightarrow$ (2): Let $P \in X(D(X))$. Consider $L = \{U_j^c \mid U_j \notin P\} \subseteq \mathcal{K}$. We note that L is dually directed because P is an irreducible deductive system. The set $Y = \bigcap\{V_i \mid V_i \in P\}$ is closed and $Y \cap U_j^c \neq \emptyset$ for each $U_j^c \in L$ because, otherwise, there exists $U_j^c \in L$ such that $U_j^c \subseteq \bigcup\{V_i^c \mid V_i \in P\}$. Since U_j^c is compact, $U_j^c \subseteq V_1^c \cup V_2^c \cup \cdots \cup V_n^c$, i.e., $(V_1 \Rightarrow (V_2 \Rightarrow \cdots (V_n \Rightarrow U_j) \cdots) = X$. It follows that $U_j \in P$, which is a contradiction. Then $Y \cap \bigcap\{U_j^c \mid U_j \notin P\} \neq \emptyset$, i.e., there exists $x \in \bigcap\{V_i \mid V_i \in P\} \cap \bigcap\{U_j^c \mid U_j \notin P\}$, which implies that $P = \varepsilon_X(x)$.

(2) $\Rightarrow$ (3): Let Y be an irreducible closed subset of X. Let us consider the set $P_Y = \{U \in D(X) \mid Y \subseteq U\}$. It is easy to see that P_Y is a deductive system of $D(X)$. We prove that P_Y is irreducible. Let $U, V \notin P_Y$. Hence, $Y \nsubseteq U$ and $Y \nsubseteq V$; as Y is irreducible, $Y \nsubseteq U \cup V$. So, there exists $x \in Y$ such that $x \in U^c$ and $x \in V^c$. As $U^c, V^c \in \mathcal{K}$ and $\mathcal{K}$ is a base of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$, there exists $W \in D(X)$ such that $x \in W^c \subseteq U^c \cap V^c$. So, $Y \nsubseteq W$. Thus, $W \notin P_Y$, and P_Y is an irreducible deductive system of $D(X)$. Since X is T_0, ε_X is injective, and as ε_X is onto, there exists a unique $y \in X$

such that $\varepsilon_X(y) = P_Y$. Now it is easy to check that $Y = \mathrm{cl}(y)$. Thus, $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is sober.

(3) $\Rightarrow$ (1): Let Y be a closed subset of X and let $L = \{U_i \mid i \in I\}$ be a dually directed subfamily of $\mathcal{K}$ such that $Y \cap U_i \neq \emptyset$ for all $i \in I$. Since Y^c is an open subset and $\mathcal{K}$ is a basis, $Y = \bigcap\{V \mid V \in B \subseteq D(X)\}$. Let us consider the set $H = \{U_i^c \mid U_i \in L\} \subseteq D(X)$. Since L is dually directed, the subset $(H] = \{W \in D(X) \mid W \subseteq U_i^c \text{ for some } U_i^c \in H\}$ is an order-ideal of $D(X)$. Let $\langle B \rangle$ be the deductive system generated by B. We prove that $\langle B \rangle \cap (H] = \emptyset$. Suppose the contrary. Then there exists $U_k^c \in H$ and there exist $V_1, \ldots, V_n \in B$ such that $V_1 \Rightarrow (V_2 \Rightarrow \cdots (V_n \Rightarrow U_k^c) \cdots) = X$. Since $Y \cap U_k \neq \emptyset$, there exists $x \in X$ such that $x \in Y$ and $x \in U_k$. As $x \in V_1, \ldots, V_n$ and $V_1 \Rightarrow (V_2 \Rightarrow \cdots (V_n \Rightarrow U_k^c) \cdots) = X$, we deduce that $x \in U_k^c$, which is a contradiction. Thus, there exists $P \in X(D(X))$ such that $\langle B \rangle \subseteq P$ and $P \cap (H] = \emptyset$. Consider the set $Z = \bigcap\{V \mid V \in P\}$. Then $Z \subseteq Y$. It is easy to see that Z is an irreducible set. As $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is sober, there exists $x \in X$ such that $Z = \mathrm{cl}(x)$. Thus, $x \in \bigcap\{U \mid U \in L\} \cap Y$. □

Definition 3.4. A *Hilbert space* or H-space is a topological space $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ such that:

- H1. $\mathcal{K}$ is a base of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$ on X.
- H2. For every $A, B \in \mathcal{K}$, $\mathrm{sat}(A \cap B^c) \in \mathcal{K}$.
- H3. $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is sober.

Remarks 3.5. (1) The order $\leq$ implicitly defined in every H-space is the dual of the specialization order. Since an H-space is sober and T_0, we get that for each $x \in X$, the set $[x)$ is the closure of $\{x\}$.

(2) It is clear that in every H-space $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$, every closed subset F is an increasing subset of the poset $\langle X, \leq \rangle$. Also, for the condition H2 of Definition 3.4, and since $\mathcal{K}$ is a base for the topology $\mathcal{T}_{\mathcal{K}}$, $\varepsilon_X(x) = \{U \in D(X) \mid x \in U\}$ is an irreducible deductive system for each $x \in X$.

(3) If $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is an H-space, then $D(X) = \langle D(X), \Rightarrow, X \rangle$, where $D(X)$ is defined as in Remark 3.2, is a Hilbert algebra called the *dual* Hilbert algebra of the H-space $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$. Moreover, $\langle X, \leq, \mathcal{K} \rangle$ is an H-set, and by Example 2.3, we have that $\langle H_{\mathcal{K}}(X), \Rightarrow, X \rangle$ is a Hilbert algebra. Since every element of $\mathcal{K}$ is a decreasing set, it is easy to see that $D(X) \subseteq H_{\mathcal{K}}(X) \subseteq \mathcal{P}_i(X)$.

Taking into account these facts, it is easy to see that the Definition 3.4 of H-space given here is equivalent to the Definition 3.1 given in [5].

Lemma 3.6. *Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be an H-space. Then $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is compact iff $D(X)$ is a bounded Hilbert algebra.*

Proof. We note that $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ is compact iff $X \in \mathcal{K}$ iff $\emptyset \in D(X)$. □

Let A be a Hilbert algebra. In [5], it was proved that the family

$$\mathcal{K}_A = \{\varphi(a)^c \mid a \in A\} \subseteq \mathcal{P}(X(A))$$

is a basis for a topology $\mathcal{T}_{\mathcal{K}_A}$ defined on $X(A)$. The topological space $X(A) = \langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is called the *dual space* of A. We note that $P \subseteq Q$ iff $Q \in \mathrm{cl}(\{P\}) = \bigcap\{\varphi(a) \mid a \in P\}$ for all $Q, P \in X(A)$, i.e., the dual of the specialization order of $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is the inclusion relation $\subseteq$.

Theorem 3.7 ([5])**.** *Let A be a Hilbert algebra. Then $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is an H-space and $D(X(A)) = \{\varphi(a) \mid a \in A\}$ is a Hilbert algebra isomorphic to A.*

Theorem 3.8 ([5])**.** *Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be an H-space. Then the topological spaces $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ and $\langle X(D(X)), \mathcal{T}_{\mathcal{K}_{D(X)}}\rangle$ are homeomorphic by means of the mapping $\varepsilon_X \colon X \to X(D(X))$.*

Definition 3.9 ([5])**.** Let A, B be two Hilbert algebras. A mapping $h\colon A \to B$ is a *semi-homomorphism* if for every $a, b \in A$,

(1) $h(a \to b) \leq h(a) \to h(b)$,
(2) $h(1) = 1$.

A homomorphism from a Hilbert algebra A into a Hilbert algebra B is a semi-homomorphism h such that $h(a) \to h(b) \leq h(a \to b)$ for $a, b \in A$.

Let us denote by $\mathcal{HS}$ the category whose objects are Hilbert algebras and whose morphisms are semi-homomorphisms between Hilbert algebras. Similarly, let $\mathcal{HH}$ be the category of Hilbert algebras with homomorphisms. Clearly, $\mathcal{HH}$ is a subcategory of $\mathcal{HS}$.

Definition 3.10 ([5])**.** Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be two H-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We say that R is an *H-relation* if it satisfies the following properties:

(1) $R^{-1}(U) \in \mathcal{K}_1$, for every $U \in \mathcal{K}_2$,
(2) $R(x)$ is a closed subset of X_2, for all $x \in X_1$.

We say that R is an *H-functional relation* if R is an H-relation and it satisfies the following condition:

(3) If $(x, y) \in R$, then there exists $z \in X_1$ such that $x \leq z$ and $R(z) = [y)$.

In [5], it was proved that the H-spaces as objects and the H-relations as arrows form a category denoted by $\mathcal{SR}$. Similarly, the H-spaces as objects and the H-functional relations as arrows form a category denoted by $\mathcal{SF}$.

Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be two H-spaces. Let $R \subseteq X_1 \times X_2$ be an H-relation. Then the mapping $h_R\colon D(X_2) \to D(X_1)$ defined by $h_R(U) = \{x \in X_1 \mid R(x) \subseteq U\}$ is a semi-homomorphism. If R is an H-functional relation, then h_R is a homomorphism.

Let A, B be two Hilbert algebras. Let $h\colon A \to B$ be a semi-homomorphism. Define a binary relation $R_h \subseteq X(B) \times X(A)$ by $(P, Q) \in R_h$ iff $h^{-1}(P) \subseteq Q$, where $h^{-1}(P) = \{a \in A \mid h(a) \in P\}$. In [5], we proved that R_h is an H-relation, and that if h is a homomorphism, then R_h is an H-functional relation. Moreover, the categories $\mathcal{SR}$ and $\mathcal{HS}$ are dually equivalent and the categories $\mathcal{SF}$ and $\mathcal{HH}$ are dually equivalent.

4. Hilbert algebras with supremum

Now we will study the class of Hilbert algebras where the associated order is a join-semilattice. We note that this class of algebras is a particular class of some BCK-algebras with lattice operations studied by P. M. Idziak in [9].

Definition 4.1. An algebra $\langle A, \to, \vee, 1\rangle$ of type $(2, 2, 0)$ is a *Hilbert algebra with supremum* or $H^{\vee}$-algebra if

(1) $\langle A, \to, 1\rangle$ is a Hilbert algebra,
(2) $\langle A, \vee, 1\rangle$ is a join-semilattice with greatest element 1,
(3) for all $a, b \in A$, $a \to b = 1$ if and only if $a \vee b = b$.

By $\mathcal{HS}^{\vee}$ we denote the category whose objects are $H^{\vee}$-algebras and whose morphisms are Hilbert semi-homomorphisms that preserve the operation $\vee$.

Example 4.2. Let us recall that a *Tarski algebra* is a Hilbert algebra $\langle A, \to, 1\rangle$ such that $(a \to b) \to a = (b \to a) \to a$ for all $a, b, c \in A$. It is known that A is a join-semilattice under the operation $\vee$ defined by $a \vee b = (a \to b) \to a$. Thus, A is a Hilbert algebra with supremum.

Example 4.3. In every join-semilattice $\langle A, \vee, 1\rangle$ with greatest element 1, it is possible to define the structure of a Hilbert algebra with supremum by considering the implication $\to$ defined by the order, i.e., $a \to b = b$ if $a \not\leq b$, and $a \to b = 1$ if $a \leq b$.

Example 4.4. The Boolean lattice with two atoms $B_2 = \{0, a, b, 1\}$ and with the implication $\to$ defined by the order is a Hilbert algebra where the supremum exists for any pair of elements, but is not a Heyting algebra.

The class of $H^{\vee}$-algebras is indeed a variety. This result follows from the results on BCK-algebras with lattice operations given by P. M. Idziak in [9].

Theorem 4.5. *Let us consider an algebra* $\langle A, \to, \vee, 1\rangle$ *of type* $(2, 2, 0)$*. Then* $\langle A, \to, \vee, 1\rangle$ *is an* $H^{\vee}$*-algebra if and only if*

(1) $\langle A, \to, 1\rangle$ *is a Hilbert algebra,*
(2) $\langle A, \vee, 1\rangle$ *is a join-semilattice with greatest element* 1*,*
(3) A *satisfies the following equations:*
 (a) $a \to (a \vee b) = 1$*,*
 (b) $(a \to b) \to ((a \vee b) \to b) = 1$*.*

5. Representation and duality for Hilbert algebras with supremum

In this section, we shall define the dual space of an $H^{\vee}$-algebra as an H-space with an additional condition. First, we will give a representation theorem for $H^{\vee}$-algebras, and will further develop the topological duality.

Let $A = \langle A, \to, \vee, 1\rangle$ be an $H^{\vee}$-algebra and let $D \in \mathrm{Ds}(A)$. We say that D is *prime* if and only if $D \neq A$ and for every $a, b \in A$ such that $a \vee b \in D$, either

$a \in D$ or $b \in D$. It is easy to prove that a deductive system D is irreducible if and only if D is prime.

Lemma 5.1. *Let $\langle A, \to, \vee, 1\rangle$ be an $H^{\vee}$-algebra. Then $\varphi\colon A \to \mathcal{P}_i(X(A))$, given by $\varphi(a) = \{P \in X(A) \mid a \in P\}$, is an injective homomorphism of $H^{\vee}$-algebras.*

Proof. That φ is an injective homomorphism of H-algebras was proved in [5]. Let $a, b \in A$ and $P \in X(A)$. We need to prove that $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$. Let $P \in X(A)$. Suppose that $P \in \varphi(a \vee b)$. So, $a \vee b \in P$. As P is prime, $a \in P$ or $b \in P$. Thus, $P \in \varphi(a) \cup \varphi(b)$. Now, we suppose that $P \in \varphi(a)$ or $P \in \varphi(b)$. So, $a \in P$ or $b \in P$. As $a \leq a \vee b$, $b \leq a \vee b$, and P is an increasing subset of A, so $a \vee b \in P$. Thus, $P \in \varphi(a \vee b)$. □

Now, we will introduce the definition of $H^{\vee}$-space.

Definition 5.2. Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be an H-space. Then $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H^{\vee}$-*space* if $U \cap V \in \mathcal{K}$ for all $U, V \in \mathcal{K}$.

Proposition 5.3. *If $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H^{\vee}$-space, then $D(X) = \langle D(X), \Rightarrow, \cup, X\rangle$ is a Hilbert algebra with supremum.*

Proof. This is immediate by the definition of $H^{\vee}$-space. □

We will prove a characterization of $H^{\vee}$-spaces. First, we note that if $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is a topological space with a basis $\mathcal{K}$ of compact and open subsets, then in X we can define another topology $\mathcal{T}_{D(X)}$ by taking the family $D(X) = \{U \mid U^c \in \mathcal{K}\}$ as subbasis. We will denote this space by $\langle X, \mathcal{T}_{D(X)}\rangle$. The following result is part of folklore, but for the sake of completeness we will give a proof here.

Proposition 5.4. *Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be a topological space with a base $\mathcal{K}$ of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$ such that $U \cap V \in \mathcal{K}$ for all $U, V \in \mathcal{K}$. Then the following conditions are equivalent:*

(1) *Every closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is a compact subset of the topological space $\langle X, \mathcal{T}_{D(X)}\rangle$.*
(2) *If Y is a closed subset and $L \subseteq \mathcal{K}$ is a dually directed set such that $Y \cap U \neq \emptyset$ for every $U \in L$, then $\bigcap\{U \mid U \in L\} \cap Y \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2): Let Y be a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ and let L be a dually directed subset of $\mathcal{K}$ such that $Y \cap U \neq \emptyset$ for each $U \in L$. We need to prove that $Y \cap \bigcap\{U \mid U \in L\} \neq \emptyset$. To the contrary, suppose that $Y \subseteq \bigcup\{U^c \mid U \in L\}$. As the sets U^c are open subsets of the topological space $\langle X, \mathcal{T}_{D(X)}\rangle$ and Y is compact in this space, there exists a finite set $\{U_1, \ldots, U_n\} \subseteq L$ such that $Y \subseteq U_1^c \cup \cdots \cup U_n^c$. As L is a dually directed subset of $\mathcal{K}$, there exists $U \in L$ such that $U \subseteq U_1 \cap \cdots \cap U_n$. Thus, $Y \cap U = \emptyset$, which is a contradiction.

(2) ⇒ (1): Let Y be a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$. Let $J = \{U_i \mid i \in I\} \subseteq D(X)$ such that $Y \subseteq \bigcup\{U_i \mid i \in I\}$. Let us consider the family $\tilde{J}$ of sets $V \subseteq X$ such that there exists a finite subfamily $\{U_1, \ldots, U_n\}$ of J such that $V = U_1 \cup \cdots \cup U_n$. It is clear that $J \subseteq \tilde{J}$ and that $\{V^c \mid V \in \tilde{J}\}$ is a dually directed subset of $\mathcal{K}$ because $\mathcal{K}$ is closed under $\cap$. Since $Y \cap \bigcap\{V^c \mid V \in \tilde{J}\} = \emptyset$, and as $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ satisfies condition (4) of Definition 3.1, we get that there exists $V = U_1 \cup \cdots \cup U_n \in \tilde{J}$ such that $Y \cap V^c = \emptyset$, i.e., $Y \subseteq U_1 \cup \cdots \cup U_n$ for $U_1, \ldots, U_n \in D(X)$. Thus, Y is a compact subset of the space $\langle X, \mathcal{T}_{D(X)}\rangle$. □

Corollary 5.5. *Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be a topological space with a base $\mathcal{K}$ of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$. Then $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H^{\vee}$-space iff it satisfies the following conditions:*

(1) *$A \cap B \in \mathcal{K}$ for all $A, B \in \mathcal{K}$.*
(2) *For every $A, B \in \mathcal{K}$, $\operatorname{sat}(A \cap B^c) \in \mathcal{K}$.*
(3) *Every closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is a compact subset of the topological space $\langle X, \mathcal{T}_{D(X)}\rangle$.*

Proof. This follows from Definition 5.2, Proposition 5.4, and Theorem 3.3. □

Theorem 5.6. *Let A be an $H^{\vee}$-algebra. Then $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is an $H^{\vee}$-space and the mapping $\varphi\colon A \to D(X(A))$ is an isomorphism of $H^{\vee}$-algebras.*

Proof. We know that $\langle X(A), \mathcal{K}_A\rangle$ is an H-space. From Lemma 5.1, it follows that $U \cap V \in \mathcal{K}_A$ for all $U, V \in \mathcal{K}_A$. Thus, $\langle X(A), \mathcal{K}_A\rangle$ is an $H^{\vee}$-space. By Proposition 5.3, we get that $D(X(A)) = \langle D(X(A)), \Rightarrow, \cup, X(A)\rangle$ is an $H^{\vee}$-algebra. □

Proposition 5.7. *Let X be an $H^{\vee}$-space. Then $\varepsilon_X\colon X \to X(D(X))$ is a homeomorphism between the spaces X and $X(D(X))$.*

Proof. By Theorem 3.8, the mapping ε_X is a homeomorphism between the topological spaces X and $X(D(X))$. By Proposition 5.3, if X is an $H^{\vee}$-space, then $D(X)$ is an $H^{\vee}$-algebra, and by Lemma 5.1, it follows that $X(D(X))$ is an $H^{\vee}$-space. □

Definition 5.8. Let A, B be $H^{\vee}$-algebras. A semi-homomorphism $h\colon A \to B$ is called a *join-semi-homomorphism*, or $\vee$-semi-homomorphism, if $h(a \vee b) = h(a) \vee h(b)$ for all $a, b \in A$.

Similarly, if h is a homomorphism that preserves the join, it will be called a $\vee$-homomorphism.

Now we shall study the duals of $\vee$-semi-homomorphisms.

Definition 5.9. Let X_1 and X_2 be two H-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We shall say that R is *irreducible* if for each $x \in X_1$, $R(x)$ is an irreducible closed subset of X_2 when $R(x) \neq \emptyset$.

Theorem 5.10. *Let A, B be two $H^{\vee}$-algebras and let $h\colon A \to B$ be a semi-homomorphism of Hilbert algebras. Then the following conditions are equivalent:*

(1) *h preserves the operation $\vee$.*
(2) *The relation R_h is irreducible.*
(3) *For all $P \in X(B)$, $h^{-1}(P) \in X(A)$ or $h^{-1}(P) = A$.*

Proof. (1) $\Rightarrow$ (2): Let $P \in X(B)$. Assume that $R_h(P) \neq \emptyset$. Let Z and W be two closed subsets of $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ such that $R_h(P) \subseteq Z \cup W$, $R_h(P) \nsubseteq Z$ and $R_h(P) \nsubseteq W$. So, there exist $Q, D \in X(A)$ such that $Q \in R_h(P) - Z$ and $D \in R_h(P) - W$. Thus, there exist $a, b \in A$ such that $Z \subseteq \varphi(a)$ with $Q \notin \varphi(a)$, and $W \subseteq \varphi(b)$ with $D \notin \varphi(b)$. So, $a, b \notin h^{-1}(P)$. Since $R_h(P) \subseteq \varphi(a \vee b)$, we get that $P \in h_{R_h}(\varphi(a \vee b))$. By Lemma 3.5 of [5], $h_{R_h}(\varphi(a \vee b)) = \varphi(h(a \vee b))$. So, $h(a \vee b) \in P$. As P is prime, $h(a) \in P$ or $h(b) \in P$, i.e., $a \in h^{-1}(P)$ or $b \in h^{-1}(P)$, which is a contradiction. Thus, $R_h(P)$ is irreducible.

(2) $\Rightarrow$ (3): Let $P \in X(B)$. By Theorem 3.2 of [2], $h^{-1}(P) \in \mathrm{Ds}(A)$. Suppose that $h^{-1}(P) \neq A$. Let $a \vee b \in h^{-1}(P)$. So, $a \vee b \in Q$ for all $Q \in R_h(P)$. Hence, $R_h(P) \subseteq \varphi(a \vee b) = \varphi(a) \cup \varphi(b)$, and as $R_h(P)$ is irreducible, we deduce that $R_h(P) \subseteq \varphi(a)$, or $R_h(P) \subseteq \varphi(b)$, i.e., $h(a) \in P$ or $h(b) \in P$. Thus, $h^{-1}(P) \in X(A)$.

(3) $\Rightarrow$ (1): To show that h preserves the operation $\vee$, let $a, b \in A$. As h is monotonic, we have $h(a) \vee h(b) \leq h(a \vee b)$. Suppose that $h(a \vee b) \nleq h(a) \vee h(b)$. So, there exists $P \in X(B)$ such that $h(a \vee b) \in P$ and $h(a) \vee h(b) \notin P$. Hence, $h(a) \notin P$ and $h(b) \notin P$. That is, $a, b \notin h^{-1}(P)$. By hypothesis, $h^{-1}(P) \in X(A)$. Thus, $a \vee b \notin h^{-1}(P)$, and so $h(a \vee b) \notin P$, which is a contradiction. □

Corollary 5.11. *Let A, B be Hilbert algebras with supremum. If $h\colon A \to B$ is a semi-homomorphism of Hilbert algebras, then the mapping h is a $\vee$-homomorphism iff R_h is an irreducible H-functional relation.*

Proof. This follows from Theorem 3.3 and Theorem 3.6 in [5] and from the previous Theorem. □

Theorem 5.12. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$, $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be $H^{\vee}$-spaces. Let $R \subseteq X_1 \times X_2$ be an H-relation. Then $h_R\colon D(X_2) \to D(X_1)$ preserves the operation $\cup$ if and only if R is irreducible.*

Proof. We suppose that R is an irreducible H-relation. From Theorem 3.4 in [5], we get that h_R is a semi-homomorphism of Hilbert algebras. We will prove that $h_R(U \cup V) = h_R(U) \cup h_R(V)$ for all $U, V \in D(X_2)$. Let $x \in X_1$ be such that $x \in h_R(U \cup V)$. So, $R(x) \subseteq U \cup V$. Suppose that $R(x) \neq \emptyset$. As $R(x)$ is an irreducible subset of X_2, $R(x) \subseteq U$ or $R(x) \subseteq V$. So, $x \in h_R(U) \cup h_R(V)$. The converse is immediate. Therefore, we get that h_R preserves the operation $\cup$.

Conversely, let $x \in X_1$ be such that $R(x) \neq \emptyset$. We will prove that $R(x)$ is irreducible. Let F_1, F_2 be closed subsets of X_2 such that $R(x) \subseteq F_1 \cup F_2$,

$R(x) \nsubseteq F_1$, and $R(x) \nsubseteq F_2$. So, there exists $y_1 \in R(x) - F_1$ and $y_2 \in R(x) - F_2$. Thus, there exist U_1, $U_2 \in D(X_2)$ such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$, $y_1 \notin U_1$, and $y_2 \notin U_2$. So, $R(x) \subseteq U_1 \cup U_2$ and, in consequence, $x \in h_R(U_1) \cup h_R(U_2)$. Then $R(x) \subseteq U_1$ or $R(x) \subseteq U_2$, which is a contradiction because $y_1, y_2 \in R(x)$. □

Corollary 5.13. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1} \rangle$, $\langle X_2, \mathcal{T}_{\mathcal{K}_2} \rangle$ be $H^{\vee}$-spaces. If $R \subseteq X_1 \times X_2$ is an H-relation, then $h_R \colon D(X_2) \to D(X_1)$ is a homomorphism of $H^{\vee}$-algebras if and only if R is an irreducible H-functional relation.*

Proof. This follows from Theorem 3.3 and Theorem 3.6 in [5] and from the previous Theorem. □

Let $\mathcal{HS}^{\vee}$ be the category of $H^{\vee}$-algebras with $\vee$-semi-homomorphisms, and let $\mathcal{HH}^{\vee}$ be the category of $H^{\vee}$-algebras with $\vee$-homomorphisms. Let $\mathcal{SR}^{\vee}$ be the category of $H^{\vee}$-spaces whose morphisms are irreducible H-relations. Let $\mathcal{SF}^{\vee}$ be the category of $H^{\vee}$-spaces with irreducible H-functional relations. From the previous results, we immediately see that the categories $\mathcal{SR}^{\vee}$ and $\mathcal{HS}^{\vee}$ are dually equivalent, and the categories $\mathcal{SF}^{\vee}$ and $\mathcal{HH}^{\vee}$ are dually equivalent.

6. H-partial functions

We will show now that irreducible H-functional relations between $H^{\vee}$-spaces can be characterized by means of special partial functions between $H^{\vee}$-spaces.

Definition 6.1. Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1} \rangle$, $\langle X_2, \mathcal{T}_{\mathcal{K}_2} \rangle$ be $H^{\vee}$-spaces. Let $f \colon X_1 \to X_2$ be a partial map. Let $\mathrm{dom}(f)$ be the domain of f. We shall say that f is an *H-partial function* if the following conditions are satisfied:

(1) $[f(x)) = f([x))$ for each $x \in \mathrm{dom}(f)$,
(2) $x \in \mathrm{dom}(f)$ iff there exists $y \in X_2$ such that $f([x)) = [y)$,
(3) if $U \in \mathcal{K}_2$, then $(f^{-1}(U)] \in \mathcal{K}_1$.

Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1} \rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2} \rangle$ be two $H^{\vee}$-spaces and $R \subseteq X_1 \times X_2$ be an irreducible H-functional relation. Let $x \in X_1$ such that $R(x) \neq \emptyset$. As $R(x)$ is a closed irreducible subset and $\langle X_2, \mathcal{T}_{\mathcal{K}_2} \rangle$ is sober, there exists a unique $y \in X_2$ such that $R(x) = \mathrm{cl}(y) = [y)$. We define a partial function $f_R \colon X_1 \to X_2$ as follows. We set $\mathrm{dom}(f_R) = \{x \in X_1 \mid R(x) \neq \emptyset\}$, and for each $x \in \mathrm{dom}(f_R)$, we set $f_R(x) = y$, i.e., $f_R(x) = y$ iff $R(x) = \mathrm{cl}(y) = [y)$.

Lemma 6.2. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1} \rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2} \rangle$ be two $H^{\vee}$-spaces and $R \subseteq X_1 \times X_2$ an H-relation. Then $\leq_1 \circ R = R$.*

Proof. Let $x, y \in X_1$ and $z \in X_2$. It is clear that $R \subseteq {\leq_1} \circ R$. Assume that $x \leq_1 y$ and $(y, z) \in R$. If $z \notin R(x)$, as $R(x)$ is a closed subset of $\langle X_2, \mathcal{T}_{\mathcal{K}_2} \rangle$, there exists $U \in D(X_2)$ such that $z \notin U$ and $R(x) \subseteq U$. Then $x \in h_R(U) \in D(X_1)$. So, $y \in h_R(U)$, i.e., $R(y) \subseteq U$, which is a contradiction because $z \notin U$. □

Lemma 6.3. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be two $H^{\vee}$-spaces and $R \subseteq X_1 \times X_2$ an irreducible H-functional relation. Then f_R is an H-partial function.*

Proof. We will prove that $[f_R(x)) = f_R([x))$ for each $x \in \mathrm{dom}(f_R)$. Let $z \in [f_R(x)) = R(x)$. As R is an H-functional relation, there exists $w \in X_1$ such that $x \leq_1 w$ and $R(w) = [z)$. So, $w \in [x)$ and $f_R(w) = z$. Then $z \in f_R([x))$. Conversely, let $z \in f_R([x))$. Hence, there exists $a \in [x)$ such that $f_R(a) = z$. So, $x \leq_1 a$ and $z \in \mathrm{cl}(z) = R(a)$. By Lemma 6.2, we have $\leq_1 \circ R = R$. Then $(x, z) \in R$. Thus, $z \in R(x) = [f_R(x))$.

It is clear that $x \in \mathrm{dom}(f_R)$ iff there exists $y \in X_2$ such that $f_R([x)) = [y)$. We will show that $R^{-1}(V) = (f_R^{-1}(V)]$ for all $V \in \mathcal{K}_2$. Let $x \in R^{-1}(V)$. So, there exists $w \in V$ such that $w \in R(x) = [f_R(x))$. Thus, $f_R(x) \in V$ and, consequently, $x \in f_R^{-1}(V)$. So, $x \in (f_R^{-1}(V)]$. Conversely, if $x \in (f_R^{-1}(V)]$, then $V \cap [f_R(x)) = V \cap R(x) \neq \emptyset$. So, $x \in R^{-1}(V)$. Thus, we proved that f_R is an H-partial function. □

Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be two $H^{\vee}$-spaces and $f\colon X_1 \to X_2$ be an H-partial function. The binary relation $R_f \subseteq X_1 \times X_2$ can be defined as follows:

$$R_f(x) = \begin{cases} [f(x)) & \text{if } x \in \mathrm{dom}(f), \\ \emptyset & \text{if } x \notin \mathrm{dom}(f), \end{cases}$$

for each $x \in X_1$.

Lemma 6.4. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be two $H^{\vee}$-spaces and $f\colon X_1 \to X_2$ an H-partial function. Then R_f is an irreducible H-functional relation.*

Proof. We will prove that R_f is an H-functional relation. Let $x \in \mathrm{dom}(f)$. By Remark 3.5, $R_f(x) = [f(x)) = \mathrm{cl}(f(x))$ is a closed subset of $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$. If $x \notin \mathrm{dom}(f)$, then $R_f(x) = \emptyset$. So, $R_f(x)$ is a closed subset of $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ for every $x \in X_1$. Let $U \in \mathcal{K}_2$. We are able to prove that $R_f^{-1}(U) \in \mathcal{K}_1$, showing that $R_f^{-1}(U) = (f^{-1}(U)]$. Let $x \in R_f^{-1}(U)$. So, there exists $w \in U$ such that $w \in R_f(x)$. Hence, $f(x) \leq_2 w$. Thus, $f(x) \in U$ because U is a decreasing subset of X_2. So, $x \in f^{-1}(U)$ and consequently, $x \in (f^{-1}(U)]$. Conversely, if $x \in (f^{-1}(U)]$, then $[x) \cap f^{-1}(U) \neq \emptyset$. So, $[f(x)) \cap U = U \cap R_f(x) \neq \emptyset$, i.e., $x \in R_f^{-1}(U)$. So, we get that $R_f^{-1}(U) = (f^{-1}(U)]$. Thus, by condition (3) of Definition 6.1, $R_f^{-1}(U) \in \mathcal{K}_1$ for all $U \in \mathcal{K}_2$.

Let $x \in \mathrm{dom}(f)$ and suppose that $(x, y) \in R_f$. Then $y \in R_f(x) = [f(x))$, and since f is an H-partial function, $[f(x)) = f([x))$. So, there exists $z \in [x)$ such that $y = f(z)$. So, $x \leq_1 z$ and $[y) = [f(z)) = R_f(z)$. Thus, R_f is an H-functional relation.

If $x \in \mathrm{dom}(f)$, then $R_f(x) \neq \emptyset$. So, to show that R_f is an irreducible relation, it only remains to prove that $R_f(x)$ is an irreducible subset of $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ for every $x \in \mathrm{dom}(f)$. Let Z, W be two closed subsets of $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ such that $R_f(x) \subseteq Z \cup W$. So, $[f(x)) \subseteq Z \cup W$. Therefore, $f(x) \in Z$ or $f(x) \in W$. Since Z, W are increasing subsets of X_2, $[f(x)) \subseteq Z$ or $[f(x)) \subseteq W$. So, $R_f(x) \subseteq Z$ or $R_f(x) \subseteq W$. Thus, R_f is an irreducible H-functional relation. □

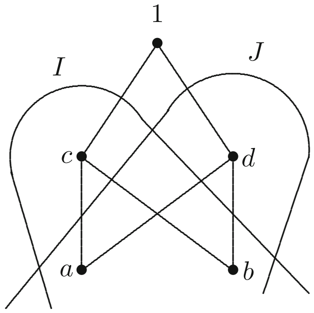

FIGURE 1

Corollary 6.5. *There exists a dual equivalence between the category of $H^{\vee}$-algebras with homomorphisms that preserve suprema and the category of $H^{\vee}$-spaces with H-partial functions.*

7. Ideals of Hilbert algebra with supremum

Let us recall that an order-ideal of a Hilbert algebra $\langle A, \rightarrow, 1\rangle$ is a decreasing subset I of A such that for each $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$. Recall that the set of all order-ideals of $\langle A, \rightarrow, 1\rangle$ will be denoted by $\mathrm{Id}(A)$. Now, if A is an $H^{\vee}$-algebra, then the usual notion of ideal in a join-semilattice coincides with the notion of order-ideal, as is established in the following lemma.

Lemma 7.1. *Let $\langle A, \rightarrow, \vee, 1\rangle$ be an $H^{\vee}$-algebra. A subset I of A is an ideal of $\langle A, \vee, 1\rangle$ iff I is an order-ideal of A.*

Let A be an $H^{\vee}$-algebra. In $\mathrm{Id}(A)$, we define the lattice operations $\sqcap$ and $\sqcup$, and an implication $\twoheadrightarrow$ in the following way:

$$I \sqcap J = I \cap J,$$
$$I \sqcup J = \{x \in A \mid \exists i \in I\ \exists j \in J\ (x \leq i \vee j)\},$$
$$I \twoheadrightarrow J = \{x \in A \mid \forall i \in I\ \exists j \in J\ (x \leq i \rightarrow j)\},$$

for all $I, J \in \mathrm{Id}(A)$.

Unlike the case of deductive systems of a Hilbert algebra, a non-empty intersection of a family of ideals may not be an ideal, as the following example shows.

Example 7.2. Figure 1 shows a finite Hilbert algebra $\langle A, \rightarrow, 1\rangle$ with five elements $A = \{a, b, c, d, 1\}$; the operation $\rightarrow$ is given by the order. The sets $(c]$ and $(d]$ are clearly ideals, but $(c] \cap (d] = \{a, b\}$ is not an ideal.

If A is bounded, i.e., there exists $0 \in A$ such that $0 \leq a$ for all $a \in A$, then the element 0 belongs to any ideal. Then $I \cap H \neq \emptyset$ for any $I, H \in \mathrm{Id}(A)$. But,

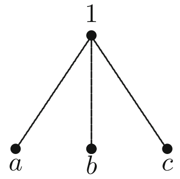

FIGURE 2

if A is not bounded, there may be ideals I and H such that $I \cap H = \emptyset$. In this case, we will consider the set $\mathrm{Id}(A) \cup \{\emptyset\}$. By known results on join-semilattices (see [6]), we get the following result.

Theorem 7.3. *Let A be an $H^{\vee}$-algebra. Then $\langle \mathrm{Id}(A) \cup \{\emptyset\}, \sqcap, \sqcup, A\rangle$ is a lattice.*

The next example shows that the lattice $\langle \mathrm{Id}(A) \cup \{\emptyset\}, \sqcap, \sqcup, A\rangle$ need not be distributive.

Example 7.4. Figure 2 shows a finite Hilbert algebra whose universe is $A = \{a, b, c, 1\}$; the operation $\rightarrow$ is given by the order. We consider the ideals $(a]$, $(b]$, and $(c]$. Then $(a] \cap ((b] \sqcup (c]) = (a] \cap (1] = (a]$, but $((a] \cap (b]) \sqcup ((a] \cap (c]) = \emptyset$.

Proposition 7.5. *Let A be an $H^{\vee}$-algebra. Then:*

(1) *$I \twoheadrightarrow J \in \mathrm{Id}(A)$ for all $I, J \in \mathrm{Id}(A)$.*
(2) *$I \twoheadrightarrow J = A$ iff $I \subseteq J$ for all $I, J \in \mathrm{Id}(A)$.*
(3) *$J \subseteq I \twoheadrightarrow J$ for all $I, J \in \mathrm{Id}(A)$.*
(4) *If $I \twoheadrightarrow J = A$ and $J \twoheadrightarrow I = A$ then, $I = J$ for all $I, J \in \mathrm{Id}(A)$.*
(5) *If $I \subseteq J \twoheadrightarrow Z$ then $I \cap J \subseteq Z$ for all $I, J, Z \in \mathrm{Id}(A)$.*

Proof. (1): First, we will prove that $I \twoheadrightarrow J$ is a decreasing subset of A. Let $a, b \in A$ such that $a \in I \twoheadrightarrow J$ and $b \leq a$. As $a \in I \twoheadrightarrow J$ for all $i \in I$, there exists $j \in J$ such that $a \leq i \rightarrow j$. As $b \leq i \rightarrow j$, so $b \in I \twoheadrightarrow J$. Let $a, b \in I \twoheadrightarrow J$. So, for every $i \in I$, there exists $j \in J$ such that $a \leq i \rightarrow j$, and for every $i \in I$, there exists $h \in J$ such that $b \leq i \rightarrow h$. As $J \in \mathrm{Id}(A)$ and $h, j \in J$, there exists $t \in J$ such that $h \leq t$ and $j \leq t$. As for all $i \in I$, $i \rightarrow h \leq i \rightarrow t$ and $i \rightarrow j \leq i \rightarrow t$, we can conclude $a \leq i \rightarrow t$ and $b \leq i \rightarrow t$. Thus, for all $i \in I$, there exists $t \in J$ such that $a \vee b \leq i \rightarrow t$. So, $a \vee b \in I \twoheadrightarrow J$ where $a \leq a \vee b$ and $b \leq a \vee b$. Therefore, we get that $I \twoheadrightarrow J$ is an ideal of A.

(2): If $I \twoheadrightarrow J = A$, then $x \in I \twoheadrightarrow J$ for all $x \in A$. In particular, $1 \in I \twoheadrightarrow J$. So, for all $i \in I$, there exists $j \in J$ such that $1 \leq i \rightarrow j$. Thus, $i \rightarrow j = 1$. Hence, $i \leq j$ for all $i \in I$. As J is a decreasing subset, $i \in J$ for all $i \in I$. Thus, $I \subseteq J$. Conversely, we suppose that $I \subseteq J$ and that $I \twoheadrightarrow J \neq A$. So, there exists $x \in A$ such that $x \notin I \twoheadrightarrow J$. That is, there exists $i_0 \in I$ such that for all $j \in J$, we have $x \not\leq i_0 \rightarrow j$. As $I \subseteq J$, we obtain that $i_0 \in J$. So, $x \not\leq i_0 \rightarrow i_0 = 1$, which is a contradiction.

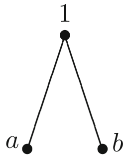

Figure 3

(3): We will prove that $J \subseteq I \twoheadrightarrow J$. Let $j \in J$. Since $j \leq i \to j$ for all $i \in I$, we have $j \in I \twoheadrightarrow J$ and consequently, $J \subseteq I \twoheadrightarrow J$.

(4): Let $I \twoheadrightarrow J = A$ and $J \twoheadrightarrow I = A$. Applying (2), we get $I = J$.

(5): Suppose that $I \cap J \neq \emptyset$. Let $I \subseteq J \twoheadrightarrow Z$ and $x \in I \cap J$. As $x \in I$, $x \in J \twoheadrightarrow Z$. So, for all $j \in J$, there exists $z \in Z$ such that $x \leq j \to z$. In particular, for $x \in J$, we get $x \leq x \to z$ and so, $1 = x \to (x \to z) = x \to z$. So, $x \leq z$. As Z is a decreasing subset of A, $x \in Z$. Thus, $I \cap J \subseteq Z$. □

Unfortunately, the set of ideals of an $H^{\vee}$-algebra with the implication $\twoheadrightarrow$ is not a Hilbert algebra, as shown in the following example.

Example 7.6. Figure 3 shows a finite Hilbert algebra whose universe is $A = \{a, b, 1\}$; the operation $\to$ is given by the order. If we consider the ideals $I = (a]$, $J = (b]$, and $Z = (a]$, then it is easy to see that $I \twoheadrightarrow J = \{b\}$, $(I \twoheadrightarrow J) \twoheadrightarrow Z = \{a\}$, and as $I \twoheadrightarrow Z = A$, $(I \twoheadrightarrow J) \twoheadrightarrow (I \twoheadrightarrow Z) = A$. So, $(I \twoheadrightarrow J) \twoheadrightarrow Z \neq (I \twoheadrightarrow J) \twoheadrightarrow (I \twoheadrightarrow Z)$.

Remark 7.7. Let A be an $H^{\vee}$-algebra. In general, $\langle \mathrm{Id}(A) \cup \{\emptyset\}, \twoheadrightarrow, \sqcap, A\rangle$ is not an implicative semilattice. We consider the Boolean lattice with two atoms $A = \{0, a, b, 1\}$; the operation $\to$ is given by the order. Considering the following ideals: $I = (a]$, $J = (b]$, and $Z = (0]$, we have that $I \cap J \subseteq Z$, but $I \nsubseteq J \twoheadrightarrow Z$.

Theorem 7.8. *Let A be an $H^{\vee}$-algebra. Then the mapping $\alpha\colon A \to \mathrm{Id}(A)$ defined by $\alpha(a) = (a]$ for each $a \in A$ is order-preserving and preserves supremum and implication.*

Proof. It is clear that α is order-preserving and preserves supremum. Let $a, b \in A$. We will prove that $\alpha(a \to b) = \alpha(a) \twoheadrightarrow \alpha(b)$. Let $x \in \alpha(a \to b)$. So, $x \leq a \to b$. As $i \leq a$ for every $i \in \alpha(a)$, we have $a \to b \leq i \to b$. Hence, for each $i \in \alpha(a)$, one has $x \leq i \to b$. So, $x \in \alpha(a) \twoheadrightarrow \alpha(b)$. Now, suppose that $x \in \alpha(a) \twoheadrightarrow \alpha(b)$. So, for every $i \in \alpha(a)$ there exists $j \in \alpha(b)$ such that $x \leq i \to j$. In particular, for $i = a$, we get that $x \leq a \to j$. Since $j \leq b$, we get that $a \to j \leq a \to b$. So, $x \in \alpha(a \to b)$. □

8. Ideals and open directed subsets

In this section, we show how the duality developed in the previous sections works by establishing a dual description of the notion of ideal.

Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be an $H^{\vee}$-space. Let $Y \subseteq X$, and let $\mathcal{B} \subseteq D(X)$. The following notations will be used. Denote $\{X - U \mid U \in \mathcal{B}\} = \{U^c \mid U \in \mathcal{B}\}$ by $\overline{\mathcal{B}}$, and $\{U \in D(X) \mid U \subseteq Y\}$ by $I(Y)$. We note that $I(Y)$ is an ideal of $D(X)$. Recall that $\langle X, \mathcal{T}_{D(X)} \rangle$ is the topological space defined on X considering the set $D(X) = \{U^c \mid U \in \mathcal{K}\}$ as subbase of the topology $\mathcal{T}_{D(X)}$. Let $\mathrm{int}_{D(X)}(Y)$ be the interior of the set $Y \subseteq X$ in the topological space $\langle X, \mathcal{T}_{D(X)} \rangle$.

Definition 8.1. Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be an $H^{\vee}$-space. We shall say that $Y \subseteq X$ is *open directed* iff there is $\mathcal{B} \subseteq D(X)$ such that $Y = \bigcup\{U \mid U \in \mathcal{B}\} = \bigcup \mathcal{B}$.

Let $\mathrm{Od}(X)$ be the set of all open directed subsets of $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$.

Remarks 8.2. (1) Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be an $H^{\vee}$-space. Let $Y \in \mathrm{Od}(X)$, and let $\mathcal{B} \subseteq D(X)$ be such that $Y = \bigcup\{U \mid U \in \mathcal{B}\} = \bigcup \mathcal{B}$. It is clear that $\mathcal{B} \subseteq I(Y)$. As $Y = \bigcup \mathcal{B} \subseteq \bigcup I(Y) \subseteq Y$, we get that $Y = \bigcup I(Y)$. We can conclude that

$$Y \subseteq X \text{ is open directed iff } Y = \bigcup I(Y).$$

(2) It is clear that $\bigcup I(O) \subseteq O$ for every open subset O of $\langle X, \mathcal{T}_{D(X)} \rangle$. However, not every open subset of $\langle X, \mathcal{T}_{D(X)} \rangle$ can be written as $\bigcup I(O)$ because $D(X)$ is only a subbase of $\langle X, \mathcal{T}_{D(X)} \rangle$.

Proposition 8.3. *Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be an $H^{\vee}$-space. Then $\mathrm{Od}(X)$ is closed under unions of sets and under the operations of infimum and implication defined by*

$$Y \mathbin{\overline{\wedge}} Z = \bigcup I(Y \cap Z),$$
$$Y \rightarrowtail Z = \bigcup\{U \in D(X) \mid U \subseteq Y^c \cup Z\} = \bigcup I(Y^c \cup Z),$$

for each $Y, Z \in \mathrm{Od}(X)$.

Proof. Let $Y, Z \in \mathrm{Od}(X)$. By definition, it is clear that $Y \rightarrowtail Z$ and $Y \mathbin{\overline{\wedge}} Z$ belong to $\mathrm{Od}(X)$. We are able to show that $Y \cup Z \in \mathrm{Od}(X)$, proving that $Y \cup Z = \bigcup I(Y \cup Z)$. Clearly, $\bigcup I(Y \cup Z) = \bigcup\{U \in D(X) \mid U \subseteq Y \cup Z\} \subseteq Y \cup Z$. As $Y \subseteq Y \cup Z$, we have that $I(Y) \subseteq I(Y \cup Z)$. So, $Y = \bigcup I(Y) \subseteq \bigcup I(Y \cup Z)$. Similarly, $Z \subseteq \bigcup I(Y \cup Z)$. Thus, $Y \cup Z \subseteq \bigcup I(Y \cup Z)$, which completes the proof. □

Theorem 8.4. *Let $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ be an $H^{\vee}$-space. Then $\langle \mathrm{Od}(X), \overline{\wedge}, \cup, X \rangle$ is a lattice, and for all $Y, Z \in \mathrm{Od}(X)$:*

(1) *$Y \subseteq Z$ iff $Y \rightarrowtail Z = X$.*
(2) *$Z \subseteq Y \rightarrowtail Z$.*
(3) *If $Y \rightarrowtail Z = X$ and $Z \rightarrowtail Y = X$, then $Y = Z$.*

Proof. It is clear that $\langle \mathrm{Od}(X), \overline{\wedge}, \cup, X \rangle$ is a lattice.

(1): If $Y \subseteq Z$, then $Y^c \cup Z = X$. So, $Y \rightarrowtail Z = \bigcup I(Y^c \cup Z) = X$. Now suppose that $Y \rightarrowtail Z = X$ and $Y \nsubseteq Z$. So, there exists $a \in Y$ such that $a \notin Z$. Since $a \in X$ and $Y \rightarrowtail Z = X$, there exists $U \in D(X)$ such that $a \in U$ and $U \subseteq Y^c \cup Z$. So, $a \in Y^c \cup Z$, which is a contradiction.

(2): If $U \in D(X)$ such that $U \subseteq Z$, then $U \subseteq Y^c \cup Z$. Consequently, $\bigcup I(Z) \subseteq \bigcup I(Y^c \cup Z)$. That is, $Z \subseteq Y \rightarrowtail Z$.

(3): If $Y \rightarrowtail Z = X$ and $Z \rightarrowtail Y = X$, (1) makes it obvious that $Y = Z$. □

Proposition 8.5. *Let A be an $H^\vee$-algebra. Then $\beta\colon \mathrm{Id}(A) \to \mathrm{Od}(X(A))$ defined by*

$$\beta(I) = \{P \in X(A) \mid P \cap I \neq \emptyset\}$$

for each $I \in \mathrm{Id}(A)$ is a lattice-isomorphism that preserves implication.

Proof. Clearly, β is well defined because $\beta(I) = \bigcup\{\varphi(a) \mid a \in I\} \in \mathrm{Od}(X(A))$ for every $I \in \mathrm{Id}(A)$.

We will prove that $I \subseteq J$ iff $\beta(I) \subseteq \beta(J)$ for each $I, J \in \mathrm{Id}(A)$. Let $I \subseteq J$. Suppose that $P \in \beta(I)$. So, there exists $a \in I$ and $a \in P$. As $I \subseteq J$, $a \in J \cap P$ and consequently, $P \in \beta(J)$. So, $\beta(I) \subseteq \beta(J)$. Now, let $\beta(I) \subseteq \beta(J)$ and we suppose that $I \nsubseteq J$. There exists $a \in I$ such that $a \notin J$. So, there exists $Q \in X(A)$ such that $a \in Q$ and $Q \cap J = \emptyset$. Then $Q \notin \beta(J)$ and $Q \in \beta(I)$. Thus, $\beta(I) \nsubseteq \beta(J)$. Thus, β is an order-isomorphism and consequently, β is injective.

By this fact, β is a lattice homomorphism, i.e., $\beta(I \cap J) = \beta(I) \barwedge \beta(J)$ and $\beta(I \sqcup J) = \beta(I) \cup \beta(J)$, for all $I, J \in \mathrm{Id}(A)$.

We prove that β is surjective. Let $Y \in \mathrm{Od}(X(A))$, i.e., there exists $B \subseteq A$ such that $Y = \bigcup\{\varphi(a) \mid a \in B\}$. We will see that there exists $I \in \mathrm{Id}(A)$ such that $\beta(I) = Y$. Consider the ideal generated by B, i.e.,

$$(B] = \{x \in A \mid \exists\{b_1, \dots, b_n\} \subseteq B \ (x \leq b_1 \vee \cdots \vee b_n)\}.$$

We will prove that $\beta((B]) = Y$. Let $P \in \beta((B]) = \bigcup\{\varphi(b) \mid b \in (B]\}$. There exists $b \in (B]$ such that $P \in \varphi(b)$. So, there exist $b_1, \dots, b_n \in B$ such that $b \leq b_1 \vee \cdots \vee b_n$. As $P \in X(A)$, $b_i \in P$ for some $b_i \in \{b_1, \dots, b_n\}$. So, $P \in \varphi(b_i)$, i.e., $P \in \bigcup\{\varphi(b) \mid b \in B\} = Y$. Now let $P \in Y$. There exists $b \in B$ such that $P \in \varphi(b)$. So, $P \cap (B] \neq \emptyset$, i.e., $P \in \beta((B])$. Thus, $Y = \beta((B])$.

We prove that $\beta(I \twoheadrightarrow J) = \beta(I) \rightarrowtail \beta(J)$ for all $I, J \in \mathrm{Id}(A)$. Let $P \in X(A)$ be such that $P \in \beta(I \twoheadrightarrow J)$. Then $P \cap (I \twoheadrightarrow J) \neq \emptyset$. There exists $a \in A$ such that $a \in P$ and $a \in I \twoheadrightarrow J$. We suppose that $P \notin \beta(I) \rightarrowtail \beta(J)$. So,

$$P \in \bigcap\{\varphi(b)^c \mid \varphi(b) \subseteq \beta(I)^c \cup \beta(J)\}.$$

As $a \in P$, $\varphi(a) \nsubseteq \beta(I)^c \cup \beta(J)$. Thus, there exists $Q \in X(A)$ such that $Q \in \varphi(a)$ and $Q \notin \beta(I)^c \cup \beta(J)$. It follows that $a \in Q$, $Q \cap I \neq \emptyset$ and $Q \cap J = \emptyset$. As $Q \cap I \neq \emptyset$, there exists $i_0 \in I$ such that $i_0 \in Q$. As $a \in I \twoheadrightarrow J$ for all $i \in I$, there exists $j \in J$ such that $a \leq i \to j$. In particular, $a \leq i_0 \to j$. Since $a \in Q$, $i_0 \to j \in Q$, and by modus ponens, $j \in Q$. This is a contradiction because $Q \cap J = \emptyset$. Thus, $\beta(I \twoheadrightarrow J) \subseteq \beta(I) \rightarrowtail \beta(J)$.

To show that $\beta(I) \rightarrowtail \beta(J) \subseteq \beta(I \twoheadrightarrow J)$, let $P \in X(A)$ be such that $P \in \beta(I) \rightarrowtail \beta(J) = \bigcup\{\varphi(a) \mid \varphi(a) \subseteq \beta(I)^c \cup \beta(J)\}$. So, there exists $a \in A$ such that $P \in \varphi(a)$ and $\varphi(a) \subseteq \beta(I)^c \cup \beta(J)$. We suppose that $P \cap (I \twoheadrightarrow J) = \emptyset$. As $a \in P$, $a \notin I \twoheadrightarrow J$. Therefore, there exists $i_0 \in I$ such that $a \nleq i_0 \to j$ for all $j \in J$. So, $i_0 \to j \notin \langle a \rangle$ for all $j \in J$. Hence, $j \notin \langle \{a, i_0\} \rangle$ for all $j \in J$. So, $\langle \{a, i_0\} \rangle \cap J = \emptyset$. Thus, there exists $Q \in X(A)$ such that $\langle \{a, i_0\} \rangle \subseteq Q$ and $Q \cap J = \emptyset$. As $a \in Q$, $Q \in \beta(I)^c \cup \beta(J)$. So, $I \cap Q = \emptyset$, which is a contradiction because $i_0 \in Q$ and $i_0 \in I$. So, we get that $\beta(I) \rightarrowtail \beta(J) = \beta(I \twoheadrightarrow J)$ for every $I, J \in \mathrm{Id}(A)$.

Thus, we have that $\langle \mathrm{Id}(A), \twoheadrightarrow, A \rangle$ is isomorphic to $\langle \mathrm{Od}(X(A)), \rightarrowtail, X(A) \rangle$ by means of β. □

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