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Natural extensions and profinite completions of algebras

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Abstract. This paper investigates profinite completions of residually finite algebras, drawing on ideas from the theory of natural dualities. Given a class $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a set, not necessarily finite, of finite algebras, it is shown that each $\mathbf{A} \in \mathcal{A}$ embeds as a topologically dense subalgebra of a topological algebra $n_{\mathcal{A}}(\mathbf{A})$ (its *natural extension*), and that $n_{\mathcal{A}}(\mathbf{A})$ is isomorphic, topologically and algebraically, to the profinite completion of $\mathbf{A}$. In addition it is shown how the natural extension may be concretely described as a certain family of relation-preserving maps; in the special case that $\mathcal{M}$ is finite and $\mathcal{A}$ possesses a single-sorted or multisorted natural duality, the relations to be preserved can be taken to be those belonging to a dualising set. For an algebra belonging to a finitely generated variety of lattice-based algebras, it is known that the profinite completion coincides with the canonical extension. In this situation the natural extension provides a new concrete realisation of the canonical extension, generalising the well-known representation of the canonical extension of a bounded distributive lattice as the lattice of up-sets of the underlying ordered set of its Priestley dual. The paper concludes with a survey of classes of algebras to which the main theorems do, and do not, apply.

1. Introduction

We study residually finite algebras and the profinite completions of such algebras. We define an algebra to be *residually finite* if it belongs to some class $\mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a set of finite algebras; we shall call such a class an *internally residually finite prevariety*. Background information on residual finiteness is given in Section 2 (the basics) and in Section 5 (major theorems). In Section 2 we also spell out precisely the definitions and elementary properties of profinite completions.

By way of introduction, we recall that we can associate with any algebra $\mathbf{A}$ an algebra $\widehat{\mathbf{A}}$, known as its profinite completion. Loosely, $\widehat{\mathbf{A}}$ is the inverse (or projective) limit algebra formed from the finite quotients $\mathbf{A}/\alpha$, where the indexing set is the set of congruences α of finite index, ordered by reverse inclusion and the bonding maps are the natural homomorphisms. There is a natural homomorphism from $\mathbf{A}$ into $\widehat{\mathbf{A}}$ and this is an embedding precisely

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when $\mathbf{A}$ is residually finite (cf. Lemma 2.1). By definition, any residually finite algebra lies in some class $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\boldsymbol{\mathcal{M}})$, where $\boldsymbol{\mathcal{M}}$ is a set of finite algebras. We adopt a notion of profinite completion of $\mathbf{A}$ relativised to the internally residually finite prevariety $\boldsymbol{\mathcal{A}}$. We denote this completion, into which $\mathbf{A}$ embeds, by $\mathrm{pro}_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ (see Section 2 for the definition). When $\boldsymbol{\mathcal{A}}$ is a variety, $\mathrm{pro}_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ coincides with $\widehat{\mathbf{A}}$.

As observed by George Grätzer [35, p. 133], inverse limits of algebras can be hard to visualise. Hence alternative concrete descriptions of profinite completions can be valuable. Our contribution is to provide a new concrete description of the profinite completion of any residually finite algebra (Theorems 3.6 and 4.1 combined). This description is uniform, in that it takes the same form for all such algebras. Interest in profinite completions has been boosted by the discovery that, for residually finite lattice-based varieties, these completions coincide with canonical extensions (see in particular J. Harding's paper [36]). Thus a by-product of our results is a new concrete characterisation of canonical extensions in such varieties.

Projective limits are, of course, defined in the setting of an arbitrary category and in particular in categories of topological algebras in which the finite objects carry the discrete topology. In this situation profinite completions are Boolean topological algebras with many desirable properties. For discussions of such completions of particular relevance to our study see Clark *et al.* [10, Section 8], Clark *et al.* [11, Sections 1 and 2], and also the papers of B. Banaschewski [3] and K. Numakura [60]. Davey, Haviar and Priestley [18] investigated profinite completions of distributive lattices from the standpoint of topological algebra, revealing, *inter alia*, the relationship between profinite completions and Priestley duality.

We shall exploit ideas from natural duality theory, as presented for example in the monograph [9] by Clark and Davey. We stress however that our principal results are applicable in a wider setting than that of [9]: we consider classes $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\boldsymbol{\mathcal{M}})$, where $\boldsymbol{\mathcal{M}}$ is a set of finite algebras, without demanding, as in [9], that $\boldsymbol{\mathcal{M}}$ be finite. Even when working in that restricted setting, we do not require dualisability; however we do get a more refined description of the profinite completions of the members of $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$ when a natural duality is available (Theorem 4.3). We do not give here the many definitions that would be required if we were to explain precisely what is meant by a natural duality, but shall otherwise make our account as self-contained as possible. The 'vanilla' version of natural duality theory is developed for classes $\mathbb{ISP}(\mathbf{M})$, for $\mathbf{M}$ a finite algebra. In this setting the determination of which quasivarieties are dualisable and which are not is a challenging problem without a definitive answer, but one on which there has been much fruitful research. More generally, a theory of multisorted dualities, initiated by Davey and Priestley in [22] and sketched in [9, Section 7.1] is available for quasivarieties $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$, where now $\boldsymbol{\mathcal{M}}$ is a finite set of finite algebras. Conceptually this theory is no more difficult than that for the single-sorted case, though at first sight the notation may look forbidding.

In outline, the basis of our approach to completions is as follows. Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathcal{M})$ be an internally residually finite prevariety. In Section 2 we define, in a completely natural way, the promised profinite completion $\mathrm{pro}_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ of $\mathbf{A}$, relative to $\boldsymbol{\mathcal{A}}$. Then, in Section 3, we construct a covariant natural extension functor $n_{\boldsymbol{\mathcal{A}}}$, derived via hom-functors, so that each $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ is a topological algebra whose underlying algebra belongs to $\boldsymbol{\mathcal{A}}$ and such that the original algebra is isomorphic to a topologically dense subalgebra of $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$. We then prove, by an argument that is categorical in style, that the natural extension of $\mathbf{A}$ is isomorphic, algebraically and topologically, to the $\boldsymbol{\mathcal{A}}$-profinite completion (Theorem 3.6). We indicate informally here how the construction of the natural extension works at the object level in the case that $\boldsymbol{\mathcal{A}}$ is a quasivariety $\mathbb{ISP}(\mathbf{M})$, for some finite algebra $\mathbf{M}$. Let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. We form the topological product $\mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})}$, where $\mathbf{M}_{\mathcal{T}}$ denotes $\mathbf{M}$ equipped with the discrete topology. Within $\mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})}$ we consider the subset $e_{\mathbf{A}}(A) = \{\, e_{\mathbf{A}}(a) \mid a \in A \,\}$ consisting of the evaluation maps. Then $e_{\mathbf{A}}(\mathbf{A})$ is an algebra isomorphic to $\mathbf{A}$, by virtue of the fact that $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$. The natural extension $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ is defined to be the topological closure in $\mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})}$ of $e_{\mathbf{A}}(A)$.

Section 3 owes relatively little to duality theory. Those familiar with the notion of an alter ego will find no mention of it there: we leapfrog the level of the first dual when constructing the natural extension. In Section 4 the focus changes. We recall what happens in the special situation that $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ is dualisable, by means of some alter ego $\underset{\sim}{\mathbf{M}}$ of $\mathbf{M}$. Then, for each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, its isomorphic copy $e_{\mathbf{A}}(\mathbf{A})$ consists exactly of the continuous maps preserving all finitary algebraic relations on $\mathbf{M}$, or equivalently those continuous maps preserving the structure derived from $\underset{\sim}{\mathbf{M}}$. Here we have an instance of entailment of relations, a topic of key importance in duality theory (see [9, Chapters 8 and 9]). We adapt to our setting ideas of entailment. This enables us to describe the natural extension as the family of (not necessarily continuous) maps preserving all finitary algebraic relations, or in the dualisable case, preserving all members of a dualising set. The precise statements, for the multisorted case, are given in Theorem 4.1 and Theorem 4.3.

The last two sections of the paper concern the applicability of our results. Section 5 draws heavily on results from universal algebra, and on the existing literature. We present a survey of classes of varieties of algebras to which the main theorems do, and do not, apply. Specifically, we highlight the extent to which residual finiteness correlates with dualisability. Classes considered include lattice-based varieties and more generally NU-varieties, those generated by monotone clones, as well as affine complete, paraprimal and discriminator varieties and varieties of groups. In all these cases the results are largely positive. We draw attention in particular to Theorems 5.1, 5.2 and 5.6. We also consider briefly semilattice-based varieties, varieties of semigroups, and varieties generated by unary algebras; here the results are less complete, but enough is known for contrasting behaviours to show up.

In Section 6 we focus on the natural extension, giving illustrations in prevarieties which are not varieties and in varieties which are not dualisable.

A comment on our policy on referencing is appropriate here. The scope of this paper is, as we have indicated above, very wide, both as regards its range of techniques and as regards the diversity of the examples. Limitations of space have dictated that not all primary sources could be cited and, where suitable secondary sources with comprehensive bibliographies were available, we have taken advantage of them, especially for peripheral material and some historical background.

2. Preliminaries: profinite completions

In this section we formally introduce profinite completions and present some of their properties, including the relationship between profiniteness and residual finiteness. Basic facts on projective, *alias* inverse, limits in categories can be found in S. Mac Lane [50]. For a discussion in the context of algebras see for example G. Grätzer [35, Section 21] and of topological spaces see R. Engelking [27, Chapter 2, Section 6]. For further background on profinite limits we recommend Ribes and Zalesskii [68, Chapter 1].

Consider a class $\boldsymbol{\mathcal{A}}$ of algebras all of the same type. We call this a *prevariety* if it is of the form $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$, the class of isomorphic copies of subalgebras of products of algebras in $\boldsymbol{\mathcal{M}}$ (where $\boldsymbol{\mathcal{M}}$ may be a proper class). According to our earlier definition, $\boldsymbol{\mathcal{A}}$ is an internally residually finite prevariety, henceforth usually abbreviated to IRF-prevariety, if in addition $\boldsymbol{\mathcal{M}}$ is a set and every algebra in $\boldsymbol{\mathcal{M}}$ can be chosen to be finite. Residual finiteness is customarily studied in the setting of varieties. There, a single algebra is called *residually finite* (the term *finitely approximable* also occurs in the literature) if it is isomorphic to a subalgebra of a product of finite algebras, and a variety is called residually finite if all its members are residually finite. Thanks to Birkhoff's Subdirect Product Theorem, a variety is residually finite in this traditional sense if and only if it is internally residually finite.

If $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$ is an IRF-prevariety in which $\boldsymbol{\mathcal{M}}$ is a finite set of finite algebras, then $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$ will be closed under ultraproducts and thus a quasivariety (see Burris and Sankapannavar [8, p. 219]). On the other hand, an IRF-prevariety need not be a quasivariety in general—see Example 6.2 below. We warn that a prevariety which is such that each of its members is residually finite need not be internally residually finite: the prevariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbb{Z})$ of abelian groups generated by the infinite cyclic group is residually finite (since $\mathbb{Z}$ is) but is not internally residually finite as it contains no non-trivial finite members.

We now fix a prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, not yet assumed to be an IRF-prevariety. Take $\mathbf{A} \in \mathcal{A}$. Consider the family of congruences

$$\mathcal{S}_{\mathbf{A}} := \{\, \alpha \in \mathrm{Con}(\mathbf{A}) \mid \mathbf{A}/\alpha \in \mathcal{A} \text{ and } \mathbf{A}/\alpha \text{ is finite}\,\}.$$

Then $\mathcal{S}_{\mathbf{A}}$ is closed under finite intersections and hence is a directed set with respect to $\supseteq$. (In fact $\mathcal{S}_{\mathbf{A}}$ is a sublattice of $\mathrm{Con}\,\mathbf{A}$ with respect to the inclusion order.) There are then natural bonding homomorphisms $\varphi_{\alpha\beta} \colon \mathbf{A}/\alpha \to \mathbf{A}/\beta$, for $\alpha \subseteq \beta$, given by $\varphi_{\alpha\beta}(a/\alpha) = a/\beta$. The algebras $\mathbf{A}/\alpha$, for $\alpha \in \mathcal{S}_{\mathbf{A}}$, together with the bonding maps $\varphi_{\alpha\beta}$ form an inverse system. The categorical inverse limit of this inverse system is denoted by $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ and is called the *$\mathcal{A}$-profinite completion of* $\mathbf{A}$. This limit is uniquely determined by $\mathcal{A}$ and is realised concretely as the subalgebra

$$\mathrm{pro}_{\mathcal{A}}(\mathbf{A}) := \Big\{\, c \in \prod_{\alpha \in \mathcal{S}_{\mathbf{A}}} \mathbf{A}/\alpha \;\Big|\; (\forall \alpha, \beta \in \mathcal{S}_{\mathbf{A}})\ \alpha \subseteq \beta \implies \varphi_{\alpha\beta}(c(\alpha)) = c(\beta) \,\Big\}$$

of the full product. There is a natural homomorphism $\mu_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ given by $\mu_{\mathbf{A}}(a)(\alpha) := a/\alpha$, for all $a \in A$ and $\alpha \in \mathcal{S}_{\mathbf{A}}$. Furthermore, the profinite completion has a universal mapping property which characterises it, namely that, given any $\mathbf{B} \in \mathcal{A}$ and any cone $(f_\alpha \colon \mathbf{B} \to \mathbf{A}/\alpha)_{\alpha \in \mathcal{S}_{\mathbf{A}}}$ of $\mathcal{A}$-homomorphisms compatible with the bonding maps, there exists a unique homomorphism $f \colon \mathbf{B} \to \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ such that $f_\alpha = \pi_\alpha \circ f$.

Consider a variety $\mathcal{V}$. Then the inverse system defining the profinite completion $\mathrm{pro}_{\mathcal{V}}(\mathbf{A})$ of an algebra $\mathbf{A} \in \mathcal{V}$ consists of all congruences α of finite index, since $\mathbf{A}/\alpha$ always belongs to $\mathcal{V}$; in this context $\mathrm{pro}_{\mathcal{V}}(\mathbf{A})$ is the traditional profinite completion $\widehat{\mathbf{A}}$. Now suppose we have an IRF-prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ which is not a variety. Let $\mathcal{V} = \mathrm{Var}(\mathcal{M})$, the variety generated by $\mathcal{M}$. We note that there is no reason to expect in general that, for $\mathbf{A} \in \mathcal{A}$, the profinite completion $\widehat{\mathbf{A}}$ (calculated relative to $\mathcal{V}$) will coincide with $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$.

The following straightforward lemma tells us that IRF-prevarieties provide exactly the appropriate setting in which to work if we wish to demand that algebras embed into their profinite completions.

Lemma 2.1. *Let $\mathcal{A}$ be a class of algebras. Then the following are equivalent:*

(i) *there exists a set $\mathcal{M}$ of finite algebras such that $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ (that is, $\mathcal{A}$ is an IRF-prevariety);*

(ii) *given $\mathbf{A} \in \mathcal{A}$ and $a \neq b$ in $\mathbf{A}$, there exists a homomorphism $h_{ab} \colon \mathbf{A} \to \mathbf{M}_{ab}$, for some finite algebra $\mathbf{M}_{ab}$ in $\mathcal{A}$, such that $h_{ab}(a) \neq h_{ab}(b)$;*

(iii) *the natural homomorphism $\mu_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ is an embedding, for all $\mathbf{A} \in \mathcal{A}$.*

We now turn to topological algebras. Again let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ be an IRF-prevariety. The $\mathcal{A}$-profinite completion $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A} \in \mathcal{A}$ carries a natural Boolean topology: for each $\alpha \in \mathcal{S}_{\mathbf{A}}$, endow the finite factor $\mathbf{A}/\alpha$ with the discrete topology, then $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ is a closed subspace of the product of these discrete spaces. We need to formalise this. For a finite algebra $\mathbf{M}$, we let $\mathbf{M}_{\mathcal{T}}$ denote $\mathbf{M}$ equipped with the discrete topology and for a set $\mathcal{M}$ of finite algebras we let $\mathcal{M}_{\mathcal{T}} := \{\, \mathbf{M}_{\mathcal{T}} \mid \mathbf{M} \in \mathcal{M} \,\}$. The *topological prevariety generated by* $\mathcal{M}_{\mathcal{T}}$ is then defined to be $\boldsymbol{\mathcal{A}}_{\mathcal{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathcal{M}_{\mathcal{T}})$, the class of isomorphic copies of topologically closed subalgebras of products of members of $\mathcal{M}_{\mathcal{T}}$. We note that $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$ is a subclass of the class of Boolean topological algebras with algebraic reduct in $\boldsymbol{\mathcal{A}}$. Here a Boolean topological algebra means an algebra equipped with a Boolean (that is, compact and zero-dimensional) topology such that the operations are continuous. We make $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$ into a category in the expected way: the morphisms are the continuous homomorphisms. This category will play an important role in the next section.

3. Natural extensions of algebras

Let $\boldsymbol{\mathcal{A}}$ be an IRF-prevariety of algebras, so $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathcal{M})$ for a set $\mathcal{M}$ of finite algebras in $\boldsymbol{\mathcal{A}}$, and let $\boldsymbol{\mathcal{A}}_{\mathcal{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathcal{M}_{\mathcal{T}})$ be the associated topological prevariety defined above. Let ${}^{\flat} \colon \boldsymbol{\mathcal{A}}_{\mathcal{T}} \to \boldsymbol{\mathcal{A}}$ denote the natural forgetful functor.

Take $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Let $X_{\mathbf{A}} := \dot{\bigcup}\{\, \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M} \,\}$. For each $\mathbf{M} \in \mathcal{M}$ and $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$, let $\mathbf{Y}_x := \mathbf{M}_{\mathcal{T}}$, that is, $\mathbf{Y}_x$ is the codomain $\mathbf{M}$ of the map x with the discrete topology $\mathcal{T}$ added. We define a map

$$e_{\mathbf{A}} \colon \mathbf{A} \to \prod \{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\}$$

by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in A$ and $x \in X_{\mathbf{A}}$. As $\mathbf{A} \in \mathbb{ISP}(\mathcal{M})$, the homomorphism $e_{\mathbf{A}}$ is an embedding. Observe that $\prod \{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\} \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}$.

We are ready to define the *natural extension* $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ of $\mathbf{A}$ in $\boldsymbol{\mathcal{A}}$ (relative to $\mathcal{M}$): it is the topological closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\prod\{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\}$. (We give an alternative but equivalent definition in Remark 3.3 below.) We have constructed a map $\mathbf{A} \mapsto n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ from $\boldsymbol{\mathcal{A}}$ into $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$. Ostensibly, this depends upon the choice of the generating set $\mathcal{M}$ for the prevariety $\boldsymbol{\mathcal{A}}$. We shall see later that $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ is in fact independent of $\mathcal{M}$.

We now define $n_{\boldsymbol{\mathcal{A}}}$ on morphisms. Let $u \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism with $\mathbf{A}, \mathbf{B} \in \boldsymbol{\mathcal{A}}$. We shall define a map $n_{\boldsymbol{\mathcal{A}}}(u) \colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{A}) \to n_{\boldsymbol{\mathcal{A}}}(\mathbf{B})$. We first note that, if $y \in X_{\mathbf{B}}$, then $y \circ u \in X_{\mathbf{A}}$; so $(y \circ u \colon \mathbf{A} \to \mathbf{Y}_y) \in X_{\mathbf{A}}$. Now, for each $y \in X_{\mathbf{B}}$, we define

$$u_y \colon \prod\{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\} \to \mathbf{Y}_y$$

by $u_y(f) := f(y \circ u)$. It is easy to see that $\mathbf{Y}_y = \mathbf{Y}_{y \circ u}$, and that u_y is continuous as it is the projection at $y \circ u$. Let us finally define

$$\widehat{u} \colon \prod\{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\} \to \prod\{\, \mathbf{Y}_y \mid y \in X_{\mathbf{B}} \,\}$$

to be the natural product map $\sqcap\{\, u_y \mid y \in X_{\mathbf{B}} \,\}$. That is,

$$(\widehat{u}(f))(y) := u_y(f) = f(y \circ u), \quad \text{for } f \in \prod\{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\} \text{ and } y \in X_{\mathbf{B}}.$$

Since each u_y is continuous, so is $\widehat{u}$. We now establish properties of $\widehat{u}$, employing little more than careful definition chasing and elementary topology.

Lemma 3.1. *For algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *in* $\boldsymbol{\mathcal{A}}$*,*

(i) $\widehat{u}(e_{\mathbf{A}}(\mathbf{A})) \subseteq e_{\mathbf{B}}(\mathbf{B})$*, and*

(ii) $\widehat{u}(\overline{e_{\mathbf{A}}(\mathbf{A})}) \subseteq \overline{e_{\mathbf{B}}(\mathbf{B})}$.

Proof. Consider (i). Let $a \in A$. Then, for all $y \in X_{\mathbf{B}}$, we have

$$\widehat{u}(e_{\mathbf{A}}(a))(y) := u_y(e_{\mathbf{A}}(a)) = e_{\mathbf{A}}(a)(y \circ u) = y(u(a)) = e_{\mathbf{B}}(u(a))(y).$$

Thus $\widehat{u}(e_{\mathbf{A}}(a)) = e_{\mathbf{B}}(u(a)) \in e_{\mathbf{B}}(\mathbf{B})$.

To prove (ii), observe that $\widehat{u}(\overline{e_{\mathbf{A}}(A)}) \subseteq \overline{\widehat{u}(e_{\mathbf{A}}(A))} \subseteq \overline{e_{\mathbf{B}}(\mathbf{B})}$, by (i) and the continuity of $\widehat{u}$. □

Thus, for each homomorphism $u\colon \mathbf{A} \to \mathbf{B}$, we may define a continuous homomorphism $n_{\boldsymbol{\mathcal{A}}}(u)\colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{A}) \to n_{\boldsymbol{\mathcal{A}}}(\mathbf{B})$ by $n_{\boldsymbol{\mathcal{A}}}(u) := \widehat{u}\restriction_{n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})}$.

The first part of the following proposition is now a routine calculation and the second is an easy consequence of Lemma 3.1.

Proposition 3.2. (i) $n_{\boldsymbol{\mathcal{A}}}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}_{\mathcal{T}}$ *is a well-defined functor.*

(ii) $e\colon \mathrm{id}_{\boldsymbol{\mathcal{A}}} \to n^\flat_{\boldsymbol{\mathcal{A}}}$ *is a natural transformation, where* $n^\flat_{\boldsymbol{\mathcal{A}}} := (n_{\boldsymbol{\mathcal{A}}})^\flat\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$.

Remark 3.3. We can elucidate the relationship between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$ by taking an alternative view of the natural extension. The codomain of the map $e_{\mathbf{A}}$, as defined above, is $\prod\{\, \mathbf{Y}_x \mid x \in X_{\mathbf{A}} \,\}$, where the indexing set $X_{\mathbf{A}}$ is the disjoint union over $\mathbf{M} \in \mathcal{M}$ of the sets $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$. We may view this as an iterated product

$$\prod\{\, \prod\{\, Y_x \mid x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \,\} \mid \mathbf{M} \in \mathcal{M} \,\}.$$

When adopting this perspective we write $e_{\mathbf{A}}(a)(\mathbf{M})(x) = x(a)$, for any fixed $a \in A$ and for $\mathbf{M} \in \mathcal{M}$ and $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$, and refer to each $e_{\mathbf{A}}(a)$ as a *multisorted evaluation map*. We have

$$e_{\mathbf{A}}\colon \mathbf{A} \to \prod\{\, \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})} \mid \mathbf{M} \in \mathcal{M} \,\}.$$

The set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ can be regarded as a subspace—in fact a closed subspace—of the topological product $\mathbf{M}_{\mathcal{T}}^{\mathbf{A}}$. In this guise we denote it by $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}}$. (Note that we are not claiming that $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}} \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}$; in general it is not a subalgebra of $\mathbf{M}_{\mathcal{T}}^{\mathbf{A}}$.) Therefore it makes sense to consider the set $\mathrm{C}(\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}}, \mathbf{M}_{\mathcal{T}})$ of continuous maps from $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}}$ into $\mathbf{M}_{\mathcal{T}}$. Since the map

$$e_{\mathbf{A}}(a)(\mathbf{M})\colon \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}} \to \mathbf{M}_{\mathcal{T}}$$

is continuous, for all $\mathbf{M} \in \mathcal{M}$, it follows that we can restrict the codomain of $e_{\mathbf{A}}$ and write

$$e_{\mathbf{A}}\colon \mathbf{A} \to \prod\{\, \mathrm{C}(\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}}, \mathbf{M}_{\mathcal{T}}) \mid \mathbf{M} \in \mathcal{M} \,\}.$$

The natural extension $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ is then obviously the topological closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\prod\{\, \mathrm{C}(\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})_{\mathcal{T}}, \mathbf{M}_{\mathcal{T}}) \mid \mathbf{M} \in \mathcal{M}\}$.

We exploit this alternative description of the natural extension when proving below that the functor $n_{\boldsymbol{\mathcal{A}}}$ is a reflection. We shall not make use of this result subsequently.

Proposition 3.4. *The natural extension functor* $n_{\boldsymbol{\mathcal{A}}} : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}_{\mathcal{T}}$ *is a reflection of* $\boldsymbol{\mathcal{A}}$ *into the (non-full) subcategory* $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$*, that is, for all* $\mathbf{A} \in \boldsymbol{\mathcal{A}}$*, each* $\mathbf{B} \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}$ *and every homomorphism* $g \colon \mathbf{A} \to \mathbf{B}^\flat$*, there exists a unique continuous homomorphism* $h \colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{A}) \to \mathbf{B}$ *such that* $h \circ e_{\mathbf{A}} = g$*.*

Proof. Let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, let $\mathbf{B} \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}$, and let $g \colon \mathbf{A} \to \mathbf{B}^\flat$ be a homomorphism. A simple diagram chase shows that to find a continuous homomorphism $h \colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{A}) \to \mathbf{B}$ with $h \circ e_{\mathbf{A}} = g$, it suffices to find a continuous homomorphism $\gamma \colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{B}) \to \mathbf{B}$ with $\gamma \circ e_{\mathbf{B}} = \mathrm{id}_B$. (We note that this is slightly stronger than asking for $\mathbf{B}^\flat$ to be a retract of $n_{\boldsymbol{\mathcal{A}}}(\mathbf{B})^\flat$ in $\boldsymbol{\mathcal{A}}$. We cannot ask for $\mathbf{B}$ to be a retract of $n_{\boldsymbol{\mathcal{A}}}(\mathbf{B})$ in $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$ as $e_{\mathbf{B}}$ need not be continuous.) The uniqueness of the continuous homomorphism h is an immediate consequence of the fact that $e_{\mathbf{A}}(A)$ is topologically dense in $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$.

Let us consider the natural map

$$c \colon \mathbf{B} \to \prod \{ \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}_{\mathcal{T}}(\mathbf{B},\mathbf{M}_{\mathcal{T}})} \mid \mathbf{M} \in \mathcal{M} \} \quad \text{given by} \quad c(b)(\mathbf{M})(x) := x(b),$$

for all $b \in B$ and $x \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}(\mathbf{B}, \mathbf{M}_{\mathcal{T}})$. Since $\mathbf{B} \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}$, the map c is a continuous embedding. Let

$$\pi \colon \prod \{ \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{B},\mathbf{M})} \mid \mathbf{M} \in \mathcal{M} \} \to \prod \{ \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}_{\mathcal{T}}(\mathbf{B},\mathbf{M}_{\mathcal{T}})} \mid \mathbf{M} \in \mathcal{M} \}$$

be the obvious projection and define

$$\rho := \pi{\restriction}_{n_{\boldsymbol{\mathcal{A}}}(\mathbf{B})} \colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{B}) \to \prod \{ \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}_{\mathcal{T}}(\mathbf{B},\mathbf{M}_{\mathcal{T}})} \mid \mathbf{M} \in \mathcal{M} \}.$$

Clearly, $\rho \circ e_{\mathbf{B}} = c$ and ρ is a bijective map from $e_{\mathbf{B}}(B)$ to $c(B)$. Since ρ is continuous and $c(B)$ is closed in $\prod \{ \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}_{\mathcal{T}}(\mathbf{B},\mathbf{M}_{\mathcal{T}})} \mid \mathbf{M} \in \mathcal{M} \}$, we have

$$\rho(n_{\boldsymbol{\mathcal{A}}}(B)) = \rho(\overline{e_{\mathbf{B}}(B)}) \subseteq \overline{\rho(e_{\mathbf{B}}(B))} = \overline{c(B)} = c(B).$$

Hence we can restrict the codomain of ρ and write $\rho \colon n_{\boldsymbol{\mathcal{A}}}(\mathbf{B}) \to c(\mathbf{B})$. Now define $\gamma := c^{-1} \circ \rho$. Then $\gamma \circ e_{\mathbf{B}} = c^{-1} \circ \rho \circ e_{\mathbf{B}} = c^{-1} \circ c = \mathrm{id}_B$, as required. □

We are now ready to prove our main theorem. But readers unfamiliar with the constructs of natural duality theory in full generality may by now be suffering from notation overload. So before proceeding with the development of our theory we want to show how what we have done looks in a particular, single-sorted, case—that of (bounded) distributive lattices.

Remark 3.5. Consider the case where $\boldsymbol{\mathcal{A}}$ is the variety $\boldsymbol{\mathcal{D}} = \mathbb{ISP}(\mathbf{2})$ of bounded distributive lattices. Here $\mathcal{M}$ is a 1-element set and $X_{\mathbf{A}} := \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{2})$ is just the set of $\boldsymbol{\mathcal{D}}$-homomorphisms of $\mathbf{A}$ into $\mathbf{2}$ (*alias* the set of prime filters of $\mathbf{A}$), for each $\mathbf{A} \in \boldsymbol{\mathcal{D}}$. Then the mapping $e_{\mathbf{A}}$ embeds $\mathbf{A}$ into $\mathbf{2}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{2})}$. Regarding this power as a member of $\boldsymbol{\mathcal{D}}_{\mathcal{T}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathbf{2}_{\mathcal{T}})$, we may view this as a topological distributive lattice. The image $e_{\mathbf{A}}(\mathbf{A})$ of $\mathbf{A}$ is the set of evaluation maps, which are certainly continuous homomorphisms; $n_{\boldsymbol{\mathcal{D}}}(\mathbf{A})$ is just the topological closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\mathbf{2}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{2})}$. As a closed sublattice of a topological lattice, $n_{\boldsymbol{\mathcal{D}}}(\mathbf{A})$ is a complete lattice. The alternative view of $n_{\boldsymbol{\mathcal{D}}}(\mathbf{A})$ comes from recognising that we obtain a topological space by equipping $\boldsymbol{\mathcal{D}}(\mathbf{A}, \mathbf{2})$ with the topology it

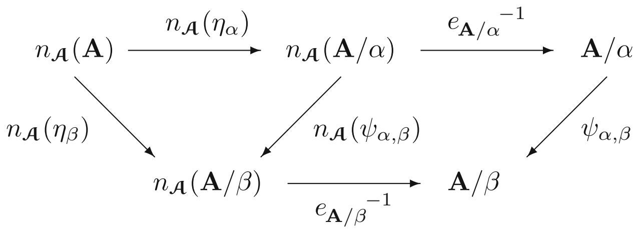

FIGURE 1. The proof of Theorem 3.6

inherits from $\mathbf{2}^A_{\mathcal{T}}$, to yield the dual space $\mathcal{D}(\mathbf{A},\mathbf{2})_{\mathcal{T}}$. Then $n_{\mathcal{D}}(\mathbf{A})$ is the closure of the set of evaluation maps, now regarded as a subset of the continuous maps from $\mathcal{A}(\mathbf{A},\mathbf{2})_{\mathcal{T}}$ into $\mathbf{2}_{\mathcal{T}}$.

We return to the general situation, where $\mathbf{A}$ is a member of the IRF-prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ generated by a set $\mathcal{M}$ of finite algebras. As promised, we shall prove that $n_{\mathcal{A}}(\mathbf{A})$ is isomorphic, both algebraically and topologically, to the $\mathcal{A}$-profinite completion of $\mathbf{A}$. As already noted in Section 2, we may regard $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ as belonging to $\mathcal{A}$ or to $\mathcal{A}_{\mathcal{T}}$, and we shall switch between the two personae as is convenient, without a change of notation. We recall from Lemma 2.1 that we have a natural embedding map $\mu_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ given by $\mu_{\mathbf{A}}(a)(\alpha) := a/\alpha$, for all $a \in A$ and $\alpha \in \mathcal{S}_{\mathbf{A}}$.

Theorem 3.6. *Let $\mathcal{M}$ be a set of finite algebras and let $\mathbf{A}$ belong to the IRF-prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$. There exists a map $v\colon n_{\mathcal{A}}(\mathbf{A}) \to \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ that is an algebraic and topological isomorphism from the natural extension of $\mathbf{A}$ in $\mathcal{A}$ to the $\mathcal{A}$-profinite completion of $\mathbf{A}$ and which satisfies $v \circ e_{\mathbf{A}} = \mu_{\mathbf{A}}$.*

Proof. We first show how to construct v and confirm that it is a well-defined continuous homomorphism.

Note that if $\mathbf{B} \in \mathcal{A}$ is finite, then $n_{\mathcal{A}}(\mathbf{B}) = e_{\mathbf{B}}(\mathbf{B})$ and hence $e_{\mathbf{B}}\colon \mathbf{B} \to n_{\mathcal{A}}(\mathbf{B})$ is an isomorphism. Let $\alpha \in \mathcal{S}_{\mathbf{A}}$ and let $\eta_\alpha\colon \mathbf{A} \to \mathbf{A}/\alpha$ be the induced homomorphism. As $\mathbf{A}/\alpha$ is finite, the map

$$\bar{\eta}_\alpha := (e_{\mathbf{A}/\alpha})^{-1} \circ n_{\mathcal{A}}(\eta_\alpha)\colon n_{\mathcal{A}}(\mathbf{A}) \to \mathbf{A}/\alpha$$

is a well-defined, continuous homomorphism. (We note that, in fact, $\bar{\eta}_\alpha$ is the unique continuous extension of η_α.)

Let $\alpha, \beta \in \mathcal{S}_{\mathbf{A}}$ with $\alpha \subseteq \beta$ and let $\psi_{\alpha,\beta}\colon \mathbf{A}/\alpha \to \mathbf{A}/\beta$ be the natural homomorphism. The diagram shown in Figure 1 commutes. For the triangle this uses the fact $n_{\mathcal{A}}$ is a functor and for the parallelogram the fact that e is a natural transformation (see Proposition 3.2).

Hence the homomorphisms $\bar{\eta}_\alpha\colon n_{\mathcal{A}}(\mathbf{A}) \to \mathbf{A}/\alpha$ are compatible with the inverse system. Consequently, by the universal mapping property applied within $\mathcal{A}$, there is a unique homomorphism

$$v\colon n_{\mathcal{A}}(\mathbf{A}) \to \mathrm{pro}_{\mathcal{A}}(\mathbf{A}) = \varprojlim\{\, \mathbf{A}/\alpha \mid \alpha \in \mathcal{S}_{\mathbf{A}} \,\}$$

satisfying $\pi_\alpha \circ v = \bar{\eta}_\alpha$, for all $\alpha \in \mathcal{S}_{\mathbf{A}}$.

We now verify that $v \circ e_{\mathbf{A}} = \mu_{\mathbf{A}}$. Let $\mathbf{A} \in \mathcal{A}$, $a \in A$, $\alpha \in \mathcal{S}_{\mathbf{A}}$ and $x \in X_{\mathbf{A}/\alpha}$, say $x \colon \mathbf{A}/\alpha \to \mathbf{M}$. Then

$$v(e_{\mathbf{A}}(a))(\alpha) = \bar{\eta}_\alpha(e_{\mathbf{A}}(a)) = {e_{\mathbf{A}/\alpha}}^{-1}(n_{\mathcal{A}}(\eta_\alpha)(e_{\mathbf{A}}(a))) = {e_{\mathbf{A}/\alpha}}^{-1}(\widehat{\eta}_\alpha(e_{\mathbf{A}}(a))).$$

So

$$\begin{aligned} e_{\mathbf{A}/\alpha}(v(e_{\mathbf{A}}(a))(\alpha))(x) &= \widehat{\eta}_\alpha(e_{\mathbf{A}}(a))(x) = e_{\mathbf{A}}(a)(x \circ \eta_\alpha) = (x \circ \eta_\alpha)(a) \\ &= x(a/\alpha) = e_{\mathbf{A}/\alpha}(a/\alpha)(x). \end{aligned}$$

Thus $e_{\mathbf{A}/\alpha}(v(e_{\mathbf{A}}(a))(\alpha)) = e_{\mathbf{A}/\alpha}(a/\alpha)$ and so $v(e_{\mathbf{A}}(a))(\alpha) = a/\alpha = \mu_{\mathbf{A}}(a)(\alpha)$, whence $v(e_{\mathbf{A}}(a)) = \mu_{\mathbf{A}}(a)$ and hence $v \circ e_{\mathbf{A}} = \mu_{\mathbf{A}}$, as required.

Since $n_{\mathcal{A}}(\mathbf{A})$ is compact and $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ is Hausdorff, it only remains to prove that v is a bijection.

Claim 1: v is one-to-one.
Let $f, g \in n_{\mathcal{A}}(\mathbf{A})$ with $f \neq g$. Thus there exists $\mathbf{M} \in \mathcal{M}$ and $x \colon \mathbf{A} \to \mathbf{M}$ such that $f(x) \neq g(x)$. Let $\alpha := \ker x \in \mathcal{S}_{\mathbf{A}}$ and let $\varphi \colon \mathbf{A}/\alpha \to \mathbf{M}$ be the unique homomorphism satisfying $\varphi \circ \eta_\alpha = x$.

To show that $v(f) \neq v(g)$, it suffices to show that $v(f)(\alpha) \neq v(g)(\alpha)$, that is, that $\bar{\eta}_\alpha(f) \neq \bar{\eta}_\alpha(g)$. Since $\bar{\eta}_\alpha = {e_{\mathbf{A}/\alpha}}^{-1} \circ n_{\mathcal{A}}(\eta_\alpha)$ and ${e_{\mathbf{A}/\alpha}}^{-1}$ is one-to-one, it suffices to prove that $n_{\mathcal{A}}(\eta_\alpha)(f) \neq n_{\mathcal{A}}(\eta_\alpha)(g)$, that is, by the definition of $n_{\mathcal{A}}(\eta_\alpha)$, it suffices to show that $\widehat{\eta}_\alpha(f) \neq \widehat{\eta}_\alpha(g)$. But

$$(\widehat{\eta}_\alpha(f))(\varphi) = f(\varphi \circ \eta_\alpha) = f(x) \neq g(x) = g(\varphi \circ \eta_\alpha) = (\widehat{\eta}_\alpha(g))(\varphi),$$

as required.

Claim 2: v is surjective.
Since $v \circ e_{\mathbf{A}} = \mu_{\mathbf{A}}$, the image of v contains $\mu_{\mathbf{A}}(A)$. As v is a closed map, to prove that v is surjective, it suffices to prove that $\mu_{\mathbf{A}}(A)$ is topologically dense in $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$.

A typical basic open set in $\prod\{\, \mathbf{A}/\alpha \mid \alpha \in \mathcal{S}_{\mathbf{A}} \,\}$ is of the form

$$V := \big\{\, f \in \prod\{\, \mathbf{A}/\alpha \mid \alpha \in \mathcal{S}_{\mathbf{A}} \,\} \bigm| f(\alpha_i) = a_i/\alpha_i, \text{ for } i = 1, \dots, n \,\big\},$$

for some $\alpha_1, \dots, \alpha_n \in \mathcal{S}_{\mathbf{A}}$ and $a_1, \dots, a_n \in A$. Assume that $f \in V \cap \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$. Let $a \in A$ with $f(\bigcap_{i=1}^n \alpha_i) = a/(\bigcap_{i=1}^n \alpha_i)$. As $f \in \mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ we have $f(\alpha_i) = a/\alpha_i$, for $i = 1, \dots, n$; hence $\mu_{\mathbf{A}}(a) \in V$. Thus, $\mu_{\mathbf{A}}(A)$ is dense in $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$. □

Corollary 3.7. *Let $\mathcal{A}$ be an IRF-prevariety. Then, for all $\mathbf{A} \in \mathcal{A}$, the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of $\mathbf{A}$ is independent of the set $\mathcal{M}$ of finite algebras chosen to generate $\mathcal{A}$.*

We now briefly consider canonical extensions of lattice-based algebras (for background and references see the work of Gehrke, Harding, Jónsson and others [32, 36, 72]). For algebras with monotone operations the theorem below follows from Theorem 3.6 and the theorem reconciling profinite and canonical extensions in [36]; the extension of Harding's result to arbitrary operations can be found in M.J. Gouveia's paper [34].

Theorem 3.8. *Let $\mathcal{V}$ be a residually finite variety of algebras having bounded lattice reducts. Then, for all $\mathbf{A} \in \mathcal{V}$, the natural extension $n_{\mathcal{V}}(\mathbf{A})$ coincides with the canonical extension of $\mathbf{A}$.*

As Harding notes in [36], in lattice-based varieties of finite type, residual finiteness occurs only for finitely generated varieties; this is an immediate consequence of a theorem of Kearnes and Willard [46].

When $\mathcal{V} = \mathfrak{D}$, the profinite completion of $\mathbf{L} \in \mathcal{V}$ is very well known to coincide with the canonical extension $\mathbf{L}^{\sigma}$ and to be identifiable with the lattice of all up-sets of its Priestley dual space, or equivalently, with the lattice of all order-preserving maps into the set $\{0, 1\}$ with order $0 < 1$ (see [18]). From Theorem 4.3 below, this lattice is exactly $n_{\mathfrak{D}}(\mathbf{L})$.

The canonical extension of a lattice-based algebra is obtained by taking a suitable completion of the underlying lattice and then extending appropriately the non-lattice operations. Unlike the canonical extension, a profinite completion, and likewise a natural extension, comes ready equipped with all the necessary fundamental operations, and automatically lies in the same variety as the original algebra (canonicity). A central tenet of the canonical extension methodology is that one aims to work with the abstract characterisation, without reference to any concrete realisation. However we contend that this can be taken too far when an amenable concrete model of the canonical extension, such as the natural extension supplies, thanks to Theorem 4.3.

4. Descriptions of the natural extension

Theorem 4.1 will allow us to give descriptions of the natural extension, and so too of the profinite completion, which are not explicitly topological. The techniques we employ draw on ideas from natural duality theory, but we emphasise that our treatment is not confined to the setting of dualisable quasivarieties. We are later, however, able to sharpen our main result in a useful way in cases where a single-sorted or multisorted duality is available.

Until further notice, we fix a set $\mathcal{M}$ of finite algebras of the same type and consider the prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$. We therefore generalize from the setting for multisorted dualities in [9, Section 7.1], where the set $\mathcal{M}$ is assumed to be finite, to our situation where $\mathcal{M}$ is in general infinite (but see also [22] where the finiteness conditions on $\mathcal{M}$ are relaxed). We shall need to consider both structures which carry a topology (as in natural duality theory) and structures which do not. A superscript $^\flat$ on a structure indicates that no topology is present; this usage is consistent with that whereby a superscript indicates the application of a functor $^\flat$ forgetting the topology on a topological structure.

We begin by highlighting a key definition. Given a (non-empty) subset s of a finite product $M_1 \times \cdots \times M_m$, (where $\mathbf{M}_i \in \mathcal{M}$, for $i = 1, \ldots, m$), we call s an *m-ary multisorted algebraic relation on* $\mathcal{M}$ if it is a subalgebra of

$\mathbf{M}_1 \times \cdots \times \mathbf{M}_m$. A *finitary* multisorted algebraic relation is one which is m-ary for some finite m. Likewise, a homomorphism from $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$ to $\mathbf{M}_{n+1}$ (where $\mathbf{M}_i \in \mathcal{M}$, for all i), will be referred to as a (*finitary*) *algebraic multisorted operation* on $\mathcal{M}$, and a homomorphism from $\mathbf{A}$ to $\mathbf{M}_{n+1}$ (where $\mathbf{A}$ is a subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$ and $\mathbf{M}_i \in \mathcal{M}$, for all i) will be called a (*finitary*) *algebraic multisorted partial operation* on $\mathcal{M}$.

We shall work until further notice with a fixed first-order language $\mathcal{L}$ whose non-logical symbols consist of sets G and H of operation symbols and a set R of relation symbols; any or all of these sets may be empty. We say that $\underset{\sim}{\mathcal{M}}^\flat$ is a *multisorted $\mathcal{L}$-structure algebraic over* $\mathcal{M}$ if the underlying set of $\underset{\sim}{\mathcal{M}}^\flat$ is $\dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$ and the members of G, H and R are interpreted, respectively, as multisorted operations, partial operations and relations, all of which are algebraic on $\mathcal{M}$.

We form an associated category in the following way. Given $\underset{\sim}{\mathcal{M}}^\flat$, by an $\mathcal{M}$-*sorted structure* (*of the same type as* $\underset{\sim}{\mathcal{M}}^\flat$) we mean an $\mathcal{L}$-structure $\mathbf{X}^\flat$ whose domain is of the form $X = \dot{\bigcup}\{\, X_{\mathbf{M}} \mid \mathbf{M} \in \mathcal{M}\,\}$ such that every $g \in G$, with $g\colon \mathbf{M}_1 \times \cdots \times \mathbf{M}_n \to \mathbf{M}_{n+1}$, for some $\mathbf{M}_i \in \mathcal{M}$ $(i = 1, \ldots, n+1)$, is interpreted as a map from $X_{\mathbf{M}_1} \times \cdots \times X_{\mathbf{M}_n}$ to $X_{\mathbf{M}_{n+1}}$, and similarly for the operation symbols in H and relation symbols in R (here we do not distinguish notationally between symbols in $\mathcal{L}$ and their interpretations). Given two such structures $\mathbf{X}^\flat$ and $\mathbf{Y}^\flat$, we say that a map

$$\alpha\colon \dot{\bigcup}\{\, X_{\mathbf{M}} \mid \mathbf{M} \in \mathcal{M}\,\} \to \dot{\bigcup}\{\, Y_{\mathbf{M}} \mid \mathbf{M} \in \mathcal{M}\,\}$$

preserves the sorts or is an $\mathcal{M}$-*sorted map* if, for every $\mathbf{M} \in \mathcal{M}$, it maps $X_{\mathbf{M}}$ into $Y_{\mathbf{M}}$. We say that α is a *multisorted homomorphism* if it is an $\mathcal{M}$-sorted map that preserves the (partial) operations and relations in the standard sense. We make the class of $\mathcal{M}$-sorted structures into a category $\mathfrak{X}^\flat$ by adding all multisorted homomorphisms as morphisms. Within this category, concepts such as *substructure, isomorphism, embedding* have their expected multisorted definitions. Also, for a non-empty set S, the *power* $(\underset{\sim}{\mathcal{M}}^\flat)^S$ of $\underset{\sim}{\mathcal{M}}^\flat$ is the $\mathcal{M}$-sorted structure with underlying set $\dot{\bigcup}\{\, M^S \mid \mathbf{M} \in \mathcal{M}\,\}$, and the (partial) operations and relations extended pointwise. (In fact it would suffice for our needs to restrict attention to structures in $\mathfrak{X}^\flat$ which are isomorphic to substructures of powers of $\underset{\sim}{\mathcal{M}}^\flat$.)

Let $X := \dot{\bigcup}\{\mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\}$. We adopt the notation $F \Subset X$ to indicate that $F \subseteq X$ with F finite. We say that an $\mathcal{M}$-sorted map

$$b\colon \dot{\bigcup}\{\, \mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\} \to \dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$$

is *locally an evaluation* if, whenever $F \Subset X$, there exists $a \in A$ such that $b(x) = x(a)$, for all $x \in F$.

We preface Theorem 4.1 with some remarks. In the theorem, the equivalence of (i), (ii), and (iii) in the single-sorted case is a long-known and very elementary result in duality theory, relying on the definition of the product topology; the equivalence appears as Exercise 2.1 in [9]. The addition of (iv) to the list of equivalences relies on the Preservation Lemma (see Pitkethly and Davey [63, 1.4.4]); even for the multisorted case it will be already familiar to those well versed in natural duality theory. For completeness and convenience we include the proof of the theorem, since the argument is short, simple and instructive.

Theorem 4.1. *Let* $\mathcal{M}$ *be a set of finite algebras, let* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathcal{M})$ *and let* $\mathbf{A} \in \boldsymbol{\mathcal{A}}$*. Assume that* $b\colon \dot{\bigcup}\{\, \boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\} \to \dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$ *is an* $\mathcal{M}$*-sorted map. Then the following are equivalent:*

(i) *b belongs to $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$, that is, b belongs to the topological closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\prod\{\, \mathbf{M}_{\mathcal{T}}^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})} \mid \mathbf{M} \in \mathcal{M}\,\}$;*

(ii) *b is locally an evaluation;*

(iii) *b preserves every finitary multisorted algebraic relation on $\mathcal{M}$;*

(iv) *b preserves every finitary multisorted algebraic relation on $\mathcal{M}$ of the form*

$$r_F := \{\, (x_1(a), \dots, x_n(a)) \mid a \in A \,\},$$

where $F = \{x_1, \dots, x_n\} \Subset \bigcup\{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\}$.

Proof. The equivalence of (i) and (ii) is an immediate consequence of the elementary fact that the closure of a subset S of a product of finite discrete spaces consists precisely of those elements of the product that are locally in S.

The proof that (ii) implies (iii) is easy since evaluation maps preserve every multisorted relation (defined pointwise on the product) because an evaluation is essentially a projection.

It is trivial that (iii) implies (iv).

Finally we shall prove that (iv) implies (ii). Let us consider an $\mathcal{M}$-sorted map $b\colon \bigcup\{\, \boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})\colon \mathbf{M} \in \mathcal{M}\,\} \to \bigcup\{\, M\colon \mathbf{M} \in \mathcal{M}\,\}$ and assume that b preserves all relations r_F. Certainly each r_F is a multisorted algebraic relation. Let us consider $F := \{x_1, \dots, x_n\} \Subset \bigcup\{\, \boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\}$. For the relation r_F assigned to F we have that $(x_1, \dots, x_n) \in r_F$ on the domain of b. Then we have by assumption that $(b(x_1), \dots, b(x_n)) \in r_F$. Hence there exists an element $a \in A$ such that $b(x_i) = x_i(a)$ for $i = 1, \dots, n$, and this says exactly that b is a local evaluation. □

The equivalence of (iii) and (iv) in Theorem 4.1 allows us to replace the 'Brute Force' condition of preservation of all multisorted algebraic relations by a condition of preservation of a smaller set of relations 'localised' to the algebra $\mathbf{A}$ whose natural extension is to be described. However there is another way in which we might expect to be able to reduce the set of relations to be preserved: by exploiting the theory of entailment. Specifically, we wish to adapt to the present context the ideas used in natural duality theory to remove redundant relations (or operations and partial operations) from a dualising set.

We now need to reconcile the topology-free $\mathcal{M}$-sorted structures we have employed above with their topological counterparts. Consider $\underset{\sim}{\mathcal{M}}^{\flat}$ as above, and equip it with the discrete topology. We denote the resulting topological structure by $\underset{\sim}{\mathcal{M}}$ (in line with the notation of [9, Chapter 7]). We shall always be looking at topologically closed subsets of powers of $\underset{\sim}{\mathcal{M}}$ and consequently we restrict our attention to Boolean topologies. Thus the topological structures we wish to consider are objects of $\mathcal{Z}^\flat$ equipped with a Boolean space topology. We refer to these as the *Boolean topological multisorted structures* (*of the same type as* $\underset{\sim}{\mathcal{M}}$) and treat them as the objects of a category $\mathcal{Z}$ in which the morphisms are the continuous multisorted homomorphisms. We now use $\underset{\sim}{\mathcal{M}}$ to build a full subcategory of the category $\mathcal{Z}$ introduced above, to serve as a potential dual category $\mathcal{X}$ for the category $\mathcal{A}$. As our objects we take the class $\mathcal{X} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathcal{M}})$ of all Boolean topological $\mathcal{M}$-sorted structures of the same type as $\underset{\sim}{\mathcal{M}}$ which are isomorphic to a (possibly empty) closed substructure of some power $\underset{\sim}{\mathcal{M}}^S$ of $\underset{\sim}{\mathcal{M}}$, for a non-empty set S. Now $\mathbb{S}^0_{\mathrm{c}}$ stands for possibly empty $\mathcal{M}$-sorted substructures with each sort topologically closed (so "$\mathbf{Y}$ is a closed substructure of $\mathbf{X}$" does not mean that Y is a closed subset of X but means that each sort of $\mathbf{Y}$ is a closed subset of the corresponding sort of $\mathbf{X}$). It is routine to verify that, for every $\mathbf{A} \in \mathcal{A}$, the *dual* of $\mathbf{A}$,

$$\mathrm{D}(\mathbf{A}) := \dot{\bigcup}\{\, \mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\},$$

is a closed substructure of $\underset{\sim}{\mathcal{M}}^A$, and for every $\mathbf{X} \in \mathcal{X}$, the hom-set $\mathrm{E}(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathcal{M}})$ forms a subalgebra of $\prod \big\{\, \mathbf{M}^{X_{\mathbf{M}}} \bigm| \mathbf{M} \in \mathcal{M} \,\big\}$; these facts stem from the assumption that the structure $\underset{\sim}{\mathcal{M}}$ is algebraic over $\mathcal{M}$, just as in the single-sorted case (see [9, 1.5.2]).

For each $\mathbf{A} \in \mathcal{A}$ we have a natural embedding $e_{\mathbf{A}}$ of $\mathbf{A}$ into $\mathrm{ED}(\mathbf{A})$, given by multisorted evaluation: $e_{\mathbf{A}}(a)(\mathbf{M})(x) := x(a)$, for all $\mathbf{M} \in \mathcal{M}$ and $x \in \mathcal{A}(\mathbf{A}, \mathbf{M})$. Similarly, for each $\mathbf{X} \in \mathcal{X}$, one can define an embedding $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ by $\varepsilon_{\mathbf{X}}(x)(\mathbf{M})(\alpha) := \alpha(x)$, for all $\mathbf{M} \in \mathcal{M}$ and $\alpha \in \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathcal{M}})$. Consequently, we have contravariant functors $\mathrm{D}\colon \mathcal{A} \to \mathcal{X}$ and $\mathrm{E}\colon \mathcal{X} \to \mathcal{A}$ (and these set up in the usual way a dual adjunction $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ between $\mathcal{A}$ and $\mathcal{X}$). If $e_{\mathbf{A}}$ is an isomorphism for each $\mathbf{A} \in \mathcal{A}$, then we say that the structure $\underset{\sim}{\mathcal{M}}$ *yields a multisorted duality on* $\mathcal{A}$ *based on* $\mathcal{M}$. More explicitly, we say that $G \cup H \cup R$ is a *dualising set* for $\mathcal{A}$ (or *yields a duality* on $\mathcal{A}$). Here, as above, we do not in our notation distinguish between symbols in $\mathcal{L}$ and their interpretations on the domain of $\underset{\sim}{\mathcal{M}}$, or more generally on members of $\mathcal{X}$, as sets of (multisorted algebraic) operations, partial operations and relations. We shall usually omit the word 'multisorted' below.

We shall now consider entailment. We cannot simply carry over the usual notions as given for example in Davey, Haviar and Priestley [17] or [9, p. 55], extended to the multisorted setting. Some care is needed: (duality-)entailment makes use of *continuous* structure-preserving maps on the dual structures of algebras in $\mathcal{A}$, whereas here we are concerned with arbitrary structure-preserving maps on (the images under $^\flat$ of) such structures. We formulate the notion of entailment that we require by omitting the word 'continuous' from the usual definitions. Specifically, we consider a fixed set of finitary algebraic multisorted (partial) operations and relations $G \cup H \cup R$ on $\mathcal{M}$. For an algebra

$\mathbf{A} \in \mathbb{ISP}(\mathcal{M})$, we say $G \cup H \cup R$ *discretely entails* a finitary algebraic relation s on $D(\mathbf{A})^\flat$ if every $\mathcal{M}$-sorted map $b\colon \dot{\bigcup}\{\, \mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\} \to \dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$ that preserves the members of $G \cup H \cup R$ also preserves s, and that $G \cup H \cup R$ *discretely entails* s if, for every $\mathbf{A} \in \mathbb{ISP}(\mathcal{M})$, it discretely entails it on $D(\mathbf{A})^\flat$.

Trivially, discrete entailment of a relation implies (duality-)entailment. At first sight it might appear that discrete entailment is a stronger notion than entailment. But it turns out that the two notions are equivalent, both globally and when localised to 'test algebras'. The proof we give of Lemma 4.2 is an easy modification of the proof of the Test Algebra Lemma [9, 8.1.3, 9.1.2]; the extension to cover multiple sorts requires only trivial, notational, modifications to the proof for the single-sorted case. (It is easy to derive from Lemma 4.2 an analogue for discrete entailment, in the multisorted setting, of the Test Algebra Lemma as given in [9]. We do not need this result and so omit it.)

Lemma 4.2. *Let $G \cup H \cup R$ be a set of finitary algebraic (partial) operations and relations on $\mathcal{M}$, where $\mathcal{M}$ is a set of finite algebras. Let s be a finitary algebraic relation on $\mathcal{M}$. Then $G \cup H \cup R$ entails s if and only if $G \cup H \cup R$ discretely entails s.*

Proof. We only need to prove that entailment implies discrete entailment.

Since s is algebraic, it is a subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_m$, where $\mathbf{M}_i \in \mathcal{M}$ for $i = 1, \dots, m$. We denote s by $\mathbf{s}$ when we are regarding it as a member of $\mathcal{A}$. We denote by ρ_i the restriction to $\mathbf{s}$ of the i-th projection map from $\mathbf{M}_1 \times \cdots \times \mathbf{M}_m$ to $\mathbf{M}_i$.

The dual of $\mathbf{s}$ is $D(\mathbf{s}) = \dot{\bigcup}\{\, \mathcal{A}(\mathbf{s}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\}$. For $x_1, \dots, x_m \in D(\mathbf{s})$ we have, by definition, $(x_1, \dots, x_m) \in s$ on $D(\mathbf{s})$ if and only if for any $a \in s$ we have $(x_1(a), \dots, x_m(a)) \in s$.

Assume that $G \cup H \cup R$ entails s. Let $\mathbf{A} \in \mathcal{A}$ and let $b\colon \dot{\bigcup}\mathcal{A}(\mathbf{A}, \mathbf{M}) \to \dot{\bigcup} M$ be an $\mathcal{M}$-sorted map which preserves $G \cup H \cup R$. Assume $(x_1, \dots, x_m) \in s$ on $D(\mathbf{s})$. Then the map $u := x_1 \sqcap \cdots \sqcap x_m\colon \mathbf{A} \to \mathbf{s}$ is a homomorphism. Consider the associated dual map $D(u)\colon D(\mathbf{s}) \to D(\mathbf{A})$ given by $D(u)(x) = x \circ u$ (for $x \in D(\mathbf{s})$). Now

$$b \circ D(u)\colon D(\mathbf{s}) \to \dot{\bigcup}\{\, M \mid M \in \mathcal{M}\,\}$$

preserves $G \cup H \cup R$, and so preserves s, since $G \cup H \cup R$ entails s. (Note that, as a map whose domain is finite, and hence discretely topologised, $b \circ D(u)$ is necessarily continuous.) We have $(\rho_1, \dots, \rho_m) \in s$ on $D(\mathbf{s})$ and hence

$$(b(x_1), \dots, b(x_m)) = ((b \circ D(u))(\rho_1), \dots, (b \circ D(u))(\rho_m)) \in s.$$

Therefore b preserves s. $\square$

When $\mathcal{M}$ is a finite set of finite algebras and $\mathbb{ISP}(\mathcal{M})$ possesses a multisorted duality, we can combine Lemma 4.2 with Theorem 4.1 to add a further condition to the list of equivalences in that theorem.

Theorem 4.3. *Let $\mathcal{M}$ be a finite set of finite algebras, let $\mathcal{A} := \mathbb{ISP}(\mathcal{M})$ and assume in addition that $G \cup H \cup R$ yields a multisorted duality on $\mathcal{A}$. Let $\mathbf{A} \in \mathcal{A}$. Assume that $b\colon \dot{\bigcup}\{\mathcal{A}(\mathbf{A},\mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\} \to \dot{\bigcup}\{ M \mid \mathbf{M} \in \mathcal{M}\}$ is an $\mathcal{M}$-sorted map. Then b belongs to $n_{\mathcal{A}}(\mathbf{A})$ (equivalently, satisfies any of* (ii)–(iv) *in Theorem 4.1) if and only if it satisfies*

(v) *b preserves every element of $G \cup H \cup R$.*

Proof. If $G \cup H \cup R$ is a dualising set for $\mathbb{ISP}(\mathcal{M})$ then $G \cup H \cup R$ entails, and hence also discretely entails, every finitary algebraic relation on $\mathcal{M}$. □

An interesting special case arises when the set $G \cup H \cup R$ in Theorem 4.3 can be taken to contain just the family of all isomorphisms between subalgebras of members of $\mathcal{M}$. In this situation, the natural extension turns out to be a full direct product (Theorem 5.7). As we shall prove, this occurs in particular when $\mathcal{A}$ is a discriminator variety (see Theorem 5.6).

We conclude this section with some comments on multisorted dualisability. Corollary 3.7 confirmed that the natural extension of an algebra in an IRF-prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ is independent of the choice of generating set $\mathcal{M}$. There is a corresponding independence result concerning dualisability. Suppose that we have a quasivariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras. Then, as shown by the present authors [15], whether or not $\mathcal{A}$ has a multisorted duality based on $\mathcal{M}$ is independent of the choice of $\mathcal{M}$; this generalises an earlier result for the single-sorted case. We thereby resolve a natural question. Suppose we have a non-dualisable quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ and suppose that this is alternatively expressed as $\mathbb{ISP}(\mathcal{N})$, where $\mathcal{N}$ is a finite set of finite algebras. Could $\mathcal{A}$ have a multisorted duality based on $\mathcal{N}$? The independence result for the multisorted case tells us that it cannot.

5. Applications: residually finite varieties

In Section 3 we adopted a categorical approach and in Section 4 we made use of ideas from duality theory. In this section the flavour of our presentation changes again, to that of universal algebra. Basic background in universal algebra for this section can be found in the text by Burris and Sankappanavar [8]. For a convenient and informative summary concentrating on the major developments of the past twenty years we recommend R. Willard's survey [74]. We shall focus on varieties. This gives access to deep and powerful results, many of them recent. In addition, for a residually finite variety $\mathcal{V}$ and $\mathbf{A} \in \mathcal{V}$ we may identify $\mathrm{pro}_{\mathcal{V}}(\mathbf{A})$, as studied in Sections 3 and 4, with the 'traditional' profinite completion $\widehat{\mathbf{A}}$.

We have two objectives. First of all, we wish to look at the applicability of our results. We concentrate on those classes of varieties and prevarieties which have received major attention in other contexts: these include classes arising in 'classical' algebra (groups and semigroups, for example) and classes of lattice-based algebras, within which profiniteness has hitherto been much studied. Secondly, we seek to reveal the extent of the gap, or lack of a gap, between the scope of Theorem 4.1, applicable to any IRF-prevariety, and of Theorem 4.3, applicable to varieties and prevarieties which have a single-sorted or multisorted duality. To this end, we survey the known theorems about varieties, both general and specific, which conspire in interesting ways to restrict the diversity of examples to which our results apply. We emphasise however that our theorems do apply not only to dualisable varieties and quasivarieties, which are prolific and varied, but also to some well-studied varieties which have no natural duality. This can occur because either the variety is finitely generated but non-dualisable or the variety is not finitely generated and so is beyond the reach of duality theory. We make one disclaimer: our survey does not seek to be exhaustive and we do not always strive for best possible results.

We need a few preliminaries. We recall that a variety $\mathcal{V}$ is of *finite type* if the number of fundamental operations is finite (all operations are assumed to be finitary). In that situation, there are, up to isomorphism, only finitely many members of $\mathcal{V}$ of cardinality less than N, for any finite N. Our theorems in Sections 3 and 4 do not require an assumption of finite type, so we shall not impose this unless it is expedient to do so. Often we shall require that $\mathcal{V}$ be finitely generated; such a variety $\mathcal{V}$ is necessarily locally finite.

Now consider any given variety $\mathcal{V}$, not necessarily of finite type, and let $\operatorname{Si}\mathcal{V}$ denote a transversal of the isomorphism classes of subdirectly irreducible algebras in $\mathcal{V}$ and, likewise, let $\operatorname{Si}_{\text{fin}}\mathcal{V}$ denote a set of representatives of the finite subdirectly irreducibles. Our focus here is on the residual character of $\mathcal{V}$. We recall that $\mathcal{V}$ is *residually large* if there is no bound on the cardinalities of members of $\operatorname{Si}\mathcal{V}$ or, equivalently, if $\operatorname{Si}\mathcal{V}$ is a proper class, and is *residually small* if $\operatorname{Si}\mathcal{V}$ is a set. We are interested in particular in knowing whether

(1) $\mathcal{V}$ is residually finite, that is, all members of $\operatorname{Si}\mathcal{V}$ are finite; or
(2) $\mathcal{V}$ is *residually very finite*, that is, there is a finite bound on the cardinalities of the members of $\operatorname{Si}\mathcal{V}$; or
(3) $\operatorname{Si}\mathcal{V}$ is a finite set of finite algebras.

Trivially (3) $\Rightarrow$ (2) $\Rightarrow$ (1), and if $\mathcal{V}$ is of finite type, then (2) $\Rightarrow$ (3). To illustrate how conditions (1), (2) and (3) compare, we note two contrasting old results. With no assumption on the type, R.W. Quackenbush [66] proved that a finitely generated variety $\mathcal{V}$ for which $\operatorname{Si}_{\text{fin}}\mathcal{V}$ is finite must be residually finite; in this case, (2) and (3) hold. Following on from work of Baldwin and Berman [2], J.T. Baldwin [1] constructed a locally finite, congruence distributive variety $\mathcal{V}$ of finite type for which $\operatorname{Si}_{\text{fin}}\mathcal{V}$ is infinite and which has, up to isomorphism, exactly one infinite subdirectly irreducible algebra.

The issue of when (1) forces (2) takes us into the realm of the famous RS (gap) conjecture, originating with R.W. Quackenbush [66] (here RS is shorthand for residually small). This asserted that, for any finite algebra $\mathbf{A}$, either $\mathcal{V} = \operatorname{Var}(\mathbf{A})$ is residually very finite or is residually large. Freese and

McKenzie proved the RS conjecture for any congruence modular variety [28, Theorem 8]. However the RS conjecture in general was, famously, refuted in a very strong sense by R. McKenzie [55]: for contextual introductions see K.A. Kearnes [44, Section 1] and [74, Section 6]; note in particular Theorem 6.3 of [74]. McKenzie's example of a finitely generated variety which is residually finite but not residually very finite was not of finite type. The Restricted Quackenbush Conjecture—that a finitely generated residually finite variety of finite type must be residually very finite—remains open, even for locally finite varieties.

We now look more closely at varieties whose congruence lattices satisfy some nontrivial lattice identity (a *congruence identity*). We have already noted that a residually small and finitely generated congruence modular variety is forced to be residually very finite. Hobby and McKenzie showed that a locally finite variety $\boldsymbol{\mathcal{V}}$ satisfying a nontrivial congruence identity is already congruence modular [38, Theorems 9.18 and 10.4] (or see [74, Section 5.7]). This result has since been sharpened to remove the locally finiteness requirement (see Kearnes and Kiss [45, Theorem 9.5]). Thus, informally, the presence of a congruence identity, however weak, may serve to rule out scenarios which might otherwise arise.

Other congruence properties are also of interest to us, as regards both residual finiteness and, shortly, dualisability. Let us consider (a) congruence distributivity; (b) congruence join semidistributivity; and (c) congruence meet semidistributivity. Trivially (a) implies (b) and (c). Kearnes and Kiss [45, Theorem 8.14] prove that a variety $\boldsymbol{\mathcal{V}}$ is congruence join semidistributive if and only if it is congruence meet semidistributive and satisfies a nontrivial congruence identity, so (b) implies (c). Kearnes and Willard [46] proved that any residually finite variety $\boldsymbol{\mathcal{V}}$ of finite type which is congruence meet semidistributive is necessarily residually very finite. Much more recently, they have announced [47] the same conclusion can be drawn if the assumption that $\boldsymbol{\mathcal{V}}$ be congruence meet semidistributive is replaced by the assumption that $\boldsymbol{\mathcal{V}}$ satisfy a nontrivial congruence identity. The results of Kearnes and Kiss subsume the earlier verification of the Restricted Quackenbush Conjecture in particular classes: groups, rings, lattices, and more, but not that obtained for semigroups by Golubov and Sapir, and independently by R. McKenzie. (We briefly discuss their results below.)

Now assume that $\boldsymbol{\mathcal{V}}$ is a congruence distributive variety, not necessarily of finite type, and that $\boldsymbol{\mathcal{V}}$ is generated by a class $\boldsymbol{\mathcal{K}}$ of finite algebras. By Jónsson's Lemma, $\mathrm{Si}\,\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{M}} := \mathbb{HSP}_{\mathrm{u}}(\boldsymbol{\mathcal{K}})$; see for example [8, Chapter IV, Section 6, Theorem 6.8]. If $\boldsymbol{\mathcal{K}}$ is finite or forms an elementary class, then $\boldsymbol{\mathcal{M}} = \mathbb{HS}(\boldsymbol{\mathcal{K}})$ and Birkhoff's Subdirect Product Theorem implies that $\boldsymbol{\mathcal{V}}$ is residually very finite. More particularly, if $\boldsymbol{\mathcal{K}}$ is finite, then $\mathrm{Si}\,\boldsymbol{\mathcal{V}}$ is a finite set.

Some additional comments on the role of local finiteness are appropriate; for criteria for this to hold, and examples, see G. Bezhanishvili [4]. As already noted, finite generation implies local finiteness; the converse fails, for example for monadic Boolean algebras and for relative Stone Heyting algebras. A locally finite lattice-based variety of finite type need not be residually finite; again relative Stone Heyting algebras provide an example. A locally finite variety which is residually small is forced to be residually very finite if it has definable principal congruences (DPC), and in particular if the congruence extension property holds; see Baldwin and Berman [2], where examples of varieties both meeting and failing these conditions are given, and also R. McKenzie [52].

Tables 1 and 2 summarise, for ease of reference, results on residual character noted above, and draw attention to some landmark papers.

Residually small	W. Taylor's seminal paper [71] gives many conditions equivalent to residual smallness	[71] (and see also [74])
Residually finite	$\mathcal{V}$ finitely generated and $\lvert \mathrm{Si}_{\mathrm{fin}}\,\mathcal{V}\rvert < \infty$ $\mathcal{V}$ is locally finite and $\lvert \mathrm{Si}_{\mathrm{fin}}\,\mathcal{V}\rvert < \infty$	R.W. Quackenbush [66] Baldwin & Berman [2] and W. Dziobiak [26] (independently)
Residually very finite	$\mathcal{V}$ is of finite type and congruence meet semidistributive (in particular congruence distributive)	Kearnes & Willard [46]

Table 1. Sufficient conditions for residual character properties to hold for a variety $\mathcal{V}$

Residually small +	$\mathcal{V}$ satisfies DPC	Baldwin & Berman [2, Theorem 4]
Residually small +	$\mathcal{V}$ is finitely generated and congruence modular	Freese & McKenzie [28]
Residually small +	$\mathcal{V}$ is locally finite and satisfies a nontrivial congruence identity	Hobby & McKenzie [38]
Residually finite +	$\mathcal{V}$ is of finite type and satisfies a nontrivial congruence identity	Kearnes & Willard [47]

Table 2. Conditions forcing a variety $\mathcal{V}$ to be residually very finite

We now turn to issues of dualisability. Consider a variety $\mathcal{V}$, not assumed to be of finite type. The condition $\mathcal{V} = \mathbb{ISP}(\mathcal{M})$, with $\mathcal{M}$ a finite set of finite algebras is a prerequisite for $\mathcal{V}$ to have a multisorted duality (as defined in Section 4).

We first review the role of near-unanimity terms. Recall that a term m of arity at least 3 is a *near-unanimity term* or an *NU-term* for a class $\mathcal{A}$ of algebras if $\mathcal{A}$ satisfies

$$m(x,\dots,x,y) \approx m(x,\dots,x,y,x) \approx \cdots \approx m(y,x,\dots,x) \approx x.$$

A 3-ary NU-term is usually known as a majority term. Any lattice or lattice-based algebra has such a term, *viz.* the median. We say a variety is an NU-variety if it has an NU-term in its clone. Many alternative characterisations of such varieties are known, with a central result in the area being the interpolation theorem due to Baker and Pixley; for a full discussion see for example Kaarli and Pixley [43, Section 3.2]. Any NU-variety is congruence distributive but the converse fails; see Mitsckhe [57].

The NU Duality Theorem, obtained by Davey and Werner, was one of the most useful results available in the early days of natural duality theory. Given a prevariety $\mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite algebra with a $(k+1)$-ary NU-term, the theorem asserts not only that $\mathbb{ISP}(\mathbf{M})$ is dualisable but also asserts that $\mathbb{S}(\mathbf{M}^k)$ provides a dualising set. A corresponding result is true in the multisorted case: see the Multisorted NU Duality Theorem [9, 7.1.1].

The result of A. Mitschke [57] mentioned above establishes that the presence of an NU-term in a variety forces it to be congruence distributive. This may be viewed as an obstacle to dualisability: Davey, Heindorf and McKenzie [16, Theorem 1.2] showed that if a quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ ($\mathbf{M}$ finite) is dualisable then $\mathbf{M}$ has an NU-term if and only if every finite algebra in $\mathcal{A}$ is congruence join semidistributive. There are now known to be precisely eight two-element non-dualisable algebras $\mathbf{M}$; each generates a congruence distributive variety having, necessarily, no NU-term (see [16, 56, 57] and [9, Chapter 10] and also Example 6.3 below).

We can now record the following theorem summarising properties of an NU-variety.

Theorem 5.1. *Let $\mathcal{V}$ be an NU-variety and consider the following conditions:*

(1) *$\mathcal{V}$ is residually finite;*
(2) *$\mathcal{V}$ is residually very finite;*
(3) *$\mathcal{V}$ has a multisorted duality;*
(4) *$\mathcal{V}$ is finitely generated.*

Assume that $\mathcal{V}$ is of finite type. Then (1)–(4) *are equivalent.*

Assume that $\mathcal{V}$ is not necessarily of finite type. Then (2)–(4) *are equivalent and imply* (1). *Moreover they imply that $\mathcal{V}$ is term equivalent to a finitely generated variety of finite type.*

Proof. Without restriction on the type, (3) perforce implies (4) and the Multisorted NU Strong Duality Theorem gives the converse.

The existence of an NU-term gives the congruence distributivity of $\mathcal{V}$. Assume $\mathcal{V} = \mathbb{HSP}(\mathcal{K})$, where $\mathcal{K}$ is a finite set of finite algebras. Then by Jónsson's Lemma every subdirectly irreducible algebra has cardinality at most $\max\{|A| : \mathbf{A} \in \mathcal{K}\}$, so (2) holds. Hence (2) and (4) are equivalent and, consequently, (2)–(4) are equivalent and imply (1).

Now assume $\mathcal{V}$ is of finite type. Then (1) and (2) are equivalent because $\mathcal{V}$ is congruence distributive [46]. But then also (1)–(4) are equivalent.

Finally, if $\mathcal{V}$ is finitely generated by an algebra $\mathbf{A}$ not of finite type, then the existence of an NU-term implies that $\mathbf{A}$ is term-equivalent to an algebra $\mathbf{B}$ which is of finite type (see for example [43, Theorem 3.2.5]). Consequently $\mathcal{V}$ is term equivalent to $\mathbb{HSP}(\mathbf{B})$, which is of finite type. □

Residual finiteness and dualisability do not always go hand-in-hand. Davey, Pitkethly and Willard [21] exhibit a four-element algebra $\mathbf{M}$ which is dualisable yet generates a residually large variety; the algebra is a term reduct of a commutative ring and the variety is congruence permutable. On the other hand, residual largeness frequently does entail non-dualisability. For example, a finite algebra $\mathbf{M}$ which generates a variety that is residually large and meet semidistributive must be non-dualisable [21, Corollary 3.3]. In fact a stronger statement is proved: the algebra $\mathbf{M}$ is *inherently non-dualisable*, meaning that there is no dualisable algebra $\mathbf{N}$ such that $\mathbf{M} \in \mathbb{ISP}(\mathbf{N})$; equivalently, $\mathbf{M}$ is not a subalgebra of a dualisable algebra.

We are now ready to explore particular classes of varieties.

Varieties of lattices and lattice-based algebras. Let $\mathcal{V}$ be a variety of lattices or of lattice-based algebras. Then $\mathcal{V}$ is congruence distributive and has a majority term, so Theorem 5.1 applies. If $\mathcal{V}$ is residually finite, then Theorems 4.3 and 3.8 give access to the natural extension, *alias* profinite completion, *alias* canonical extension, of any member of $\mathcal{V}$.

In important particular cases, explicit descriptions are known for the residually finite subvarieties of a lattice-based variety which is not itself residually finite. Consider, as an example, the variety of Heyting algebras. Drawing on results of L. Maximova [51] and Bezhanishvili and Grigolia [5, Theorem 4.1], we obtain that a variety $\mathcal{V}$ of Heyting algebras satisfies (the equivalent) conditions (1)–(4) of Theorem 5.1 if and only if $\mathcal{V}$ contains none of

(a) the variety generated by all finite Heyting chains, that is, the variety $\mathcal{L}$ of relative Stone algebras;
(b) the variety generated by algebras $\mathbf{B} \oplus \mathbf{1}$, where $\mathbf{B}$ is a Boolean algebra;
(c) the variety generated by algebras $\mathbf{1} \oplus \mathbf{B} \oplus \mathbf{1}$, where $\mathbf{B}$ is a Boolean algebra.

Varieties determined by monotone clones. These are varieties which are, loosely speaking, 'order-generated'. A clone C on a set P is a *monotone clone* if there exists an order $\leqslant$ on P such that C is the clone of finitary order-preserving functions on the ordered set $\langle P; \leq \rangle$. A finite nontrivial algebra $\mathbf{P}$ is *order primal* if its clone of finitary term functions is monotone.

Varieties $\mathrm{Var}(\mathbf{P})$, where $\mathbf{P}$ is order-primal with respect to an order $\leqslant$, are of intrinsic interest in clone theory, and reveal a fascinating interplay between the order structure of $\langle P, \leqslant \rangle$ and properties of $\mathrm{Var}(\mathbf{P})$. But they were studied most intensively, as a test case for the RS Conjecture, when the quest was at its height for a proof or refutation outside the congruence modular case.

Assume henceforth that $\mathbf{P}$ is an order-primal algebra with underlying ordered set $\langle P; \leqslant \rangle$. R. McKenzie [54] proved that, provided P is bounded, the following conditions are equivalent: (a) $\mathrm{Var}(\mathbf{P})$ is residually small, (b) $\mathrm{Var}(\mathbf{P})$ is congruence modular, (c) $\mathrm{Var}(\mathbf{P})$ is congruence distributive, (d) the subdirectly irreducibles in $\mathrm{Var}(\mathbf{P})$ have size at most $|P|$. He also gave order-theoretic conditions under which this occurs. (See also B.A. Davey [14].)

Around the same time, Davey, Quackenbush and Schweigert [23, Lemma 1.1] showed that $\mathbb{HS}(\mathbf{P}) = \mathbb{I}(\mathbf{P})$ if P is an antichain or connected; in general $\mathbb{HS}(\mathbf{P}) = \mathbb{I}(\mathbf{P}, \mathbf{Q})$, where $\mathbf{Q} = \mathbf{P}/\theta$ and θ identifies the points of the connected order-components of $\mathbf{P}$. They then proved that $\mathbb{ISP}(\mathbf{P}, \mathbf{Q})$ has a generalised form of multisorted duality in which infinitary algebraic relations are allowed [23, Theorem 1.4]; by exploiting the relative congruence distributivity this duality entails [23, Theorem 1.6], they were able to prove that $\mathrm{Var}(\mathbf{P})$ is congruence distributive if and only if $\mathrm{Si}(\mathrm{Var}(\mathbf{P})) \subseteq \{\mathbf{P}, \mathbf{Q}\}$ [23, Theorem 1.7], in which case $\mathrm{Var}(\mathbf{P}) = \mathbb{ISP}(\mathbf{P}, \mathbf{Q})$, so $\mathrm{Var}(\mathbf{P})$ is residually very finite.

Now assume that P is both finite and bounded. L. Zádori [75, Theorem 4.1] gave necessary and sufficient conditions for $\mathrm{Var}(\mathbf{P})$ to have an NU-term. One of these is the condition that $\mathbf{P}$ has the Jónsson terms supplied by congruence distributivity. Hence CD implies NU here. Other conditions provide characterisations of the ordered sets for which an NU term exists. See [16, 54, 75] for further information including references to papers analysing the ordered sets meeting these criteria.

We can now record the following portmanteau theorem. In (6), the dualising set consists of finitary algebraic relations, as in Section 4. The implication (6) $\Rightarrow$ (5) follows from the fact that $\mathbb{ISP}(\mathbf{P})$ is relatively congruence distributive [16, Theorem 1.2].

Theorem 5.2. *Let* $\mathrm{Var}(\mathbf{P})$ *be the variety generated by an order-primal algebra, where P is a finite bounded ordered set. Then the following are equivalent:*

(1) $\mathrm{Var}(\mathbf{P})$ *is residually small;*
(2) $\mathrm{Var}(\mathbf{P})$ *is residually very finite, with the subdirectly irreducibles in* $\mathrm{Var}(\mathbf{P})$ *of size at most $|P|$;*
(3) $\mathrm{Var}(\mathbf{P})$ *is congruence modular;*
(4) $\mathrm{Var}(\mathbf{P})$ *is congruence distributive;*
(5) $\mathrm{Var}(\mathbf{P})$ *has an NU-term;*
(6) $\mathbb{ISP}(\mathbf{P})$ *has a natural duality.*

Moreover, these conditions imply that

(7) Var(**P**) *has a multisorted natural duality.*

Affine complete varieties. We refer to the monograph of Kaarli and Pixley [43] for background, recalling only that an algebra **A** is *affine complete* if every congruence-compatible operation on **A** is a polynomial function of **A** and that a variety is *affine complete* if every member is affine complete. Classic examples are the variety of Boolean algebras, and more generally any arithmetical variety generated by a finite set of primal algebras [43, p. 158]. (For a full discussion of the close relationship between arithmeticity and affine completeness, see [43].)

Deep results of Kaarli and Pixley [41] and Kaarli and McKenzie [42] establish the following facts about any affine complete variety $\mathcal{V}$: it is residually finite ([41, Theorem 2.1] and [43, Theorem 4.2.1]), congruence distributive ([42, Theorem 4.1] and [43, Theorem 4.2.2]), and every member of $\mathrm{Si}\,\mathcal{V}$ has no proper subalgebras ([41, Theorem 3.4] and [43, Corollary 4.2.6]).

The following surprising result [43, Theorem 4.4.3] is due to K. Kaarli [40].

Theorem 5.3. *For an affine complete variety* $\mathcal{V}$ *the following are equivalent:*

(1) $\mathcal{V}$ *is locally finite;*

(2) $\mathcal{V}$ *is finitely generated;*

(3) $\mathcal{V}$ *is term equivalent to a variety of finite type;*

(4) $\mathcal{V}$ *has only finitely many subvarieties.*

In practice we mostly deal with affine complete varieties of finite type. Such varieties have very strong properties: in the theorem below, (i)–(iii) come from [40, 42, 41], see [43, 4.4.1, 4.3.7, 4.2.6, 4.2.7], while (iv) follows from the Multisorted NU Duality Theorem [9, Theorem 7.1.1], see also [9, p. 192].

Theorem 5.4. *Every affine complete variety* $\mathcal{V}$ *of finite type satisfies the following conditions:*

(i) $\mathcal{V}$ *is an NU-variety and so is congruence distributive;*

(ii) $\mathcal{V}$ *is residually very finite* (*up to isomorphism* $\mathrm{Si}\,\mathcal{V}$ *consists of a finite set of finite algebras each of which has no proper subalgebras*)*;*

(iii) $\mathcal{V} = \mathbb{HSP}(\mathbf{M})$, *where* **M** *is finite and has no proper subalgebras;*

(iv) $\mathcal{V}$ *has a multisorted duality.*

Paraprimal and discriminator varieties. We take as our principal references here the paper of R. McKenzie [52] and H. Werner's monograph [73], but acknowledge also the important contributions of Clark and Krauss, Quackenbush, Pixley and others in relation to paraprimality and the special cases of quasiprimality and primality. An overview of primality and its generalisations is provided by R. Quackenbush [35, Appendix 5].

Throughout this subsection we restrict to algebras of finite type. A nontrivial algebra **A** is *paraprimal* if it is finite, every subalgebra of **A** is simple and Var(**A**) is congruence permutable. If, in addition, Var(**A**) is congruence distributive, then $\mathbf{A}$ is *quasiprimal*. We note that a paraprimal algebra $\mathbf{A}$ is quasiprimal if and only if $\mathrm{Var}(\mathbf{A})$ contains no non-trivial affine algebras (R. McKenzie [52, Theorem 21]). (An algebra $\mathbf{A}$ is *affine* if there is an abelian group structure on the underlying set of $\mathbf{A}$ such all of the fundamental operations of $\mathbf{A}$ are translates of group homomorphisms.) Paraprimal algebras hybridise quasiprimal algebras and affine algebras in a precise way which leads to a very satisfactory structure theory for the varieties they generate (see [52, Section 2]).

R. McKenzie [52] defined a variety to be *paraprimal*, respectively *quasiprimal*, if it is congruence permutable and generated by a finite set of paraprimal, respectively quasiprimal, algebras, and proved that a variety is quasiprimal if and only if it is congruence distributive and paraprimal [52, Theorem 21]. Our interest in paraprimal varieties stems from a result of McKenzie [52, Theorem 17]: a paraprimal variety is directly representable by its set of finite simple members and this set has only finitely many members up to isomorphism. Explicitly, if $\mathcal{V} = \mathbb{HSP}(\{\mathbf{P}_1, \dots, \mathbf{P}_n\})$, where $\mathbf{P}_1, \dots, \mathbf{P}_n$ are paraprimal, then every finite subdirectly irreducible algebra in $\mathcal{V}$ is simple and of cardinality at most $\max\{|\mathbf{P}_1|, \dots, |\mathbf{P}_n|\}$. Moreover, because $\mathcal{V}$ is congruence permutable, every subdirect product of a finite system of (necessarily finite) simple algebras is isomorphic to a finite product of simple algebras (see the proof of Theorem 17 in [52] for the earlier results in the paper on which these claims rely). Therefore, by Quackenbush's theorem [66], we have $\mathcal{V} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite simple algebras and every finite algebra in $\mathcal{V}$ is isomorphic to a direct product of algebras belonging to $\mathcal{M}$. There are two natural choices for the set $\mathcal{M}$: the largest one being (a transversal of the set of isomorphism classes of) the set of *all* simple algebras, and the smallest being the set of maximal simple algebras, that is, those that do not embed into a strictly larger simple algebra. Either of these sets may therefore be used to construct the natural extension of an algebra in $\mathcal{V}$.

We now specialise to the quasiprimal case. Let $\mathcal{V}$ be a quasiprimal variety. Then $\mathcal{V} = \mathrm{Var}(\mathcal{M})$, for some finite set $\mathcal{M}$ of quasiprimal algebras. By a well-known characterisation of quasiprimality (see [8, IV.10.7]), for each algebra $\mathbf{M}$ in $\mathcal{M}$, there is a term $t_{\mathbf{M}}$ which yields the *ternary discriminator* function on $\mathbf{M}$, that is, $\mathbf{M}$ satisfies $t_{\mathbf{M}}(x, y, z) = x$ if $x \neq y$ and $t_{\mathbf{M}}(x, x, z) = z$. In fact, as we shall soon see, there is a single term that yields the ternary discriminator on every algebra in $\mathcal{M}$.

Werner's single-sorted natural duality for the variety generated by a quasiprimal algebra (see [9, 3.3.13]) extends easily to a multisorted duality for $\mathcal{V}$ based on $\mathcal{M}$. In order to state the result, and the lemma which precedes it, we require some terminology and notation. Every n-ary term induces a term function $t^{\mathbf{M}} \colon M^n \to M$ on each algebra $\mathbf{M} \in \mathcal{M}$. We refer to the corresponding $\mathcal{M}$-sorted map as an n-ary *term function on* $\mathcal{M}$. Define

$$\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}} := \left\langle \dot{\bigcup} \{ M \mid \mathbf{M} \in \mathcal{M} \}; \mathrm{Iso}(\mathcal{M}), \mathcal{T} \right\rangle,$$

where $\mathrm{Iso}(\mathcal{M})$ is the set of isomorphisms between subalgebras of algebras in $\mathcal{M}$. Note that $\mathrm{Iso}(\mathcal{M})$ forms an inverse semigroup when endowed with the usual composition of partial maps: for all $g, h \in \mathrm{Iso}(\mathcal{M})$, we define $g \circ h$ to be the map with domain $\{\, a \in \mathrm{dom}(h) \mid g(a) \in \mathrm{dom}(g) \,\}$ given, of course, by $(g \circ h)(a) := g(h(a))$, for all $a \in \mathrm{dom}(g \circ h)$.

Lemma 5.5. *Let $\mathcal{V} = \mathrm{Var}(\mathcal{M})$ with $\mathcal{M}$ a finite set of finite algebras and assume that $\mathcal{V}$ is congruence permutable and that each $\mathbf{M}$ in $\mathcal{M}$ is such that each of its subalgebras is simple. Then $\mathrm{Iso}(\mathcal{M})$ entails (and so also discretely entails) every binary multisorted algebraic relation on $\mathcal{M}$.*

Proof. Let $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{M}$ and let $\mathbf{B}$ be a subalgebra of $\mathbf{M}_1 \times \mathbf{M}_2$. By Fleischer's Theorem (see [8, IV.10.1]), there exist subalgebras $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively, and surjective homomorphisms $u_1 \colon \mathbf{A}_1 \to \mathbf{C}$ and $u_1 \colon \mathbf{A}_1 \to \mathbf{C}$ such that

$$B = \{\, (a_1, a_2) \in A_1 \times A_2 \mid u_1(a_1) = u_2(a_2) \,\}.$$

As $\mathbf{A}_1$ and $\mathbf{A}_2$ are simple, it follows that either $|C| = 1$, in which case $B = A_1 \times A_2$, or u_1 and u_2 are isomorphisms, in which case B is the graph of the isomorphism $u_2^{-1} \circ u_1 \colon \mathbf{A}_1 \to \mathbf{A}_2$. It follows immediately that $\mathrm{Iso}(\mathcal{M})$ entails the algebraic relation B (via the constructs domain, product and graph—see [9, 2.4.5]). □

Theorem 5.6. *Let $\mathcal{V} = \mathrm{Var}(\mathcal{M})$ with $\mathcal{M}$ a finite set of finite algebras. The following are equivalent:*

1. *$\mathcal{V}$ is arithmetical and every nontrivial subalgebra of each algebra in $\mathcal{M}$ is simple;*
2. *there is a term that yields the ternary discriminator on every algebra in $\mathcal{M}$;*
3. *every $\mathcal{M}$-sorted morphism from $(\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}})^n$ to $\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}}$ is a term function on $\mathcal{M}$;*
4. *the structure $\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}}$ yields a multisorted duality on $\mathcal{V}$ based on $\mathcal{M}$.*

Moreover, the duality given in (4) can be upgraded to a strong duality by adding each element that forms a one-element subalgebra of an algebra in $\mathcal{M}$ as a nullary operation to the type of $\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}}$.

Proof. The implication (4) ⇒ (3) is a basic fact from duality theory: a dualising structure determines the term functions. It is trivial that the $\mathcal{M}$-sorted function built from ternary discriminator functions on (the underlying sets of) the algebras in $\mathcal{M}$ preserves isomorphisms between subalgebras of members of $\mathcal{M}$; hence, (3) ⇒ (2). Now assume that (2) holds and let t be a term that yields the ternary discriminator on every algebra in $\mathcal{M}$. It follows at once that each member of $\mathcal{M}$ is quasiprimal. Since t is a Pixley (two-thirds minority) term on $\mathcal{M}$, that is, $\mathcal{M}$ satisfies the equations

$$t(x, y, y) = t(x, y, x) = t(y, y, x) = x,$$

it follows that t is a Pixley term on $\mathcal{V} = \mathrm{Var}(\mathcal{M})$ and hence $\mathcal{V}$ is arithmetical (see [8, II.12.5]). Hence, (1) holds. Finally, assume that (1) holds. By Lemma 5.5 and the Multisorted NU Duality Theorem [9, 7.1.1], (4) holds.

The claim regarding the upgrade of the duality in (4) to a strong duality comes straight from the Multisorted NU Strong Duality Theorem [9, 7.1.2]. □

Werner [73] defines a variety $\mathcal{V}$ to be a *discriminator variety* if $\mathcal{V} = \mathrm{Var}(\mathcal{M})$ for some class $\mathcal{M}$ of algebras such that there is a term t that yields the ternary discriminator on every algebra in $\mathcal{M}$. It follows from Theorem 5.6 that quasiprimal varieties and finitely generated discriminator varieties are one and the same. Indeed, assume that $\mathcal{V}$ is a finitely generated discriminator variety. Then Jónsson's Lemma implies that $\mathcal{V} = \mathrm{Var}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of quasiprimal algebras. As a discriminator variety is arithmetical (see the proof above), it follows that $\mathcal{V}$ is a quasiprimal variety. For the converse, note that $\mathcal{V}$ quasiprimal implies $\mathcal{V}$ is arithmetical, by [52, Theorem 21]; then (1) $\Rightarrow$ (2) of Theorem 5.6 applies.

By way of illustration, we highlight lattice-based discriminator varieties arising in algebraic logic. At the heart of this application lie residuated lattices, which have been intensively studied in recent years in connection with substructural logics; see, for example, Galatos et al. [30]. Residuated lattices serve as models for the full Lambek calculus without contraction; they generalise Heyting algebras, which model intuitionistic propositional calculus. A *residuated lattice* (or *FL_{we}-algebra*) is an algebra $\mathbf{A} = (A; \wedge, \vee, \cdot, \to, 0, 1)$ such that (i) $(A; \wedge, \vee, 0, 1)$ is a bounded lattice, (ii) $(A; \cdot, 1)$ is a commutative monoid and (iii) $x \cdot y \leqslant z$ if and only if $x \leqslant y \to z$ for all $x, y, z \in A$. The nontrivial finitely generated varieties of residuated lattices which are discriminator varieties have been completely characterised by T. Kowalski [48] (or see [30, Chapter 11]): they are exactly those which are semisimple, and are equationally characterised as those satisfying $x \vee \neg x^n \approx 1$ for some $n \geqslant 1$. Under this umbrella falls (to within term-equivalence) any MV-algebra variety generated by a single finite chain. For these particular varieties, Theorem 5.6 gives nothing new, but does set well-known facts in a wider context. Now consider a variety $\mathcal{V}$ whose members are Boolean algebras with (finitely many) modal operators. Semisimplicity is again a necessary and sufficient condition for $\mathcal{V}$ to be a discriminator variety, and an intrinsic characterisation has also been found (Kowalski and Kracht [49]). We note too the intimate connection between the presence of a discriminator term, semisimplicity and the property EDPC (equationally definable principal congruences) which commonly occurs in varieties arising from logics (see Blok and Pigozzi [6]).

Finally in this subsection we show that in finitely generated discriminator varieties the natural extension has a particularly simple description.

Theorem 5.7. *Let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras, let $\mathbf{A} \in \mathcal{A}$ and let $\mathbf{B}$ be the algebra of all $\mathcal{M}$-sorted maps*

$$b \colon \dot{\bigcup}\{\, \mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\} \to \dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$$

that preserve $\mathrm{Iso}(\mathcal{M})$*. Then* $\mathbf{B}$ *is isomorphic to a full direct product of algebras from* $\mathbb{S}(\mathcal{M})$*. In particular, the natural extensions of algebras in a finitely generated discriminator variety are, up to isomorphism, direct products of quasi-primal algebras.*

Proof. Let $X := \dot{\bigcup}\mathcal{A}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\}$ and note that $\mathrm{Iso}(\mathcal{M})$ acts on X via composition: for $h \in \mathrm{Iso}(\mathcal{M})$ and $x \in X$ we have $x \in \mathrm{dom}(h^X)$ if and only if $x(A) \subseteq \mathrm{dom}(h)$ and $h(x) := h \circ x$, for all $x \in \mathrm{dom}(h^X)$. Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if there exists $h \in \mathrm{Iso}(\mathcal{M})$ such that $y \in \mathrm{dom}(h^X)$ and $x = h^X(y)$. Take Y to be a set of representatives of the equivalence classes.

As each $b \in B$ preserves $\mathrm{id}_{\mathbf{N}}$, for each $\mathbf{N} \in \mathbb{S}(\mathcal{M})$, we have $b(x) \in x(A)$, for all $x \in X$. Hence the map $\varphi \colon \mathbf{B} \to \prod_{y \in Y} y(\mathbf{A})$, given by $\varphi(b) = (b(y))_{y \in Y}$, for all $b \in B$, is a well-defined homomorphism. We claim that φ is an isomorphism. Let $b_1, b_2 \in B$ with $b_1 \neq b_2$. Then there exists $x \in X$ such that $b_1(x) \neq b_2(x)$. Choose $y \in Y$ and $h \in \mathrm{Iso}(\mathcal{M})$ such that $x = h^X(y)$. Since b_1 and b_2 preserve h, we have $h(b_1(y)) = b_1(h^X(y)) = b_1(x) \neq b_2(x) = b_2(h^X(y)) = h(b_2(y))$, so that $b_1(y) \neq b_2(y)$. Hence φ is one-to-one.

Now assume that $(y(a_y))_{y \in Y} \in \prod_{y \in Y} y(\mathbf{A})$, where $a_y \in A$, for all $y \in Y$. Let $x \in X$, and let $y_1, y_2 \in Y$ and $h_1, h_2 \in \mathrm{Iso}(\mathcal{M})$, with $h_1^X(y_1) = x = h_2^X(y_2)$. Then $y_1 = y_2$ (since Y is a transversal of the $\sim$-equivalence classes) and $h_1(y_1(a)) = h_2(y_2(a))$, for all $a \in A$, by the definition of $h_i^X(y_i)$. Hence we may unambiguously define an $\mathcal{M}$-sorted map b from X to $\dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$ by $b(x) = h(y(a_y))$, where $y \in Y$ and $h \in \mathrm{Iso}(\mathcal{M})$ are such that $x = h^X(y)$. We shall prove that b preserves each $k \in \mathrm{Iso}(\mathcal{M})$. Assume that $x \in X$ with $x \in \mathrm{dom}(k^X)$; hence $x(a) \in \mathrm{dom}(k)$, for all $a \in A$. We have $x = h^X(y)$, for some $h \in \mathrm{Iso}(\mathcal{M})$ and $y \in Y$. We also have $k^X(x) = k \circ (h \circ y) = (k \circ h) \circ y = (k \circ h)^X(y)$, and $k \circ h \in \mathrm{Iso}(\mathcal{M})$. Therefore, by the definition of b, we have

$$\begin{aligned} b(k^X(x)) &= b(k^X(h^X(y))) = b((k \circ h)^X(y)) \\ &= (k \circ h)(y(a_y)) = k(h(y(a_y))) = k(b(x)); \end{aligned}$$

so b preserves k. Finally note that $\varphi(b) = (y(a_y))_{y \in Y}$, whence φ is surjective.

The final assertions of the theorem follow from Theorems 4.1 and 5.6. □

Remark 5.8. By combining Theorem 5.7 with Theorem 3.8 we obtain a description of the canonical extension $\mathbf{A}^\sigma$ of any algebra $\mathbf{A}$ in a finitely generated discriminator variety of algebras having a (bounded) lattice reduct. In the case that the reduct is distributive, such a description was obtained by Gehrke and Jónsson [32, Section 3]; their result can now be seen in a wider context. We remark that $\prod_{h \in H} h(\mathbf{A})$ is a dense and compact completion of

$\mathbf{A}$ via the embedding $\varphi \circ j$, where j is the natural embedding of $\mathbf{A}$ into $\mathbf{B}$ and $\varphi \colon \mathbf{B} \to \prod_{y \in Y} y(\mathbf{A})$ is the isomorphism given by Theorem 5.7.

Semilattice-based varieties. Semilattice-based algebras have become increasingly prominent, due particularly to their role in R. McKenzie's groundbreaking undecidability results from the late 1990s, following on from [55]; these are discussed in [74].

Semilattices fail to satisfy any nontrivial congruence identity (Freese and Nation [29]). However every semilattice-based algebra is congruence meet semidistributive: a direct proof appears in D. Papert [62] but the result also follows from [38, 9.10]. Therefore a semilattice-based variety of finite type which is residually finite is residually very finite [46].

Particular attention has been given to *flat algebras*, that is, those with an underlying meet-semilattice of height 1, as these are the algebras that occur in McKenzie's work. In this case, more can be said. Davey, Jackson, Pitkethly and Talukder [19] prove the equivalence of the following conditions for a finite flat algebra $\mathbf{M}$ whose bottom is absorbing (that is, the zero of the semilattice acts as a zero for every operation):

(i) $\mathrm{Var}(\mathbf{M})$ is residually small;
(ii) $\mathrm{Var}(\mathbf{M})$ is residually very finite;
(iii) $\mathbf{M}$ is dualisable;
(iv) each fundamental operation of $\mathbf{M}$ is compatible with the semilattice operation.

In [19] extensive classes of algebras $\mathbf{M}$ which satisfy these conditions are presented, as well as many classes for which they fail, with $\mathbf{M}$ not merely non-dualisable but inherently non-dualisable. The case where the bottom of $\mathbf{M}$ is not absorbing is necessarily more complex as it includes the algebras determined by the Turing machines in McKenzie's work on undecidability. Nevertheless, Clark, Davey, Pitkethly and Rifqui [12] succeed in characterising completely those finite flat unars $\mathbf{M}$ with non-absorbing bottom which are dualisable.

Varieties of groups and semigroups. McKenzie, in [53], verified the RS Conjecture for semigroups. The class $\mathbb{S}_{\text{fin}}$ of finite semigroups therefore splits into two classes:

- $\mathcal{L} = \{\, \mathbf{M} \in \mathbb{S}_{\text{fin}} \mid \mathrm{Var}(\mathbf{M}) \text{ is residually large} \,\}$, and
- $\mathcal{F} = \{\, \mathbf{M} \in \mathbb{S}_{\text{fin}} \mid \mathrm{Var}(\mathbf{M}) \text{ is residually very finite} \,\}$.

The class $\mathcal{F}$, which coincides with the class of semigroups that generate a residually finite variety, has been completely described. In 1979, Golubov and Sapir [33] proved that every residually finite variety of semigroups must be a subvariety of some member of a particular class of varieties of semigroups $\{\mathcal{V}_i^n \mid i \in \{1,2,3\} \text{ and } n > 1\}$. They also showed that a residually finite variety of semigroups is residually very finite and that every such variety is generated by a member of a distinguished list of twenty types of finite semigroups (for a convenient summary see A.C.J. Bonato [7]). Using their characterisation Golubov and Sapir were able to identify all the subdirectly irreducible semigroups in a residually finite semigroup variety. The characterisation was, essentially, also obtained independently by McKenzie as a corollary of the main result in [53]; this same result enabled him to verify the RS Conjecture for semigroups.

We now turn to duality issues, and ask how far it is known that dualisability of a finite semigroup $\mathbf{M}$ correlates with the residual character of $\mathbb{HSP}(\mathbf{M})$. Complete results are not available, but the theorems which have been proved so far indicate uniform behaviour: a dichotomy between the properties of dualisability and residual finiteness on the one hand and those of inherent non-dualisability and residual largeness on the other.

Groups illustrate this well. A finite group is congruence permutable and in particular congruence modular. Therefore it generates a variety which is either residually large or residually very finite. A. Ju. Ol'shankskii [61] proved that a variety of groups is residually finite if and only if it is generated by a finite group whose Sylow subgroups are all abelian. Quackenbush and Szabó [67] proved that a finite group is dualisable if all its Sylow subgroups are cyclic and conjectured that 'cyclic' here could be replaced by 'abelian'. A positive solution to this—the Quackenbush–Szabó-conjecture (QSC)—was announced by M. H. Nickodemus [58]. It is also known that a finite group with a non-abelian Sylow subgroup is inherently non-dualisable (see [39]). We conclude that a finitely generated variety of groups is residually finite if and only if it is generated by a dualisable finite group and that this holds if and only if it is generated by a finite group with abelian Sylow subgroups.

We now turn to monoids and to semigroups in general. It follows from Jackson [39, Theorem 15] that, since QSC is true, a finite inverse semigroup is inherently non-dualisable if and only if it generates a variety which is not residually finite. Moreover this implies that the class of inherently non-dualisable finite algebras includes finite monoids containing a non-group element, completely regular non-Clifford monoids and finite non-Clifford inverse semigroups (see [39]).

Like the finite semigroups, the finite monoids split into two classes, which we denote by $\mathcal{F}^1$ and $\mathcal{L}^1$; these classes consist, respectively, of monoids which are residually very finite and residually large. No non-dualisable monoid in $\mathcal{F}^1$ has yet been found. For finite monoids of various particular types, dualisability is equivalent to membership of $\mathcal{F}^1$ (this happens, for example, for aperiodic monoids) and inherent non-dualisability to membership of $\mathcal{L}^1$ (true for commutative monoids); see [39] for details. The only members of $\mathcal{F}^1$ whose dualisability status remains unresolved are the semilattices of dualisable groups.

For finite semigroups in general the picture is less complete. There are members $\mathbf{M}$ of $\mathcal{F}$ and of $\mathcal{L}$ for which it is unknown whether $\mathbf{M}$ is dualisable, inherently non-dualisable, or neither. Nevertheless, the uniform pattern seems to persist: all semigroups in $\mathcal{F}$, where the answer is known, are dualisable; all semigroups in $\mathcal{L}$, where the answer is known, are inherently non-dualisable.

There is one niggling issue on which we should comment. The classification of finite semigroups concerns varieties $\mathbb{HSP}(\mathbf{M})$, whereas when we refer to dualisability of $\mathbf{M}$, we are referring to a duality for the prevariety $\mathbb{ISP}(\mathbf{M})$. The classes $\mathbb{HSP}(\mathbf{M})$ and $\mathbb{ISP}(\mathbf{M})$ do not always coincide; note that $\mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$ if and only if each subdirectly irreducible of $\mathbb{HSP}(\mathbf{M})$ embeds into $\mathbf{M}$. The studies of residually finite and residually small semigroups in [33, 53] provide many examples of finite semigroups $\mathbf{M}$ generating varieties that do not satisfy $\mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$. Clark *et al.* [10] ask if there is an algorithm to decide when it is the case that $\mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$. They observe that we can decide if $\mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$ for algebras in a class $\mathcal{K}$ whenever, within $\mathcal{K}$, we have

(1) an effective method to determine if $\mathbb{HSP}(\mathbf{M})$ is residually finite, and
(2) in case $\mathbb{HSP}(\mathbf{M})$ is residually finite, an effective method to compute a number $f(|M|) \in \mathbb{N}$ such that each member of $\mathrm{Si}(\mathbb{HSP}(\mathbf{M}))$ has size at most $f(|M|)$.

(Given (1) and (2), we may check if each subdirectly irreducible quotient of the free algebra in $\mathbb{HSP}(\mathbf{M})$ on $f(M)$ generators embeds into $\mathbf{M}$.) For semigroups, algorithms for (1) and (2) can be extracted from [33] and [53].

Unary algebras. Finally in this section we turn to a class of algebras that exhibits a wealth of different behaviours, both good and very bad.

Results on the residual character of varieties of unary algebras date back to the seventies. W. Taylor [71, p. 39] argued why every variety of unary algebras is residually small. Since unary algebras have the congruence extension property, the results of Baldwin and Berman [2] then show that every locally finite variety of unary algebras has DPC and so is residually very finite. Even for unars the assumption of local finiteness cannot be dropped here. The variety of all unars $\langle A; f\rangle$ is not locally finite (the natural numbers with f the successor function is one-generated) and also not residually finite (take f to be the predecessor function on the natural numbers with $f(1) = 1$, then every non-trivial subalgebra is subdirectly irreducible).

The monograph of Pitkethly and Davey [63] contains a detailed analysis of dualisability for unary algebras. The issue of dualisability is revealed to be surprisingly subtle and complex. Every finite unar is dualisable [63, 3.5.1], but not every three-element unary algebra is dualisable. In addition, there are no inherently non-dualisable unary algebras ([63, 2.1.4]).

6. Applications: internally residually finite prevarieties

In this section we venture into the territory where Theorem 4.1 applies but Theorem 4.3 does not, and present some examples.

In certain circumstances, useful information can be derived from Theorem 4.3 even for prevarieties $\mathbb{ISP}(\mathcal{M})$ for which $\mathcal{M}$ is infinite. Assume $\mathcal{M}$ has the property that every finite subset $\mathcal{N}$ of $\mathcal{M}$ is such that $\mathbb{ISP}(\mathcal{N})$ is dualisable; this occurs for example whenever the algebras in $\mathcal{M}$ have an NU-term. The following result can be viewed as being in the spirit of a compactness theorem, but we note that there is no connection with the Duality Compactness Theorem (see for example [9, Theorem 2.2.11]).

Theorem 6.1. *Let $\mathcal{F}$ be a directed family of finite sets of finite algebras of the same type. For each $\mathcal{N} \in \mathcal{F}$, assume that $\boldsymbol{\mathcal{A}}_{\mathcal{N}} := \mathbb{ISP}(\mathcal{N})$ is dualisable, with a dualising set $\mathcal{S}^{\mathcal{N}}$. Let $\mathcal{M} := \bigcup \mathcal{F}$, $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathcal{M})$ and let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Then an $\mathcal{M}$-sorted map $b \colon \dot{\bigcup}\{\, \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\} \to \dot{\bigcup}\{\, M \mid \mathbf{M} \in \mathcal{M}\,\}$ belongs to $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ if and only if b preserves $\mathcal{S} := \bigcup\{\, \mathcal{S}^{\mathcal{N}} \mid \mathcal{N} \in \mathcal{F}\,\}$.*

Proof. The idea of the proof is to localise to a finitely generated quasivariety within $\boldsymbol{\mathcal{A}}$ and to exploit (duality-)entailment there. We know, by Theorem 4.1, that the map b belongs to $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ if and only if b preserves all finitary multisorted algebraic relations on $\mathcal{M}$. So the forward implication is trivial.

For the backward implication, let s be a finitary multisorted algebraic relation on $\mathcal{M}$ of arity m. Since the family $\mathcal{F}$ is directed we can choose $\mathcal{N} \in \mathcal{F}$ such that $\mathbf{s} \in \boldsymbol{\mathcal{A}}_{\mathcal{N}}$. Assume that the $\mathcal{M}$-sorted map b preserves $\mathcal{S}$ and let $(x_1, \dots, x_m) \in s_{D(\mathbf{s})}$, where $D(\mathbf{s}) = \dot{\bigcup}\{\, \boldsymbol{\mathcal{A}}(\mathbf{s}, \mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\,\}$, the dual space of $\mathbf{s}$, viewed as a member of $\boldsymbol{\mathcal{A}}$.

Define c to be the restriction of b to the subset $\dot{\bigcup}\{\, \boldsymbol{\mathcal{A}}_{\mathcal{N}}(\mathbf{s}, \mathbf{N}) \mid \mathbf{N} \in \mathcal{N}\,\}$ of its domain. We know that $\mathcal{S}^{\mathcal{N}}$ dualises $\boldsymbol{\mathcal{A}}_{\mathcal{N}}$ and that s is a finitary algebraic relation on $\mathcal{N}$. Working with (duality-)entailment as it applies to $\boldsymbol{\mathcal{A}}_{\mathcal{N}}$, we deduce that $\mathcal{S}^{\mathcal{N}}$ entails s. By Lemma 4.2, $\mathcal{S}^{\mathcal{N}}$ discretely entails s. In particular c, which by assumption preserves $\mathcal{S}^{\mathcal{N}}$, must preserves s. Since we know that $(x_1, \dots, x_m) \in s$ on the dual space of $\mathbf{s}$, derived as for a member of $\boldsymbol{\mathcal{A}}_{\mathcal{N}}$, we deduce that $(c(x_1), \dots, c(x_m)) \in s$. Therefore $(b(x_1), \dots, b(x_m)) \in s$ and we have proved that b preserves s. $\square$

A few comments should be made here. By insisting that the generating sets for the prevarieties, rather than the prevarieties themselves, be directed, we have ensured that each relation we have to consider lies in a single dualisable sub-prevariety and that the set of relations determining the natural extension can be simply described. There are situations where directedness is automatic or can be engineered. If $\mathcal{M}$ is a family of finite algebras, directed with respect to 'is isomorphic to a subalgebra of', then the family of prevarieties $\{\mathbb{ISP}(\mathbf{M})\}_{\mathbf{M} \in \mathcal{M}}$ satisfies the condition demanded of the family $\boldsymbol{\mathcal{A}}_{\mathcal{N}}$ of prevarieties in Theorem 6.1. Given a variety $\boldsymbol{\mathcal{V}}$ which is not residually finite, one may consider $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is the family of finite algebras in $\boldsymbol{\mathcal{V}}$; then $\boldsymbol{\mathcal{A}}$ is properly contained in $\boldsymbol{\mathcal{V}}$ since $\boldsymbol{\mathcal{V}}$ contains infinite subdirectly algebras while $\boldsymbol{\mathcal{A}}$ contains none.

Now assume $\mathcal{V}$ is congruence distributive and let $\mathcal{F} := \{\mathfrak{F}_n\}_{n=2}^{\infty}$, where $\mathfrak{F}_n$ consists of the members of $\mathcal{V}$ of cardinality less than n; this is certainly directed. By Jónsson's Lemma, $\mathbb{ISP}(\mathfrak{F}_n) = \mathbb{HSP}(\mathfrak{F}_n)$ and, since $\mathfrak{F}_n$ is an elementary class, this variety contains, and is generated by, all subdirectly irreducible members of $\mathcal{V}$ of size less than n. If in addition $\mathcal{V}$ is of finite type then, up to isomorphism, $\mathbb{HSP}(\mathfrak{F}_n)$ is generated by a finite set of finite algebras. This is the scenario in the examples we present.

As a first application of Theorem 6.1 we consider a particular example.

Example 6.2. Let $\mathbf{C}_n$ be the n-element chain, regarded as a Heyting algebra and let $\mathcal{M} := \{\mathbf{C}_n\}_{n=1}^{\infty}$. Then $\mathbb{ISP}(\mathcal{M})$ is a prevariety properly contained in the variety $\mathcal{H}$ of Heyting algebras (proper containment because $\mathcal{H}$ is not residually finite). Indeed, $\mathrm{Var}(\mathcal{M})$ is the variety $\mathcal{L}$ of relative Stone algebras and $\mathbb{ISP}(\mathcal{M})$ is the prevariety generated by the finite members of $\mathcal{L}$. Let $\mathbf{C}$ be the chain $\{0\} \cup \{1/m\}_{m \geqslant 1}$ and embed each chain $\mathbf{C}_n$ into $\mathbf{C}$ so that $C_n \backslash \{0\}$ is an up-set of $C \backslash \{0\}$. We can therefore assume that we have

$$\mathbf{C}_1 \subset \mathbf{C}_2 \subset \mathbf{C}_3 \subset \cdots \subset \mathbf{C}.$$

Furthermore, with this representation, $\mathbf{C}_n$ is a Heyting subalgebra of $\mathbf{C}_{n+1}$ for each n. Here $\mathbb{ISP}(\{\mathbf{C}_1, \ldots, \mathbf{C}_N\})$ is simply $\mathbb{ISP}(\mathbf{C}_N)$ and this is dualised by $\mathrm{End}\,\mathbf{C}_N$—for a very short proof of this fact due to Davey and Priestley, see [9, Theorem 8.1.5]. Theorem 6.1 implies that the natural extension $n_{\mathcal{A}}(\mathbf{A})$, for any $\mathbf{A} \in \mathcal{A}$, is the family of $\mathcal{M}$-sorted maps preserving the set $\bigcup_{N=1}^{\infty} \mathrm{End}\,\mathbf{C}_N$.

Varieties within the scope of Theorem 6.1 are plentiful. We draw attention in particular to varieties $\mathcal{V}$ whose lattice of subvarieties is an $(\omega + 1)$-chain $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}$, where $\mathcal{V}_0$ is the trivial variety and each $\mathcal{V}_n$ is finitely generated and dualisable, but $\mathcal{V}$ is not residually finite. Here we have an IRF-prevariety $\mathcal{W} := \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M} := \bigcup_{n \geqslant 1} \mathrm{Si}\,\mathcal{V}_n$, which is properly contained in $\mathcal{V}$. Examples of this type include:

- relative Stone Heyting algebras, the variety $\mathcal{L}$ from Example 6.2;
- monadic Boolean algebras (here we have a discriminator variety in which $\mathcal{V}_n$ for $n \geqslant 1$ is the quasiprimal variety generated by the simple monadic algebra with reduct $\mathbf{2}^n$);
- monadic distributive lattices;
- modular ortholattices;
- distributive p-algebras.

For definitions and properties of these varieties, see for example [13, 4] and for dualities for their finitely generated subvarieties see [9, 65, 37].

We observe that changing the type can radically alter the picture. The variety of p-semilattices behaves quite differently from the variety $\mathcal{B}_\omega$ of distributive p-algebras: in both cases the finite subdirectly irreducibles are the algebras $\mathbf{2}^n \oplus \mathbf{1}$. But, *qua* p-semilattices, $\mathbb{HSP}(\mathbf{2}^n \oplus \mathbf{1})$ is the same for every $n \geqslant 1$ and is residually large. The associated quasivarieties $\mathbb{ISP}(\mathbf{2}^n \oplus \mathbf{1})$ are distinct, but fail to be dualisable for $n > 1$. See [20, 69, 70] for details.

Examples where Theorem 6.1 applies to the prevariety generated by all finite subdirectly irreducible algebras but where we do not have an $(\omega+1)$-chain of subvarieties are supplied by the variety $\mathcal{MV}$ of MV-algebras. Here the ordered set of finitely generated subvarieties is isomorphic to the finite down-sets of $\mathbb{N}\cup\{0\}$ under divisibility (see for example P. Niederkorn [59, Theorem 1.1]). As noted earlier, Theorem 5.6 applies to each variety generated by a finite $\mathcal{MV}$-chain. More generally we may consider the variety $\mathcal{BL}$ of BL-lattices (see Di Nola and Niederkorn [25, Section 2]); $\mathcal{BL}$ is a subvariety of the variety of residuated lattices and its members model Hájek's Basic Logic. The finite $\mathcal{BL}$-chains are ordinal sums of finite $\mathcal{MV}$-chains, and are directed, by [25, Lemma 2.1]. Hence we may apply Theorem 6.1 to the prevariety generated by all finite $\mathcal{BL}$-chains. Each single chain is dualised by its endomorphisms and partial endomorphisms (see [25, Theorems 2.6 and 4.1] and [9, Theorem 6.4.2]).

We now move on from consideration of varieties which fail to be finitely generated and discuss briefly finitely generated (quasi)varieties which fail to be dualisable.

A very important technique in duality theory for proving that a quasivariety $\mathbb{ISP}(\mathbf{\mathcal{M}})$, where $\mathbf{\mathcal{M}}$ is finite, is non-dualisable is the Ghost Element Method (see [9, Section 10.5] and [63, 1.4.6 and Section 3.4]). The underlying idea is to manufacture continuous maps which lie in the second dual of an algebra $\mathbf{A}$, but which cannot be evaluations. We remark that such ghost elements witnessing non-dualisability do lie in the cloud of elements in $n_{\mathcal{A}}(\mathbf{A})\backslash e_{\mathbf{A}}(\mathbf{A})$.

By way of illustration we present an example of the natural extension of an algebra in a non-dualisable quasivariety.

Example 6.3. Let $\mathfrak{I}$ be the variety of implication algebras. Then $\mathfrak{I}=\mathbb{ISP}(\mathbf{I})$. Here $\mathbf{I}=\langle\{0,1\};\to\rangle$ is the two-element implication algebra: $x\to y=0$ if and only if $x=1$ and $y=0$. It was the first non-dualisable algebra to be discovered by Davey and Werner ([24] and [9, 10.5.4]). The implication algebra they used [24] to witness non-dualisability was the algebra $\mathbf{A}^*$ consisting of the *non-empty* finite or cofinite subsets of $\mathbb{N}$. They showed that the double dual $\mathrm{ED}(\mathbf{A}^*)$, that is, the algebra consisting of all continuous maps from $\mathfrak{I}(\mathbf{A}^*,\mathbf{I})$ to $\{0,1\}$ that are locally evaluations, has a bottom and so can't be isomorphic to $\mathbf{A}^*$. We now show, with very little extra work, that $\mathrm{ED}(\mathbf{A}^*)$ is isomorphic to the algebra $\mathbf{A}$ of all finite or cofinite subsets of $\mathbb{N}$, and that the natural extension of $\mathbf{A}^*$ is isomorphic to the powerset algebra $\wp(\mathbb{N})$.

Since congruences on an implication algebra correspond to filters, we see easily that

$$\mathfrak{I}(\mathbf{A}^*,\mathbf{I})=\{\,x_n\mid n\in\mathbb{N}\,\}\cup\{z,\underline{1}\},$$

where $x_n:=\chi_{\uparrow\{n\}}$ is the characteristic function of the principal ultrafilter, $z:=\chi_{\mathcal{C}}$ is the characteristic function of the ultrafilter $\mathcal{C}$ of cofinite subsets of $\mathbb{N}$, and $\underline{1}$ is the constant map onto $\{1\}$. Topologically, $\mathfrak{I}(\mathbf{A}^*,\mathbf{I})$ is the one-point compactification of a countably infinite discrete space with z as the compactification point. For each $a \in A^*$, the evaluation map $e_{\mathbf{A}}(a)$ is given by

$$e_{\mathbf{A}}(a)(y) = \begin{cases} 1, & \text{if } y = \underline{1} \text{ or } (y = x_n \text{ and } n \in a) \text{ or } (y = z \text{ and } a \text{ is cofinite}), \\ 0, & \text{if } (y = x_n \text{ and } n \notin a) \text{ or } (y = z \text{ and } a \text{ is finite}). \end{cases}$$

A map from $\mathfrak{I}(\mathbf{A}^*, \mathbf{I})$ to $\{0, 1\}$ is locally an evaluation if and only if it maps $\underline{1}$ to 1. Consequently, by Theorem 4.1, $n_{\mathfrak{I}}(\mathbf{A})$ is isomorphic to the powerset algebra $\wp(\mathbb{N})$. The only continuous map from $\mathfrak{I}(\mathbf{A}^*, \mathbf{I})$ to $\{0, 1\}$ that is locally an evaluation but not an evaluation is the map $\hat{0}$ given by $\hat{0}(y) = 1 \Leftrightarrow y = \underline{1}$. It follows that $\mathrm{ED}(\mathbf{A}^*)$ is isomorphic to $\mathbf{A}^*$ with a least element added.

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