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Some applications of higher commutators in Mal'cev algebras

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Abstract. We establish several properties of Bulatov's higher commutator operations in congruence permutable varieties. We use higher commutators to prove that for a finite nilpotent algebra of finite type that is a product of algebras of prime power order and generates a congruence modular variety, affine completeness is a decidable property. Moreover, we show that in such algebras, we can check in polynomial time whether two given polynomial terms induce the same function.

1. Introduction

Many properties of a universal algebra can be seen from its congruence lattice and the binary commutator operation on this lattice. The binary commutator operation for arbitrary universal algebras was introduced in [19] and studied further in [8] and [15]. However, even in a congruence permutable variety, an algebra need not be determined up to polynomial equivalence by its unary polynomial functions, its congruences, and the commutator operation on these congruences. In [4], A. Bulatov generalized the binary commutator operation by introducing n-ary commutator operations for all $n \in \mathbb{N}$ and thereby provided a finer tool to distinguish between polynomially inequivalent algebras.

In this paper, we use these n-ary commutator operations to give a common generalization of the theorems in [9, 6] that on finite nilpotent groups and finite nilpotent rings, it can be checked in polynomial time whether two given terms induce the same function (Theorem 2.2). In [2], it was proved that there is an algorithm that decides whether a given finite nilpotent group is affine complete. Using higher commutators, we can generalize this result to certain finite nilpotent Mal'cev algebras (Theorem 2.1). However, the main contribution of the present paper is to prove several properties of Bulatov's higher commutator operations in congruence permutable varieties. While these properties seem quite natural, our proofs require a rather technical tool that we develop here,

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namely the difference operator on polynomial functions. In Corollary 6.12, we give a description of the higher commutator operations for expanded groups that resembles the description of the binary commutator operation stated in [3, p. 273].

2. Notation and results

A *Mal'cev term (polynomial)* on an algebra $\mathbf{A}$ is a ternary term (polynomial) function m on A that satisfies $m(x, x, y) = m(y, x, x) = y$ for every $x, y \in A$. A *Mal'cev algebra* is an algebra that has a Mal'cev term. An algebra $\mathbf{A}$ is k-*affine complete* if every k-ary congruence preserving function is a polynomial function. An algebra $\mathbf{A}$ is *affine complete* if it is k-affine complete for every $k \geq 1$. We recall that an algebra $\mathbf{A}$ from a congruence modular variety is *nilpotent* if there exists an $n \in \mathbb{N}$ such that $\underbrace{[1, \ldots, [1}_{n}, 1]] = 0$ (see [8, p. 58]), where $[\cdot, \cdot]$ denotes the usual (binary) commutator operation (see [15, p. 252]), $\underbrace{[1, \ldots, [1}_{n}, 1]] = [1, 1]$ for $n = 1$ and $\underbrace{[1, \ldots, [1}_{n}, 1]] = [1, \underbrace{[1, \ldots, [1}_{n-1}, 1]]]$ for $n \geq 2$. We say that an algebra $\mathbf{A}$ from a congruence modular variety is nilpotent of class n if $n \in \mathbb{N}$ is the smallest number such that $\underbrace{[1, \ldots, [1}_{n}, 1]] = 0$. In Section 7 we prove the following theorem.

Theorem 2.1. *There is an algorithm that decides whether a finite nilpotent algebra of finite type that is a product of algebras of prime power order and generates a congruence modular variety is affine complete.*

The *polynomial equivalence problem*, also called the *identity checking problem*, for a Mal'cev algebra $\mathbf{A}$ is the problem of deciding whether the identity $s \approx t$ is satisfied by $\mathbf{A}$ for given polynomial terms s and t of $\mathbf{A}$. Here, a *polynomial term* is a term that is built from variables and the elements of $\mathbf{A}$ using the operation symbols of $\mathbf{A}$. We have the following theorem, which is proved in Section 7.

Theorem 2.2. *The polynomial equivalence problem for a finite nilpotent algebra $\mathbf{A}$ of finite type that is a product of algebras of prime power order and generates a congruence modular variety has polynomial time complexity in the length of the input terms.*

For proving these results, we use the *higher commutator operations* as they were introduced by A. Bulatov in [4]. We will usually denote a tuple (vector) $(x_0, \ldots, x_k)$ by $\mathbf{x}$ and its ith component by x_i or $\mathbf{x}^{(i)}$. For arbitrary tuples $\mathbf{x}, \mathbf{y} \in A^k$ and a congruence α on an algebra $\mathbf{A}$ we write $\mathbf{x} \equiv_\alpha \mathbf{y}$ if $\mathbf{x}^{(i)} \equiv_\alpha \mathbf{y}^{(i)}$ for every $i \in \{0, \ldots, k-1\}$. Also, for $f\colon A^{k+m+n} \to A$ and tuples $\mathbf{x} = (x_0, \ldots, x_{k-1}) \in A^k$, $\mathbf{y} = (y_0, \ldots, y_{m-1}) \in A^m$ and $\mathbf{z} = (z_0, \ldots, z_{n-1}) \in A^n$, we write $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ instead of $f(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{m-1}, z_0, \ldots, z_{n-1})$. For

$k \geq 0$, we denote the set of all k-ary polynomials on an algebra $\mathbf{A}$ by $\mathsf{Pol}_k\, \mathbf{A}$, and $\mathsf{Pol}\, \mathbf{A} := \bigcup_{k\geq 0} \mathsf{Pol}_k\, \mathbf{A}$.

3. Higher commutators

In this section, we investigate the higher commutators that were introduced by A. Bulatov in [4]. These higher commutators are a generalization of the term condition commutator widely used in universal algebra [15].

Definition 3.1 (cf. [4])**.** Let $\mathbf{A}$ be an algebra, let $n \in \mathbb{N}$, and let $\alpha_1, \ldots, \alpha_n, \beta, \delta$ be congruences of $\mathbf{A}$. Then we say that $\alpha_1, \ldots, \alpha_n$ *centralize* β *modulo* δ if for all polynomials $f(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y})$ and vectors $\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_n, \mathbf{b}_n, \mathbf{c}, \mathbf{d}$ from $\mathbf{A}$ satisfying

(1) $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for all $i \in \{1, 2, \ldots, n\}$,
(2) $\mathbf{c} \equiv_\beta \mathbf{d}$, and
(3) $f(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{c}) \equiv_\delta f(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{d})$ for all $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \{\mathbf{a}_1, \mathbf{b}_1\} \times \cdots \times \{\mathbf{a}_n, \mathbf{b}_n\} \backslash \{(\mathbf{b}_1, \ldots, \mathbf{b}_n)\}$

we have

$$f(\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{c}) \equiv_\delta f(\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{d}).$$

We abbreviate this property by $C(\alpha_1, \ldots, \alpha_n, \beta; \delta)$.

It follows immediately from the definition that for congruences $\alpha_1, \ldots, \alpha_n, \beta$, $\{\delta_i \mid i \in I\}$, we have: if $C(\alpha_1, \ldots, \alpha_n, \beta; \delta_i)$ for each $i \in I$, then

$$C(\alpha_1, \ldots, \alpha_n, \beta; \bigwedge_{i\in I} \delta_i).$$

This justifies the following definition.

Definition 3.2 (cf. [4])**.** Let $\mathbf{A}$ be an algebra, let $n \geq 2$, and let $\alpha_1, \ldots, \alpha_n$ be congruences of $\mathbf{A}$. The smallest congruence δ such that $C(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n; \delta)$ holds is called the *(n-ary) commutator of* $\alpha_1, \ldots, \alpha_n$. We abbreviate it by $[\alpha_1, \ldots, \alpha_n]$.

Notice that for $n = 1$ in Definition 3.1 we obtain the definition of the (binary) centralizing relation that is used in [8]. In [1, Proposition 2.1] a proof is stated that the centralizing relation defined in [8] is the same as the centralizing relation of [15]. For $n = 2$, Definition 3.2 yields the binary term-condition commutator ([15, Definition 4.150]).

Let $k \geq 1$ and let $\alpha_0, \ldots, \alpha_k, \eta$ be congruences of an algebra $\mathbf{A}$. The following properties can be shown directly from the definition of higher commutators and are stated in [4, Proposition 1]:

(HC1) $[\alpha_0, \ldots, \alpha_k] \leq \bigwedge_{0\leq i\leq k} \alpha_i$;
(HC2) for all $\beta_0, \ldots, \beta_k \in \mathsf{Con}\, \mathbf{A}$ such that $\alpha_0 \leq \beta_0, \ldots, \alpha_k \leq \beta_k$, we have

$$[\alpha_0, \ldots, \alpha_k] \leq [\beta_0, \ldots, \beta_k];$$

(HC3) $[\alpha_0, \ldots, \alpha_k] \leq [\alpha_1, \ldots, \alpha_k]$.

Furthermore, we will show that if $\mathbf{A}$ generates a congruence permutable variety, we have:

(HC4) $[\alpha_0, \dots, \alpha_k] = [\alpha_{\pi(0)}, \dots, \alpha_{\pi(k)}]$ for each permutation π of $\{0, \dots, k\}$;

(HC5) $[\alpha_0, \dots, \alpha_k] \leq \eta$ if and only if $C(\alpha_0, \dots, \alpha_k; \eta)$;

(HC6) if $\eta \leq \alpha_0, \dots, \alpha_k$, then in $\mathbf{A}/\eta$, we have

$$[\alpha_0/\eta, \dots, \alpha_k/\eta] = ([\alpha_0, \dots, \alpha_k] \vee \eta)/\eta;$$

(HC7) if I is a nonempty set, $j \in \{0, \dots, k\}$, and $\{\rho_i \mid i \in I\} \subseteq \mathsf{Con}\,\mathbf{A}$, then

$$\bigvee_{i \in I} [\alpha_0, \dots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \dots, \alpha_k] = [\alpha_0, \dots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \dots, \alpha_k];$$

(HC8) $[\alpha_0, [\alpha_1, \dots, \alpha_k]] \leq [\alpha_0, \alpha_1, \dots, \alpha_k]$, and more generally

$$[\alpha_0, \dots, \alpha_{i-1}, [\alpha_i, \dots, \alpha_k]] \leq [\alpha_0, \dots, \alpha_k]$$

for all $i \in \{1, \dots, k\}$.

The proofs of properties (HC4)–(HC8) are given in Section 6. Actually, for $k = 1$, we obtain as a special case several properties of the binary commutator operation on Mal'cev algebras that have been listed in [15, Exercises 4.156(1), (11), (13)] and [3, Proposition 2.3].

We notice that the higher commutator operations of an algebra are not determined by its binary commutator operation. As examples, we consider the expansions of the cyclic group $(\mathbb{Z}_4, +)$ that were studied in [5]. For $n \geq 2$, let $\mathbf{A}_n$ be the algebra $(\mathbb{Z}_4, +, f_n)$, where f_n is the n-ary operation defined by $f_n(x_1, \dots, x_n) := 2x_1 \cdots x_n$. $\mathbf{A}_n$ has exactly three congruences; we denote them by 0, α, and 1. Then from Lemma 2.4 of [3], one can easily infer that for $n \geq 2$, $\mathbf{A}_n$ satisfies $[1, 1] = \alpha$ and $[1, \alpha] = 0$. Furthermore, in $\mathbf{A}_2$ we have $[1, 1, 1] = 0$, but in $\mathbf{A}_3$, we have $[1, 1, 1] = \alpha$. The property $[1, 1, 1]_{\mathbf{A}_2} = 0$ can be proved by observing that all ternary polynomial functions of $\mathbf{A}_2$ are of the form $(x, y, z) \mapsto a_0 + a_1 x + a_2 y + a_3 z + 2a_4 xy + 2a_5 xz + 2a_6 yz$ with $a_0, \dots, a_6 \in \mathbb{Z}_4$. Now one can use Corollary 6.12 to show that $[1, 1, 1]_{\mathbf{A}_2} = 0$. The property $[1, 1, 1]_{\mathbf{A}_3} = \alpha$ is easier to show: Since $[1, 1, 1]_{\mathbf{A}_3} \leq [1, 1]_{\mathbf{A}_3}$ by (HC3), we have $[1, 1, 1]_{\mathbf{A}_3} \neq 1$. Now we show $[1, 1, 1]_{\mathbf{A}_3} \neq 0$. Seeking a contradiction, we assume $C(1, 1, 1; 0)$. Since $f_3(\alpha_0, \alpha_1, 0) = f_3(\alpha_0, \alpha_1, 3)$ for all $(\alpha_0, \alpha_1) \in \{(0, 0), (0, 3), (3, 0)\}$, $C(1, 1, 1; 0)$ yields $f_3(3, 3, 0) = f_3(3, 3, 3)$, a contradiction. Thus $[1, 1, 1]_{\mathbf{A}_3} = \alpha$.

Similarly, if $k \geq 2$ and $n \geq 2$, then one obtains

$$[1, [1, 1]]_{\mathbf{A}_n} = 0, \quad [\underbrace{1, \dots, 1}_{k}]_{\mathbf{A}_n} = \alpha \text{ if } k \leq n, \text{ and } [\underbrace{1, \dots, 1}_{k}]_{\mathbf{A}_n} = 0 \text{ if } k > n.$$

The higher commutator operation is particularly interesting for algebras that have a group reduct; we call an algebra $\mathbf{V}$ an *expanded group* iff it has the operation symbols $+$ (binary), $-$ (unary) and 0 (nullary) and its reduct $(V, +, -, 0)$ is a group. When we speak about the Mal'cev term of $\mathbf{V}$, we mean the operation $m(x, y, z) := x - y + z$, despite the fact that other ternary term functions satisfying $m(x, x, y) = m(y, x, x) = y$ may exist.

In order to keep the paper self-contained, we shall now list a few additional properties of the binary commutator that are proved in [1]. The following lemma is a special case of [14, Theorem 3.8 (iii)].

Lemma 3.3. *Let $k \in \mathbb{N}$, let $\mathbf{A}$ be an algebra with a Mal'cev term m, let $\alpha, \beta \in \mathsf{Con}\,\mathbf{A}$ and let $p \in \mathsf{Pol}_k\,\mathbf{A}$. If $[\alpha, \beta] = 0$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A^k$ such that $\mathbf{a} \equiv_\alpha \mathbf{b} \equiv_\beta \mathbf{c}$, then we have*

$$m(p(\mathbf{a}), p(\mathbf{b}), p(\mathbf{c})) = p(m(\mathbf{a}^{(1)}, \mathbf{b}^{(1)}, \mathbf{c}^{(1)}), \ldots, m(\mathbf{a}^{(k)}, \mathbf{b}^{(k)}, \mathbf{c}^{(k)})).$$

Proof. The statement can be obtained directly from [1, Proposition 2.6]. □

Lemma 3.4. *Let $\mathbf{A}$ be an algebra with a Mal'cev term m, let $\alpha, \beta \in \mathsf{Con}\,\mathbf{A}$ and let $o, a, b \in A$. If $[\alpha, \beta] = 0$ and $a \equiv_\alpha b \equiv_\beta o$, then we have*

$$m(m(a, b, o), o, b) = a.$$

Proof. Using Lemma 3.3 we obtain

$$\begin{aligned} m(m(a, b, o), o, b) &= m(m(a, b, o), m(b, b, o), m(b, o, o)) \\ &= m(m(a, b, b), m(b, b, o), m(o, o, o)) = m(a, o, o) = a. \end{aligned}$$ □

Lemma 3.5. *Let $\mathbf{A}$ be an algebra with a Mal'cev term m, let $\alpha, \beta \in \mathsf{Con}\,\mathbf{A}$ and let $o, a, b \in A$. If $[\alpha, \beta] = 0$ and $a \equiv_\alpha b \equiv_\beta o$, then we have*

$$m(a, b, o) = m(a, o, m(o, b, o)).$$

Proof. Using Lemma 3.3 we obtain

$$\begin{aligned} m(a, b, o) &= m(m(a, o, o), m(o, o, b), m(o, o, o)) \\ &= m(m(a, o, o), m(o, o, o), m(o, b, o)) = m(a, o, m(o, b, o)). \end{aligned}$$ □

4. The difference operator

The main tool for proving the properties of higher commutators will be the difference operator D defined in this section.

Definition 4.1. Let $\mathbf{A}$ be an algebra. Then for each $k \in \mathbb{N}_0$, $i \in \{0, 1, \ldots, k\}$, and $y \in A$, we define a mapping $\mathsf{E}_y^{(i)} \colon \mathsf{Pol}_{k+1}\,\mathbf{A} \to \mathsf{Pol}_k\,\mathbf{A}$ by

$$\mathsf{E}_y^{(i)}(p)\,(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) := p(x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$$

for all $p \in \mathsf{Pol}_{k+1}\,\mathbf{A}$ and $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \in A$.

Example 4.2. Let $p(x_0, x_1, x_2) = x_0 x_1^2 x_2$ be a polynomial of the ring $\mathbb{Z}_8$. Then $\mathsf{E}_5^{(1)}(p)(x_0, x_2) = x_0 5^2 x_2 = x_0 x_2$.

Definition 4.3. For a Mal'cev algebra $\mathbf{A}$ with a Mal'cev term m, for $o \in A$, for $k \in \mathbb{N}$ and for a vector $(a_0, \dots, a_k) \in A^{k+1}$, we define a mapping $\mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})} \colon \mathsf{Pol}_k\, \mathbf{A} \to \mathsf{Pol}_k\, \mathbf{A}$ by

$$\mathsf{D}^{(1)}_{o,a_0}(f)(x_0) := m(f(x_0), f(a_0), o)$$

for every $f \in \mathsf{Pol}_1\, \mathbf{A}$, $x_0 \in A$ and

$$\mathsf{D}^{(k+1)}_{o,(a_0,\dots,a_k)}(p)(x_0,\dots,x_k) := m\begin{pmatrix} \mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\big(\mathsf{E}^{(k)}_{x_k}(p)\big)(x_0,\dots,x_{k-1}) \\ \mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\big(\mathsf{E}^{(k)}_{a_k}(p)\big)(x_0,\dots,x_{k-1}) \\ o \end{pmatrix}$$

for every $p \in \mathsf{Pol}_{k+1}\, \mathbf{A}$, $x_0, \dots, x_k \in A$.

Note that the definition of D depends on the Mal'cev term m, which will always be clear from the context. Also, we use $m\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$ instead of $m(a, b, c)$ if this improves readability.

Example 4.4. Let p be a ternary polynomial of an expanded group $(V, +, -, 0, F)$. For $o = 0$ and $a_0, a_1, a_2 \in V$ we obtain:

$$\begin{aligned}
&\mathsf{D}^{(3)}_{0,(a_0,a_1,a_2)}(p)(x_0,x_1,x_2) \\
&= \mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{x_2}(p)\big)(x_0,x_1) - \mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{a_2}(p)\big)(x_0,x_1) \\
&= \mathsf{D}^{(1)}_{0,a_0}\big(\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{x_2}(p))\big)(x_0) - \mathsf{D}^{(1)}_{0,a_0}\big(\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{x_2}(p))\big)(x_0) \\
&\qquad - \Big(\mathsf{D}^{(1)}_{0,a_0}\big(\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{a_2}(p))\big)(x_0) - \mathsf{D}^{(1)}_{0,a_0}\big(\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{a_2}(p))\big)(x_0)\Big) \\
&= \Big((\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{x_2}(p)))(x_0) - (\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{x_2}(p)))(a_0)\Big) \\
&\qquad - \Big((\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{x_2}(p)))(x_0) - (\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{x_2}(p)))(a_0)\Big) \\
&\qquad\quad + \Big((\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{a_2}(p)))(x_0) - (\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{a_2}(p)))(a_0)\Big) \\
&\qquad\qquad - \Big((\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{a_2}(p)))(x_0) - (\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{a_2}(p)))(a_0)\Big) \\
&= (p(x_0,x_1,x_2) - p(a_0,x_1,x_2)) - (p(x_0,a_1,x_2) - p(a_0,a_1,x_2)) \\
&\qquad + (p(x_0,a_1,a_2) - p(a_0,a_1,a_2)) - (p(x_0,x_1,a_2) - p(a_0,x_1,a_2)) \\
&= p(x_0,x_1,x_2) - p(a_0,x_1,x_2) + p(a_0,a_1,x_2) - p(x_0,a_1,x_2) \\
&\qquad + p(x_0,a_1,a_2) - p(a_0,a_1,a_2) + p(a_0,x_1,a_2) - p(x_0,x_1,a_2).
\end{aligned}$$

Lemma 4.5. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, $o \in A$, $k \in \mathbb{N}$, and let α be a congruence of $\mathbf{A}$. Let $q \in \mathsf{Pol}_k\, \mathbf{A}$ such that $q(A^k) \subseteq o/\alpha$. Then for every $(x_0, \dots, x_{k-1}) \in A^k$, we have $\mathsf{D}^{(k)}_{o,(o,\dots,o)}(q)(x_0,\dots,x_{k-1}) \in o/\alpha$.*

Proof. We show the statement by induction on k. For $k = 1$ and $x_0 \in A$, we have $q(x_0) \in o/\alpha$ and $q(o) \in o/\alpha$, and thus $\mathsf{D}^{(1)}_{o,o}(q)(x_0) = m(q(x_0), q(o), o) \equiv_\alpha m(o, o, o) = o \in o/\alpha$. Now let $k \geq 2$ and $x_0, \dots, x_{k-1} \in A$. Since $q(A^k) \subseteq o/\alpha$, we clearly have $\mathsf{E}^{(k-1)}_{x_{k-1}}(q)(A^{k-1}) \subseteq o/\alpha$ and $\mathsf{E}^{(k-1)}_{o}(q)(A^{k-1}) \subseteq o/\alpha$. Thus, by the induction hypothesis, both $a := \mathsf{D}^{(k-1)}_{o,(o,\dots,o)}(\mathsf{E}^{(k-1)}_{x_{k-1}}(q))(x_0,\dots,x_{k-2})$ and $b := \mathsf{D}^{(k-1)}_{o,(o,\dots,o)}(\mathsf{E}^{(k-1)}_{o}(q))(x_0,\dots,x_{k-2})$ lie in o/α. Hence we have $m(a,b,o) \in o/\alpha$. This completes the proof. □

Definition 4.6. Let $\mathbf{A}$ be an algebra with a Mal'cev term m and let $k \in \mathbb{N}$. Then we define a mapping $\mathsf{F}_{o,u}\colon \mathsf{Pol}_k\,\mathbf{A} \to \mathsf{Pol}_k\,\mathbf{A}$ by

$$\mathsf{F}_{o,u}(p)(x_0,\dots,x_{k-1}) := m(p(x_0,\dots,x_{k-2},x_{k-1}), p(x_0,\dots,x_{k-2},u), o)$$

for all $p \in \mathsf{Pol}_k\,\mathbf{A}$ and $x_0,\dots,x_{k-1},o,u \in A$.

Example 4.7. Let p be a ternary polynomial of an expanded group $(V,+,-,0,F)$. For $o = 0$ and $u \in V$ we obtain:

$$\mathsf{F}_{0,u}(p)(x_0,x_1,x_2) = p(x_0,x_1,x_2) - p(x_0,x_1,u).$$

We need the following technical lemma.

Lemma 4.8. *Let $\mathbf{A}$ be an algebra with a Mal'cev term m and $k \in \mathbb{N}$. Then for all $q \in \mathsf{Pol}_{k+1}\,\mathbf{A}$ and $x_0,\dots,x_{k-2},o,t,u,v \in A$, we have*

$$\mathsf{E}^{(k-1)}_t(\mathsf{E}^{(k)}_v(\mathsf{F}_{o,u}(q)))(x_0,\dots,x_{k-2}) = \mathsf{E}^{(k-1)}_v(\mathsf{F}_{o,u}(\mathsf{E}^{(k-1)}_t(q)))(x_0,\dots,x_{k-2}).$$

Proof. Let $q \in \mathsf{Pol}_{k+1}\,\mathbf{A}$. We calculate the left-hand side:

$$\begin{aligned}
&\mathsf{E}^{(k-1)}_t(\mathsf{E}^{(k)}_v(\mathsf{F}_{o,u}(q)))(x_0,\dots,x_{k-2})\\
&\quad= \mathsf{E}^{(k)}_v(\mathsf{F}_{o,u}(q))(x_0,\dots,x_{k-2},t) = \mathsf{F}_{o,u}(q)(x_0,\dots,x_{k-2},t,v)\\
&\qquad= m(q(x_0,\dots,x_{k-2},t,v), q(x_0,\dots,x_{k-2},t,u), o).
\end{aligned}$$

Now, we compute the right-hand side:

$$\begin{aligned}
&\mathsf{E}^{(k-1)}_v(\mathsf{F}_{o,u}(\mathsf{E}^{(k-1)}_t(q)))(x_0,\dots,x_{k-2}) = \mathsf{F}_{o,u}(\mathsf{E}^{(k-1)}_t(q))(x_0,\dots,x_{k-2},v)\\
&\quad= m(\mathsf{E}^{(k-1)}_t(q)(x_0,\dots,x_{k-2},v), \mathsf{E}^{(k-1)}_t(q)(x_0,\dots,x_{k-2},u), o)\\
&\qquad= m(q(x_0,\dots,x_{k-2},t,v), q(x_0,\dots,x_{k-2},t,u), o).
\end{aligned}$$
□

Definition 4.9. Let $\mathbf{A}$ be an algebra. Now let $k \in \mathbb{N}$, let $p\colon A^k \to A$, let $(a_0,\dots,a_{k-1}) \in A^k$, and let $o \in A$. Then p is *absorbing at* $(a_0,\dots,a_{k-1})$ *with value* o if for all $(x_0,\dots,x_{k-1}) \in A^k$ we have: if there is an $i \in \{0,1,\dots,k-1\}$ with $x_i = a_i$, then $p(x_0,\dots,x_{k-1}) = p(a_0,\dots,a_{k-1})$, and $p(a_0,\dots,a_{k-1}) = o$.

Example 4.10. In a ring $\mathbf{R}$, for $a,b,c,d \in R$, the function $f(x,y,z) := (x-a)(y-b)(z-c)+d$ is absorbing at (a,b,c) with value d.

Lemma 4.11. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $k \geq 1$, let $(a_0,\dots,a_{k-1}) \in A^k$ and let $o \in A$. If $q \in \mathsf{Pol}_k\,\mathbf{A}$, then $\mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}(q)$ is absorbing at $(a_0,\dots,a_{k-1})$ with value o.*

Proof. We prove the statement by induction on k. Using Definition 4.3, we see that $\mathsf{D}^{(1)}_{o,a_0}(q)(a_0) = m(q(a_0),q(a_0),o) = o$. For the induction step, let $k \geq 2$,

$(x_0, \ldots, x_{k-1}) \in A^k$ such that there is an $i \in \{0, 1, \ldots, k-1\}$ with $x_i = a_i$, and let $q \in \mathsf{Pol}_k\, \mathbf{A}$. We want to prove

$$\mathsf{D}^{(k)}_{o,(a_0,\ldots,a_{k-1})}(q)(x_0, \ldots, x_{k-1}) = o. \tag{4.1}$$

If $x_{k-1} = a_{k-1}$, then we obtain (4.1) directly from Definition 4.3. If there exists an $i \in \{0, \ldots, k-2\}$ such that $x_i = a_i$, then we reason as follows: Since $\mathsf{E}^{(k-1)}_{x_{k-1}}(q), \mathsf{E}^{(k-1)}_{a_{k-1}}(q) \in \mathsf{Pol}_{k-1}\, \mathbf{A}$, we have

$$\begin{aligned}&\mathsf{D}^{(k-1)}_{o,(a_0,\ldots,a_{k-2})}(\mathsf{E}^{(k-1)}_{x_{k-1}}(q))(x_0, \ldots, x_{k-2})\\ &\qquad = \mathsf{D}^{(k-1)}_{o,(a_0,\ldots,a_{k-2})}(\mathsf{E}^{(k-1)}_{a_{k-1}}(q))(x_0, \ldots, x_{k-2}) = o\end{aligned}$$

by the induction hypothesis. Now, by Definition 4.3 we obtain (4.1). □

Example 4.12. Let p be a ternary polynomial of an expanded group $(V, +, -, 0, F)$. We already calculated in Example 4.4:

$$\begin{aligned}&\mathsf{D}^{(3)}_{0,(a_0,a_1,a_2)}(p)(x_0, x_1, x_2)\\ &\quad = p(x_0, x_1, x_2) - p(a_0, x_1, x_2) + p(a_0, a_1, x_2) - p(x_0, a_1, x_2)\\ &\qquad + p(x_0, a_1, a_2) - p(a_0, a_1, a_2) + p(a_0, x_1, a_2) - p(x_0, x_1, a_2)\end{aligned}$$

and hence,

$$\begin{aligned}\mathsf{D}^{(3)}_{0,(a_0,a_1,a_2)}(p)(a_0, x_1, x_2) &= \mathsf{D}^{(3)}_{0,(a_0,a_1,a_2)}(p)(x_0, a_1, x_2)\\ &= \mathsf{D}^{(3)}_{0,(a_0,a_1,a_2)}(p)(x_0, x_1, a_2) = 0.\end{aligned}$$

Definition 4.13. Let $\mathbf{A}$ be an algebra and let $o \in A$. For each $n \in \mathbb{N}$ and $I \subseteq \{0, \ldots, n-1\}$, we define a function $S^{(n)}_{I,o} \colon A^n \to A^n$ by

$$S^{(n)}_{I,o}(x_0, \ldots, x_{n-1}) := (y_0, \ldots, y_{n-1})$$

for all $x_0, \ldots, x_{n-1} \in A$, where $y_j := x_j$ if $j \in I$, and $y_j := o$ if $j \notin I$.

In the previous definition, all entries whose indices are not listed in I are replaced with o.

Definition 4.14. Let $\mathbf{A}$ be an algebra and let $o \in A$. For each $n \in \mathbb{N}$ and $I \subseteq \{0, \ldots, n-1\}$, we define a function $H^{(n)}_{I,o} \colon \mathsf{Pol}_n\, \mathbf{A} \to \mathsf{Pol}_n\, \mathbf{A}$ by

$$(H^{(n)}_{I,o}(p))(x_0, \ldots, x_{n-1}) := p(S^{(n)}_{I,o}(x_0, \ldots, x_{n-1}))$$

for all $(x_0, \ldots, x_{n-1}) \in A^n$.

Example 4.15. For $a_0, a_1, a_2 \in A$ we have $S^{(3)}_{\{0,2\},o}(a_0, a_1, a_2) = (a_0, o, a_2)$ and $H^{(3)}_{\{0,2\},o}(p)(a_0, a_1, a_2) = p(a_0, o, a_2)$.

Proposition 4.16. *Let $\mathbf{A}$ be an algebra, let $o \in A$, let $\alpha \in \mathsf{Con}\, \mathbf{A}$ and let $k \geq 1$. If $p \in \mathsf{Pol}_k\, \mathbf{A}$ such that $p(A^k) \subseteq o/\alpha$, then $H^{(k)}_{I,o}(p)(A^k) \subseteq o/\alpha$ for every $I \subseteq \{0, \ldots, k-1\}$.*

Proof. Obviously, we have $H^{(k)}_{I,o}(p)(A^k) = p(S^{(k)}_{I,o}(A^k)) \subseteq p(A^k)$. □

In a Mal'cev algebra $\mathbf{A}$ with a Mal'cev term m and $o \in A$ we define a binary polynomial $+_o$ and a unary polynomial $-_o$ by

$$a +_o b := m(a, o, b) \quad \text{and} \quad -_o(a) := m(o, a, o)$$

for all $a, b \in A$. We abbreviate $((a_1 +_o a_2) +_o \cdots +_o a_{n-1}) +_o a_n$ by ${}_o\sum_{i=1}^{n} a_i$. By $(-1)^k \cdot a$, we mean $-_o(a)$ if k is odd, and a if k is even.

Lemma 4.17. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $o \in A$, $k \in \mathbb{N}$, $\alpha \in \mathsf{Con}\,\mathbf{A}$ and $p \in \mathsf{Pol}_k\,\mathbf{A}$ such that $[\alpha, \alpha] = 0$ and $p(A^k) \subseteq o/\alpha$. Then*

1. *$(o/\alpha, +_o, -_o, o)$ is an abelian group and $m(a, b, o) = a +_o (-_o(b))$ for all $a, b \in o/\alpha$;*
2. *there exists a bijective mapping $\varphi_k \colon \{1, \ldots, 2^k\} \to \mathcal{P}(\{0, \ldots, k-1\})$ such that $\varphi_k(1) = \{0, \ldots, k-1\}$ and*

$$\mathsf{D}^{(k)}_{o,(o,\ldots,o)}(p) = {}_o\sum_{i=1}^{2^k} (-1)^{k-|\varphi_k(i)|} \cdot H^{(k)}_{\varphi_k(i),o}(p).$$

Proof. (1) Since $a +_o b \in o/\alpha$ and $-_o(a) \in o/\alpha$ for all $a, b \in o/\alpha$, we know that $(o/\alpha, +_o, -_o, o)$ is an abelian group by Lemma 3.3 and the calculations in [15, p. 256]. By Lemma 3.5 we obtain $m(a, b, o) = m(a, o, m(o, b, o)) = a +_o (-_o(b))$ for all $a, b \in o/\alpha$.

(2) We proceed by induction on k. For $k = 1$ we define $\varphi_1(1) := \{0\}$ and $\varphi_1(2) := \emptyset$. Then, by Definition 4.3 and (1), for $x_0 \in A$, we have

$$\begin{aligned}\mathsf{D}^{(1)}_{o,o}(p)(x_0) &= m(p(x_0), p(o), o) = p(x_0) +_o (-_o(p(o))) \\ &= {}_o\sum_{i=1}^{2} (-1)^{1-|\varphi_1(i)|} \cdot H^{(1)}_{\varphi_1(i),o}(p)(x_0).\end{aligned}$$

For $k > 1$ we define $\varphi_k \colon \{1, \ldots, 2^k\} \to \mathcal{P}(\{0, \ldots, k-1\})$ by $\varphi_k(i) := \varphi_{k-1}(i) \cup \{k-1\}$ and $\varphi_k(2^k + 1 - i) := \varphi_{k-1}(i)$ for $i \in \{1, \ldots, 2^{k-1}\}$. Now, let $(x_0, \ldots, x_{k-1}) \in A^k$. Using item (1) we compute

$$\begin{aligned}\mathsf{D}^{(k)}_{o,(o,\ldots,o)}(p)(x_0, \ldots, x_{k-1}) &= \mathsf{D}^{(k-1)}_{o,(o,\ldots,o)}\big(\mathsf{E}^{(k-1)}_{x_{k-1}}(p)\big)(x_0, \ldots, x_{k-2}) \\ &\quad +_o \Big(-_o\Big(\mathsf{D}^{(k-1)}_{o,(o,\ldots,o)}\big(\mathsf{E}^{(k-1)}_{o}(p)\big)(x_0, \ldots, x_{k-2})\Big)\Big). \quad (4.2)\end{aligned}$$

Since $\mathsf{E}^{(k-1)}_{x_{k-1}}(p)(A^{k-1}) \subseteq o/\alpha$ and $\mathsf{E}^{(k-1)}_{o}(p)(A^{k-1}) \subseteq o/\alpha$, we may use the induction hypothesis and obtain that the last expression is equal to

$$\begin{aligned}&{}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-1-|\varphi_{k-1}(i)|} \cdot \Big(H^{(k-1)}_{\varphi_{k-1}(i),o}(\mathsf{E}^{(k-1)}_{x_{k-1}}(p))\Big)(x_0, \ldots, x_{k-2}) \\ &\quad +_o \Big(-_o\Big({}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-1-|\varphi_{k-1}(i)|} \Big(H^{(k-1)}_{\varphi_{k-1}(i),o}(\mathsf{E}^{(k-1)}_{o}(p))\Big)(x_0, \ldots, x_{k-2})\Big)\Big).\end{aligned}$$

The last expression is equal to

$$\begin{aligned}
&{}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-1-|\varphi_{k-1}(i)|} \cdot \left(\mathsf{E}^{(k-1)}_{x_{k-1}}(p)\right)\left(S^{(k-1)}_{\varphi_{k-1}(i),o}(x_0,\dots,x_{k-2})\right) \\
&\qquad +_o \left({}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-|\varphi_{k-1}(i)|} \cdot \left(\mathsf{E}^{(k-1)}_{o}(p)\right)\left(S^{(k-1)}_{\varphi_{k-1}(i),o}(x_0,\dots,x_{k-2})\right)\right) \\
&= {}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-1-|\varphi_{k-1}(i)|} \cdot p(S^{(k-1)}_{\varphi_{k-1}(i),o}(x_0,\dots,x_{k-2}),x_{k-1}) \\
&\qquad +_o \left({}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-|\varphi_{k-1}(i)|} \cdot p(S^{(k-1)}_{\varphi_{k-1}(i),o}(x_0,\dots,x_{k-2}),o)\right) \\
&= {}_o\sum_{i=1}^{2^{k-1}} (-1)^{k-|\varphi_k(i)|} \cdot p(S^{(k)}_{\varphi_k(i),o}(x_0,\dots,x_{k-1})) \\
&\qquad +_o \left({}_o\sum_{i=2^{k-1}+1}^{2^{k}} (-1)^{k-|\varphi_k(i)|} \cdot p(S^{(k)}_{\varphi_k(i),o}(x_0\dots,x_{k-1}))\right) \\
&= {}_o\sum_{i=1}^{2^{k}} (-1)^{k-|\varphi_k(i)|} \cdot H^{(k)}_{\varphi_k(i),o}(p)(x_0,\dots,x_{k-1}). \qquad \square
\end{aligned}$$

In an algebra $\mathbf{A}$ we say that a function $p\colon A^n \to A$ *depends on its ith argument* if there are $a, b \in A$ and $(x_1,\dots,x_n) \in A^n$ such that

$$p(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_n) \neq p(x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_n).$$

The number of arguments on which p depends is called the *essential arity* of p.

Definition 4.18. Let $\mathbf{A}$ be an algebra. For $n \in \mathbb{N}$ and $o \in A$ we call a polynomial $p \in \mathsf{Pol}_n\,\mathbf{A}$ an *o-polynomial* if for each $i \in \{1,\dots,n\}$ at least one of the following two conditions holds:

(1) p does not depend on its ith argument;
(2) $p(x_1,\dots,x_{i-1},o,x_{i+1},\dots,x_n) = p(o,\dots,o)$ for all $(x_1,\dots,x_n) \in A^n$.

Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m and let $o \in A$. Then, for every $k \geq 0$, $\mathbf{P}(A,k,m,o) := (\mathsf{Pol}_k\,\mathbf{A}, m, \overline{o})$ is an algebra of type $(3,0)$ with Mal'cev operation m and a constant polynomial $\overline{o} \in \mathsf{Pol}_k\,\mathbf{A}$. For a nonempty set P of polynomials of $\mathbf{A}$, we denote the subuniverse of $\mathbf{P}(A,k,m,o)$ generated by P by $\mathrm{Sg}^{\mathbf{P}(A,k,m,o)}(P)$.

Proposition 4.19. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $o \in A$, $n \in \mathbb{N}$, $\alpha \in \mathsf{Con}\,\mathbf{A}$ and $p \in \mathsf{Pol}_n\,\mathbf{A}$. If $[\alpha,\alpha] = 0$ and $p(A^n) \subseteq o/\alpha$, then $p \in \mathrm{Sg}^{\mathbf{P}(A,n,m,o)}(P)$, where $P := \{f \in \mathsf{Pol}_n\,\mathbf{A} \mid f$ is an o-polynomial and the essential arity of f is at most the essential arity of $p\}$.*

Proof. Let $k \in \mathbb{N}$ be the essential arity of the polynomial p. Then there exists a polynomial $q \in \mathsf{Pol}_k\,\mathbf{A}$ such that the essential arity of q is k and

$p(x_0, \dots, x_{n-1}) = q(x_{i_0}, \dots, x_{i_{k-1}})$ for all $(x_0, \dots, x_{n-1}) \in A^n$. To simplify the notation let us denote the arguments of q by $x_0, \dots, x_{k-1}$. We prove the statement of the proposition by induction on k. For $k = 1$ the statement is obvious because every unary polynomial function is an o-polynomial function of essential arity at most 1. Now let $k \geq 2$. Since $[\alpha, \alpha] = 0$ and $q(A^k) \subseteq o/\alpha$, Lemma 4.17 yields a bijective mapping $\varphi_k\colon \{1, \dots, 2^k\} \to \mathcal{P}(\{0, \dots, k-1\})$ such that $\varphi_k(1) = \{0, \dots, k-1\}$ and

$$\mathsf{D}^{(k)}_{o,(o,\dots,o)}(q) = {}_o\sum_{i=1}^{2^k} (-1)^{k-|\varphi_k(i)|} \cdot H^{(k)}_{\varphi_k(i),o}(q). \tag{4.3}$$

We observe that

$$(-1)^{k-|\varphi_k(1)|} \cdot H^{(k)}_{\varphi_k(1),o}(q)(x_0, \dots, x_{k-1}) = q(x_0, \dots, x_{k-1})$$

and hence we obtain

$$q = \mathsf{D}^{(k)}_{o,(o,\dots,o)}(q) +_o (-_o \left({}_o\sum_{i=2}^{2^k} (-1)^{k-|\varphi_k(i)|} \cdot H^{(k)}_{\varphi_k(i),o}(q) \right))$$

from (4.3). Therefore

$$q \in \mathrm{Sg}^{\mathbf{P}(A,k,m,o)}(\{\mathsf{D}^{(k)}_{o,(o,\dots,o)}(q)\} \cup \{H^{(k)}_{\varphi_k(i),o}(q) \mid 2 \leq i \leq 2^k\}).$$

Now, by Lemma 4.11, we know that $\mathsf{D}^{(k)}_{o,(o,\dots,o)}(q)$ is a k-ary o-polynomial. Obviously, its essential arity is at most k. Furthermore, for all $i \in \{2, 3, \dots, 2^{k-1}\}$, the polynomials $H^{(k)}_{\varphi_k(i),o}(q)$ depend on at most $k-1$ arguments and, by Proposition 4.16, $H^{(k)}_{\varphi_k(i),o}(q)(A^k) \subseteq o/\alpha$. Hence by the induction hypothesis $H^{(k)}_{\varphi_k(i),o}(q) \in \mathrm{Sg}^{\mathbf{P}(A,k-1,m,o)}(P)$ for $i \in \{2, 3, \dots, 2^{k-1}\}$, where P is the set of o-polynomials of essential arities at most $k-1$. Therefore,

$$q \in \mathrm{Sg}^{\mathbf{P}(A,k,m,o)}(\{\mathsf{D}^{(k)}_{o,(o,\dots,o)}(q)\} \cup P).$$

This completes the induction step. □

Lemma 4.20. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m *and let* $k \geq 1$. *Let* $q \in \mathsf{Pol}_{k+1} \mathbf{A}$, $(a_0, \dots, a_{k-1}) \in A^k$, $o, u \in A$. *Let* $f \in \mathsf{Pol}_{k+1} \mathbf{A}$ *be defined by*

$$f(x_0, \dots, x_k) := \mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}(\mathsf{E}^{(k)}_{x_k}(\mathsf{F}_{o,u}(q)))(x_0, \dots, x_{k-1})$$

for all $x_0, \dots, x_k \in A$. *Then* f *is absorbing at* $(a_0, \dots, a_{k-1}, u)$ *with value* o.

Example 4.21. Let $k \geq 1$ and $q \in \mathsf{Pol}_{k+1} \mathbf{A}$ with $(y_0, \dots, y_k) \mapsto q(y_0, \dots, y_k)$ for all $(y_0, \dots, y_k) \in A^{k+1}$. If $o, u, x_k \in A$, then we have $\mathsf{F}_{o,u}(q)\colon A^{k+1} \to A$,

$$(y_0, \dots, y_k) \mapsto m(q(y_0, \dots, y_k), q(y_0, \dots, y_{k-1}, u), o),$$

and $\mathsf{E}^{(k)}_{x_k}(\mathsf{F}_{o,u}(q))\colon A^k \to A$,

$$(y_0, \dots, y_{k-1}) \mapsto m(q(y_0, \dots, y_{k-1}, x_k), q(y_0, \dots, y_{k-1}, u), o),$$

for $(y_0, \ldots, y_{k-1}) \in A^k$. To obtain $f(x_0, \ldots, x_{k-1}, x_k)$ for given $x_0, \ldots, x_{k-1} \in A$, the operator $\mathsf{D}^{(k)}_{o,(a_0,\ldots,a_{k-1})}$ is applied to the k-ary function $\mathsf{E}^{(k)}_{x_k}(\mathsf{F}_{o,u}(q))$, and the resulting function is evaluated at $(x_0, \ldots, x_{k-1})$.

Proof of Lemma 4.20. Let $(x_0, \ldots, x_k) \in A^{k+1}$ such that $x_i = a_i$ for an $i \in \{0, 1, \ldots, k-1\}$ or $x_k = u$. We first consider the case $x_k = u$. By the definitions of the operators E and F we know that $\mathsf{E}^{(k)}_u(\mathsf{F}_{o,u}(q))(y_0, \ldots, y_{k-1}) = o$ for all $(y_0, \ldots, y_{k-1}) \in A^k$. Then, $f(x_0, \ldots, x_k) = o$ because the operator D, acting on a constant function, produces the constant function with value o. In the case that there is an i with $a_i = x_i$, the assertion follows from Lemma 4.11. □

Example 4.22. In this example we start with a ternary polynomial p of an expanded group $(V, +, -, 0, F)$. Since we have already computed $\mathsf{F}_{0,u}(p)$ in Example 4.7, we have

$$\begin{aligned}
&\mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p))\big)(x_0, x_1)\\
&= \mathsf{D}^{(1)}_{0,a_0}\Big(\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p)))\Big)(x_0) - \mathsf{D}^{(1)}_{0,a_0}\Big(\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p)))\Big)(x_0)\\
&= \Big(\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p)))\Big)(x_0) - \Big(\mathsf{E}^{(1)}_{x_1}(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p)))\Big)(a_0)\\
&\qquad - \bigg(\Big(\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p)))\Big)(x_0) - \Big(\mathsf{E}^{(1)}_{a_1}(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p)))\Big)(a_0)\bigg)\\
&= (p(x_0, x_1, x_2) - p(x_0, x_1, u)) - (p(a_0, x_1, x_2) - p(a_0, x_1, u))\\
&\qquad - \big((p(x_0, a_1, x_2) - p(x_0, a_1, u)) - (p(a_0, a_1, x_2) - p(a_0, a_1, u))\big)\\
&= p(x_0, x_1, x_2) - p(x_0, x_1, u) + p(a_0, x_1, u) - p(a_0, x_1, x_2)\\
&\qquad + p(a_0, a_1, x_2) - p(a_0, a_1, u) + p(x_0, a_1, u) - p(x_0, a_1, x_2).
\end{aligned}$$

Clearly,

$$\begin{aligned}
\mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{u}(\mathsf{F}_{0,u}(p))\big)(x_0, x_1) &= \mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p))\big)(a_0, x_1)\\
&= \mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p))\big)(x_0, a_1) = 0.
\end{aligned}$$

Lemma 4.23. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*, let* $o \in A$*, and let* $\eta \in \mathsf{Con}\,\mathbf{A}$*. If* $k \geq 1$*,* $(a_0, \ldots, a_{k-1})$*,* $(b_0, \ldots, b_{k-1}) \in A^k$*,* $u, v \in A$ *and* $q \in \mathsf{Pol}_{k+1}\,\mathbf{A}$ *such that*

$$q(\alpha_0, \ldots, \alpha_{k-1}, u) \equiv_\eta q(\alpha_0, \ldots, \alpha_{k-1}, v)$$

for every $(\alpha_0, \ldots, \alpha_{k-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$*, then*

$$\mathsf{D}^{(k)}_{o,(a_0,\ldots,a_{k-1})}\big(\mathsf{E}^{(k)}_v(\mathsf{F}_{o,u}(q))\big)(b_0, \ldots, b_{k-1}) \equiv_\eta o.$$

Proof. We prove the statement by induction on k. For $k = 1$, we have

$$\begin{aligned}
&\mathsf{D}^{(1)}_{o,a_0}\big(\mathsf{E}^{(1)}_v(\mathsf{F}_{o,u}(q))\big)(b_0)\\
&\qquad = m(m(q(b_0, v), q(b_0, u), o), m(q(a_0, v), q(a_0, u), o), o) \equiv_\eta o,
\end{aligned}$$

using the assumptions on q. For $k \geq 2$ let $q \in \mathsf{Pol}_{k+2}\,\mathbf{A}$ such that

$$q(\alpha_0, \dots, \alpha_k, u) \equiv_\eta q(\alpha_0, \dots, \alpha_k, v)$$

for every $(\alpha_0, \dots, \alpha_k) \in \{a_0, b_0\} \times \cdots \times \{a_k, b_k\}$. Now we divide all possible choices of $(\alpha_0, \dots, \alpha_k)$ in two groups: $\{(\alpha_0, \dots, \alpha_k) \in \{a_0, b_0\} \times \cdots \times \{a_k, b_k\} \mid \alpha_k = a_k\}$ and $\{(\alpha_0, \dots, \alpha_k) \in \{a_0, b_0\} \times \cdots \times \{a_k, b_k\} \mid \alpha_k = b_k\}$. Hence we have

$$\mathsf{E}^{(k)}_{a_k}(q)(\alpha_0, \dots, \alpha_{k-1}, u) \equiv_\eta \mathsf{E}^{(k)}_{a_k}(q)(\alpha_0, \dots, \alpha_{k-1}, v)$$

and

$$\mathsf{E}^{(k)}_{b_k}(q)(\alpha_0, \dots, \alpha_{k-1}, u) \equiv_\eta \mathsf{E}^{(k)}_{b_k}(q)(\alpha_0, \dots, \alpha_{k-1}, v)$$

for every $(\alpha_0, \dots, \alpha_{k-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$. By the induction hypothesis we obtain

$$\mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\Big(\mathsf{E}^{(k)}_v(\mathsf{F}_{o,u}(\mathsf{E}^{(k)}_{a_k}(q)))\Big)(b_0, \dots, b_{k-1}) \equiv_\eta o,$$

and then by Lemma 4.8 we have

$$\mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\Big(\mathsf{E}^{(k)}_{a_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{o,u}(q)))\Big)(b_0, \dots, b_{k-1}) \equiv_\eta o, \tag{4.4}$$

and in the same way we have

$$\mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\Big(\mathsf{E}^{(k)}_{b_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{o,u}(q)))\Big)(b_0, \dots, b_{k-1}) \equiv_\eta o. \tag{4.5}$$

Now, using equations (4.4) and (4.5) and the definition of the operator D for $p = \mathsf{E}^{(k+1)}_v(\mathsf{F}_{o,u}(q))$, we obtain

$$\begin{aligned}
&\mathsf{D}^{(k+1)}_{o,(a_0,\dots,a_k)}\big(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{o,u}(q))\big)(b_0, \dots, b_k) \\
&\quad = m\begin{pmatrix} \mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\big(\mathsf{E}^{(k)}_{b_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{o,u}(p)))\big)(b_0, \dots, b_{k-1}) \\ \mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\big(\mathsf{E}^{(k)}_{a_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{o,u}(p)))\big)(b_0, \dots, b_{k-1}) \\ o \end{pmatrix} \equiv_\eta o. \quad \square
\end{aligned}$$

Example 4.24. Let $\mathbf{V} = (V, +, -, 0, F)$ be an expanded group and choose $a_0, a_1, u, b_0, b_1, v \in V$, $o = 0$. The relation η is the equality relation on V. Now, let p be a polynomial of $\mathbf{V}$ such that $p(a_0, a_1, u) = p(a_0, a_1, v)$, $p(a_0, b_1, u) = p(a_0, b_1, v)$, $p(b_0, a_1, u) = p(b_0, a_1, v)$, $p(b_0, b_1, u) = p(b_0, b_1, v)$. In Example 4.22, we have already calculated $\mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p))\big)(x_0, x_1)$, and thus $\mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_v(\mathsf{F}_{0,u}(p))\big)(b_0, b_1) = 0$.

Lemma 4.25. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $\eta \in \mathsf{Con}\,\mathbf{A}$ and let $k \geq 1$. If $(a_0, \dots, a_{k-1})$, $(b_0, \dots, b_{k-1})$ are vectors in A, $u, v \in A$ and $q \in \mathsf{Pol}_{k+1}\,\mathbf{A}$ such that*

$$q(\alpha_0, \dots, \alpha_{k-1}, u) \equiv_\eta q(\alpha_0, \dots, \alpha_{k-1}, v)$$

for every $(\alpha_0, \dots, \alpha_{k-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{k-1}, b_{k-1}\} \backslash \{(b_0, \dots, b_{k-1})\}$, then for $o = q(b_0, \dots, b_{k-1}, u)$ we have

$$\mathsf{D}^{(k)}_{o,(a_0,\dots,a_{k-1})}\big(\mathsf{E}^{(k)}_v(\mathsf{F}_{o,u}(q))\big)(b_0, \dots, b_{k-1}) \equiv_\eta q(b_0, \dots, b_{k-1}, v).$$

Proof. By induction on k. For $k = 1$, by the assumption $q(a_0, u) \equiv_\eta q(a_0, v)$, we have

$$\begin{aligned}
&\mathsf{D}^{(1)}_{q(b_0,u),a_0}\big(\mathsf{E}^{(1)}_v(\mathsf{F}_{q(b_0,u),u}(q)\big)(b_0)\\
&= m\big(m(q(b_0,v), q(b_0,u), q(b_0,u)), m(q(a_0,v), q(a_0,u), q(b_0,u)), q(b_0,u)\big)\\
&\qquad = m\big(q(b_0,v), m\big(q(a_0,v), q(a_0,u), q(b_0,u)\big), q(b_0,u)\big) \equiv_\eta q(b_0,v).
\end{aligned}$$

For the induction step we let $k \geq 2$. We will now compute

$$\mathsf{D}^{(k+1)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_k)}\Big(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(q))\Big)(b_0,\ldots,b_k).$$

According to Definition 4.3 we have to compute

$$\mathsf{D}^{(k)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_{k-1})}\Big(\mathsf{E}^{(k)}_{a_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(q))\Big)(b_0,\ldots,b_{k-1})$$

and

$$\mathsf{D}^{(k)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_{k-1})}\Big(\mathsf{E}^{(k)}_{b_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(q))\Big)(b_0,\ldots,b_{k-1}).$$

We assume that

$$q(\alpha_0,\ldots,\alpha_k,u) \equiv_\eta q(\alpha_0,\ldots,\alpha_k,v),$$

for every $(\alpha_0,\ldots,\alpha_k) \in \{a_0,b_0\} \times \cdots \times \{a_k,b_k\} \backslash \{(b_0,\ldots,b_k)\}$. Using Definition 4.1 we obtain $\mathsf{E}^{(k)}_{a_k}(q)(\alpha_0,\ldots,\alpha_{k-1},u) \equiv_\eta \mathsf{E}^{(k)}_{a_k}(q)(\alpha_0,\ldots,\alpha_{k-1},v)$ for all $(\alpha_0,\ldots,\alpha_{k-1}) \in \{a_0,b_0\} \times \cdots \times \{a_{k-1},b_{k-1}\}$. Thus, by Lemma 4.8 and Lemma 4.23 (for $o := q(b_0,\ldots,b_k,u)$ and $\mathsf{E}^{(k)}_{a_k}(q)$), we have

$$\begin{aligned}
&\mathsf{D}^{(k)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_{k-1})}\Big(\mathsf{E}^{(k)}_{a_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(q))\Big)(b_0,\ldots,b_{k-1})\\
&\quad = \mathsf{D}^{(k)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_{k-1})}\Big(\mathsf{E}^{(k)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(\mathsf{E}^{(k)}_{a_k}(q)))\Big)(b_0,\ldots,b_{k-1})\\
&\qquad\qquad \equiv_\eta q(b_0,\ldots,b_k,u). \qquad (4.6)
\end{aligned}$$

From the assumptions we know that

$$\mathsf{E}^{(k)}_{b_k}(q)(\alpha_0,\ldots,\alpha_{k-1},u) \equiv_\eta \mathsf{E}^{(k)}_{b_k}(q)(\alpha_0,\ldots,\alpha_{k-1},v)$$

for all $(\alpha_0,\ldots,\alpha_{k-1}) \in \{a_0,b_0\} \times \cdots \times \{a_{k-1},b_{k-1}\} \backslash \{(b_0,\ldots,b_{k-1})\}$. By Lemma 4.8 and the induction hypothesis (for $o := \mathsf{E}^{(k)}_{b_k}(q)(b_0,\ldots,b_{k-1},u)$ and $\mathsf{E}^{(k)}_{b_k}(q)$) we obtain

$$\begin{aligned}
&\mathsf{D}^{(k)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_{k-1})}\Big(\mathsf{E}^{(k)}_{b_k}(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(q))\Big)(b_0,\ldots,b_{k-1})\\
&= \mathsf{D}^{(k)}_{\mathsf{E}^{(k)}_{b_k}(q)(b_0,\ldots,b_{k-1},u),(a_0,\ldots,a_{k-1})}\Big(\mathsf{E}^{(k)}_v(\mathsf{F}_{\mathsf{E}^{(k)}_{b_k}(q)(b_0,\ldots,b_{k-1},u),u}(\mathsf{E}^{(k)}_{b_k}(q)))\Big)(b_0,\ldots\\
&\qquad \ldots,b_{k-1}) \equiv_\eta \mathsf{E}^{(k)}_{b_k}(q)(b_0,\ldots,b_{k-1},v) = q(b_0,\ldots,b_k,v). \qquad (4.7)
\end{aligned}$$

Now using Definition 4.3 and the congruences (4.6) and (4.7), we compute

$$\mathsf{D}^{(k+1)}_{q(b_0,\ldots,b_k,u),(a_0,\ldots,a_k)}\left(\mathsf{E}^{(k+1)}_v(\mathsf{F}_{q(b_0,\ldots,b_k,u),u}(q))\right)(b_0,\ldots,b_k)$$
$$\equiv_\eta m\begin{pmatrix} q(b_0,\ldots,b_k,v) \\ q(b_0,\ldots,b_k,u) \\ q(b_0,\ldots,b_k,u)\end{pmatrix} = q(b_0,\ldots,b_k,v). \qquad \square$$

Example 4.26. Let $\mathbf{V} = (V,+,-,0,F)$ be an expanded group and choose $a_0, a_1, u, b_0, b_1, v \in V$, $o = 0$. The relation η is the equality relation on V. Now let p be a polynomial of $\mathbf{V}$ such that $p(a_0,a_1,u) = p(a_0,a_1,v)$, $p(a_0,b_1,u) = p(a_0,b_1,v)$, $p(b_0,a_1,u) = p(b_0,a_1,v)$ and $p(b_0,b_1,u) = 0$. In Example 4.22, we have already calculated $\mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{x_2}(\mathsf{F}_{0,u}(p))\big)(x_0,x_1)$, and thus obtain $\mathsf{D}^{(2)}_{0,(a_0,a_1)}\big(\mathsf{E}^{(2)}_{v}(\mathsf{F}_{0,u}(p))\big)(b_0,b_1) = p(b_0,b_1,v)$.

Remark 4.27. Definitions 4.1, 4.3 and 4.6 and Lemmas 4.11, 4.20 and 4.25 can be formulated and proved analogously for arbitrary vectors, not just elements of the algebra. As an illustration we give the analogon of Lemma 4.20.

Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $k \geq 1$, and let $n_0,\ldots,n_k \in \mathbb{N}$. Let $q \in \mathsf{Pol}_{n_0+\cdots+n_k}\mathbf{A}$, let $\mathbf{a}_i \in A^{n_i}$ for each $i \in \{0,1,\ldots,k-1\}$, let $\mathbf{u} \in A^{n_k}$, and let $o \in A$. Let $f \in \mathsf{Pol}_{n_0+\cdots+n_k}\mathbf{A}$ be defined by

$$f(\mathbf{x}_0,\ldots,\mathbf{x}_k) := \mathsf{D}^{(k)}_{o,(\mathbf{a}_0,\ldots,\mathbf{a}_{k-1})}\big(\mathsf{E}^{(k)}_{\mathbf{x}_k}(\mathsf{F}_{o,\mathbf{u}}(q))\big)(\mathbf{x}_0,\ldots,\mathbf{x}_{k-1})$$

for all $\mathbf{x}_0 \in A^{n_0},\ldots,\mathbf{x}_k \in A^{n_k}$. Then f is absorbing at $(\mathbf{a}_0,\ldots,\mathbf{a}_{k-1},\mathbf{u})$ with value o.

5. Some properties of the centralizing relation

In Bulatov's definition of the n-ary commutator operation $[\cdot,\cdot,\ldots,\cdot]$, polynomials of arbitrary arity are used. We will now show that in Mal'cev algebras, n-ary polynomials are enough. For the binary case, this is well known and has been stated in [15]; proofs can be found in [1].

Definition 5.1. Let $\mathbf{A}$ be an algebra, $n_0,\ldots,n_k \in \mathbb{N}$, $k \geq 0$ and let $\alpha_0,\ldots,\alpha_k$, η be arbitrary congruences of $\mathbf{A}$. Then we say that $C(n_0,\ldots,n_k;\alpha_0,\ldots,\alpha_k;\eta)$ holds if for all polynomials $p \in \mathsf{Pol}_{n_0+\cdots+n_k}\mathbf{A}$ and vectors $\mathbf{a}_0,\mathbf{b}_0 \in A^{n_0}$, $\ldots$, $\mathbf{a}_{k-1},\mathbf{b}_{k-1} \in A^{n_{k-1}}$, $\mathbf{u},\mathbf{v} \in A^{n_k}$ that satisfy

(1) $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for all $i \in \{0,1,\ldots,k-1\}$,
(2) $\mathbf{u} \equiv_{\alpha_k} \mathbf{v}$,
(3) $p(\mathbf{x}_0,\ldots,\mathbf{x}_{k-1},\mathbf{u}) \equiv_\eta p(\mathbf{x}_0,\ldots,\mathbf{x}_{k-1},\mathbf{v})$ for all $(\mathbf{x}_0,\ldots,\mathbf{x}_{k-1}) \in \{\mathbf{a}_0,\mathbf{b}_0\}\times\cdots\times\{\mathbf{a}_{k-1},\mathbf{b}_{k-1}\}\setminus\{(\mathbf{b}_0,\ldots,\mathbf{b}_{k-1})\}$,

we have

$$p(\mathbf{b}_0,\ldots,\mathbf{b}_{k-1},\mathbf{u}) \equiv_\eta p(\mathbf{b}_0,\ldots,\mathbf{b}_{k-1},\mathbf{v}).$$

Lemma 5.2. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*, let* $k \geq 0$*, let* $n_0, \ldots, n_k, n_0', \ldots, n_k' \in \mathbb{N}$ *and let* $\alpha_0, \alpha_1, \ldots, \alpha_k, \eta$ *be arbitrary congruences of* $\mathbf{A}$*. Then*

(a) *if* $C(n_0, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$ *and* $n_0' \leq n_0, \ldots, n_k' \leq n_k$*, then* $C(n_0', \ldots, n_k'; \alpha_0, \ldots, \alpha_k; \eta)$*;*

(b) *if* $C(1, n_1, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$ *and* $n_0 \geq 1$*, then* $C(n_0, n_1, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$*.*

Proof. (a) follows from the fact that every $(n_0' + \cdots + n_k')$-ary polynomial function can be seen as an $(n_0 + \cdots + n_k)$-ary polynomial function that depends on the same arguments as the polynomial function we have started with.

In order to prove (b), we assume that $C(1, n_1, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$ holds and we show by induction that $C(n_0, n_1, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$ holds for all $n_0 \geq 1$. Let $p \in \mathsf{Pol}_{(n_0+1)+n_1+\cdots+n_k} \mathbf{A}$. Furthermore, take any $a, c \in A$, $\mathbf{b}, \mathbf{d} \in A^{n_0}$, $\mathbf{e}_i, \mathbf{f}_i \in A^{n_i}$, $i \in \{1, \ldots, k-1\}$, and $\mathbf{u}, \mathbf{v} \in A^{n_k}$ such that

(1) $a \equiv_{\alpha_0} c$,
(2) $\mathbf{b} \equiv_{\alpha_0} \mathbf{d}$,
(3) $\mathbf{e}_i \equiv_{\alpha_i} \mathbf{f}_i$ for all $i \in \{1, \ldots, k-1\}$,
(4) $\mathbf{u} \equiv_{\alpha_k} \mathbf{v}$,
(5) $p(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{u}) \equiv_\eta p(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{v})$ for all $(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}) \in \{(a, \mathbf{b}), (c, \mathbf{d})\} \times \{\mathbf{e}_1, \mathbf{f}_1\} \times \cdots \times \{\mathbf{e}_{k-1}, \mathbf{f}_{k-1}\} \backslash \{((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1})\}$.

We want to show that

$$p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \equiv_\eta p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}).$$

Now we define the polynomial $h \in \mathsf{Pol}_{(n_0+1)+n_1+\cdots+n_k} \mathbf{A}$ such that

$$h(\mathbf{x}_0, \ldots, \mathbf{x}_k) := \mathsf{D}^{(k)}_{p((c,\mathbf{d}),\mathbf{f}_1,\ldots,\mathbf{f}_{k-1},\mathbf{u}),((a,\mathbf{b}),\mathbf{e}_1,\ldots,\mathbf{e}_{k-1})} \Big(\mathsf{E}^{(k)}_{\mathbf{x}_k} (\mathsf{F}_{p((c,\mathbf{d}),\mathbf{f}_1,\ldots,\mathbf{f}_{k-1},\mathbf{u}),\mathbf{u}}(p)) \Big) (\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}).$$

We have

$$\begin{aligned} h((a, \mathbf{b}), \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{u}) &= p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \\ &= h((a, \mathbf{b}), \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{v}), \end{aligned} \tag{5.1}$$

for all $(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) \in \{\mathbf{e}_1, \mathbf{f}_1\} \times \cdots \times \{\mathbf{e}_{k-1}, \mathbf{f}_{k-1}\}$. This can be obtained from the analogon of Lemma 4.20 for vectors, by setting $o = p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u})$, $a_i = \mathbf{e}_i$, $b_i = \mathbf{f}_i$, for $i \in \{1, \ldots, k-1\}$, $a_0 = (a, \mathbf{b})$ and $b_0 = (c, \mathbf{d})$. In the same way we obtain

$$\begin{aligned} h((a, \mathbf{d}), \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{u}) &= p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \\ &= h((a, \mathbf{d}), \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{v}), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} h((c, \mathbf{d}), \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{u}) &= p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \\ &= h((c, \mathbf{d}), \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{v}), \end{aligned} \tag{5.3}$$

for all $(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) \in \{\mathbf{e}_1, \mathbf{f}_1\} \times \cdots \times \{\mathbf{e}_{k-1}, \mathbf{f}_{k-1}\} \backslash \{(\mathbf{f}_1, \ldots, \mathbf{f}_{k-1})\}$. Now, we define a polynomial $q \in \mathsf{Pol}_{n_0+n_1+\cdots+n_k} \mathbf{A}$ by

$$q(\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_k) := h((a, \mathbf{y}), \mathbf{x}_1, \ldots, \mathbf{x}_k).$$

Obviously, by (5.1) and (5.2) we have

$$q(\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{u}) \equiv_\eta q(\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{v})$$

for $(\mathbf{y}, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) \in \{\mathbf{b}, \mathbf{d}\} \times \{\mathbf{e}_1, \mathbf{f}_1\} \times \cdots \times \{\mathbf{e}_{k-1}, \mathbf{f}_{k-1}\} \backslash \{(\mathbf{d}, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1})\}$. From the induction hypothesis we obtain

$$q(\mathbf{d}, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \equiv_\eta q(\mathbf{d}, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}),$$

or in other words

$$h((a, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \equiv_\eta h((a, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}). \tag{5.4}$$

If we introduce

$$s(x, \mathbf{x}_1, \ldots, \mathbf{x}_k) := h((x, \mathbf{d}), \mathbf{x}_1, \ldots, \mathbf{x}_k),$$

where $s \in \mathsf{Pol}_{1+n_1+\cdots+n_k} \mathbf{A}$, then (5.2), (5.3) and (5.4) yield

$$s(x, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{u}) \equiv_\eta s(x, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{v}),$$

for all $(x, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) \in \{a, c\} \times \{\mathbf{e}_1, \mathbf{f}_1\} \times \cdots \times \{\mathbf{e}_{k-1}, \mathbf{f}_{k-1}\} \backslash \{(c, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1})\}$. Using the assumption $C(1, n_0, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$ we conclude

$$s(c, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \equiv_\eta s(c, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}),$$

or in other words

$$h((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) \equiv_\eta h((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}).$$

We know from the analogon of Lemma 4.20 for vectors, where $a_i = \mathbf{e}_i$, $b_i = \mathbf{f}_i$, $1 \leq i \leq k-1$, $a_0 = (a, \mathbf{b})$, $b_0 = (c, \mathbf{d})$ and $o = p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u})$, that

$$h((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u}) = p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{u})$$

and, for the same parameters, from the analagon of Lemma 4.25 that

$$h((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}) \equiv_\eta p((c, \mathbf{d}), \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, \mathbf{v}).$$

This completes the induction step. □

Lemma 5.3. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*,* $\alpha_0, \ldots, \alpha_k$*,* $\eta \in \mathsf{Con}\,\mathbf{A}$*,* $k \geq 0$*,* $n_0, \ldots, n_k \in \mathbb{N}$*, and let* π *be a permutation of* $\{0, \ldots, k\}$*. Then if* $C(n_0, \ldots, n_k; \alpha_0, \ldots, \alpha_k; \eta)$*, we have*

$$C(n_{\pi(0)}, \ldots, n_{\pi(k)}; \alpha_{\pi(0)}, \ldots, \alpha_{\pi(k)}; \eta).$$

Proof. Since every permutation of $\{0, \ldots, k\}$ is generated by the transpositions, it suffices to consider the following two cases.

Case (i): $\pi = (i\ j)$, where $i, j \neq k$. Without loss of generality we can assume that $\pi = (0\ 1)$. Choose $p \in \mathsf{Pol}_{n_1+n_0+n_2\cdots+n_k} \mathbf{A}$, $\mathbf{a}_0, \mathbf{b}_0 \in A^{n_1}$, $\mathbf{a}_1, \mathbf{b}_1 \in A^{n_0}$, $\mathbf{a}_i, \mathbf{b}_i \in A^{n_i}$, $i \in \{2, \ldots, k-1\}$, and $\mathbf{u}, \mathbf{v} \in A^{n_k}$ so that

(1) $\mathbf{a}_0 \equiv_{\alpha_1} \mathbf{b}_0$,

(2) $\mathbf{a}_1 \equiv_{\alpha_0} \mathbf{b}_1$,
(3) $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for $i \in \{2, \dots, k-1\}$,
(4) $\mathbf{u} \equiv_{\alpha_k} \mathbf{v}$,
(5) $p(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}, \mathbf{u}) \equiv_\eta p(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}, \mathbf{v})$ for all $(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}) \in \{\mathbf{a}_0, \mathbf{b}_0\} \times \cdots \times \{\mathbf{a}_{k-1}, \mathbf{b}_{k-1}\} \backslash \{(\mathbf{b}_0, \dots, \mathbf{b}_{k-1})\}$.

Next, consider the polynomial $q \in \mathsf{Pol}_{n_0+\cdots+n_k} \mathbf{A}$ defined as follows:

$$q(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}, \mathbf{t}) := p(\mathbf{x}_1, \mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_{k-1}, \mathbf{t}).$$

Now, we have

$$q(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}, \mathbf{u}) \equiv_\eta q(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}, \mathbf{v})$$

for all $(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}) \in \{\mathbf{a}_1, \mathbf{b}_1\} \times \{\mathbf{a}_0, \mathbf{b}_0\} \times \{\mathbf{a}_2, \mathbf{b}_2\} \times \cdots \times \{\mathbf{a}_{k-1}, \mathbf{b}_{k-1}\} \setminus \{(\mathbf{b}_1, \mathbf{b}_0, \mathbf{b}_2, \dots, \mathbf{b}_{k-1})\}$. From the assumption $C(n_0, \dots, n_k; \alpha_0, \dots, \alpha_k; \eta)$ we conclude that

$$q(\mathbf{b}_1, \mathbf{b}_0, \mathbf{b}_2, \dots, \mathbf{b}_{k-1}, \mathbf{u}) \equiv_\eta q(\mathbf{b}_1, \mathbf{b}_0, \mathbf{b}_2, \dots, \mathbf{b}_{k-1}, \mathbf{v})$$

and hence, we have $p(\mathbf{b}_0, \dots, \mathbf{b}_{k-1}, \mathbf{u}) \equiv_\eta p(\mathbf{b}_0, \dots, \mathbf{b}_{k-1}, \mathbf{v})$.

Case (ii): $\pi = (i\ j)$, where $i = k$ or $j = k$. Without loss of generality we can assume that $\pi = (0\ k)$. Let $p \in \mathsf{Pol}_{n_k+n_1+\cdots+n_{k-1}+n_0} \mathbf{A}$, $\mathbf{a}_0, \mathbf{b}_0 \in A^{n_k}$, $\mathbf{a}_0 \equiv_{\alpha_k} \mathbf{b}_0$, $\mathbf{a}_i, \mathbf{b}_i \in A^{n_i}$, $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$, $i \in \{1, \dots, k-1\}$, and $\mathbf{u}, \mathbf{v} \in A^{n_0}$ such that $\mathbf{u} \equiv_{\alpha_0} \mathbf{v}$ and

$$p(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}, \mathbf{u}) \equiv_\eta p(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}, \mathbf{v})$$

for all $(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}) \in \{\mathbf{a}_0, \mathbf{b}_0\} \times \cdots \times \{\mathbf{a}_{k-1}, \mathbf{b}_{k-1}\} \backslash \{(\mathbf{b}_0, \dots, \mathbf{b}_{k-1})\}$. Next, we define the polynomial $q \in \mathsf{Pol}_{n_0+\cdots+n_k} \mathbf{A}$ as follows:

$$q(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{t}) := m \begin{pmatrix} p(\mathbf{t}, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{u}) \\ p(\mathbf{t}, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_0) \\ p(\mathbf{b}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_0) \end{pmatrix}.$$

Then we calculate

$$\begin{aligned} q(\mathbf{u}, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{a}_0) &= p(\mathbf{b}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{u}) \\ &= q(\mathbf{u}, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{b}_0), \end{aligned} \tag{5.5}$$

$$q(\mathbf{v}, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{b}_0) = p(\mathbf{b}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{u}), \tag{5.6}$$

and by the assumption

$$p(\mathbf{a}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{v}) \equiv_\eta p(\mathbf{a}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{u}) \tag{5.7}$$

for all $(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}) \in \{\mathbf{a}_1, \mathbf{b}_1\} \times \cdots \times \{\mathbf{a}_{k-1}, \mathbf{b}_{k-1}\}$.

Finally, if $(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}) \neq (\mathbf{b}_1, \dots, \mathbf{b}_{k-1})$, then we have

$$p(\mathbf{b}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{v}) \equiv_\eta p(\mathbf{b}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{u}), \tag{5.8}$$

by the assumption. Thus, using (5.7), (5.8) and (5.6) we obtain

$$\begin{aligned} q(\mathbf{v},\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{a}_0) &= m\begin{pmatrix} p(\mathbf{a}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{u}) \\ p(\mathbf{a}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{v}) \\ p(\mathbf{b}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{v}) \end{pmatrix} \\ &\equiv_\eta p(\mathbf{b}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{v}) \\ &\equiv_\eta p(\mathbf{b}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{u}) \\ &= q(\mathbf{v},\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{b}_0) \end{aligned}$$

for all $(\mathbf{x}_1,\ldots,\mathbf{x}_{k-1}) \in \{\mathbf{a}_1,\mathbf{b}_1\} \times \cdots \times \{\mathbf{a}_{k-1},\mathbf{b}_{k-1}\} \backslash \{(\mathbf{b}_1,\ldots,\mathbf{b}_{k-1})\}$. Together with (5.5) we have

$$q(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{a}_0) \equiv_\eta q(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{k-1},\mathbf{b}_0)$$

for $(\mathbf{x}_0,\ldots,\mathbf{x}_{k-1}) \in \{\mathbf{u},\mathbf{v}\} \times \{\mathbf{a}_1,\mathbf{b}_1\} \times \cdots \times \{\mathbf{a}_{k-1},\mathbf{b}_{k-1}\} \backslash \{(\mathbf{v},\mathbf{b}_1,\ldots,\mathbf{b}_{k-1})\}$ and we can conclude

$$q(\mathbf{v},\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{a}_0) \equiv_\eta q(\mathbf{v},\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{b}_0) \tag{5.9}$$

by the assumption $C(n_0,\ldots,n_k;\alpha_0,\ldots,\alpha_k;\eta)$. Finally, using (5.6), (5.9) and (5.7) we obtain

$$\begin{aligned} p(\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{u}) &= q(\mathbf{v},\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{b}_0) \equiv_\eta q(\mathbf{v},\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{a}_0) \\ &= m\begin{pmatrix} p(\mathbf{a}_0,\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{u}) \\ p(\mathbf{a}_0,\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{v}) \\ p(\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{v}) \end{pmatrix} \equiv_\eta p(\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_{k-1},\mathbf{v}). \end{aligned}$$

This proves the statement. □

Proposition 5.4. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m, *let* $\alpha_0,\ldots,\alpha_k$ *and* η *be congruences of* $\mathbf{A}$ *and* $k \geq 0$. *Then* $C(\alpha_0,\ldots,\alpha_k;\eta)$ *if and only if* $C(1,\ldots,1;\alpha_0,\ldots,\alpha_k;\eta)$.

Proof. If $C(\alpha_0,\ldots,\alpha_k;\eta)$, then clearly from Definition 3.1 we have $C(n_0,\ldots,n_k;\alpha_0,\ldots,\alpha_k;\eta)$ for all $n_0,\ldots,n_k \in \mathbb{N}$, thus for $n_0 = \cdots = n_k = 1$ we obtain $C(1,\ldots,1;\alpha_0,\ldots,\alpha_k;\eta)$. To prove the opposite direction suppose that $C(1,\ldots,1;\alpha_0,\ldots,\alpha_k;\eta)$. Let $n_0,\ldots,n_k \geq 1$. Then by Lemma 5.3 for $\pi = (0\ k)$ we obtain $C(1,\ldots,1;\alpha_k,\alpha_1,\ldots,\alpha_{k-1},\alpha_0;\eta)$ and by Lemma 5.2(b), we obtain $C(n_k,1,\ldots,1;\alpha_k,\alpha_1,\ldots,\alpha_{k-1},\alpha_0;\eta)$. When we apply Lemma 5.3 one more time for $\pi = (0\ k)$ we obtain $C(1,\ldots,1,n_k;\alpha_0,\ldots,\alpha_k;\eta)$. We can repeat the same procedure for each of the places from $k-1$ to 0 and obtain $C(n_0,\ldots,n_k;\alpha_0,\ldots,\alpha_k;\eta)$. Thus we have $C(\alpha_0,\ldots,\alpha_k;\eta)$. □

6. Properties and characterizations of higher commutators

Let $n \in \mathbb{N}$, $n \geq 2$. The aim of this section is to give a necessary and sufficient condition for $[\underbrace{1,\ldots,1}_{n}] \neq 0$ in Mal'cev algebras (Proposition 6.16) and to prove that a polynomial Mal'cev clone on a finite set is finitely generated whenever there exists an $n \in \mathbb{N}$ such that $[\underbrace{1, \ldots, 1}_{n}] = 0$ (Proposition 6.18). Both results will be essential for proving Theorems 2.1 and 2.2 in Section 7 of the paper.

Let $(a, b) \in A^2$. We denote the congruence of $\mathbf{A}$ that is generated by (a, b) by $\Theta_{\mathbf{A}}(a, b)$. A congruence that is generated by a single pair in A^2 is also called a *principal congruence* of $\mathbf{A}$ (see [7, Definition 5.6]).

Proposition 6.1 (HC4)**.** *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*,* $\alpha_0, \ldots, \alpha_k$ *congruences of* $\mathbf{A}$*,* $k \geq 0$ *and* π *a permutation of* $\{0, \ldots, k\}$*. Then*

$$[\alpha_0, \ldots, \alpha_k] = [\alpha_{\pi(0)}, \ldots, \alpha_{\pi(k)}].$$

Proof. From Definition 3.2 we know that $C(\alpha_0, \ldots, \alpha_k; [\alpha_0, \ldots, \alpha_k])$ holds and thus $C(1, \ldots, 1; \alpha_0, \ldots, \alpha_k; [\alpha_0, \ldots, \alpha_k])$ holds by Proposition 5.4. Now, from Lemma 5.3 we obtain $C(1, \ldots, 1; \alpha_{\pi(0)}, \ldots, \alpha_{\pi(k)}; [\alpha_0, \ldots, \alpha_k])$ and therefore $C(\alpha_{\pi(0)}, \ldots, \alpha_{\pi(k)}; [\alpha_0, \ldots, \alpha_k])$, again by Proposition 5.4. Now, by Definition 3.2 we have $[\alpha_{\pi(0)}, \ldots, \alpha_{\pi(k)}] \leq [\alpha_0, \ldots, \alpha_k]$. In order to prove the other inequality we start with $\alpha_{\pi(0)}, \ldots, \alpha_{\pi(k)}$ and the permutation π^{-1}, and reason in the same way. □

Lemma 6.2 (HC5)**.** *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*. Let* $\alpha_0, \ldots, \alpha_k$ *and* η *be arbitrary congruences of* $\mathbf{A}$ $(k \geq 0)$*. Then* $[\alpha_0, \ldots, \alpha_k] \leq \eta$ *if and only if* $C(\alpha_0, \ldots, \alpha_k; \eta)$*.*

Proof. If $C(\alpha_0, \ldots, \alpha_k; \eta)$, then by Definition 3.2, we have $[\alpha_0, \ldots, \alpha_k] \leq \eta$. Now, suppose that $[\alpha_0, \ldots, \alpha_k] \leq \eta$ for $\alpha_0, \ldots, \alpha_k \in \operatorname{\mathsf{Con}} \mathbf{A}$. We will show that $C(1, \ldots, 1; \alpha_0, \ldots, \alpha_k; \eta)$ by Definition 5.1. Choose $p \in \operatorname{\mathsf{Pol}}_{k+1} \mathbf{A}$ and $a_0, \ldots, a_{k-1}, u, b_0, \ldots, b_{k-1}, v \in A$ so that

(1) $a_i \equiv_{\alpha_i} b_i$ for $i \in \{0, \ldots, k-1\}$,
(2) $u \equiv_{\alpha_k} v$,
(3) $p(x_0, \ldots, x_{k-1}, u) \equiv_\eta p(x_0, \ldots, x_{k-1}, v)$ for all $(x_0, \ldots, x_{k-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{k-1}, b_{k-1}\} \backslash \{(b_0, \ldots, b_{k-1})\}$.

We want to show that $p(b_0, \ldots, b_{k-1}, u) \equiv_\eta p(b_0, \ldots, b_{k-1}, v)$. We start by introducing a polynomial $s \in \operatorname{\mathsf{Pol}}_{k+1} \mathbf{A}$ by

$$s(x_0, \ldots, x_k) := \mathsf{D}^{(k)}_{p(b_0, \ldots, b_{k-1}, u), (a_0, \ldots, a_{k-1})} \Big(\mathsf{E}^{(k)}_{x_k} (\mathsf{F}_{p(b_0, \ldots, b_{k-1}, u), u}(p)) \Big) (x_0, \ldots, x_{k-1})$$

where $(x_0, \ldots, x_k) \in A^{k+1}$. Then we observe the following:

$$s(x_0, \ldots, x_{k-1}, u) = p(b_0, \ldots, b_{k-1}, u) = s(x_0, \ldots, x_{k-1}, v),$$

for all $(x_0, \ldots, x_{k-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{k-1}, b_{k-1}\} \backslash \{(b_0, \ldots, b_{k-1})\}$, by Lemma 4.20 and thus

$$s(x_0, \ldots, x_{k-1}, u) \equiv_{[\alpha_0, \ldots, \alpha_k]} s(x_0, \ldots, x_{k-1}, v),$$

for all $(x_0,\dots,x_{k-1}) \in \{a_0,b_0\}\times\cdots\times\{a_{k-1},b_{k-1}\}\backslash\{(b_0,\dots,b_{k-1})\}$. Now, we can conclude

$$s(b_0,\dots,b_{k-1},u) \equiv_{[\alpha_0,\dots,\alpha_k]} s(b_0,\dots,b_{k-1},v),$$

and by the assumption we have

$$s(b_0,\dots,b_{k-1},u) \equiv_\eta s(b_0,\dots,b_{k-1},v).$$

The left side of the last congruence is equal to $p(b_0,\dots,b_{k-1},u)$ by Lemma 4.20 and the right side is congruent modulo η to $p(b_0,\dots,b_{k-1},v)$ by Lemma 4.25. Now, by Proposition 5.4 we obtain $C(\alpha_0,\dots,\alpha_k;\eta)$. □

Recall that for an algebra $\mathbf{A}$ and $\alpha,\beta \in \mathsf{Con}\,\mathbf{A}$ such that $\alpha \geq \beta$, α/β denotes the congruence of the factor algebra $\mathbf{A}/\beta$ which corresponds to α in $\mathbf{A}$.

Corollary 6.3 (HC6)**.** *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $n \geq 2$, and let $\alpha_1,\dots,\alpha_n,\eta \in \mathsf{Con}\,\mathbf{A}$ such that $\eta \leq \alpha_1,\dots,\alpha_n$. Then*

$$[\alpha_1/\eta,\dots,\alpha_n/\eta] = ([\alpha_1,\dots,\alpha_n]\vee\eta)/\eta.$$

Proof. We will show that $([\alpha_1,\dots,\alpha_n]\vee\eta)/\eta$ is the smallest congruence θ/η with the property

$$C(\alpha_1/\eta,\dots,\alpha_n/\eta;\theta/\eta).$$

Directly using Definition 3.2 we can check that for every $\eta \in \mathsf{Con}\,\mathbf{A}$, $\eta \leq \alpha_1,\dots,\alpha_n,\theta$, we have

$$C(\alpha_1,\dots,\alpha_n;\theta) \Leftrightarrow C(\alpha_1/\eta,\dots,\alpha_n/\eta;\theta/\eta). \tag{6.1}$$

Since $[\alpha_1,\dots,\alpha_n] \leq [\alpha_1,\dots,\alpha_n]\vee\eta$ we have $C(\alpha_1,\dots,\alpha_n;[\alpha_1,\dots,\alpha_n]\vee\eta)$ by Lemma 6.2, and thus $C(\alpha_1/\eta,\dots,\alpha_n/\eta;([\alpha_1,\dots,\alpha_n]\vee\eta)/\eta)$ using (6.1) for $\theta = [\alpha_1,\dots,\alpha_n]\vee\eta$. Let us assume now

$$C(\alpha_1/\eta,\dots,\alpha_n/\eta;\theta/\eta),$$

for a $\theta \in \mathsf{Con}\,\mathbf{A}$ such that $\eta \leq \theta$. Then by (6.1) we have $C(\alpha_1,\dots,\alpha_n;\theta)$ and thus $[\alpha_1,\dots,\alpha_n] \leq \theta$ by Lemma 6.2, whence $[\alpha_1,\dots,\alpha_n]\vee\eta \leq \theta$. Using the Correspondence Theorem we have $([\alpha_1,\dots,\alpha_n]\vee\eta)/\eta \leq \theta/\eta$. □

Lemma 6.4. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $\rho_1,\dots,\rho_n$, $\alpha_1,\dots,\alpha_k$ and η be congruences of $\mathbf{A}$ and $k,n \geq 1$. If $C(\rho_i,\alpha_1,\dots,\alpha_k;\eta)$ for every $i \in \{1,\dots,n\}$, then $C(\bigvee_{1\leq i\leq n}\rho_i,\alpha_1,\dots,\alpha_k;\eta)$.*

Proof. We know that $\bigvee_{1\leq i\leq n}\rho_i = \rho_1\circ\cdots\circ\rho_n$, since $\mathbf{A}$ is congruence permutable. We will prove the statement by induction. For $n=1$ the statement is obvious. Let $n \geq 2$. We put $\theta_1 = \rho_1\circ\cdots\circ\rho_{n-1}$ and $\theta_2 = \rho_n$. Now $\bigvee_{1\leq i\leq n}\rho_i = \theta_1\circ\theta_2$. We will prove that $C(1,\dots,1;\theta_1\circ\theta_2,\alpha_1,\dots,\alpha_k;\eta)$ by Definition 5.1. Let $p \in \mathsf{Pol}_{k+1}\,\mathbf{A}$ and choose $a_0,\dots,a_{k-1},u,b_0,\dots,b_{k-1},v \in A$ so that $a_0 \equiv_{\theta_1\circ\theta_2} b_0$, $a_i \equiv_{\alpha_i} b_i$ for $i \in \{1,\dots,k-1\}$, $u \equiv_{\alpha_k} v$ and

$$p(x_0,\dots,x_{k-1},u) \equiv_\eta p(x_0,\dots,x_{k-1},v),$$

for all $(x_0, \ldots, x_{k-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{k-1}, b_{k-1}\} \backslash \{(b_0, \ldots, b_{k-1})\}$. We have to show $p(b_0, \ldots, b_{k-1}, u) \equiv_\eta p(b_0, \ldots, b_{k-1}, v)$. From the assumption $a_0 \equiv_{\theta_1 \circ \theta_2} b_0$ we know that there exists a $c \in A$ such that $a_0 \equiv_{\theta_1} c$ and $c \equiv_{\theta_2} b_0$. We introduce a polynomial $s \in \mathsf{Pol}_{k+1} \mathbf{A}$ as follows:

$$s(x_0, \ldots, x_k) := \\ \mathsf{D}^{(k)}_{p(b_0,\ldots,b_{k-1},u),(a_0,\ldots,a_{k-1})} \left(\mathsf{E}^{(k)}_{x_k} (\mathsf{F}_{p(b_0,\ldots,b_{k-1},u),u}(p)) \right) (x_0, \ldots, x_{k-1}).$$

Observe that, by Lemma 4.20,

$$s(x_0, \ldots, x_{k-1}, u) = p(b_0, \ldots, b_{k-1}, u) = s(x_0, \ldots, x_{k-1}, v),$$

for all $(x_0, x_1, \ldots, x_{k-1}) \in \{a_0, c\} \times \{a_1, b_1\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$ such that $(x_0, x_1, \ldots, x_{k-1}) \neq (c, b_1, \ldots, b_{k-1})$, whence

$$s(x_0, \ldots, x_{k-1}, u) \equiv_\eta s(x_0, \ldots, x_{k-1}, v),$$

for all $(x_0, x_1, \ldots, x_{k-1}) \in \{a_0, c\} \times \{a_1, b_1\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$ such that $(x_0, x_1, \ldots, x_{k-1}) \neq (c, b_1, \ldots, b_{k-1})$. Using the induction hypothesis we know that $C(\theta_1, \alpha_1, \ldots, \alpha_k; \eta)$ and therefore we obtain

$$s(c, b_1, \ldots, b_{k-1}, u) \equiv_\eta s(c, b_1, \ldots, b_{k-1}, v). \tag{6.2}$$

Also, by Lemma 4.20 we have

$$s(x_0, \ldots, x_{k-1}, u) = p(b_0, \ldots, b_{k-1}, u) = s(x_0, \ldots, x_{k-1}, v),$$

for all $(x_0, x_1, \ldots, x_{k-1}) \in \{c, b_0\} \times \{a_1, b_1\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$ such that $(x_0, x_1, \ldots, x_{k-1}) \notin \{(c, b_1, \ldots, b_{k-1}), (b_0, b_1, \ldots, b_{k-1})\}$, and thus

$$s(x_0, \ldots, x_{k-1}, u) \equiv_\eta s(x_0, \ldots, x_{k-1}, v), \tag{6.3}$$

for all $(x_0, x_1, \ldots, x_{k-1}) \in \{c, b_0\} \times \{a_1, b_1\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$ such that $(x_0, x_1, \ldots, x_{k-1}) \notin \{(c, b_1, \ldots, b_{k-1}), (b_0, b_1, \ldots, b_{k-1})\}$. From (6.2) and (6.3) we have

$$s(x_0, \ldots, x_{k-1}, u) \equiv_\eta s(x_0, \ldots, x_{k-1}, v),$$

for all $(x_0, x_1, \ldots, x_{k-1}) \in \{c, b_0\} \times \{a_1, b_1\} \times \cdots \times \{a_{k-1}, b_{k-1}\}$ such that $(x_0, \ldots, x_{k-1}) \neq (b_0, \ldots, b_{k-1})$. Using the assumption $C(\theta_2, \alpha_1, \ldots, \alpha_k; \eta)$ we obtain

$$s(b_0, \ldots, b_{k-1}, u) \equiv_\eta s(b_0, \ldots, b_{k-1}, v).$$

The left side of the last congruence is equal to $p(b_0, \ldots, b_{k-1}, u)$ by Lemma 4.20 and the right side is congruent modulo η to $p(b_0, \ldots, b_{k-1}, v)$ by Lemma 4.25. Now, by Proposition 5.4, we have $C(\theta_1 \circ \theta_2, \alpha_1, \ldots, \alpha_k; \eta)$. □

Proposition 6.5. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*, and let* $\rho_1, \ldots, \rho_n$*,* $\alpha_1, \ldots, \alpha_k$ *be congruences of* $\mathbf{A}$*,* $k, n \geq 1$*. Then*

$$\bigvee_{1 \leq i \leq n} [\rho_i, \alpha_1, \ldots, \alpha_k] = [\bigvee_{1 \leq i \leq n} \rho_i, \alpha_1, \ldots, \alpha_k].$$

Proof. Since $\rho_j \leq \bigvee_{1\leq i\leq n} \rho_i$ we know from (HC2) that

$$[\rho_j, \alpha_1, \ldots, \alpha_k] \leq [\bigvee_{1\leq i\leq n} \rho_i, \alpha_1, \ldots, \alpha_k],$$

for every $j \in \{1, \ldots, n\}$. Thus,

$$\bigvee_{1\leq i\leq n} [\rho_i, \alpha_1, \ldots, \alpha_k] \leq [\bigvee_{1\leq i\leq n} \rho_i, \alpha_1, \ldots, \alpha_k].$$

Let us show the other inequality. By Definition 3.2 we know that

$$C(\rho_j, \alpha_1, \ldots, \alpha_k; [\rho_j, \alpha_1, \ldots, \alpha_k]),$$

for every $j \in \{1, \ldots, n\}$, and thus using the inequality

$$[\rho_j, \alpha_1, \ldots, \alpha_k] \leq \bigvee_{1\leq i\leq n} [\rho_i, \alpha_1, \ldots, \alpha_k]$$

and Lemma 6.2 we have

$$C(\rho_j, \alpha_1, \ldots, \alpha_k; \bigvee_{1\leq i\leq n} [\rho_i, \alpha_1, \ldots, \alpha_k]),$$

for every $j \in \{1, \ldots, n\}$. By Lemma 6.4 we obtain

$$C(\bigvee_{1\leq i\leq n} \rho_i, \alpha_1, \ldots, \alpha_k; \bigvee_{1\leq i\leq n} [\rho_i, \alpha_1, \ldots, \alpha_k]).$$

Finally, by Definition 3.2 we have

$$[\bigvee_{1\leq i\leq n} \rho_i, \alpha_1, \ldots, \alpha_k] \leq \bigvee_{1\leq i\leq n} [\rho_i, \alpha_1, \ldots, \alpha_k]. \qquad \square$$

Corollary 6.6. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*, and let* $\rho_1, \ldots, \rho_n$*,* $\alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k$ *be congruences of* $\mathbf{A}$*,* $j, k, n \geq 1$*. Then*

$$\bigvee_{1\leq i\leq n} [\alpha_0, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_k] = [\alpha_0, \ldots, \alpha_{j-1}, \bigvee_{1\leq i\leq n} \rho_i, \alpha_{j+1}, \ldots, \alpha_k].$$

Proof. We obtain the statement directly from Propositions 6.1 and 6.5. $\square$

As a consequence we immediately obtain the following lemma which claims that distributivity holds for higher commutators in Mal'cev algebras.

Lemma 6.7 (HC7)**.** *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*. Let* $j, k \geq 1$*, let* $I \neq \emptyset$ *be a set and* $\{\alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k\} \cup \{\rho_i \mid i \in I\} \subseteq \mathsf{Con}\,\mathbf{A}$*. Then*

$$\bigvee_{i\in I} [\alpha_0, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_k] = [\alpha_0, \ldots, \alpha_{j-1}, \bigvee_{i\in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_k].$$

Proof. Obviously, $\rho_i \leq \bigvee_{i\in I} \rho_i$, for every $i \in I$ and thus we have

$$[\alpha_0, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_k] \leq [\alpha_0, \ldots, \alpha_{j-1}, \bigvee_{i\in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_k],$$

for every $i \in I$, by (HC2). Then

$$\bigvee_{i\in I}[\alpha_0,\dots,\alpha_{j-1},\rho_i,\alpha_{j+1},\dots,\alpha_k] \leq [\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in I}\rho_i,\alpha_{j+1},\dots,\alpha_k].$$

To show the other inequality we put $\eta = \bigvee_{i\in I}[\alpha_0,\dots,\alpha_{j-1},\rho_i,\alpha_{j+1},\dots,\alpha_k]$. Let us show that

$$C(\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in I}\rho_i,\alpha_{j+1},\dots,\alpha_k;\eta). \tag{6.4}$$

We use Proposition 5.4 to show (6.4). Thus, let $(a_0,\dots,a_{k-1}),(b_0,\dots,b_{k-1}) \in A^k$, $u,v \in A$ and $p \in \mathsf{Pol}_{k+1}\,\mathbf{A}$ such that

(1) $a_i \equiv_{\alpha_i} b_i$ for every $i \in \{0,\dots,j-1,j+1,\dots,k-1\}$,
(2) $a_j \equiv_{\bigvee_{i\in I}\rho_i} b_j$,
(3) $u \equiv_{\alpha_k} v$,
(4) $p(x_0,\dots,x_{k-1},u) \equiv_\eta p(x_0,\dots,x_{k-1},v)$ for all $(x_0,\dots,x_{k-1}) \in \{a_0,b_0\} \times \cdots \times \{a_{k-1},b_{k-1}\}\backslash\{(b_0,\dots,b_{k-1})\}$.

We have to show $p(b_0,\dots,b_{k-1},u) \equiv_\eta p(b_0,\dots,b_{k-1},v)$. It is well known that the join of an arbitrary set of congruences is the union of joins of its finite subsets. Condition (4) actually consists of 2^k-1 formulas, one for each choice of $(x_0,\dots,x_{k-1})$. If we number all 2^k-1 choices of the vector $(x_0,\dots,x_{k-1})$ with $1,\dots,2^k-1$, then for the ℓth choice there is a *finite* subset J_ℓ of I such that the congruence (4) is true for $\bigvee_{i\in J_\ell}[\alpha_0,\dots,\alpha_{j-1},\rho_i,\alpha_{j+1},\dots,\alpha_k]$ instead of η. Also there exists a finite set J_0, $J_0 \subseteq I$ such that $a_j \equiv_{\bigvee_{i\in J_0}\rho_i} b_j$. We take $J = \bigcup_{0\leq\ell\leq 2^k-1} J_\ell$. The set J is a finite subset of I. By Corollary 6.6 we know

$$\bigvee_{i\in J}[\alpha_0,\dots,\alpha_{j-1},\rho_i,\alpha_{j+1},\dots,\alpha_k] = [\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in J}\rho_i,\alpha_{j+1},\dots,\alpha_k]$$

and since $J_\ell \subseteq J$ for all $\ell \geq 1$ we obtain

$$\bigvee_{i\in J_\ell}[\alpha_0,\dots,\alpha_{j-1},\rho_i,\alpha_{j+1},\dots,\alpha_k] \leq [\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in J}\rho_i,\alpha_{j+1},\dots,\alpha_k], \tag{6.5}$$

for all $\ell \geq 1$. Now, we have

(1) $a_i \equiv_{\alpha_i} b_i$ for every $i \in \{0,\dots,j-1,j+1,\dots,k-1\}$,
(2) $a_j \equiv_{\bigvee_{i\in J}\rho_i} b_j$ (we know $\bigvee_{i\in J_0}\rho_i \subseteq \bigvee_{i\in J}\rho_i$),
(3) $u \equiv_{\alpha_k} v$,
(4) $p(x_0,\dots,x_{k-1},u) \equiv_\theta p(x_0,\dots,x_{k-1},v)$ for all $(x_0,\dots,x_{k-1}) \in \{a_0,b_0\} \times \cdots\times\{a_{k-1},b_{k-1}\}\backslash\{(b_0,\dots,b_{k-1})\}$ where $\theta = [\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in J}\rho_i,\alpha_{j+1},\dots,\alpha_k]$, because of inequality (6.5).

Thus, we obtain

$$p(b_0,\dots,b_{k-1},u) \equiv_\theta p(b_0,\dots,b_{k-1},v),$$

because

$$C(\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in J}\rho_i,\alpha_{j+1},\dots,\alpha_k;[\alpha_0,\dots,\alpha_{j-1},\bigvee_{i\in J}\rho_i,\alpha_{j+1},\dots,\alpha_k]).$$

Since

$$[\alpha_0, \ldots, \alpha_{j-1}, \bigvee_{i \in J} \rho_i, \alpha_{j+1}, \ldots, \alpha_k] = \bigvee_{i \in J} [\alpha_0, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_k] \leq \eta,$$

we have

$$p(b_0, \ldots, b_{k-1}, u) \equiv_\eta p(b_0, \ldots, b_{k-1}, v).$$

This proves $C(1, \ldots, 1; \alpha_0, \ldots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_k; \eta)$, and hence concludes the proof of (6.4). □

Corollary 6.8. *Let* $\mathbf{A}$ *be a Mal'cev algebra, let* α *be a congruence of* $\mathbf{A}$ *and let* $n \in \mathbb{N}$. *If* $\underbrace{[1, \ldots, [1}_{n}, 1]] \leq \alpha$, *then* $\mathbf{A}/\alpha$ *is nilpotent of class at most* n.

Proof. We prove

$$\underbrace{[1/\alpha, \ldots, [1/\alpha}_{k}, 1/\alpha]] \leq (\underbrace{[1, \ldots, [1}_{k}, 1]] \vee \alpha)/\alpha \text{ for all } k \in \mathbb{N} \tag{6.6}$$

by induction on k. For $k = 1$ the statement is a consequence of (HC6). Let $k > 1$. By the induction hypothesis we have

$$[1/\alpha, \underbrace{[1/\alpha, \ldots, [1/\alpha}_{k-1}, 1/\alpha]]] \leq [1/\alpha, (\underbrace{[1, \ldots, [1}_{k-1}, 1]] \vee \alpha)/\alpha].$$

We now compute the right-hand side of the last inequality. Applying (HC6), we obtain

$$[1/\alpha, (\underbrace{[1, \ldots, [1}_{k-1}, 1]] \vee \alpha)/\alpha] = ([1, \underbrace{[1, \ldots, [1}_{k-1}, 1]] \vee \alpha] \vee \alpha)/\alpha.$$

Using the distributivity of higher commutators (HC7), the last expression is equal to

$$([1, \underbrace{[1, \ldots, [1}_{k-1}, 1]]] \vee [1, \alpha] \vee \alpha)/\alpha.$$

Since $[1, \alpha] \leq \alpha$, this is equal to

$$(\underbrace{[1, \ldots, [1}_{k}, 1]] \vee \alpha)/\alpha.$$

This completes the induction step. From (6.6), we obtain

$$\underbrace{[1/\alpha, \ldots, [1/\alpha}_{n}, 1/\alpha]]_{\mathbf{A}/\alpha} = 0_{\mathbf{A}/\alpha}$$

and hence $\mathbf{A}/\alpha$ is nilpotent of class at most n. □

Lemma 6.9. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m, $\alpha_0, \ldots, \alpha_n$ *congruences of* $\mathbf{A}$ *and* $n \geq 0$. *Then* $[\alpha_0, \ldots, \alpha_n]$ *is generated as a congruence by the set*

$$R = \big\{ \big(c(b_0, \ldots, b_n), c(a_0, \ldots, a_n)\big) \;\big|\; b_0, \ldots, b_n, a_0 \ldots, a_n \in A,\ \forall i : a_i \equiv_{\alpha_i} b_i, \\ c \in \mathsf{Pol}_{n+1}\,\mathbf{A} \text{ and } c \text{ is absorbing at } (a_0, \ldots, a_n) \big\}. \tag{6.7}$$

Proof. To prove the statement we will first show $R \subseteq [\alpha_0, \dots, \alpha_n]$. Let $n \geq 0$, $b_0, \dots, b_n, a_0, \dots, a_n \in A$ such that $b_i \equiv_{\alpha_i} a_i$, $i \in \{0, \dots, n\}$, and let $c \in \mathsf{Pol}_{n+1}\,\mathbf{A}$ be absorbing at $(a_0, \dots, a_n)$. Now it is clear that

$$c(x_0, \dots, x_{n-1}, a_n) = c(a_0, \dots, a_n) = c(x_0, \dots, x_{n-1}, b_n)$$

for all $(x_0, \dots, x_{n-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{n-1}, b_{n-1}\} \backslash \{(b_0, \dots, b_{n-1})\}$ and thus we have

$$c(x_0, \dots, x_{n-1}, a_n) \equiv_{[\alpha_0, \dots, \alpha_n]} c(x_0, \dots, x_{n-1}, b_n)$$

for all $(x_0, \dots, x_{n-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{n-1}, b_{n-1}\} \backslash \{(b_0, \dots, b_{n-1})\}$. Thus $c(b_0, \dots, b_{n-1}, a_n) \equiv_{[\alpha_0, \dots, \alpha_n]} c(b_0, \dots, b_n)$. Since c is absorbing at $(a_0, \dots, a_n)$, we obtain

$$(c(b_0, \dots, b_n), c(a_0, \dots, a_n)) \in [\alpha_0, \dots, \alpha_n].$$

This proves that every element of R is contained in $[\alpha_0, \dots, \alpha_n]$.

Now, let γ be a congruence of $\mathbf{A}$ such that $R \subseteq \gamma$. To finish the proof it will be enough to prove $[\alpha_0, \dots, \alpha_n] \leq \gamma$, which is equivalent to $C(\alpha_0, \dots, \alpha_n; \gamma)$ by Lemma 6.2. To this end, we take $b_0, \dots, b_n, a_0, \dots, a_n \in A$ such that $a_i \equiv_{\alpha_i} b_i$ for all $i \in \{0, \dots, n\}$ and $p \in \mathsf{Pol}_{n+1}\,\mathbf{A}$ such that

$$p(x_0, \dots, x_{n-1}, a_n) \equiv_\gamma p(x_0, \dots, x_{n-1}, b_n)$$

for all $(x_0, \dots, x_{n-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{n-1}, b_{n-1}\} \backslash \{(b_0, \dots, b_{n-1})\}$. We will show $p(b_0, \dots, b_{n-1}, a_n) \equiv_\gamma p(b_0, \dots, b_{n-1}, b_n)$. We define a polynomial $t \in \mathsf{Pol}_{n+1}\,\mathbf{A}$ as follows:

$$t(x_0, \dots, x_n) := \mathsf{D}^{(n)}_{p(b_0, \dots, b_{n-1}, a_n), (a_0, \dots, a_{n-1})} \Big(\mathsf{E}^{(n)}_{x_n} (\mathsf{F}_{p(b_0, \dots, b_{n-1}, a_n), a_n}(p)) \Big) (x_0, \dots, x_{n-1}).$$

We can observe that by Lemma 4.20, t is absorbing at $(a_0, \dots, a_n)$, and hence

$$t(x_0, \dots, x_n) = p(b_0, \dots, b_{n-1}, a_n) = t(a_0, \dots, a_n) \tag{6.8}$$

for every $(x_0, \dots, x_n) \in \{a_0, b_0\} \times \cdots \times \{a_n, b_n\} \backslash \{(b_0, \dots, b_n)\}$. Therefore

$$(t(b_0, \dots, b_n), t(a_0, \dots, a_n)) \in R$$

and thus $(t(b_0, \dots, b_n), t(a_0, \dots, a_n)) \in \gamma$. By Lemma 4.25 we know that $t(b_0, \dots, b_n) \equiv_\gamma p(b_0, \dots, b_n)$, so we obtain

$$(p(b_0, \dots, b_n), t(a_0, \dots, a_n)) \in \gamma.$$

Therefore, using (6.8) we have $p(b_0, \dots, b_{n-1}, a_n) \equiv_\gamma p(b_0, \dots, b_n)$. □

Corollary 6.10. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, $\alpha_0, \dots, \alpha_n$ congruences of $\mathbf{A}$ and $n \geq 0$. Then $[\alpha_0, \dots, \alpha_n]$ is generated as a congruence by the set*

$$T = \Big\{ \big(c(b_0, \dots, b_n), c(a_0, \dots, a_n) \big) \;\Big|\; b_0, \dots, b_n, a_0 \dots, a_n \in A,\ \forall i : a_i \equiv_{\alpha_i} b_i, \\ c \in \mathsf{Pol}_{n+1}\,\mathbf{A} \textit{ and } c|_{\{a_0, b_0\} \times \cdots \times \{a_n, b_n\}} \textit{ is absorbing at } (a_0, \dots, a_n) \Big\}. \tag{6.9}$$

Proof. Denote the congruence generated by the set R in (6.7) by ρ_1 and the congruence generated by the set T in (6.9) by ρ_2. Clearly, $R \subseteq T$ and hence $\rho_1 \subseteq \rho_2$. We have shown that $\rho_1 = [\alpha_0, \dots, \alpha_n]$ in Lemma 6.9. To prove that $\rho_2 \subseteq \rho_1$ we have to show that $T \subseteq [\alpha_0, \dots, \alpha_n]$. Let $b_0, \dots, b_n, a_0 \dots, a_n \in A$ and $c \in \mathsf{Pol}_{n+1} \mathbf{A}$ such that $\forall i : a_i \equiv_{\alpha_i} b_i$ and $c|_{\{a_0,b_0\}\times\cdots\times\{a_n,b_n\}}$ is absorbing at $(a_0, \dots, a_n)$. Then we have

$$c(x_0, \dots, x_{n-1}, a_n) \equiv_{[\alpha_0,\dots,\alpha_n]} c(x_0, \dots, x_{n-1}, b_n) \tag{6.10}$$

for all $(x_0, \dots, x_{n-1}) \in \{a_0, b_0\} \times \cdots \times \{a_{n-1}, b_{n-1}\} \setminus \{(b_0, \dots, b_{n-1})\}$, because

$$c(x_0, \dots, x_n) = c(b_0, \dots, b_{n-1}, a_n) = c(a_0, \dots, a_n) \tag{6.11}$$

for every $(x_0, \dots, x_n) \in \{a_0, b_0\} \times \cdots \times \{a_n, b_n\} \setminus \{(b_0, \dots, b_n)\}$. Therefore,

$$c(b_0, \dots, b_{n-1}, a_n) \equiv_{[\alpha_0,\dots,\alpha_n]} c(b_0, \dots, b_n)$$

using $C(\alpha_0, \dots, \alpha_n; [\alpha_0, \dots, \alpha_n])$. Hence,

$$(c(b_0, \dots, b_n), c(a_0, \dots, a_n)) \in [\alpha_0, \dots, \alpha_n]. \qquad \square$$

Corollary 6.11. *Let $n \geq 2$. Let $\mathbf{A}$ and $\mathbf{B}$ be Mal'cev algebras, on the same set (with possibly different Mal'cev terms). If $\mathsf{Pol}_n \mathbf{A} = \mathsf{Pol}_n \mathbf{B}$, then $\mathbf{A}$ and $\mathbf{B}$ have the same n-ary commutator operation.*

Proof. If $\mathsf{Pol}_n \mathbf{A} = \mathsf{Pol}_n \mathbf{B}$, then $\mathsf{Con}\, \mathbf{A} = \mathsf{Con}\, \mathbf{B}$. Hence every n-ary commutator is generated by the same set on both $\mathbf{A}$ and $\mathbf{B}$. Since $\mathsf{Con}\, \mathbf{A} = \mathsf{Con}\, \mathbf{B}$, this set generates the same congruence on both $\mathbf{A}$ and $\mathbf{B}$. $\square$

As another consequence of Lemma 6.9, we obtain a description of the commutator operation for expanded groups. Actually, for expanded groups, this description can be taken as a definition of the higher commutator operations. A similar description for the binary commutator operation in ideal determined varieties is given in [10] (cf. also [18]).

Corollary 6.12. *Let $\mathbf{V}$ be an expanded group, let $n \in \mathbb{N}$, let $\alpha_0, \dots, \alpha_n \in \mathsf{Con}\, \mathbf{V}$, and let $\gamma := [\alpha_0, \dots, \alpha_n]$. For $i \in \{0, \dots, n\}$, let A_i be the class $0/\alpha_i$, and let $C := 0/\gamma$. Then C is the subgroup of $(V, +, -, 0)$ that is generated by*

$$S := \big\{c(a_0, \dots, a_n) \bigm| a_0 \in A_0, \dots, a_n \in A_n, c \in \mathsf{Pol}_{n+1} \mathbf{V}, \\ \textit{and } c \textit{ is absorbing at } (0, \dots, 0) \textit{ with value } 0\big\}.$$

Proof. Let S' be the subgroup of $(V, +, -, 0)$ that is generated by S. Since for all $p \in \mathsf{Pol}_1 \mathbf{V}$ with $p(0) = 0$, we have $p(S) \subseteq S$, it is easy to show that for all $p \in \mathsf{Pol}_1 \mathbf{V}$ with $p(0) = 0$, we have $p(S') \subseteq S'$. By [17, Theorem 7.123], S' is an ideal of $\mathbf{V}$, and thus the relation σ' defined by

$$\sigma' := \{(v_0, v_1) \in V \times V \mid v_0 - v_1 \in S'\}$$

is a congruence of $\mathbf{V}$.

We will now prove $S' = C$. For proving $C \subseteq S'$, it is sufficient to prove $\gamma \subseteq \sigma'$. To prove this inclusion, we show that all of the generators of γ that are given in Lemma 6.9 lie in σ'. To this end, let $c \in \mathsf{Pol}_{n+1}\mathbf{V}$, $\mathbf{a} = (a_0, \dots, a_n) \in V^{n+1}$, and $\mathbf{b} = (b_0, \dots, b_n) \in V^{n+1}$ be such that c is absorbing at $\mathbf{a}$ and for all $i \in \{0, \dots, n\}$, we have $(a_i, b_i) \in \alpha_i$. We define $d \in \mathsf{Pol}_{n+1}\mathbf{V}$ by

$$d(\mathbf{x}) := c(\mathbf{a} + \mathbf{x}) - c(\mathbf{a}) \text{ for all } \mathbf{x} \in V^{n+1}.$$

Then d is absorbing at 0 with value 0, and therefore $d(-\mathbf{a} + \mathbf{b}) \in S$. Hence $(0, d(-\mathbf{a}+\mathbf{b})) \in \sigma'$, and thus $(0 + c(\mathbf{a}), d(-\mathbf{a}+\mathbf{b}) + c(\mathbf{a})) = (c(\mathbf{a}), c(\mathbf{b})) \in \sigma'$, since σ' is a congruence of $\mathbf{V}$. This completes the proof of $\gamma \subseteq \sigma'$.

For proving $S' \subseteq C$, we first prove $S \subseteq C$. Let $s \in S$. Then there is a $c \in \mathsf{Pol}_{n+1}\mathbf{V}$ such that c is absorbing at 0 with value 0, and there are $a_0 \in A_0, \dots, a_n \in A_n$ such that $c(a_0, \dots, a_n) = s$. Lemma 6.9 yields $(0, c(a_0, \dots, a_n)) \in \gamma$, and hence $s \in 0/\gamma = C$. Since C is a subgroup of $(V, +, -, 0)$, we have $S' \subseteq C$. □

For the commutator of principal congruences we can avoid the congruence generation involved in Lemma 6.9.

Lemma 6.13. *Let $\mathbf{A}$ be a Mal'cev algebra with a Mal'cev term m, let $n \geq 0$, let $(u_0, \dots, u_n), (v_0, \dots, v_n) \in A^{n+1}$ and for all $i \in \{0, \dots, n\}$ let $\alpha_i = \Theta_{\mathbf{A}}(u_i, v_i)$. Then*

$$[\alpha_0, \dots, \alpha_n] = \big\{ \big(c(v_0, \dots, v_n), c(u_0, \dots, u_n)\big) \mid c \in \mathsf{Pol}_{n+1}\mathbf{A},\ c \text{ absorbing at } (u_0, \dots, u_n) \big\}.$$

Proof. We denote the set on the right side of the equality by S. It is well known that in Mal'cev algebras we have

$$\Theta_{\mathbf{A}}(u, v) = \{(p(u), p(v)) \mid p \in \mathsf{Pol}_1\mathbf{A}\}, \tag{6.12}$$

$u, v \in A$. First, we prove that the set of generators of $[\alpha_0, \dots, \alpha_n]$ from Lemma 6.9 is a subset of S. Take $a_0, \dots, a_n, b_0, \dots, b_n \in A$, $n \geq 0$, so that $a_i \equiv_{\alpha_i} b_i$, $i \in \{0, \dots, n\}$, and $c \in \mathsf{Pol}_{n+1}\mathbf{A}$ so that c is absorbing at $(a_0, \dots, a_n)$. Using statement (6.12) for $\alpha_i = \Theta_{\mathbf{A}}(u_i, v_i)$ we know that there exist polynomials $p_i \in \mathsf{Pol}_1\mathbf{A}$ such that $a_i = p_i(u_i)$ and $b_i = p_i(v_i)$, for every $i \in \{0, \dots, n\}$. Thus

$$\big(c(b_0, \dots, b_n), c(a_0, \dots, a_n)\big) = \big(c\big(p_0(v_0), \dots, p_n(v_n)\big), c\big(p_0(u_0), \dots, p_n(u_n)\big)\big).$$

Since c is absorbing at $(a_0, \dots, a_{n-1})$, $c(p_0(x_0), \dots, p_n(x_n))$ is absorbing at $(u_0, \dots, u_n)$. Then we know that

$$(c(b_0, \dots, b_n), c(a_0, \dots, a_n)) = \big(c\big(p_0(v_0), \dots, p_n(v_n)\big), c\big(p_0(u_0), \dots, p_n(u_n)\big)\big)$$

belongs to S. Since S is obviously a subset of the generating set of $[\alpha_0, \dots, \alpha_n]$ from Lemma 6.9, we conclude that S generates $[\alpha_0, \dots, \alpha_n]$. We will now show that S is a congruence relation of $\mathbf{A}$. Clearly, S is reflexive, since we can substitute constant functions for c. To prove the symmetry of S let

$(c(v_0, \dots, v_n), c(u_0, \dots, u_n)) \in S$ for a polynomial $c \in \mathsf{Pol}_{n+1} \mathbf{A}$ absorbing at $(u_0, \dots, u_n)$. Now, we define the polynomial $e \in \mathsf{Pol}_{n+1} \mathbf{A}$ as follows:

$$e(x_0, \dots, x_n) := m\big(c(u_0, \dots, u_n), c(x_0, \dots, x_n), c(v_0, \dots, v_n)\big).$$

We have that $(e(v_0, \dots, v_n), e(u_0, \dots, u_n)) \in S$, by the definition of S. Therefore $(c(u_0, \dots, u_n), c(v_0, \dots, v_n)) \in S$. To prove the transitivity of S we assume that $c, d \in \mathsf{Pol}_{n+1} \mathbf{A}$ are such that $(c(v_0, \dots, v_n), c(u_0, \dots, u_n)) \in S$, $(d(v_0, \dots, v_n), d(u_0, \dots, u_n)) \in S$ and $c(u_0, \dots, u_n) = d(v_0, \dots, v_n)$ for c and d absorbing at $(u_0, \dots, u_n)$, and we show

$$(c(v_0, \dots, v_n), d(u_0, \dots, u_n)) \in S.$$

We introduce the polynomial $e \in \mathsf{Pol}_{n+1} \mathbf{A}$ as follows:

$$e(x_0, \dots, x_n) := m\big(c(x_0, \dots, x_n), c(u_0, \dots, u_n), d(x_0, \dots, x_n)\big).$$

It is not hard to see that e is absorbing at $(u_0, \dots, u_n)$. Thus, we conclude that $(e(v_0, \dots, v_n), e(u_0, \dots, u_n)) \in S$. Since $e(v_0, \dots, v_n) = c(v_0, \dots, v_n)$ and $e(u_0, \dots, u_n) = d(u_0, \dots, u_n)$, we have $(c(v_0, \dots, v_n), d(u_0, \dots, u_n)) \in S$. It remains to prove the compatibility property for S. As it is mentioned in [11, p. 9] it is enough to check the compatibility for unary polynomials. Let $f \in \mathsf{Pol}_1 \mathbf{A}$ and $(c(v_0, \dots, v_n), c(u_0, \dots, u_n)) \in S$ for a polynomial $c \in \mathsf{Pol}_{n+1} \mathbf{A}$ absorbing at $(u_0, \dots, u_n)$. Then for a polynomial $t \in \mathsf{Pol}_{n+1} \mathbf{A}$, defined by

$$t(x_0, \dots, x_n) := f\big(c(x_0, \dots, x_n)\big),$$

we have that t is absorbing at $(u_0, \dots, u_n)$. We conclude that $(t(v_0, \dots, v_n), t(u_0, \dots, u_n)) \in S$ or, in other words, $(f\big(c(v_0, \dots, v_n)\big), f\big(c(u_0, \dots, u_n)\big)) \in S$. This completes the proof. □

Proposition 6.14. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m*, let* $n, k \in \mathbb{N}$ *be such that* $k < n$*, and let* $\alpha_0, \dots, \alpha_n$ *be congruences of* $\mathbf{A}$*. Then*

$$[\alpha_0, \dots, \alpha_{k-1}, [\alpha_k, \dots, \alpha_n]] \leq [\alpha_0, \dots, \alpha_n].$$

Proof. Since we know that every congruence is a join of principal congruences, it suffices to consider the case where $\alpha_k, \dots, \alpha_n$ are principal congruences. The general inequality then follows from Lemma 6.7. We will prove that each of the generators of $[\alpha_0, \dots, \alpha_{k-1}, [\alpha_k, \dots, \alpha_n]]$ given in Lemma 6.9 belongs to $[\alpha_0, \dots, \alpha_n]$. Assume that $\alpha_i = \Theta_{\mathbf{A}}(a_i, b_i)$, where $(a_i, b_i) \in A^2$, $i \in \{k, \dots, n\}$. Let $(c(v_0, \dots, v_k), c(u_0, \dots, u_k))$ be an element in the generating set of $[\alpha_0, \dots, \alpha_{k-1}, [\alpha_k, \dots, \alpha_n]]$ as in Lemma 6.9. Then $v_i \equiv_{\alpha_i} u_i$ for all $i \in \{0, \dots, k-1\}$, $v_k \equiv_{[\alpha_k, \dots, \alpha_n]} u_k$, and c is a k-ary polynomial of $\mathbf{A}$ that is absorbing at $(u_0, \dots, u_k)$. From Lemma 6.13 we know that there exists a $d \in \mathsf{Pol}_{n-k+1} \mathbf{A}$ such that $v_k = d(b_k, \dots, b_n)$ and $u_k = d(a_k, \dots, a_n)$ and d is absorbing at $(a_k, \dots, a_n)$. Now, we observe that the polynomial $e \in \mathsf{Pol}_{n+1} \mathbf{A}$ defined by

$$e(x_0, \dots, x_n) := c(x_0, \dots, x_{k-1}, d(x_k, \dots, x_n))$$

is absorbing at $(u_0, \ldots, u_{k-1}, a_k, \ldots, a_n)$. Thus, from Lemma 6.9 we obtain that $(e(v_0, \ldots, v_{k-1}, b_k, \ldots, b_n), e(u_0, \ldots, u_{k-1}, a_k, \ldots, a_n))$ belongs to the generating set of the commutator $[\alpha_0, \ldots, \alpha_n]$ or, in other words,

$$(c(v_0, \ldots, v_k), c(u_0, \ldots, u_k)) \in [\alpha_0, \ldots, \alpha_n].$$

This completes the proof. □

Corollary 6.15 (HC8)**.** *Let* **A** *be a Mal'cev algebra with a Mal'cev term* m*, let* $n \in \mathbb{N}$*, and let* $\alpha_0, \ldots, \alpha_n$ *be congruences of* **A***. Then*

$$[\alpha_0, [\alpha_1, \ldots, \alpha_n]] \leq [\alpha_0, \alpha_1, \ldots, \alpha_n].$$

Proof. We obtain the inequality directly from Proposition 6.14 if we choose $k := 1$. □

Proposition 6.16. *Let* **A** *be a Mal'cev algebra with a Mal'cev term* m *and let* $n \geq 2$*. Then*

$$[\underbrace{1, \ldots, 1}_{n}] > 0$$

if and only if there exists a $c \in \mathsf{Pol}_n\, \mathbf{A}$ *and* $o, o_0, \ldots, o_{n-1} \in A$ *such that*

(1) c *is absorbing at* $(o_0, \ldots, o_{n-1})$ *with value* o*, and*
(2) *there exists a vector* $(a_0, \ldots, a_{n-1}) \in A^n$ *such that* $c(a_0, \ldots, a_{n-1}) \neq o$*.*

Proof. (⇒) Let $[\underbrace{1, \ldots, 1}_{n}] > 0$. Then $C(\underbrace{1, \ldots, 1}_{n}; 0)$ is not true. By Proposition 5.4 and Definition 5.1 there exist $o_0, \ldots, o_{n-1}, a_0, \ldots, a_{n-1} \in A$ and $p \in \mathsf{Pol}_n\, \mathbf{A}$ such that

$$p(x_0, \ldots, x_{n-2}, o_{n-1}) = p(x_0, \ldots, x_{n-2}, a_{n-1}),$$

for all $(x_0, \ldots, x_{n-2}) \in \{o_0, a_0\} \times \cdots \times \{o_{n-2}, a_{n-2}\} \backslash \{(a_0, \ldots, a_{n-2})\}$ and

$$p(a_0, \ldots, a_{n-2}, o_{n-1}) \neq p(a_0, \ldots, a_{n-2}, a_{n-1}).$$

Now, we put $o = p(a_0, \ldots, a_{n-2}, o_{n-1})$ and define $c \in \mathsf{Pol}_n\, \mathbf{A}$ as follows:

$$c(x_0, \ldots, x_{n-1}) :=$$
$$\mathsf{D}^{(n-1)}_{p(a_0,\ldots,a_{n-2},o_{n-1}),(o_0,\ldots,o_{n-2})}\big(\mathsf{E}^{(n-1)}_{x_{n-1}}(\mathsf{F}_{p(a_0,\ldots,a_{n-2},o_{n-1}),o_{n-1}}(p))\big)(x_0, \ldots, x_{n-2}).$$

By Lemma 4.20 we have that c is absorbing at $(o_0, \ldots, o_{n-1})$ with value o, and by Lemma 4.25 we know that $c(a_0, \ldots, a_{n-1}) = p(a_0, \ldots, a_{n-1})$ and thus $c(a_0, \ldots, a_{n-1}) \neq o$.

(⇐) Since

$$c(x_0, \ldots, x_{n-2}, o_{n-1}) = c(x_0, \ldots, x_{n-2}, a_{n-1}),$$

for all $(x_0, \ldots, x_{n-2}) \in \{o_0, a_0\} \times \cdots \times \{o_{n-2}, a_{n-2}\} \backslash \{(a_0, \ldots, a_{n-2})\}$ and

$$c(a_0, \ldots, a_{n-2}, o_{n-1}) = o \neq c(a_0, \ldots, a_{n-1}),$$

the condition $C(\underbrace{1, \ldots, 1}_{n}; 0)$ is false by Definition 3.1. Thus $[\underbrace{1, \ldots, 1}_{n}] = 0$ does not hold, by Definition 3.2. □

Note that the polynomial that satisfies the conditions (1) and (2) of Proposition 6.16 depends on each of its arguments, or, in other words, its essential arity equals its arity. In the sequel we will need o-polynomials, which we have introduced in Definition 4.18.

Corollary 6.17. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m *and* $n \geq 2$. *If* $[\underbrace{1,\dots,1}_{n}] = 0$, *then for every* $o \in A$, *every* o*-polynomial* p *of* $\mathbf{A}$ *has essential arity at most* $n-1$.

Proof. Let $o \in A$, and let $p \in \mathsf{Pol}_k\,\mathbf{A}$ be an o-polynomial with essential arity k. Then p satisfies (1) from Proposition 6.16 for $(o', o'_0, \dots, o'_{k-1}) := (p(o,\dots,o), o, \dots, o)$. Since p depends on x_{k-1}, there exist $(a_0,\dots,a_{k-1})$, $(a_0,\dots,a_{k-2},b_{k-1}) \in A^k$ such that $p(a_0,\dots,a_{k-1}) \neq p(a_0,\dots,a_{k-2},b_{k-1})$. Clearly, $p(a_0,\dots,a_{k-1}) \neq p(o,\dots,o)$ or $p(a_0,\dots,a_{k-2},b_{k-1}) \neq p(o,\dots,o)$. Thus, p satisfies also (2) from Proposition 6.16. Thus we have $[\underbrace{1,\dots,1}_{k}] > 0$, and hence $k \leq n-1$. □

Let us recall that a *clone* on a set A is a collection of finitary functions on A that contains all projections and is closed under all compositions. If a clone on a set A contains all constant unary operations we call such clone a *polynomial clone*. Let F be a set of polynomials of an algebra $\mathbf{A}$. We denote the clone generated by F by $\mathrm{Clo}(F)$.

Proposition 6.18. *Let* $\mathbf{A}$ *be a Mal'cev algebra with a Mal'cev term* m *and* $n \geq 2$. *If* $[\underbrace{1,\dots,1}_{n}] = 0$, *then* $\mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i\,\mathbf{A} \cup \{m\}) = \mathsf{Pol}\,\mathbf{A}$.

Proof. By (HC8), $\mathbf{A}$ is nilpotent. We proceed by induction on the nilpotency class of $\mathbf{A}$.

In the case that $\mathbf{A}$ is abelian, Gumm's Theorem [15, Theorem 4.155] yields that the clone of all polynomials of $\mathbf{A}$ is generated by m and all the unary polynomials of $\mathbf{A}$.

For the induction step, we let $r \in \mathbb{N}$ such that $\mathbf{A}$ is of nilpotency class $r+1$. Then, we have $\underbrace{[1,\dots,[1,1]]}_{r+1} = 0$, and for $\alpha := \underbrace{[1,\dots,[1,1]]}_{r}$, we have $\alpha > 0$. Hence

$$[1,\alpha] = 0. \tag{6.13}$$

By Corollary 6.8, $\mathbf{A}/\alpha$ is nilpotent of class at most r. Furthermore, by (HC6) we have $[\underbrace{1,\dots,1}_{n}]_{\mathbf{A}/\alpha} = 0_{\mathbf{A}/\alpha}$. We fix $p \in \mathsf{Pol}_k\,\mathbf{A}$, $k \geq n$, and let p_α be the corresponding polynomial from $\mathsf{Pol}(\mathbf{A}/\alpha)$ such that $p_\alpha(\mathbf{x}/\alpha) = p(\mathbf{x})/\alpha$ for all $\mathbf{x} \in A^k$. Thus, by the induction hypothesis we know that

$$p_\alpha \in \mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i(\mathbf{A}/\alpha) \cup \{m\}).$$

In other words, there exists a $p' \in \mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i\, \mathbf{A} \cup \{m\})$ such that $p_\alpha(\mathbf{x}/\alpha) = p'(\mathbf{x})/\alpha$. Now, we obtain $p(\mathbf{x}) \equiv_\alpha p'(\mathbf{x})$, for every $\mathbf{x} \in A^k$. We choose $o \in A$, and define $t \in \mathsf{Pol}_k\, \mathbf{A}$ as follows:

$$t(\mathbf{x}) := m(p(\mathbf{x}), p'(\mathbf{x}), o) \text{ for every } \mathbf{x} \in A^k.$$

We want to show $t \in \mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i\, \mathbf{A} \cup \{m\})$. First, we observe that $t(\mathbf{x}) \in o/\alpha$, for all $\mathbf{x} \in A^k$. From (6.13) we have $[\alpha, \alpha] = 0$ and thus we can apply Proposition 4.19 and obtain that $t \in \mathrm{Sg}^{\mathbf{P}(A,k,m,o)}(P)$, where P is a set of o-polynomial functions of essential arities at most the essential arity of t. Using the assumption $[\underbrace{1, \ldots, 1}_{n}] = 0$ in $\mathbf{A}$ and Corollary 6.17 we conclude that all such o-polynomial functions are of essential arities at most $n-1$ and thus we have $t \in \mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i\, \mathbf{A} \cup \{m\})$. Since $p(\mathbf{x}) \equiv_\alpha p'(\mathbf{x})$ and $[\alpha, 1] = 0$, Lemma 3.4 yields $p(\mathbf{x}) = m(t(\mathbf{x}), o, p'(\mathbf{x}))$. Since $t, p' \in \mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i\, \mathbf{A} \cup \{m\})$, we have $p \in \mathrm{Clo}(\bigcup_{i=0}^{n-1} \mathsf{Pol}_i\, \mathbf{A} \cup \{m\})$. $\square$

7. Supernilpotent algebras

In this section we prove Theorems 2.1 and 2.2.

Definition 7.1. Let $k \in \mathbb{N}$. An algebra is called *k-supernilpotent* if

$$[\underbrace{1, \ldots, 1}_{k+1}] = 0.$$

An algebra $\mathbf{A}$ is called *supernilpotent* if there exists a $k \in \mathbb{N}$ such that $\mathbf{A}$ is k-supernilpotent.

Definition 7.2. (cf. [13, p. 179]) Let $\mathbf{A}$ be an algebra, and let $k \in \mathbb{N}$. The function $c \in \mathsf{Pol}_{k+1}\, \mathbf{A}$ is a *commutator polynomial of rank k* if the following conditions hold:

(1) For all $x_0, \ldots, x_{k-1}, z \in A$, if $z \in \{x_0, \ldots, x_{k-1}\}$, then

$$c(x_0, \ldots, x_{k-1}, z) = z.$$

(2) There exist $y_0, \ldots, y_{k-1}, u \in A$ such that

$$c(y_0, \ldots, y_{k-1}, u) \neq u.$$

Proposition 7.3. *Let $k \in \mathbb{N}$ and let $\mathbf{A}$ be a k-supernilpotent Mal'cev algebra. If $\mathbf{A}$ is $(k+1)$-affine complete, then $\mathbf{A}$ is affine complete.*

Proof. We define an algebra $\mathbf{B}$ by $\mathbf{B} = (A, \mathcal{C})$ where $\mathcal{C}$ is the set of all functions on $\mathbf{A}$ that preserve all congruences of $\mathbf{A}$. We want to show that $\mathsf{Pol}\, \mathbf{B} = \mathsf{Pol}\, \mathbf{A}$. Since $\mathbf{A}$ is $(k+1)$-affine complete by the assumptions we have $\mathsf{Pol}_s\, \mathbf{B} = \mathsf{Pol}_s\, \mathbf{A}$ for every $s \leq k+1$. It is not hard to see that $\mathsf{Con}\, \mathbf{A} = \mathsf{Con}\, \mathbf{B}$. Then, from Corollary 6.11 we know that $[\underbrace{1,\dots,1}_{k+1}] = 0$ is true in $\mathbf{B}$. Finally, from Proposition 6.18 we have

$$\mathsf{Pol}\,\mathbf{B} = \mathrm{Clo}(\bigcup_{i=0}^{k} \mathsf{Pol}_i\,\mathbf{B} \cup \{m\}) = \mathrm{Clo}(\bigcup_{i=0}^{k} \mathsf{Pol}_i\,\mathbf{A} \cup \{m\}) = \mathsf{Pol}\,\mathbf{A}. \qquad \square$$

Corollary 7.4. *There is an algorithm that decides whether a supernilpotent finite Mal'cev algebra of finite type, given by its operation tables, is affine complete.*

Proof. From Proposition 6.16, we obtain a way to compute a $k \in \mathbb{N}$ such that $\mathbf{A}$ is k-supernilpotent. Once such a k is known, it remains to check whether every $(k+1)$-ary congruence preserving function is a polynomial function. $\square$

Lemma 7.5. *Let $\mathbf{A}$ be an algebra that generates a congruence permutable variety and let $k \in \mathbb{N}$. Then the following are equivalent:*

(1) $[\underbrace{1,\dots,1}_{k+1}] = 0$;

(2) $\mathbf{A}$ *is nilpotent and all commutator polynomials have rank at most* k.

Proof. (1) $\Rightarrow$ (2) Let c be a commutator polynomial of rank $t \geq k+1$. From $[\underbrace{1,\dots,1}_{k+1}] = 0$ we have $[\underbrace{1,\dots,1}_{t}] = 0$ by (HC3), and thus

$$C(\underbrace{1,1,\dots,1}_{t}; 0), \tag{7.1}$$

by Definition 3.2. Let $(y_0,\dots,y_{t-1},u) \in A^{t+1}$. We want to show that $c(y_0,\dots,y_{t-1},u) = u$. For all $(x_0,\dots,x_{t-2}) \in \{u,y_0\} \times \dots \times \{u,y_{t-2}\} \backslash \{(y_0,\dots,y_{t-2})\}$ we have $c(x_0,\dots,x_{t-2},y_{t-1},u) = c(x_0,\dots,x_{t-2},u,u)$, by Definition 7.2. Thus, by Definition 3.1, we have $c(y_0,\dots,y_{t-1},u) = c(y_0,\dots,y_{t-2},u,u) = u$ because of (7.1). This contradicts the fact that c is a commutator polynomial. Clearly, by (HC8), $\mathbf{A}$ is nilpotent.

(2) $\Rightarrow$ (1) Let $c \in \mathsf{Pol}_n\,\mathbf{A}$, $o, o_0,\dots,o_{n-1} \in A$ and $(a_0,\dots,a_{n-1}) \in A^n$ be such that the following is satisfied:

(i) c is absorbing at $(o_0,\dots,o_{n-1})$ with value o;

(ii) there exists a vector $(a_0,\dots,a_{n-1}) \in A^n$ such that $c(a_0,\dots,a_{n-1}) \neq o$.

By the assumptions of the lemma, $\mathbf{A}$ has a Mal'cev term. Let us denote it by m. By [8, Corollary 7.4], since $\mathbf{A}$ is nilpotent we know that the functions $f_i\colon A \to A$ defined by

$$f_i(x) := m(x, o, o_i),$$

for every $i \in \{0,\dots,n-1\}$ are bijections. Thus there exist $b_0,\dots,b_{n-1} \in A$ such that $f_i(b_i) = a_i$, for every $i \in \{0,\dots,n-1\}$. Let us define a polynomial $d \in \mathsf{Pol}_{n+1}\,\mathbf{A}$ by

$$d(x_0,\dots,x_{n-1},z) := m(c(m(x_0,z,o_0),\dots,m(x_{n-1},z,o_{n-1})), o, z).$$

Clearly, we have $d(x_0,\ldots,\overset{i}{\overset{\downarrow}{z}},\ldots,x_{n-1},z) = z$, for every $i \in \{0,\ldots,n-1\}$. Also,

$$d(x_0,\ldots,x_{n-1},o) = c(f_0(x_0),\ldots,f_{n-1}(x_{n-1})).$$

Therefore $d(b_0,\ldots,b_{n-1},o) = c(a_0,\ldots,a_{n-1}) \neq o$. Now, the conditions (1) and (2) of Definition 7.2 are satisfied. Thus, d is a commutator polynomial of rank n. By the assumptions $n \leq k$, and thus by Proposition 6.16 we obtain $[\underbrace{1,\ldots,1}_{k+1}] = 0$. □

Lemma 7.6. *Let* **A** *be a finite nilpotent algebra of finite type that generates a congruence modular variety. Then,* **A** *factors as a direct product of algebras of prime power cardinality if and only if* **A** *is a supernilpotent Mal'cev algebra.*

Proof. Let us suppose that a finite nilpotent algebra **A** factors as a direct product of algebras of prime power cardinality. We define a new algebra $\mathbf{A}^\star$ in the following way: for each $c \in A$, we add a nullary operation $c^{\mathbf{A}^\star}$ defined by $c() := c$. Since A is finite we have that $\mathbf{A}^\star$ is a finite algebra of finite type. Furthermore Gumm's terms from [8, Theorem 6.4, p. 60] for **A** are also terms in $\mathbf{A}^\star$ and thus $\mathbf{A}^\star$ generates a congruence modular variety. By [1, Lemma 2.2] we know that the binary commutator in **A** and $\mathbf{A}^\star$ is the same. Thus we have that $\mathbf{A}^\star$ is nilpotent. By assumption we know that there is a natural number n such that $A = A_1 \times \cdots \times A_n$, where $|A_i| = p_i^{\alpha_i}$ for some primes p_i and $\alpha_i \in \mathbb{N}$, $i \in \{1,\ldots,n\}$. We define an algebra $\mathbf{A}_i^\star$ in the following way: for each $c \in A$, we add a nullary operation $c^{\mathbf{A}_i{}^\star}$ defined by $c() := \pi_i(c)$ for every $i \in \{1,\ldots,n\}$. Now, $\mathbf{A}^\star = \mathbf{A}_1^\star \times \cdots \times \mathbf{A}_n^\star$ and $\mathbf{A}_i^\star$ has prime power cardinality. By [13, Theorem 3.14 (3), (4)] we have an $m \in \mathbb{N}$ such that the rank of every non-trivial commutator term of $\mathbf{A}^\star$ is at most m. Since these terms are precisely the commutator polynomials of **A**, and since by [13, Theorem 2.7] the algebra **A** generates a congruence permutable variety, Lemma 7.5 yields that **A** is m-supernilpotent.

Now, suppose that **A** is a supernilpotent Mal'cev algebra. By Lemma 7.5, we know that all commutator polynomials of **A** are of bounded rank. Hence, all nontrivial commutator terms of **A** have bounded rank. Applying [13, Theorem 3.14 (3), (4)] we obtain that **A** factors as a direct product of algebras of prime power cardinality. □

Now we can give the proof of Theorem 2.1.

Proof of Theorem 2.1. From Lemma 7.6 we know that a finite nilpotent algebra **A** of finite type which is a product of algebras of prime power order and generates a congruence modular variety is supernilpotent and generates a congruence permutable variety. Therefore **A** has a Mal'cev term. Then by Corollary 7.4 we have that the property of affine completeness for **A** is decidable. □

In a Mal'cev algebra $\mathbf{A}$, for $(x_0, \dots, x_{k-1}) \in A^k$ and $o \in A$ we introduce the following notation:

$$\omega_o(x_0, \dots, x_{k-1}) := |\{i : x_i \neq o\}|.$$

Proposition 7.7. *Let $n, k \in \mathbb{N}$, let $\mathbf{A}$ be a k-supernilpotent Mal'cev algebra, and let $o \in A$. Let $p \in \mathsf{Pol}_n \mathbf{A}$ such that for all $(x_0, \dots, x_{n-1}) \in A^n$ with $\omega_o(x_0, \dots, x_{n-1}) \leq k$, we have $p(x_0, \dots, x_{n-1}) = o$. Then p is the constant function with value o.*

Proof. Let $p \in \mathsf{Pol}_n \mathbf{A}$ with the given property and let $(x_0, \dots, x_{n-1}) \in A^n$. We shall prove that $p(x_0, \dots, x_{n-1}) = o$ by induction on $\omega_o(x_0, \dots, x_{n-1})$. If $\omega_o(x_0, \dots, x_{n-1}) < k+1$, then the statement is true by the assumption. Let us suppose that $\omega_o(x_0, \dots, x_{n-1}) = m \geq k+1$. Let $\{i_1, \dots, i_m\} = \{i \mid x_i \neq o\}$. We define a new polynomial q by

$$q(y_1, \dots, y_{k+1}, z_{k+2}, \dots, z_m) :=$$
$$p(o, \dots, o, \overset{\overset{i_1}{\downarrow}}{y_1}, o, \dots, o, \overset{\overset{i_{k+1}}{\downarrow}}{y_{k+1}}, o, \dots, o, \overset{\overset{i_{k+2}}{\downarrow}}{z_{k+2}}, o, \dots, o, \overset{\overset{i_m}{\downarrow}}{z_m}, o, \dots, o)$$

for $y_1, \dots, y_{k+1}, z_{k+2}, \dots, z_m \in A^m$. By the induction hypothesis we have $q(y_1, \dots, y_{k+1}, x_{i_{k+2}}, \dots, x_{i_m}) = o$ for every $(y_1, \dots, y_{k+1}) \in \{x_{i_1}, o\} \times \dots \times \{x_{i_{k+1}}, o\} \backslash \{(x_{i_1}, \dots, x_{i_{k+1}})\}$. If we introduce a polynomial q' in the following way

$$q'(y_1, \dots, y_{k+1}) := q(y_1, \dots, y_{k+1}, x_{i_{k+2}}, \dots, x_{i_m}),$$

we have $q'(y_1, \dots, y_{k+1}) = o$ whenever there exists an $i \in \{1, \dots, k+1\}$ such that $y_i = o$. Therefore q' is an o-polynomial, and hence the essential arity of q' is 0 or $k+1$. Since $\mathbf{A}$ is k-supernilpotent we know that $[\underbrace{1, \dots, 1}_{k+1}] = 0$, and thus the essential arity of q' is at most k by Corollary 6.17. Thus q is constant, and therefore this yields $q(x_{i_1}, \dots, x_{i_m}) = o$ and thus, $p(x_1, \dots, x_n) = o$. ☐

Lemma 7.8. *Let $\mathbf{A}$ be a nilpotent Mal'cev algebra $\mathbf{A}$ with a Mal'cev term m and let $x, y, o \in A$. Then if $m(x, y, o) = o$, we have $x = y$.*

Proof. Suppose $m(x, y, o) = o$. By [8, Corollary 7.4] we know that the function $f : A \to A$ defined by $f(t) := m(t, y, o)$ for $t \in A$ is one to one. Therefore if $x \neq y$, then $f(x) \neq f(y)$. Then we have $m(x, y, o) \neq m(y, y, o) = o$. This contradicts the assumption. ☐

On supernilpotent Mal'cev algebras, these results provide a method to determine whether two polynomial terms induce the same function. In particular, we can now give a proof of Theorem 2.2.

Proof of Theorem 2.2. Suppose that $s(x_0, \dots, x_{n-1}), t(x_0, \dots, x_{n-1})$ are polynomial terms of $\mathbf{A}$. By Lemma 7.6, there is a $k \in \mathbb{N}$ such that $\mathbf{A}$ is k-supernilpotent, and $\mathbf{A}$ has a Mal'cev term m. Let o be an element of $\mathbf{A}$; since

$\mathbf{A}$ is nilpotent, by Lemma 7.8, it suffices to check whether

$$m(s(x_0, \dots, x_{n-1}), t(x_0, \dots, x_{n-1}), o) \approx o$$

holds in $\mathbf{A}$. We define a polynomial term p of $\mathbf{A}$ by

$$p(x_0, \dots, x_{n-1}) := m(s(x_0, \dots, x_{n-1}), t(x_0, \dots, x_{n-1}), o).$$

By Proposition 7.7 we have to check $p^{\mathbf{A}}(a_0, \dots, a_{n-1}) = o$ only for those n-tuples from A^n that satisfy $\omega_o(a_0, \dots, a_{n-1}) < k + 1$. There are precisely

$$1 + (|A| - 1)n + (|A| - 1)^2 \binom{n}{2} + \cdots + (|A| - 1)^k \binom{n}{k}$$

such n-tuples. Clearly, this expression is a polynomial in n. Since n is the number of variables that occur in s and t, n is obviously bounded by the length of these terms.

Therefore the polynomial equivalence problem has polynomial complexity in the length of the input terms. □

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