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Characteristic triangles of closure operators with applications in general algebra

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Abstract. Our aim is to investigate groups and their weak congruence lattices in the abstract setting of lattices L with (local) closure operators C in the categorical sense, where L is regarded as a small category and C is a family of closure maps on the principal ideals of L. A useful tool for structural investigations of such "lattices with closure" is the so-called characteristic triangle, a certain substructure of the square L^2. For example, a purely order-theoretical investigation of the characteristic triangle shows that the Dedekind groups (alias Hamiltonian groups) are precisely those with modular weak congruence lattices; similar results are obtained for other classes of algebras.

1. Introduction

Modern mathematics has some powerful tools that allow us to eliminate elementwise calculations. Prominent disciplines in that area are order and lattice theory (as applied in universal algebra, in pointfree topology, or in the abstract treatment of geometry) and, of course, category theory — which encompasses, under suitable identification, the theory of ordered sets. Such a framework often provides the most transparent reason "why a theorem is true". In the present paper, we prove the following theorem (formulated already in [28], however with an incorrect proof):

Theorem 1.1. *A group is a Dedekind group if and only if its weak congruence lattice is modular.*

Here, by a *Dedekind group* we mean a group in which all subgroups are normal. Sometimes, such groups are also called *Hamiltonian* (see e.g., [2]), but often the latter name is reserved for the non-abelian case (cf. [25], [26]).

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Theorem 1.1, which is easily established in the finite case (cf. [32]), is mainly a group-theoretical statement but involves certain lattices that gave rise to the pointfree, i.e., purely lattice-theoretical treatment of the *Congruence Intersection Property* (CIP) discussed extensively in [27] (cf. Obraztsov [23] and Traustason [30]). The CIP combined with the classical *Congruence Extension Property* (CEP) provides a useful tool in universal algebra (see [27] again).

To accomplish our goal, we study certain closure operators on lattices L, i.e., families of closure maps on the principal ideals

$$L_x = {\downarrow}x = \{\, y \in L \mid y \le x \} \quad (x \in L)\,.$$

We translate the situation of weak congruence lattices into the abstract model, establish a much more general analogue in the lattice environment (not dealing with group elements any longer), and obtain Theorem 1.1 as a special instance. Our construction mimics abstractly the formation of the weak congruence lattice $\mathrm{Con_w}(G)$ of a group G by means of the subgroup lattice $\mathrm{Sub}(G)$. Below, we give a survey of the involved notions.

In a complete lattice L, an element y is *way-below* x, denoted $y \ll x$, if for all directed subsets D of L, $x \le \bigvee D$ implies that y belongs to the downset

$${\downarrow}D = \bigcup \{\, {\downarrow}z \mid z \in D \}\,.$$

The elements x with $x \ll x$ are the *compact* elements. The ideal $\{\, y \in L \mid y \ll x\}$ is called the *way-below ideal* of x. A *continuous lattice* is a complete lattice in which each element is the join of its way-below ideal (see [16] and, for more general continuity structures, [10] and [12]). A special class of continuous lattices is that of *algebraic lattices*, in which the compact elements are join-dense; that is, each element is a join of compact elements. For more background concerning algebraic lattices and their generalizations, see [2], [3], [11] and [16]. Prominent examples of algebraic lattices are the lattices $\mathrm{Sub}(A)$ of all subuniverses (carriers of subalgebras) of general (finitary) algebras A, and their congruence lattices $\mathrm{Con}(A)$. In fact, any algebraic lattice arises as an isomorphic copy of one in either of these two classes; the second, harder representation is the classical Grätzer–Schmidt theorem, cf. [19]. By a much stronger result due to Lampe (see [21]), for any two nontrivial algebraic lattices L, K and any group G there is an algebra A whose subalgebra and congruence lattice is isomorphic to L and K, respectively, and whose automorphism group is isomorphic to G. Moreover, Tuma [31] has shown that every algebraic lattice is isomorphic to an interval of a subgroup lattice $\mathrm{Sub}(G)$.

A *weak congruence* on an algebra A is a symmetric and transitive subuniverse of A^2. The weak congruences on A form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(A)$; indeed, as in the congruence case, $\mathrm{Con_w}(A)$ is closed under arbitrary intersections and under directed unions. The congruence lattice $\mathrm{Con}(A)$ of A is a principal filter in $\mathrm{Con_w}(A)$, generated by the diagonal (= identity) relation Δ of A. Moreover, the congruence lattice of any subalgebra of A is an interval sublattice of $\mathrm{Con_w}(A)$. On the other hand, the subalgebra lattice Sub(A) is isomorphic to the principal ideal generated by Δ, by sending each weak congruence θ contained in Δ to its domain

$$A\theta = \{\, a \mid a\,\theta\, a\} = \{b \mid \exists\, a\ (a\,\theta\, b)\}.$$

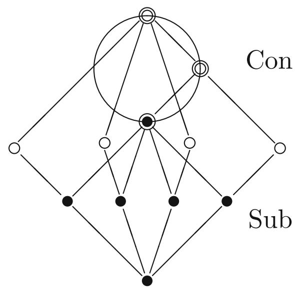
Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice. More about weak congruences and the corresponding lattices can be found in [27] (see also [33]).

In the case of a group G, a particular construction of the weak congruence lattice $\mathrm{Con}_{\mathrm{w}}(G)$ is possible inside of the square $L(G)^2 = L(G) \times L(G)$ of the subgroup lattice $L(G) = \mathrm{Sub}(G)$. Writing $N(X)$ for the lattice of normal subgroups of $X \in L(G)$, we see that the set

$$L(G)^C_{\geq} = \{\, (X,Y) \mid X \in L(G),\ Y \in N(X)\}$$

is closed under arbitrary meets in $L(G)^2$, hence is a complete lattice, and the map $\theta \mapsto (G\theta,\, e\theta)$ (where e is the neutral element) turns out to be an isomorphism between $\mathrm{Con}_{\mathrm{w}}(G)$ and $L(G)^C_{\geq}$. As demonstrated in [7] and [27], weak congruence lattices of groups are quite useful for various group-theoretical investigations.

Every group G has a modular congruence lattice isomorphic to $N(G)$, whence every Dedekind group has a modular subgroup lattice; however, there are also many other groups G with modular $L(G)$ but non-modular $L(G)^C_{\geq}$, the simplest example being the symmetric group $G = S_3$.



The point is that, by Theorem 1.1, Dedekind groups are characterized by the modularity of their *weak* congruence lattices $\mathrm{Con}_{\mathrm{w}}(G) \subseteq L(G^2)$; compare this with the result of Lukács and Pálfy [20] that the whole of $L(G^2)$ is modular if and only if G is abelian. For a comprehensive investigation of subgroup lattices and their properties like modularity, distributivity etc., the reader is referred to the monograph by R. Schmidt [26] (see also Birkhoff [2, Ch. VII], Ore [24], and Suzuki [29] for earlier sources).

Replacing the normal closure of subgroups with a general categorical closure operator $C = (C_x \mid x \in L)$ on any lattice L (where each C_x is a closure map on the principal ideal L_x, see Section 2), we shall construct a certain lattice contained in the square L^2, viz. the *characteristic triangle*

$$L^C_{\geq} = \{(x,y) \in L^2 \mid x \geq y = C_x(y)\} = \{(x,y) \in L^2 \mid y \in L^C_x\},$$

where L^C_x denotes the range (= fixpoint set) of the closure map C_x on the principal ideal generated by x. As we shall see, important and valuable information about the closure operator is coded in the characteristic triangle. This will enable us to prove an element-free generalization of Theorem 1.1, saying that L is modular and equal to L^C_1 (where 1 is the top element of L) if and only if $L^C_{\geq}$ is modular and the *"discrete"* elements, i.e., the elements x with $L^C_x = L_x$, are join-dense in L. This and related results on the *"corner element"* $(1,0)$ of $L^C_{\geq}$ (the abstract counterpart of the diagonal element Δ of the weak congruence lattice $\mathrm{Con_w}(A)$) will apply not only to groups but also to more general *group-like algebras*.

2. Lattices with closure operators

We shall make use of the fact that the class of algebraic lattices and that of continuous lattices are closed under the formation of direct products, complete sublattices (closed under arbitrary joins and meets) and intervals (see, e.g., [16, Ch. I]). In particular, for any algebraic or continuous lattice L, each principal ideal $L_x = {\downarrow}x$, the *square* $L^2 = L \times L$, and the *triangle*

$$L_{\geq} = \{(x,y) \in L^2 \mid x \geq y\},$$

which is closed under arbitrary (coordinatewise) joins and meets in the square, are again algebraic or continuous lattices, respectively.

Henceforth, let L be a lattice. A *closure range* in L is a subset M such that for each $x \in L$ there is a least $y \in M$ with $x \leq y$; in case L is complete, the latter means that M is closed under arbitrary meets in L. A *closure map* (or *closure operation*) on L is an extensive, isotone (= order preserving) and idempotent self-map of L, or equivalently, a map $c\colon L \to L$ such that $y \leq c(z) \Leftrightarrow c(y) \leq c(z)$. Associating with any such closure map its range $c[L]$, one obtains a dual isomorphism Φ between the pointwise ordered set of all closure maps (which is complete if L is) and that of all closure ranges (ordered by inclusion). We avoid here the terms *closure operator* and *closure system*, because on the one hand, they are often reserved to the classical set-theoretical case where L is a power set lattice, and on the other hand, we wish to prevent confusion with the categorical notion of closure operator (see, e.g., [8]).

In order to ensure that a subposet M of an algebraic or continuous lattice L is an algebraic or continuous lattice, too, it suffices to require that M be closed under arbitrary meets and under *directed* joins (*up-closed*); although the compact elements of M may differ from those of L, they are just the *closures* of the compact elements of L (see [16, Ch. I–4]). A map $f\colon L \to M$ between complete lattices is called *(Scott) continuous* if it preserves directed joins, i.e., $f(\bigvee D) = \bigvee f[D]$ whenever D is directed (see [15], [16]). Notice that every continuous map f is isotone, i.e., $x \leq y$ implies $f(x) \leq f(y)$. It is straightforward to check that the above dual isomorphism Φ induces a one-to-one correspondence between continuous closure maps and up-closed closure ranges in a complete lattice. In particular, the range of any continuous closure map on an algebraic or continuous lattice is again algebraic or continuous, respectively (cf. [16, Ch. I–4]).

Observe that for any closure map $c\colon L \to L$, joins in the range $c[L]$ are given by $\bigvee_{c[L]} Y = c(\bigvee_L Y)$, and the surjective corestriction of c from L onto $c[L]$ preserves arbitrary joins, whereas in general, c itself neither preserves finite joins (as in the topological case) nor directed joins (as in most algebraic situations).

Now, before introducing a new and central notion, we briefly outline its categorical background. As it is well known, any lattice or ordered set $(L, \leq)$ may be regarded as a small category $\mathbf{L}$, with L as the set of objects and all pairs in the order relation as morphisms. Under that categorical perspective, a *closure operator* on L or, more precisely, on the isomorphic category of all principal ideals of L, is a family $C = (C_x \mid x \in L)$ of maps $C_x\colon L_x \to L_x$ such that for all $x, y, z \in L$,

$$y \leq z \leq x \ \text{ implies } \ y \leq C_z(y) \leq C_x(y) \leq C_x(z).$$

In order to avoid confusion with closure maps, one could speak of *local closure operators*, but we follow the general convention of category theorists and omit the word *"local"*. The reader may refer to [8] for the theory of categorical closure operators and to [1] for more categorical background.

Deviating from [8], we shall assume throughout that each C_x is a closure map in the previous sense. In other words, for us, a *closure operator* on a lattice L is a family $C = (C_x \mid x \in L)$ of isotone maps $C_x\colon L_x \to L_x$ such that

C1: $y \leq C_x(y) = C_x(C_x(y))$ for all $y \leq x$ in L;
C2: $C_z(y) \leq C_x(y)$ for all $y \leq z \leq x$ in L.

Under that hypothesis, we call (L, C) a *lattice with closure* and put

$$L_x^C = C_x[L_x] = \{y \in L_x \mid C_x(y) = y\}\,.$$

If the lattice L is bounded by a least element 0 and a greatest element 1, a closure operator C on L is said to be *grounded* (see [8]) if

C0: $C_x(0) = 0$ for all $x \in L$.

In categorical contexts, Axiom **C2** is often referred to as the continuity axiom, but in order to make the machinery work in the desired area, we have to consider here the stronger notion of Scott continuity. Namely, by a *continuous closure operator* on a complete lattice L we mean a family of closure maps C_x on the principal ideals L_x such that, instead of **C2**, the following two conditions are fulfilled:

C3: each C_x is continuous, i.e., C_x preserves directed joins;

C4: $C_{\bigvee D}(y) = \bigvee\{\, C_x(y) \mid x \in D,\ x \geq z\}$, for any directed subset D of L, any $z \in D$ and each element $y \in L_z$.

A pair (L, C) consisting of a complete lattice L and a closure operator C on L satisfying **C1**, **C3** and **C4** will be referred to as a *complete lattice with continuous closure*. In order to see that **C4** entails **C2**, consider $D = \{x, z\}$. Notice that for any closure operator C on a complete lattice L and each $x \in L$,

$$C_x(y) = \bigwedge\{\, z \in L_x^C \mid y \leq z\}.$$

The structure of (L, C) may be recovered from the *characteristic triangle*

$$L_{\geq}^C = \{\, (x, y) \mid x \in L,\ y \in L_x^C\} = \{\, (x, y) \in L_{\geq} \mid C_x(y) = y\}\,.$$

By definition, a closure operator is grounded if and only if $(1, 0) \in L_{\geq}^C$. In that case, the principal ideal generated by $(1, 0)$ in $L_{\geq}^C$ is isomorphic to L via projection onto the first coordinate. On the other hand, the principal filter generated by $(1, 0)$ is isomorphic to L_1^C, and more generally, the interval $[(x, 0), (x, x)]$ in $L_{\geq}^C$ is isomorphic to L_x^C via projection onto the second coordinate. Hence, any lattice identity carries over from $L_{\geq}^C$ to each of the lattices L_x^C, whereas the converse fails (see Example 2.3).

Theorem 2.1. *If (L, C) is a lattice with closure, then $L_{\geq}^C$ is a closure range in $L_{\geq}$ (and so in L^2). The corresponding closure map on $L_{\geq}$ is determined by*

$$C^*(x, y) = (x, C_x(y)) \text{ for } (x, y) \in L_{\geq}.$$

If L is complete, then $L_{\geq}^C$ is closed under arbitrary meets, and if C is continuous, then $L_{\geq}^C$ is also closed under directed joins in $L_{\geq}$ (and in L^2). The assignment $C \mapsto C^*$ yields a one-to-one correspondence between the (continuous) closure operators on L and the (continuous) closure maps on $L_{\geq}$ keeping the first coordinate fixed.

Proof. For $(x, y) \in L_{\geq}$ and $(u, v) \in L_{\geq}^C$, we have $(x, y) \leq (u, v) \Leftrightarrow (x, C_x(y)) \leq (u, v)$ (indeed, $y \leq x \leq u$ and $y \leq v \leq u$ imply $C_x(y) \leq C_u(y) \leq C_u(v) = v$ if $v \in L_u^C$), showing that $(x, C_x(y))$ is the closure of (x, y) with respect to the closure range $L_{\geq}^C$. In particular, $L_{\geq}^C$ is closed under arbitrary meets in $L_{\geq}$ if L is complete.

Now consider a directed family of elements (x_i, y_i) in $L_{\geq}^C$ $(i \in I)$. The index set I may be directed by $i \leq j \Leftrightarrow (x_i, y_i) \leq (x_j, y_j)$. Forming the directed joins $x_\vee = \bigvee_{i \in I} x_i$, $y_\vee = \bigvee_{i \in I} y_i$ and using first **C3** and then **C4**, we obtain

$$C_{x_\vee}(y_\vee) = \bigvee_{j \in I} C_{x_\vee}(y_j) = \bigvee_{j \in I} \bigvee_{x_i \geq x_j} C_{x_i}(y_j) \leq \bigvee_{k \in I} C_{x_k}(y_k) = \bigvee_{k \in I} y_k = y_\vee$$

(since $C_{x_i}(y_j) \leq C_{x_k}(y_k)$ for $i, j \leq k$ by **C2**), and therefore, $y_\vee \in L_{x_\vee}^C$, $(x_\vee, y_\vee) \in L_{\geq}^C$. Thus, if C is a continuous closure operator, then $L_{\geq}^C$ is up-closed in $L_{\geq}$, and hence, C^* is a continuous closure map.

Let p_1 and p_2 denote the first and second projection from $L_{\geq}$ onto L, respectively. By definition, $p_1 \circ C^*(x, y) = x = p_1(x, y)$, and the original closure operator C is obtained by $C_x(y) = p_2 \circ C^*(x, y)$. Conversely, let

c be a closure map on $L_{\geq}$ with $p_1 \circ c = p_1$, and define $C_x\colon L_x \to L_x$ by $C_x(y) = p_2 \circ c(x,y)$. Then

$$\begin{aligned} y \leq C_x(z) &\Leftrightarrow (x,y) \leq (x, C_x(z)) = c(x,z) \Leftrightarrow c(x,y) \leq c(x,z) \\ &\Leftrightarrow C_x(y) \leq C_x(z), \end{aligned}$$

showing that each C_x is a closure map with $c(x,y) = (x, C_x(y))$, since $p_1 \circ c = p_1$. If c is continuous, then for directed $D \subseteq L_x$, respectively $D \subseteq L$ and $y \leq z \in D$, we get

$$\begin{aligned} C_x(\bigvee D) &= p_2(c(x,\bigvee D)) = p_2(\bigvee c[\{x\}\times D]) = \bigvee p_2[\,c[\{x\}\times D]] = \bigvee C_x[D]; \\ C_{\bigvee D}(y) &= p_2 \circ c(\bigvee D, y) = p_2 \circ c(\bigvee\{(x,y) \mid x \in D,\, x \geq z\}) \\ &= p_2(\bigvee\{c(x,y) \mid x \in D,\, x \geq z\}) = \bigvee\{C_x(y) \mid x \in D,\, x \geq z\}. \end{aligned}$$

Thus, $C = (C_x \mid x \in L)$ is a continuous closure operator on L. □

The first projection from $L^C_{\geq}$ onto L preserves arbitrary joins and meets. Hence, it transfers any finite or infinite lattice identity from $L^C_{\geq}$ to L and to each of the principal ideals L_x. For example, if $L^C_{\geq}$ is modular then so is L — but not conversely (see the introduction and Example 2.3 below). However, there are important lattice properties that are transferred from L to $L^C_{\geq}$ and vice versa:

Corollary 2.2. *Let (L, C) be any complete lattice with continuous closure. Then $L^C_{\geq}$ is continuous or algebraic, respectively, if and only if so is L.*

In general, a complete homomorphism need not preserve compactness, nor algebraicity: for example, the real unit interval is the image of the algebraic Cantor discontinuum under the complete homomorphism identifying any two adjacent endpoints—but $[\,0,1\,]$ has no compact elements except 0. However, it can be shown that the first projection from $L^C_{\geq}$ onto L always preserves compactness and the way-below relation.

We conclude this section with a few (large classes of) instructive examples.

Example 2.3. The primary situation we are concerned with in the present note is that of a group G and its subgroup lattice $L = L(G)$. For $Y \leq X$ in L, let $C_X(Y)$ denote the normal subgroup of X generated by Y. Then L^C_X is the lattice of normal subgroups of X; (L, C) is an algebraic lattice with continuous closure, and

$$L^C_{\geq} = \{(X,Y) \mid X \in L,\, Y \in L^C_X\}$$

is an algebraic lattice isomorphic to the weak congruence lattice $\mathrm{Con_w}(G)$ (see the introduction and Proposition 5.3). Moreover, C is clearly grounded. Whereas the normal subgroup lattice L^C_X is modular for each $X \in L$, Theorem 1.1 states that the characteristic triangle $L^C_{\geq}$ is modular if and only if G is a Dedekind group.

Example 2.4. Let L be a *meet-continuous* [2] or *upper continuous* [3] lattice, i.e., a complete lattice enjoying the following identity for all $x \in L$ and all directed $Y \subseteq L$:

(d) $x \wedge \bigvee Y = \bigvee\{x \wedge y \mid y \in Y\}$.

If c is any continuous closure map on L, then the equation $C_x(y) = x \wedge c(y)$ defines a set of continuous closure maps $C_x \colon L_x \to L_x$ with $C = (C_x \mid x \in L)$ being a continuous closure operator.

Note that any continuous (and so any algebraic) lattice is meet-continuous. Hence, in case L is algebraic (or continuous), the characteristic triangle

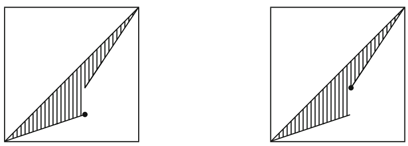
$$L^C_{\geq} = \{(x, y) \in L^2 \mid y = c(y) \leq x\}$$

is again algebraic (or continuous), and $C^*(x, y) = (x, x \wedge c(y))$ defines a continuous closure map on $L_{\geq}$. In particular, this applies to any set-theoretical algebraic closure operator like the subalgebra or congruence generator of an arbitrary algebra.

Example 2.5. As we saw, a typical continuous but not algebraic lattice is the unit interval $L = [0, 1]$. Let $f \colon L \to L$ be an isotone contraction (i.e., $x \leq y$ implies $f(x) \leq f(y) \leq y$). Then $C_x(y) = \max\{y, f(x)\}$ defines a closure operator on L, and

f preserves arbitrary nonempty joins
$\Leftrightarrow$ f is continuous from the left (i.e., Scott continuous)
$\Leftrightarrow$ C is continuous
$\Leftrightarrow$ the closure map c_f on $L_{\geq}$ with $c_f(x, y) = (x, C_x(y))$ is continuous.

But C is grounded only for the zero map $f(x) = 0$, where c_f is the identity map.

Continuous  *Not continuous*

Two characteristic triangles.

Example 2.6. A *nucleus* on a lattice is a closure map c with $c(x \wedge y) = c(x) \wedge c(y)$; for closure maps, this equation is equivalent to the formally weaker condition $x \wedge c(y) \leq c(x \wedge y)$. In view of Example 2.4, any continuous nucleus c on a meet-continuous lattice induces a continuous closure operator C by $C_x(y) = x \wedge c(y)$. Nuclei play an important role in the theory of *locales* or *frames* (enjoying the distributive law (d) for arbitrary $Y \subseteq L$) and their applications in logic and pointfree topology (see, e.g., Johnstone [20]). As we shall see in Proposition 4.1, a Boolean frame has only one grounded closure operator — but there are many nuclei $\vee_a \colon x \mapsto x \vee a$.

Of course, nuclei also occur in other parts of algebra. For example, an algebra A has the *Congruence Intersection Property* if and only if the closure map c associated with $\mathrm{Con}(A)$ induces a nucleus on the weak congruence lattice $\mathrm{Con_w}(A)$ (see, e.g., [27]). The corresponding closure operator C given by $C_\theta(\rho) = \theta \cap c(\rho)$ is then continuous.

Or, let A be an algebra in a congruence modular variety; let L be the lattice of tolerance relations (i.e., reflexive and symmetric relations compatible with the operations) of A, and as before, let $c(\rho)$ stand for the congruence generated by $\rho \in L$. Then (L, c) is an algebraic closure lattice, c is a nucleus (cf. [4, 5, 6]), and putting $C_\theta(\rho) = \theta \cap c(\rho)$ again yields a continuous closure operator C on L.

3. Distributive, standard, neutral, and modular elements

Recall from [17, Ch. III] the following notions which play a fundamental role in the structure and decomposition theory of lattices: an element a of a lattice L is

distributive if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$,
standard if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$,
neutral if $(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$

for all $x, y \in L$. It is known that each of the following properties equivalently characterizes neutral elements a (see Grätzer and Schmidt [18] and [17, Ch. III. 2, Theorems 3 and 4]):

- a is a codistributive (i.e., dually distributive) standard element;
- a together with any two other elements generates a distributive sublattice;
- a is mapped onto $(1, 0)$ by an embedding of L in a product lattice $A \times B$.

In modular lattices, the notions of (co)distributive, standard and neutral elements coincide (see Birkhoff [2, Ch. II, Theorem 12]). More specifically, call an element a of an arbitrary lattice L *s-modular* if

$$x \wedge (a \vee y) = (x \wedge a) \vee y \text{ for all } x, y \in L \text{ with } x \geq y\,,$$

or equivalently, if

$$a \wedge x = a \wedge y\,,\ a \vee x = a \vee y \text{ and } x \geq y \text{ imply } x = y.$$

From the cited sources, one easily derives that *an element is standard if and only if it is distributive and s-modular.* For various aspects of the above kinds of special elements in the theory of weak congruence lattices, refer to [27].

Now, let C be a grounded closure operator on a lattice L and recall that the $L^C_\geq$-closure of elements $(x, y) \in L_\geq$ is $(x, C_x(y))$. A "central" role in the structure theory of (L, C) is played by the element $(1, 0)$ of $L^C_\geq$, the abstract counterpart of the diagonal $\Delta \in \mathrm{Con_w}(G)$ in the group case. Let us state some of its properties.

Proposition 3.1. *Let L be a bounded lattice with grounded closure C. Then the "corner element" $(1, 0)$ generates a principal ideal isomorphic to L and a* *principal filter isomorphic to* L_1^C*. The element* $(1,0)$ *is always a codistributive element of* $L_{\geq}^C$*; hence, it is neutral if and only if it is a standard element of* $L_{\geq}^C$*. Furthermore,*

(1) $(1,0)$ *is a distributive element of* $L_{\geq}^C$ *if and only if* C_1 *is a nucleus, that is,* C_1 *preserves finite meets,*
(2) $(1,0)$ *is an s-modular element of* $L_{\geq}^C$ *if and only if* C *is hereditary, that is,* $C_x(y) = x \wedge C_1(y)$ *for* $y \in L_x$*,*
(3) $(1,0)$ *is a standard (neutral) element of* $L_{\geq}^C$ *if and only if for* $y, z \in L_x$*,*

$$C_x(y \wedge z) = x \wedge C_1(y) \wedge C_1(z)$$

or, equivalently,

$$z \leq C_1(y) \text{ implies } z = C_z(y \wedge z).$$

Proof. $(1,0)$ is codistributive in $L_{\geq}^C$ on account of the equations (formed in $L_{\geq}^C$)

$$\begin{aligned}(1,0) \wedge ((x,u) \vee (y,v)) &= (x \vee y, 0) = (x \vee y, C_{x\vee y}(0)) \\ &= ((1,0)\wedge(x,u)) \vee ((1,0)\wedge(y,v)).\end{aligned}$$

(1): $(1,0)$ is distributive if and only if

$$(1,0) \vee ((x,u) \wedge (y,v)) = ((1,0)\vee(x,u)) \wedge ((1,0)\vee(y,v)),$$

i.e., $C_1(u \wedge v) = C_1(u) \wedge C_1(v)$ for all $u, v \in L$ (consider the case $u = x$, $v = y$ in order to verify the necessity of the latter condition).

(2): $(1,0)$ is an s-modular element of $L_{\geq}^C$ if and only if

$$((x,u) \wedge (1,0)) \vee (y,v) = (x,u) \wedge ((1,0) \vee (y,v)) \text{ for all } (x,u) \geq (y,v) \text{ in } L_{\geq}^C,$$

i.e., $C_x(v) = u \wedge C_1(v)$ for all $(x,u) \geq (y,v)$ in $L_{\geq}^C$, and that implies $C_x(y) = x \wedge C_1(y)$ for all $x \geq y$ in L (take again $u = x, v = y$). For the converse, note that $(x,u) \geq (y,v)$ in $L_{\geq}^C$ and $C_u(v) = u \wedge C_1(v)$ imply $C_x(v) \geq u \wedge C_1(v) \geq C_x(v)$.

(3): Suppose $(1,0)$ is standard, i.e., distributive and s-modular. Then, by the previous equivalences, $z \leq C_1(y)$ entails $C_z(y \wedge z) = z \wedge C_1(y) \wedge C_1(z) = z$. Conversely, assume the equality $z = C_z(y \wedge z)$ holds for $z \leq C_1(y)$. Given $x, y \in L$, put $z = x \wedge C_1(y)$. Then $z = C_z(y \wedge z) \leq C_1(x \wedge y)$; whence, C_1 is a nucleus. And if $y \leq x$, then $z \leq C_x(y) \leq x \wedge C_1(y)$, showing that C is hereditary. Thus, $(1,0)$ is distributive and s-modular, i.e., a standard element. □

Corollary 3.2. *Suppose* C *is a grounded closure operator on a bounded lattice* L *and* $(1,0)$ *is a standard element of* $L_{\geq}^C$*. Then* $z \leq C_1(y)$ *and* $y \wedge z \in L_z^C$ *imply* $z \leq y$*. In particular, the only* $z \leq C_1(y)$ *with* $y \wedge z = 0$ *is* $z = 0$*.*

Recall that a bounded lattice L is *disjunctive* (Wallman [34]) if

for $x \not\leq y$ in L, there is a $z \in L$ with $x \wedge z \neq 0$, but $y \wedge z = 0$.

This is equivalent to postulating that

for $y < x$ in L, there is a $z \in L$ with $0 < z \leq x$ and $y \wedge z = 0$.

Large classes of disjunctive lattices are formed by

– all sectionally complemented lattices (and so by all modular complemented lattices)
– all atomistic lattices (and so by all geometric lattices and all dual T_1-topologies).

Proposition 3.3. *A closure operator C on a disjunctive lattice L is the identity operator if and only if $(1,0)$ is a standard (neutral) element of $L^C_{\geq}$.*

Proof. Clearly, if $C_1 = \mathrm{id}_L$, then $(1,0)$ is neutral (hence, standard) in $L^C_{\geq} = L_{\geq}$. Conversely, if $(1,0)$ is a standard element of $L^C_{\geq} = L_{\geq}$, then C is grounded, and by Corollary 3.2, $z \leq C_1(y)$ and $y \wedge z = 0$ imply $z = 0$. Hence, $y < C_1(y)$ cannot occur in case L is disjunctive. Thus, C_1 is the identity on L (and so $C_x = \mathrm{id}_{L_x}$). □

Next, consider the map

$$\varphi_C \colon L^C_{\geq} \to L \times L^C_1, \text{ where } (x,u) \mapsto (x, C_1(u)) .$$

Proposition 3.4. *For any grounded closure operator C on a bounded lattice L, the map φ_C is a $\vee$-homomorphism, and*

(1) $(1,0)$ *is a distributive element of* $L^C_{\geq}$ $\Leftrightarrow$ φ_C *is a lattice homomorphism;*
(2) $(1,0)$ *is an s-modular element of* $L^C_{\geq}$ $\Leftrightarrow$ φ_C *is injective (an order embedding);*
(3) $(1,0)$ *is a standard element of* $L^C_{\geq}$ $\Leftrightarrow$ φ_C *is a lattice embedding.*

Proof. By the closure properties of C_1, it is clear that $C_1(u \vee v)$ is the join of $C_1(u)$ and $C_1(v)$ in L^C_1; consequently, φ_C is a $\vee$-homomorphism.

(1): φ_C is a $\wedge$-homomorphism (hence, a lattice homomorphism) if and only if C_1 is one, which is tantamount to distributivity of $(1,0)$, by Proposition 3.1.

(2): If φ_C is injective and $x \geq y$, then for $u = C_x(y)$ and $v = C_x(x \wedge C_1(y))$, the equation $(x, C_1(u)) = (x, C_1(y)) = (x, C_1(x \wedge C_1(y))) = (x, C_1(v))$ entails $u = v$; hence, $C_x(y) \geq x \wedge C_1(y)$, and the reverse inequality is clear. Thus, C is hereditary, i.e., $(1,0)$ is s-modular by Proposition 3.1. Conversely, if the latter holds, then for $(x,u),(y,v) \in L^C_{\geq}$, the equation $(x, C_1(u)) = (y, C_1(v))$ entails $x = y$ and then $u = C_x(u) = x \wedge C_1(u) = y \wedge C_1(v) = C_y(v) = v$, proving injectivity of φ_C.

(3) follows from the previous two equivalences. □

We are now ready for the main result of this section, providing a modularity criterion for characteristic triangles (see [27] for a similar result on weak congruences).

Theorem 3.5. *Let C be a grounded closure operator on a bounded lattice L, and let $\mathcal{V}$ be a variety of modular lattices. Then $L^C_{\geq}$ is modular (resp. a member of $\mathcal{V}$) if and only if L is modular (resp. a member of $\mathcal{V}$) and $(1,0)$ is a neutral element of $L^C_{\geq}$.*

Proof. If $L^C_{\geq}$ is modular then so is L, as observed earlier; also, $(1,0)$ is a neutral element of $L^C_{\geq}$, being codistributive (see Proposition 3.1). Conversely, if L is modular and $(1,0)$ is neutral then, by Proposition 3.1 again, L^C_1 is modular as well, being the homomorphic image of L under the nucleus C_1. By Proposition 3.4, the map φ_C is a lattice embedding of $L^C_{\geq}$ in $L \times L^C_1$; whence, $L^C_{\geq}$ is modular, too. Analogous reasoning holds for any equational property stronger than modularity. □

4. Discrete elements

As before, let C be a closure operator on a lattice L. With the obvious spatial interpretation in mind, we call an element $x \in L$ (C-)*discrete* if C_x is the identity map, i.e., $L^C_x = L_x$. Notice that if x is C-discrete, then so is each $z \leq x$, because of the inequality $C_z(y) \leq C_x(y) = y$ for $y \leq z \leq x$ (see **C2**). Thus, by definition, the following conditions are equivalent.

(1) The top element of L is C-discrete, i.e., $C_1 = \mathrm{id}_L$.
(2) Each element of L is C-discrete.
(3) C is the identity operator, i.e., $C_x = \mathrm{id}_{L_x}$ for each $x \in L$.
(4) The characteristic triangle $L^C_{\geq}$ is the whole triangle $L_{\geq}$.

Seemingly weak assumptions together with the modularity of $L^C_{\geq}$ already force C to be the identity operator. By Proposition 3.3, disjunctivity of L is such a hypothesis. In particular, we have:

Proposition 4.1. *A grounded closure operator C on a complemented lattice L has a modular characteristic triangle $L^C_{\geq}$ if and only if L is modular and C is the identity operator.*

Proof. An alternative argument is the following. If $L^C_{\geq}$ is modular, then so is L. But a $\wedge$- and 0-preserving closure map c on a complemented modular lattice (like C_1 on $L = L_1$, by virtue of Proposition 3.1) must be the identity map (cf. [9]), since for complementary elements x and x',

$$0 = c(0) = c(x \wedge x') = c(x) \wedge c(x')\,;\; 1 = x' \vee x = c(x') \vee x\,;$$

$$c(x) = c(x) \wedge (c(x') \vee x) = (c(x) \wedge c(x')) \vee x = 0 \vee x = x.$$ □

A similar result is obtained if the complementation property is replaced by a rich supply of discrete elements (see Corollary 4.3 below). The key to group-theoretical and other algebraic applications is:

Theorem 4.2. *Let C be a closure operator on a bounded lattice L whose elements are joins of C-discrete elements. Then the following statements are equivalent.*

(1) *The top element* 1 *is C-discrete, i.e., C is the identity operator.*
(2) $L^C_{\geq}$ *is a sublattice of* L^2 *containing* $(1,0)$.
(3) $(1,0)$ *is a standard (equivalently, a neutral) element of* $L^C_{\geq}$.

Proof. Notice first that each of the conditions (1)–(3) entails the groundedness of C. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are straightforward. For $(3) \Rightarrow (1)$, apply Corollary 3.2: each element $C_1(y)$ is a join of C-discrete elements z, and these satisfy $z \leq y$; whence, $C_1(y) = y$. □

Now, invoking Theorem 3.5, we arrive at

Corollary 4.3. *Let C be a grounded closure operator on a bounded lattice L. Then L is a modular lattice whose top element (and so, each element) is C-discrete if and only if the characteristic triangle $L^C_{\geq}$ is modular and the C-discrete elements of L are join-dense. In this equivalence, "modular" may be replaced by "distributive".*

Since atoms are certainly discrete for any grounded closure operator, we obtain:

Corollary 4.4. *If L is an atomistic lattice, then a grounded closure operator C on L with modular $L^C_{\geq}$ must be the identity operator. This applies to any set-theoretical closure operator on a power set. Thus, the only topological spaces whose closure operator has a modular characteristic triangle are the discrete ones.*

Note that this corollary also immediately follows from Proposition 3.3.

5. Applications to group-like algebras

We deduce now various consequences of the previous lattice-theoretical results in general algebra; some of them have been stated earlier (cf. [27]), but the original arguments relied on Theorem 1.1, whose proof in [28] was erroneous. The first complete proofs are based on Theorems 3.5 and 4.2. Let us recall from [27] a few facts about the weak congruence lattices $\mathrm{Con_w}(A)$ of arbitrary algebras A (cf. Proposition 3.1):

- The diagonal Δ is always a codistributive element of $\mathrm{Con_w}(A)$.
- Δ is a distributive element of $\mathrm{Con_w}(A)$ if and only if A has the Congruence Intersection Property (CIP), requiring that the congruence generating closure operator of A preserves finite intersections of weak congruences.
- Δ is an s-modular element of $\mathrm{Con_w}(A)$ if and only if A has the Congruence Extension Property (CEP), requiring that every congruence on a subalgebra is induced by a congruence on A.
- Δ is a standard (equivalently, a neutral) element of $\mathrm{Con_w}(A)$ if and only if A has the CIP and the CEP, while Δ is always a neutral element of $\mathrm{Con}(A)$.
- The weak congruence lattice $\mathrm{Con_w}(A)$ is modular if and only if $\mathrm{Sub}(A)$ and $\mathrm{Con}(A)$ are modular and Δ is a neutral element of $\mathrm{Con_w}(A)$.

First, we focus on the special case of groups. For each group G, the discrete elements of $L(G) = \mathrm{Sub}(G)$ with respect to the normal closure operator C

(see Example 2.3) are just the Dedekind subgroups. Each group is the union of its cyclic subgroups, which are, of course, Dedekind subgroups. Since the lattice of all normal subgroups of G is modular and there is an isomorphism between $\mathrm{Con_w}(G)$ and $L(G)^C_\geq$ sending Δ to $(1,0) = (G,\{e\})$, Corollary 4.3 applies to establish Theorem 1.1. Moreover, from Theorems 3.5 and 4.2 we derive a stronger result:

Corollary 5.1. *The following statements on a group G are equivalent.*

(1) *G is a Dedekind group.*
(2) *$\mathrm{Con_w}(G)$ is modular.*
(3) *Δ is a standard (equivalently, a neutral) element of $\mathrm{Con_w}(G)$.*
(4) *G has the CIP and the CEP.*

A further immediate consequence of Corollary 4.3 and Ore's Theorem, which says that the locally cyclic groups are exactly those with a distributive subgroup lattice (see [24] and [26, Thm 1.2.3]), is the following

Corollary 5.2. *A group is locally cyclic if and only if its weak congruence lattice is distributive.*

As the reader might guess, Theorems 3.5, 4.2 and their corollaries also apply to algebras other than groups. To extract the essential ingredient, we call a general algebra A *group-like* if it has a least subuniverse $\{e\}$ and there is some function $q\colon A^2 \to A$ (not necessarily an algebraic one) such that for all $\theta \in \mathrm{Con_w}(A)$,

$$a\,\theta\,b \;\Leftrightarrow\; e\,\theta\,q(a,b) \text{ and } a,b \in A\theta\ .$$

Of course, in groups, $q(a,b) = ab^{-1}$ is such a function (other examples will be discussed later on). As in the group case, in any algebra with a least subuniverse $\{e\}$, the congruence classes $e\theta$ are precisely the *kernels* $\varphi^{-1}(e')$ of homomorphisms φ from A to similar algebras A' with least subuniverses $\{e'\}$.

Proposition 5.3. *Let $L = \mathrm{Sub}(A)$ be the algebraic lattice of all subuniverses (subalgebras) of a group-like algebra A. For each subalgebra X, the algebraic closure system $L_X = \mathrm{Sub}(X)$ contains the algebraic closure system $L^C_X = \{e\theta \mid \theta \in \mathrm{Con}(X)\}$, which is isomorphic to $\mathrm{Con}(X)$. The corresponding closure maps C_X define a grounded closure operator C so that*

$$\Psi\colon \mathrm{Con_w}(A) \to L^C_\geq = \{(X,Y) \mid X\in L,\ Y\in L^C_X\},\ \textit{where}\ \theta \mapsto (A\theta,\ e\theta)$$

is an isomorphism of algebraic lattices. Hence, the weak congruence lattice of A is isomorphic to the characteristic triangle of $\mathrm{Sub}(A)$. If A has the CEP, then C is hereditary and continuous.

Proof. Since $\{e\}$ is a subuniverse, so is each congruence class $e\theta$ for $\theta \in \mathrm{Con}(X)$, and the equations

$$e\big(\bigcap\{\theta_i \mid i\in I\}\big) = \bigcap\{e\theta_i \mid i\in I\} \ \text{ and }\ e\big(\bigcup\{\theta_i \mid i\in I\}\big) = \bigcup\{e\,\theta_i \mid i\in I\}$$

for $\theta_i \in \mathrm{Con}(X)$ (and unions over directed systems) show that not only $\mathrm{Con}(X)$ but also $L_X^C = \{e\theta \mid \theta \in \mathrm{Con}(X)\}$ is an algebraic closure system, hence closed under arbitrary meets and directed joins in L. Therefore, the corresponding closure map C_X preserves directed joins (= unions). In order to ensure that $C = (C_x \mid x \in L)$ is a closure operator, it remains to verify **C2**. Let $Y \leq Z \leq X$ in L. Then $C_X(Y) = e\theta$ for some $\theta \in \mathrm{Con}(X)$, while $C_Z(Y) = e\rho$ for some $\rho \in \mathrm{Con}(Z)$. Since $Y \subseteq Z \cap e\theta = e\,\theta|_Z$ and $\theta|_Z \in \mathrm{Con}(Z)$, we conclude $C_Z(Y) \subseteq e\,\theta|_Z \subseteq e\theta = C_X(Y)$. The equality $C_X(\{e\}) = \{e\}$ means that the closure operator C is grounded. For $\theta, \rho \in \mathrm{Con_w}(A)$, the implications

$$\begin{aligned}\theta \subseteq \rho \;\Rightarrow\; & A\theta \subseteq A\rho \text{ and } e\theta \subseteq e\rho \\ \Rightarrow\; & \theta = \{(a,b) \in (A\theta)^2 \mid e\,\theta\, q(a,b)\} \subseteq \{(a,b) \in (A\rho)^2 \mid e\,\rho\, q(a,b)\} = \rho\end{aligned}$$

ensure that Ψ is an embedding of $\mathrm{Con_w}(A)$ in $L_{\geq}^C$, and in particular, that $\mathrm{Con}(X)$ is isomorphic to L_X^C via $\theta \mapsto e\theta$. Concerning the surjectivity of Ψ, simply observe that for $X \in L$ and $Y = e\theta \in L_X^C$ with $\theta \in \mathrm{Con}(X)$, we have $\theta \in \mathrm{Con_w}(A)$ and $X = A\theta$.

For the last statement in Proposition 5.3, see Example 2.4. □

Generalizing the group case, we call a group-like algebra A a *Dedekind algebra* if every subalgebra of A is a kernel, i.e., of the form $e\theta$ for some $\theta \in \mathrm{Con}(A)$. By the inclusion $\{e\} \subseteq X$ for $X \in \mathrm{Sub}(A)$, this is equivalent to saying that A is *Hamiltonian*, i.e., every subalgebra is a congruence class. Now we are in a position to derive from Theorems 3.5 and 4.2 the following generalization of Corollary 5.1:

Theorem 5.4. *Let A be a group-like algebra that is a join of Dedekind subalgebras. Then the following statements are equivalent.*

(1) *A is a Dedekind algebra.*
(2) $\mathrm{Con_w}(A)$ *admits an isomorphism onto* $\mathrm{Sub}(A)_{\geq}$ *sending* Δ *to* $(A, \{e\})$.
(3) $\mathrm{Con_w}(A)$ *admits a lattice embedding in* $\mathrm{Sub}(A)^2$ *sending* Δ *to* $(A, \{e\})$.
(4) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(A)$.
(5) *A has the* CIP *and the* CEP.

Moreover, the weak congruence lattice $\mathrm{Con_w}(A)$ *is modular (distributive) if and only if A is a Dedekind algebra with modular (distributive) subalgebra lattice* $\mathrm{Sub}(A)$.

Proof. By Proposition 5.3, $\mathrm{Con_w}(A)$ is isomorphic to $L_{\geq}^C$ for $L = \mathrm{Sub}(A)$ and the closure operator C with $C_X(Y) = \bigcap\{e\theta \mid \theta \in \mathrm{Con}(X),\, Y \subseteq e\theta\}$. Under the isomorphism $\theta \mapsto (A\theta, e\theta)$, the diagonal $\Delta \in \mathrm{Con_w}(A)$ is mapped onto the pair $(A, \{e\}) \in L_{\geq}^C$. Furthermore, for any subalgebra X, the closure system L_X^C of all kernel subalgebras of X is isomorphic to $\mathrm{Con}(X)$. By definition, X is a C-discrete element of L if and only if it is a Dedekind subalgebra of A. Hence, Theorem 4.2 immediately yields the equivalence of (1)–(4). Corollary 4.3 establishes the last claim in Theorem 5.4. □

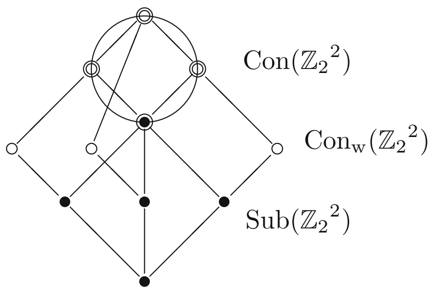
Note that by Propositions 3.3 and 5.3, the hypothesis that A is a join of Dedekind subalgebras may be substituted by disjunctivity of the subalgebra lattice in order to derive the equivalence $(1) \Leftrightarrow (4)$. Since the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious, and the equivalence $(4) \Leftrightarrow (5)$ has been shown in [27], Theorem 5.4 remains valid for group-like algebras with disjunctive subalgebra lattice.

Now, let us have a look at rings. A warning in advance: if a ring has a unit element 1, this should *not* be regarded as a constant in the present context. In any ring A, the zero element 0 constitutes the least subuniverse $\{0\}$. Clearly, A is a group-like algebra, taking $q(a, b) = a - b$. The kernels are just the (two-sided) ideals. The ideal closure defines a grounded closure operator C on the algebraic lattice L of all subrings, and C is hereditary, hence continuous, if A has the CEP. Let us call a ring *Hamiltonian* if each subring is an ideal (the name *Dedekind ring* is reserved for another class of rings). Then Theorem 5.4 amounts to:

Corollary 5.5. *A ring is Hamiltonian if and only if it is generated by Hamiltonian subrings and has a modular weak congruence lattice or* Δ *is a neutral element of it.*

Example 5.6. In the ring $\mathbb{Z}$ of all integers, the subrings coincide with the additive subgroups $n\mathbb{Z}$ and with the ideals. Thus $\mathbb{Z}$ is Hamiltonian. The weak congruence lattice $\mathrm{Con_w}(\mathbb{Z})$ is distributive, being isomorphic to $D_{\geq}$, where D is the lattice of all natural numbers (including 0), ordered by the dual of the divisibility relation.

Example 5.7. For any ring A with $1 \neq 0$, the ring A^2 has the diagonal subring Δ whose ideal closure $c(\Delta)$ is the whole A^2. Since for the ideal $A_0 = A \times \{0\}$, one obtains $c(\Delta) \cap c(A_0) = A_0 \neq \{(0,0)\} = c(\Delta \cap A_0)$, the CIP fails, and in particular, $\mathrm{Con_w}(A^2)$ cannot be modular.



Example 5.8. For any Boolean ring (in which all elements are idempotent), the subrings generated by single elements $a \neq 0$ have two elements only, so their subrings are ideals. Thus, Corollary 5.5 tells us that for Boolean rings A, the weak congruence lattice $\mathrm{Con_w}(A)$ is never modular unless A has at most two elements.

Example 5.9. An analogous phenomenon occurs with lattices, although they need not be group-like: the weak congruence lattice of a lattice A is modular only if A has at most two elements (see [27]). Moreover, if a lattice A contains three elements $a < b < c$, then Δ is not distributive in $\mathrm{Con_w}(A)$, since $B = \{a, b\}$ and $C = \{a, c\}$ are sublattices with $\Delta \vee (B^2 \wedge C^2) = \Delta \neq \Delta \vee B^2 = (\Delta \vee B^2) \wedge (\Delta \vee C^2)$.

Example 5.10. Let A be a sectionally complemented lattice (that is, all principal ideals of A are complemented). If we pass to the augmented algebra A^+ obtained by adding all unary operations $\wedge_a : x \mapsto a \wedge x$, then the resulting subalgebras of A^+ are just the ideals of A, while the kernels are exactly the *standard ideals*. Moreover, the map $\theta \mapsto 0\theta$ is an embedding of the congruence lattice in the ideal lattice of A and induces an isomorphism between $\mathrm{Con}(A) = \mathrm{Con}(A^+)$ and the lattice of all standard ideals (see [2, II, Theorem 6] and [17, III. 3, Theorem 10]). Furthermore, A^+ is a group-like algebra: for $q(a, b)$ one may take any relative complement of $a \wedge b$ in the interval $[\,0, a \vee b\,] = {\downarrow}(a \vee b)$.

Corollary 5.11. *A sectionally complemented lattice A with no infinite chains gives rise to a modular (equivalently, distributive) weak congruence lattice* $\mathrm{Con_w}(A^+)$ *if and only if A is distributive, i.e., a finite Boolean lattice, while* $\mathrm{Con_w}(A)$ *is modular only if* $|A| \leq 2$.

Proof. For generalized Boolean lattices B (and only for these), the assignment $\theta \mapsto 0\theta$ is an isomorphism between the congruence lattice $\mathrm{Con}(B^+) = \mathrm{Con}(B)$ and the ideal lattice $\mathrm{Sub}(B^+)$ of B, and both are distributive. Now let A be a chain-finite sectionally complemented lattice. Since the augmented algebra A^+ is a Dedekind algebra if and only if each ideal is a kernel, we infer from Theorem 5.4 that $\mathrm{Con_w}(A^+)$ is modular if and only if A is a (generalized) Boolean lattice; for join-density of the “discrete” members of $\mathrm{Sub}(A^+)$ (i.e., those ideals which are generalized Boolean lattices), use Corollary 4.4 and the fact that sectionally complemented chain-finite lattices are atomistic and isomorphic to their own ideal lattices. □

Corollary 5.11 applies, for example, to all finite-dimensional geometric lattices (see [2, IV] and [17, IV. 3]). On the other hand, we have:

Example 5.12. Every vector space is a Dedekind algebra with a modular geometric (hence complemented and atomistic) subalgebra lattice. Therefore, the weak congruence lattice of any vector space is modular, too.

6. Prospect: closure operators as diagrams

This final section contains a few thoughts aiming towards a more general categorical perspective for the previous considerations. In the language of category theory, a *diagram* is merely a functor between two categories. In most cases, the domain (called the *scheme* of the diagram) is a poset or lattice L, regarded as a category $\mathbf{L}$. Directed colimits in $\mathbf{L}$ are just directed joins in L.

Consider the category **CCL** of *complete closure lattices* (cf. [14]): its objects are pairs (L, c) where L is a complete lattice and c is a closure map on L; morphisms are the *"continuous"* maps $f\colon (L, c) \to (L', c')$, preserving directed joins and satisfying $f(c(z)) \leq c'(f(z))$. Now, any closure operator C on a complete lattice L naturally extends to a diagram, i.e., a functor $\tilde{C}$ from **L** to **CCL**. On the object level, $\tilde{C}$ assigns to each $x \in L$ the closure lattice (L_x, C_x); on the morphism level, one takes for $\tilde{C}_{xy}$ $(x \leq y)$ simply the inclusion map from L_x into L_y (the condition $C_x(z) \leq C_y(z)$ ensures that each $\tilde{C}_{xy}$ is a morphism). Now the following result justifies our notion of continuous closure operators from a categorical point of view (a proof and related material is deferred to [13]):

Theorem 6.1. *A closure operator C on an algebraic lattice L is continuous if and only if it naturally extends to a continuous diagram $\tilde{C}$ of algebraic closure lattices.*

The term "algebraic" may be substituted by "continuous" in that theorem, working with way-below ideals instead of compact elements.

A variant of Theorem 6.1 is obtained by replacing the hypothesis of algebraicity or continuity with a related (but incomparable) property. Let us call a complete closure lattice (L, c) *meet-continuous* if each unary meet operation $\wedge_x$ is a **CCL**-morphism from (L, c) to (L, c); explicitly, this condition means that L is meet-continuous in the usual sense and c is a nucleus (see Example 2.6). Call a closure operator C *strongly continuous* if it is continuous and each of the closure lattices (L_x, C_x) is meet-continuous. The closure operators mentioned in Example 2.6 are not only continuous but even strongly continuous. Now, one can show [13]:

Theorem 6.2. *A closure operator C on a complete lattice L is strongly continuous if and only if it naturally extends to a continuous diagram $\tilde{C}$ of meet-continuous closure lattices.*

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