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# Definability in substructure orderings, IV: Finite lattices

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Abstract. Let $\mathcal{L}$ be the ordered set of isomorphism types of finite lattices, where the ordering is by embeddability. We study first-order definability in this ordered set. Our main result is that for every finite lattice $\mathbf{L}$, the set $\{\ell, \ell^{\mathrm{opp}}\}$ is definable, where $\ell$ and $\ell^{\mathrm{opp}}$ are the isomorphism types of $\mathbf{L}$ and its opposite ($\mathbf{L}$ turned upside down). We shall show that the only non-identity automorphism of $\mathcal{L}$ is the map $\ell \mapsto \ell^{\mathrm{opp}}$.

## 1. Introduction and first concepts

This paper is the last in a series of four exploring definability in substructure orderings. The paper [2] dealt with finite semilattices, [3] deals with finite ordered sets, [4] treats finite distributive lattices, and here we deal with finite lattices. The set $\mathcal{L}$ of isomorphism types of finite lattices is denumerable. This set becomes a poset under the order induced by the substructure relation: we put $l_0 \leq l_1$, where $l_i$ is the type of the finite lattice $\mathbf{L}_i$, iff $\mathbf{L}_0$ is isomorphic to a sublattice of $\mathbf{L}_1$. In this way we obtain a poset $\langle \mathcal{L}, \leq \rangle$. We explore the scope of first-order definitions in the structure $\langle \mathcal{L}, \leq \rangle$.

Every lattice has its opposite. For a lattice $\mathbf{A} = \langle A, \wedge, \vee \rangle$ we denote by $\mathbf{A}^{\mathrm{opp}}$ the lattice $\langle A, \vee, \wedge \rangle$. The map $\mathbf{A} \mapsto \mathbf{A}^{\mathrm{opp}}$ induces an automorphism, $\ell \mapsto \ell^{\mathrm{opp}}$, of the ordered set $\langle \mathcal{L}, \leq \rangle$. Our goal here is to show that this is the only non-identity automorphism of $\langle \mathcal{L}, \leq \rangle$, and that $\{\ell, \ell^{\mathrm{opp}}\}$ is first-order definable in this structure, for every $\ell \in \mathcal{L}$. For this purpose, it proves to be convenient to fix a constant $p_1 \in \mathcal{L}$. With the proper choice of $p_1 \neq p_1^{\mathrm{opp}}$, we shall be able to prove that $\{\ell\}$ is first-order definable in the pointed ordered set $\langle \mathcal{L}, \leq, p_1 \rangle$ for every $\ell \in \mathcal{L}$.

Our principal object of investigation will actually be the quasi-ordered set Latt, whose members are all the lattices $\mathbf{A}$ whose members constitute a finite set of non-negative integers. The quasi-ordering is the substructure ordering, so that $\mathbf{A} \leq \mathbf{B}$ means that $\mathbf{A}$ is isomorphic to a sublattice of $\mathbf{B}$. Denote by $\mathbf{P}_1$ a lattice belonging to Latt with elements $a_0, a_1, a_2, a_3, a_4$ and covers $a_0 < a_1 < a_2 < a_4$ and $a_1 < a_3 < a_4$, and by $p_1$ the isomorphism type of $\mathbf{P}_1$.

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We define $\text{L}\textsc{att}'$ to be the pointed quasi-ordered set $\langle \text{L}\textsc{att}, \leq, \mathbf{P}_1 \rangle$, and $\mathcal{L}'$ to be the pointed ordered set $\langle \mathcal{L}, \leq, p_1 \rangle$.

When we say that a subset of $\text{L}\textsc{att}$ or a relation over $\text{L}\textsc{att}$ is definable in $\text{L}\textsc{att}'$, we shall mean definable by a formula in the first-order language with two non-logical symbols, $\leq$ and $\mathbf{P}_1$, and without the equality symbol. To denote that two lattices are isomorphic, we write $\mathbf{A} \cong \mathbf{B}$. The relation $\{(\mathbf{A}, \mathbf{B}) : \mathbf{A} \cong \mathbf{B}\}$ is definable in $\text{L}\textsc{att}'$ (since $\mathbf{A} \cong \mathbf{B}$ iff $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A}$ for finite $\mathbf{A}$ and $\mathbf{B}$) and it is easily proved (say by induction on the complexity of formulas) that for every formula $\varphi(x_0, \dots, x_{n-1})$ in this language and for $\mathbf{A}_0, \mathbf{B}_0, \dots, \mathbf{A}_{n-1}, \mathbf{B}_{n-1} \in \text{L}\textsc{att}$ with $\mathbf{A}_i \cong \mathbf{B}_i$ for $i < n$ we have $\text{L}\textsc{att}' \models \varphi(\mathbf{A}_0, \dots, \mathbf{A}_{n-1})$ if and only if $\text{L}\textsc{att}' \models \varphi(\mathbf{B}_0, \dots, \mathbf{B}_{n-1})$. Thus with our convention about the language (omitting equality) first-order definability in $\text{L}\textsc{att}'$ is only "up to isomorphism". In particular, $\{\mathbf{P}_1\}$ is not definable, although $\{\mathbf{A} : \mathbf{A} \cong \mathbf{P}_1\}$ is definable. However, we write that "$\mathbf{P}_1$ is a definable member of $\text{L}\textsc{att}'$", meaning that it is definable up to isomorphism; and we shall generally use this language with respect to all definable elements, definable subsets and definable relations over $\text{L}\textsc{att}'$.

The relation of isomorphism, definable in $\text{L}\textsc{att}'$, is an equivalence relation over $\text{L}\textsc{att}$ that gives rise to the pointed ordered set $\mathcal{L}'$ of isomorphism types. Via the map sending $\mathbf{A} \in \text{L}\textsc{att}$ to $\mathbf{A}/\cong \in \mathcal{L}$, definable relations over $\text{L}\textsc{att}'$ become definable relations over $\mathcal{L}'$, and conversely. Thus working over $\text{L}\textsc{att}'$ is simply a convenient means to give a more concrete feel to the study of definability over $\mathcal{L}'$.

We now introduce some very basic concepts for our study. We use $\mathbf{A} < \mathbf{B}$ to denote that $\mathbf{A} \leq \mathbf{B}$ and the two lattices are not isomorphic. The least and greatest elements of $\mathbf{A}$ will be denoted $0_\mathbf{A}$ and $1_\mathbf{A}$. For every $n \geq 0$, we denote by $\mathbf{C}_n$ the chain of height $n$ (so that $|\mathbf{C}_n| = n + 1$). The height $\text{ht}(\mathbf{A})$ of a finite lattice $\mathbf{A}$ is the greatest $n$ for which $\mathbf{C}_n \leq \mathbf{A}$. For every $n \geq 1$ we denote by $\mathbf{M}_n$ the only lattice of height 2 with $n$ atoms. Thus we have $\mathbf{C}_2 \cong \mathbf{M}_1$.

By a *cut-point* in a lattice $\mathbf{A}$ we mean an element $c \in A$ that is comparable to all elements of $\mathbf{A}$.

For two finite lattices $\mathbf{A}, \mathbf{B}$ we denote by $\mathbf{A} \oplus \mathbf{B}$ the lattice $\mathbf{C}$ that has a cut-point $c$ such that the interval $I[0_\mathbf{C}, c]$ is isomorphic to $\mathbf{A}$ and the interval $I[c, 1_\mathbf{C}]$ is isomorphic to $\mathbf{B}$. For two non-trivial finite lattices $\mathbf{A}, \mathbf{B}$ we denote by $\mathbf{A} + \mathbf{B}$ the lattice with the underlying set the disjoint union of the universes $A$ and $B$ of the lattices, but with $0_\mathbf{A}$ identified with $0_\mathbf{B}$ and $1_\mathbf{A}$ identified with $1_\mathbf{B}$, such that $\mathbf{A}$ and $\mathbf{B}$ are sublattices and there are no order relations between elements of $A - \{0_\mathbf{A}, 1_\mathbf{A}\}$ and $B - \{0_\mathbf{B}, 1_\mathbf{B}\}$. (Observe that $\mathbf{A} + \mathbf{C}_1 \cong \mathbf{A}$ and $|\mathbf{A} + \mathbf{C}_2| = |\mathbf{A}| + 1$.)

## 2. Definability of chains, flat lattices and some small lattices

An element $\mathbf{A}$ of $\text{L}\textsc{att}$ is said to be covered by an element $\mathbf{B}$ of $\text{L}\textsc{att}$ if $\mathbf{A} < \mathbf{B}$ and there is no $\mathbf{C} \in \text{L}\textsc{att}$ with $\mathbf{A} < \mathbf{C} < \mathbf{B}$. We write $\mathbf{A} \prec \mathbf{B}$ and also say that $\mathbf{B}$ is a cover of $\mathbf{A}$, or that $\mathbf{A}$ is a subcover of $\mathbf{B}$. Clearly, if $\mathbf{A} < \mathbf{B}$ and $|\mathbf{B}| = |\mathbf{A}| + 1$ then $\mathbf{A} \prec \mathbf{B}$.

**Lemma 2.1.** *Let $n \geq 0$. The only covers of $\mathbf{C}_n$ in* LATT *are $\mathbf{C}_{n+1}$ and the lattices $\mathbf{C}_k \oplus (\mathbf{C}_l + \mathbf{C}_2) \oplus \mathbf{C}_m$ for $k + l + m = n$ with $l \geq 2$.*

*Proof.* Clearly, all these lattices are covers of $\mathbf{C}_n$. Let $\mathbf{L}$ be a cover of $\mathbf{C}_n$. Clearly, $\mathbf{L}$ is of height either $n$ or $n + 1$, and if it is of height $n + 1$ then $\mathbf{L} \cong \mathbf{C}_{n+1}$. Let $\mathbf{L}$ be of height $n$ and let $a_0 < a_1 < \cdots < a_n$ be a chain in $\mathbf{L}$ (necessarily, this is a maximal chain). There is an index $i \leq n - 2$ such that $a_i$ has a cover $b$ different from $a_{i+1}$. The sublattice $\{a_0, \ldots, a_i\} \cup \uparrow a_{i+1}$ is above $\mathbf{C}_n$ and does not contain the element $b$; consequently, it coincides with $\{a_0, a_1, \ldots, a_n\}$. Thus $a_{i+1} \vee b = a_j$ for some $j \geq i + 2$. Now $\{a_0, \ldots, a_n, b\}$ is a sublattice of $\mathbf{L}$ isomorphic to $\mathbf{C}_i \oplus (\mathbf{C}_{j-i} + \mathbf{C}_2) \oplus \mathbf{C}_{n-j}$. □

**Lemma 2.2.** *The lattice $\mathbf{M}_2$ has four covers: $\mathbf{M}_3$, $\mathbf{N}_5 = \mathbf{C}_3 + \mathbf{C}_2$, $\mathbf{P} = \mathbf{M}_2 \oplus \mathbf{C}_1$ and $\mathbf{P}^{\mathrm{opp}} = \mathbf{C}_1 \oplus \mathbf{M}_2$. The lattice $\mathbf{M}_3$ has infinitely many covers.*

*Proof.* The first statement is easy. A slight modification of a construction from [1] gives a lattice $\mathbf{L}_n$ with $12 + 2n$ elements, for any $n \geq 2$, such that $\mathbf{L}_n$ contains a single copy of $\mathbf{M}_3$ as a sublattice, and this sublattice is maximal in $\mathbf{L}_n$. For $n = 6$ the lattice $\mathbf{L}_n$ is pictured in Figure 1. □

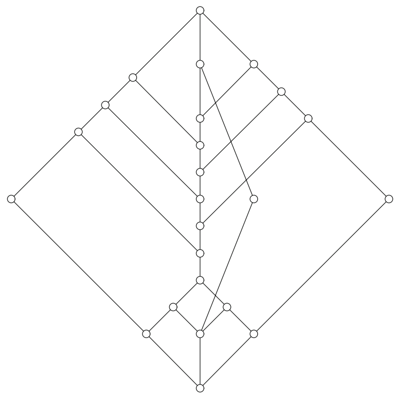

FIGURE 1

**Theorem 2.3.** *The set of finite chains is definable. The set $\{\mathbf{M}_n : n \geq 1\}$ is definable. Every finite chain and every lattice $\mathbf{M}_n$ is a definable element of* LATT′.

*Proof.* It is easy to see that an element $\mathbf{L}$ of LATT′ has the property that the principal ideal generated by $\mathbf{L}$ is a chain, i.e., for all $\mathbf{A} \leq \mathbf{L}$ and $\mathbf{B} \leq \mathbf{L}$ either $\mathbf{A} \leq \mathbf{B}$ or $\mathbf{B} \leq \mathbf{A}$, if and only if $\mathbf{L}$ is either a chain or $\mathbf{M}_n$ for some $n$. Thus the set $U = \{\mathbf{C}_n : n \geq 0\} \cup \{\mathbf{M}_n : n \geq 1\}$ is definable.

(According to our convention, this language just means to assert that $\{\mathbf{A} \in \textsc{Latt} : \mathbf{A} \cong \mathbf{U}$ for some $\mathbf{U} \in U\}$ is definable.) Also, it follows that the set $\{\mathbf{C}_4, \mathbf{M}_3\}$ is definable: $\mathbf{C}_4$ and $\mathbf{M}_3$ are the only two lattices $\mathbf{Q} \in U$ such that $\left|\{\mathbf{R} \in \textsc{Latt}' : \mathbf{R} < \mathbf{Q}\}/{\cong}\right| = 4$. By 2.1 and 2.2, $\mathbf{C}_4$ has only seven covers, while $\mathbf{M}_3$ has more than seven covers in $\textsc{Latt}'$. Thus both $\mathbf{C}_4$ and $\mathbf{M}_3$ are definable elements. A finite lattice is a chain if and only if it belongs to $\{\mathbf{C}_n : n \geq 0\} \cup \{\mathbf{M}_n : n \geq 1\}$ and is comparable with $\mathbf{C}_4$. □

It also follows that the mapping $\mathbf{L} \mapsto \mathbf{C}$, where $\mathbf{L} \in \textsc{Latt}'$ and $\mathbf{C}$ is the chain of height $\mathrm{ht}(\mathbf{L})$, is definable.

**Lemma 2.4.** *The lattices* $\mathbf{N}_5$, $\mathbf{P}_1$, $\mathbf{P}_0 = \mathbf{P}_1^{\mathrm{opp}}$ *and* $\mathbf{N}_6 = \mathbf{C}_3 + \mathbf{C}_3$ *are definable.*

*Proof.* It follows from 2.2 that the set consisting of the first three lattices is definable. Of these three lattices, $\mathbf{N}_5$ is the only one that has a cover of height 3 that is not above any of the remaining two lattices and also is not above $\mathbf{M}_3$; it has only one such cover and this cover is isomorphic to $\mathbf{N}_6$. Thus $\mathbf{N}_5$, and also $\mathbf{N}_6$, are definable elements. Consequently, the set $\{\mathbf{P}_0, \mathbf{P}_1\}$ is definable. But $\mathbf{P}_1$ is definable in $\textsc{Latt}'$ by definition, so that $\mathbf{P}_0$ is definable as well. □

Since the opposite of $\mathbf{P}_1$ is definable, it follows that whenever a relation is definable in $\textsc{Latt}'$ then also its opposite is definable in $\textsc{Latt}'$.

Since $\mathbf{N}_5$ and $\mathbf{M}_3$ are definable, the set of finite modular lattices and the set of finite distributive lattices are definable.

## 3. Definability of the relation $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$

**Lemma 3.1.** *Let $n \geq 2$ and $k \geq 0$. Then $\mathbf{C}_k \oplus \mathbf{M}_n$ is the least lattice $\mathbf{L}$ with the following properties:*

(1) *$\mathbf{L}$ is modular;*
(2) *$\mathbf{M}_n \leq \mathbf{L}$;*
(3) *$\mathrm{ht}(\mathbf{L}) = k + 2$;*
(4) $\mathbf{M}_2 \oplus \mathbf{C}_1 \not\leq \mathbf{L}$.

*Proof.* Let $\mathbf{L}$ have these properties. There exists a sublattice $\mathbf{M}$ of $\mathbf{L}$ isomorphic to $\mathbf{M}_n$. By (4), $1_\mathbf{M} = 1_\mathbf{L}$. Put $o = 0_\mathbf{M}$ and denote by $a_1, \ldots, a_n$ the atoms of $\mathbf{M}$. For every $i = 1, \ldots, n$ there is a cover $b_i$ of $o$ in $\mathbf{L}$ with $b_i \leq a_i$. For $i \neq j$ we have $b_i \vee b_j = 1_\mathbf{L}$ by (4). Since $\mathbf{L}$ is modular, it follows that $b_i = a_i$. Thus $o \prec a_i$ in $\mathbf{L}$ for all $i$. But then, again since $\mathbf{L}$ is modular, also $a_i \prec 1_\mathbf{L}$ in $\mathbf{L}$ for all $i$. Thus $o$ is of height $k$ in $\mathbf{L}$. But then $\mathbf{C}_k \oplus \mathbf{M}_n \leq \mathbf{L}$. □

**Lemma 3.2.** *Let $n \geq 2$ and $k, l \geq 0$. Then $\mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l$ is the least lattice $\mathbf{L}$ with the following properties:*

(1) *$\mathbf{L}$ is modular;*
(2) *$\mathbf{M}_n \leq \mathbf{L}$;*
(3) *$\mathrm{ht}(\mathbf{L}) = k + 2 + l$;*

(4) $\mathbf{C}_k \oplus \mathbf{M}_n \leq \mathbf{L}$;
(5) $\mathbf{M}_n \oplus \mathbf{C}_l \leq \mathbf{L}$;
(6) $\mathbf{C}_{k+1} \oplus \mathbf{M}_2 \nleq \mathbf{L}$;
(7) $\mathbf{M}_2 \oplus \mathbf{C}_{l+1} \nleq \mathbf{L}$.

*Consequently, the mapping* $\langle \mathbf{C}_k, \mathbf{M}_n, \mathbf{C}_l \rangle \mapsto \mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l$ *is definable and every* $\mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l$ *is a definable element of* LATT$'$.

*Proof.* Let $\mathbf{L}$ have these properties. There exists a sublattice $M$ of $\mathbf{L}$ isomorphic to $\mathbf{M}_n$. Denote by $o$ the least element, by $I$ the greatest element, and by $a_1, \ldots, a_n$ the remaining elements of $M$. By (4) we can assume that $\mathrm{ht}(o) \geq k$. By (6), $\mathrm{ht}(o) = k$. For every $i = 1, \ldots, n$ there is a sub-cover, $b_i$ of $I$ in $\mathbf{L}$ with $b_i \geq a_i$. If $i \neq j$ and $b_i \wedge b_j > o$ then we obtain a contradiction by (3) and (6), since $\mathbf{L}$ is modular. Thus it follows (by modularity) that the interval $I[o, I]$ has height two, and consequently, by (3) and modularity, the interval $I[I, 1_{\mathbf{L}}]$ has height $l$. But this means that $\mathbf{C}_k \oplus \mathbf{M}_n \oplus \mathbf{C}_l \leq \mathbf{L}$. That concludes our proof.

For the definability of the mapping we need to apply Lemma 3.1 and its opposite. □

**Lemma 3.3.** *The following ternary relation* $R$ *on* LATT$'$ *is definable:* $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ *if and only if* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are chains and* $\mathrm{ht}(\mathbf{C}) = \mathrm{ht}(\mathbf{A}) + \mathrm{ht}(\mathbf{B})$.

*Proof.* It follows from 3.2. □

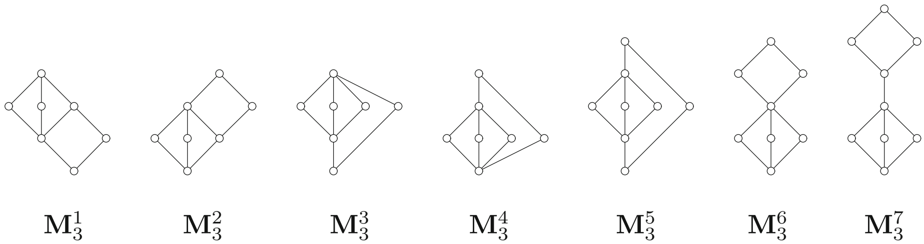


FIGURE 2

For $n \geq 3$ denote by $\mathbf{M}_n^1$ the lattice $\mathbf{C}_1 \oplus \mathbf{M}_n$ with one element added, this element being an atom of $\mathbf{L}$ below one of the atoms of $\mathbf{M}_n$. Denote by $\mathbf{M}_n^2$ the opposite of $\mathbf{M}_n^1$. Put $\mathbf{M}_n^3 = (\mathbf{C}_1 \oplus \mathbf{M}_n) + \mathbf{C}_2$, $\mathbf{M}_n^4 = (\mathbf{M}_n \oplus \mathbf{C}_1) + \mathbf{C}_2$, $\mathbf{M}_n^5 = (\mathbf{C}_1 \oplus \mathbf{M}_n \oplus \mathbf{C}_1) + \mathbf{C}_2$, $\mathbf{M}_n^6 = \mathbf{M}_n \oplus \mathbf{M}_2$ and $\mathbf{M}_n^7 = \mathbf{M}_n \oplus \mathbf{C}_1 \oplus \mathbf{M}_2$. These lattices are pictured in Figure 2 for $n = 3$.

**Lemma 3.4.** *Let* $n \geq 3$. *Then* $\mathbf{M}_n^1$ *and* $\mathbf{M}_n^3$ *are the only covers of* $\mathbf{C}_1 \oplus \mathbf{M}_n$ *of height* 3 *that are not above* $\mathbf{M}_{n+1}$*; the first is modular, the second is not. Consequently, the mappings* $\mathbf{M}_n \mapsto \mathbf{M}_n^1$ *and* $\mathbf{M}_n \mapsto \mathbf{M}_n^3$ *are definable. Similarly, the mappings* $\mathbf{M}_n \mapsto \mathbf{M}_n^2$ *and* $\mathbf{M}_n \mapsto \mathbf{M}_n^4$ *are definable.*

*Proof.* Let $\mathbf{L}$ be a cover of $\mathbf{C}_1 \oplus \mathbf{M}_n$ such that $\mathrm{ht}(\mathbf{L}) = 3$ and $\mathbf{M}_{n+1} \not\leq \mathbf{L}$. Then $\mathbf{L}$ has a proper sublattice $K = \{0_{\mathbf{L}}, o, a_1, \dots, a_n, 1_{\mathbf{L}}\}$ isomorphic to $\mathbf{C}_1 \oplus \mathbf{M}_n$.

Suppose that there is an element $b \in L - K$ comparable with $o$. Clearly, $b > o$. But then $b$ is of height 2 and $\mathbf{L}$ has a sublattice $(K - \{0_{\mathbf{L}}\}) \cup \{b\} \cong \mathbf{M}_{n+1}$, a contradiction.

Thus all elements of $L - K$ are incomparable with $o$. Consider first the case when there is an element $c \in L - K$ that is below at least one of the coatoms $a_i$ of $\mathbf{L}$. Clearly, the index $i$ is unique. Without loss of generality, $c < a_1$ and $c \| a_i$ for all $i > 1$. Clearly, $c$ is an atom of $\mathbf{L}$. Thus $c \wedge a_i = 0_{\mathbf{L}}$ and $c \vee a_i = 1_{\mathbf{L}}$ for $i > 1$. It follows that $K \cup \{c\}$ is a sublattice of $\mathbf{L}$ isomorphic to $\mathbf{M}_n^1$.

It remains to consider the case when $c \| a_i$ for all $i$ and all $c \in L - K$. Take one element $c \in L - K$. Since $a_i$ are coatoms, we have $c \vee a_i = 1_{\mathbf{L}}$. If $c \wedge a_i > 0_{\mathbf{L}}$ for some $i$ then $c \wedge a_i$ is an element of $L - K$ below $a_i$, a contradiction. Thus $c \wedge a_i = 0_{\mathbf{L}}$ for all $i$ and $K \cup \{c\}$ is a sublattice of $\mathbf{L}$ isomorphic to $\mathbf{M}_n^3$. □

**Lemma 3.5.** *Let $n \geq 3$. Then $\mathbf{M}_n^5$ is the only cover of $\mathbf{C}_1 \oplus \mathbf{M}_n \oplus \mathbf{C}_1$ of height 4 that is not above any of the lattices $\mathbf{M}_n^1$, $\mathbf{M}_n^2$, $\mathbf{M}_n^3$, $\mathbf{M}_n^4$ and $\mathbf{M}_{n+1}$. Consequently, the mapping $\mathbf{M}_n \mapsto \mathbf{M}_n^5$ is definable.*

*Proof.* Let $\mathbf{L}$ be such a cover. Denote by $K$ a copy of $\mathbf{C}_1 \oplus \mathbf{M}_n \oplus \mathbf{C}_1$ in $\mathbf{L}$; denote by $o$ the only atom and by $I$ the only coatom in this copy. Clearly, $\downarrow I \cup \{1_{\mathbf{L}}\}$ is a sublattice of $\mathbf{L}$; if it contains an element not in $K$ then $\downarrow I$ is a sublattice properly extending $\mathbf{C}_1 \oplus \mathbf{M}_n$, so that (according to 3.4) it contains either $\mathbf{M}_n^1$ or $\mathbf{M}_n^3$, a contradiction. Thus $\downarrow I = \mathbf{M}_n \cup \{0_{\mathbf{L}}\}$ and similarly $\uparrow o = \mathbf{M}_n \cup \{1_{\mathbf{L}}\}$. Thus for any element $c \in L - K$, $K \cup \{c\}$ is a sublattice of $\mathbf{L}$ isomorphic to $\mathbf{M}_n^5$. □

**Lemma 3.6.** *Let $n \geq 3$. $\mathbf{M}_n^6$ is the only cover of $\mathbf{M}_n \oplus \mathbf{C}_2$ of height 4 that is modular and is not above any of the lattices $\mathbf{M}_{n+1}$ and $\mathbf{M}_n^2$. Also, $\mathbf{M}_n^7$ is the only cover of $\mathbf{M}_n \oplus \mathbf{C}_3$ of height 5 that is modular and is not above any of the lattices $\mathbf{M}_{n+1}$, $\mathbf{M}_n^2$ and $\mathbf{M}_n^6$. Consequently, the mappings $\mathbf{M}_n \mapsto \mathbf{M}_n^6$ and $\mathbf{M}_n \mapsto \mathbf{M}_n^7$ are definable.*

The proof is easy.

**Lemma 3.7.** *Let $\mathbf{B}$ be a finite lattice of height $n+m$ where $n \geq 0$ and $m > 0$. Then $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_m$ for a lattice $\mathbf{A}$ of height $n$ if and only if for every positive integer $l$ there exist a positive integer $k > l$ and a finite lattice $\mathbf{C}$ such that the following conditions are satisfied:*

(1) $\mathbf{B} < \mathbf{C}$;
(2) $\mathrm{ht}(\mathbf{C}) = n + m + 1$;
(3) $\mathbf{C}_n \oplus \mathbf{M}_k \oplus \mathbf{C}_{m-1} \leq \mathbf{C}$;
(4) $\mathbf{C}$ *is not above any of the lattices* $\mathbf{M}_{k+1}$, $\mathbf{M}_k^1$, $\mathbf{M}_k^2$, $\mathbf{M}_k^3$, $\mathbf{M}_k^4$, $\mathbf{M}_k^5$, $\mathbf{M}_k^6$, $\mathbf{M}_k^7$;
(5) *If $n = 0$ then $\mathbf{B}$ is a chain. If $n > 0$ then $\mathbf{C}_{n-1} \oplus \mathbf{M}_2 \not\leq \mathbf{B}$.*

*Proof.* If $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_m$ then the lattice $\mathbf{C} = \mathbf{A} \oplus \mathbf{M}_k \oplus \mathbf{C}_{m-1}$ has all these properties for every sufficiently large $k$. Conversely, let $k$ and $\mathbf{C}$ exist for every $l$. By (2) and (3) there exists a sublattice $\mathbf{M}$ of $\mathbf{C}$ isomorphic to $\mathbf{M}_k$ with the least element $o$, the largest element $I$, and the remaining elements $a_1, \dots, a_k$ such that $o$ is of height $n$, $I$ is of height $n+2$ in $\mathbf{C}$, and $\uparrow I$ is of height $m-1$; all covers in $\mathbf{M}$ are covers in $\mathbf{C}$. Since $\mathbf{M}_k^6 \not\leq \mathbf{C}$ and $\mathbf{M}_k^7 \not\leq \mathbf{C}$, $\uparrow I$ is a chain; denote it by $\mathbf{D}$; it must be a chain of height $m-1$.

Since $\mathbf{M}_{k+1} \not\leq \mathbf{C}$, $\mathbf{M}_k^2 \not\leq \mathbf{C}$ and $\mathbf{M}_k^4 \not\leq \mathbf{C}$, it is easy to see that every element of $\uparrow o$ belongs to $\mathbf{M} \cup \mathbf{D}$. Since $\mathbf{M}_k^1 \not\leq \mathbf{C}$, $\mathbf{M}_k^3 \not\leq \mathbf{C}$ and $\mathbf{M}_k^5 \not\leq \mathbf{C}$, it is easy to see that $o$ is a cut-point in $\mathbf{C}$. Thus $\mathbf{C} = \downarrow o \oplus \mathbf{M} \oplus \mathbf{D}$. Since $\mathbf{B}$ is a sublattice of $\mathbf{C}$ of height $\mathrm{ht}(\mathbf{C}) - 1$, it follows from (5) that $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_m$ for some $\mathbf{C}$. □

**Lemma 3.8.** *The set of finite lattices with precisely one coatom* (*or precisely one atom, respectively*) *is definable. Consequently, also the set of finite lattices with at least two coatoms* (*or atoms, respectively*) *is definable.*

*Proof.* It follows from 3.7, since a finite lattice $\mathbf{B}$ has precisely one coatom if and only if $\mathbf{B} \cong \mathbf{A} \oplus \mathbf{C}_1$ for a finite lattice $\mathbf{A}$. □

**Lemma 3.9.** *The following ternary relation $R$ on* LATT *is definable in* LATT′*:* $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ *if and only if* $\mathbf{B}$ *is a chain and* $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$.

*Proof.* Let $\mathbf{B}$ be a chain. If $\mathbf{A}$ has at least two coatoms (this being definable by 3.8), then $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$ if and only if $\mathbf{C} \cong \mathbf{A}' \oplus \mathbf{B}$ for some $\mathbf{A}'$ of the same height as $\mathbf{A}$ (this being definable by 3.7) and $\mathbf{A}$ is up to isomorphism the largest element of LATT′ below $\mathbf{C}$ with at least two coatoms. If $\mathbf{A}$ is arbitrary (and not a chain) then $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{B}_0$ for a lattice $\mathbf{A}_0$ with at least two coatoms and a chain $\mathbf{B}_0$; we have $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$ if and only if $\mathbf{C} \cong \mathbf{A}_0 \oplus (\mathbf{B}_0 \oplus \mathbf{B})$. (We need to apply 3.3.) □

**Lemma 3.10.** *Let* $\mathbf{A}$ *be a finite lattice of height $m$; let $2 \leq n < m$. Then* $\mathbf{A}$ *has a cut-point at height $n$ if and only if there exist* $\mathbf{B}, \mathbf{C} \in$ LATT′ *with the following properties:*

(1) $\mathbf{A} \leq \mathbf{B}$;
(2) $\mathrm{ht}(\mathbf{B}) = m$;
(3) $\mathrm{ht}(\mathbf{C}) = n$;
(4) $\mathbf{C} \oplus \mathbf{C}_{m-n} \leq \mathbf{B}$;
(5) *if* $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \leq \mathbf{B}$, *then* $\mathbf{C}'' \cong \mathbf{C} \oplus \mathbf{C}_2$.

*Proof.* Let $\mathbf{A}$ have a cut-point at height $n$, so that $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ where $\mathrm{ht}(\mathbf{A}_1) = n$ and $\mathrm{ht}(\mathbf{C}_2) = m - n$; the element $1_{\mathbf{A}_1} = 0_{\mathbf{A}_2}$ is the cut-point. Let $k > 0$ be sufficiently large so that $\mathbf{C} = \mathbf{M}_l + \mathbf{A}_1 \not\leq \mathbf{A}_2$. Put $\mathbf{B} = \mathbf{C} \oplus \mathbf{A}_2$. Since $n \geq 2$, then with this choice of $\mathbf{B}$ and $\mathbf{C}$ we have $\mathbf{A} \leq \mathbf{B}$, and in fact conditions (1) through (4) are obviously true.

To check that (5) is satisfied, suppose that $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \leq \mathbf{B}$. We claim that the only sublattice of $\mathbf{B}$ isomorphic to $\mathbf{C}$ is $C$, equal to the interval $I[0_{\mathbf{B}}, 0_{\mathbf{A}_2}]$ in $\mathbf{B}$. Indeed, let $E$ be a sublattice of $\mathbf{B}$ isomorphic to $\mathbf{C}$. Then

$E \not\subseteq A_2$. It easily follows that the $0,1$-sublattice of $E$ isomorphic to $\mathbf{M}_k$ is contained in $C$, and thus $E \subseteq C$, yielding that $E = C$ by cardinality. Thus we have sublattices $\mathbf{C} \subset \mathbf{C}' \subset \mathbf{C}'' \subseteq \mathbf{B}$ with $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}''$, and we are to show that $\mathbf{C}'' \cong \mathbf{C} \oplus \mathbf{C}_2$.

Now every element of $B \setminus C$ is greater than all elements of $C$. Thus clearly, $C' = C \cup \{p\}$ for an element $p \in A_2 - \{0_{\mathbf{A}_2}\}$. The sublattice $C''$ contains an element $q \notin C'$. If $p, q$ are incomparable then $C \cup \{p, p \vee q\}$ is a sublattice, so that $C'' = C \cup \{p, p \vee q\}$ and $q \notin C''$, a contradiction. Thus $p, q$ are comparable and $C'' = C \cup \{p, q\} \cong \mathbf{C} \oplus \mathbf{C}_2$. Condition (5) is therefore satisfied.

To prove the reverse implication, suppose that $2 \le n < m$, $\mathrm{ht}(\mathbf{A}) = m$, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfy (1) through (5). By (4) we can assume that $\mathbf{C} \oplus \mathbf{C}_{m-n}$ is a sublattice of $\mathbf{B}$.

Suppose that $C$ is a proper sublattice of the interval $I[0_{\mathbf{B}}, 1_{\mathbf{C}}]$ in $\mathbf{B}$. Then there is a lattice $\mathbf{C}' \succ \mathbf{C}$ with $\mathbf{C}' \subseteq \mathbf{B}$ and, in fact, $C \subset C' \subseteq I[0_{\mathbf{B}}, 1_{\mathbf{C}}]$. Clearly, $\mathrm{ht}(\mathbf{C}') = n$. Let $C'' = C' \cup \{p\}$ with $p \in B$ and $p > 1_{\mathbf{C}}$. This gives a lattice $\mathbf{C}''$ with $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \le \mathbf{B}$ and $\mathrm{ht}(\mathbf{C}'') = n+1$. Clearly, $\mathbf{C}'' \ncong \mathbf{C} \oplus \mathbf{C}_2$. This contradicts (5), so it follows that $C$ is identical to the interval $I[0_{\mathbf{B}}, 1_{\mathbf{C}}]$ in $\mathbf{B}$.

We claim that $1_{\mathbf{C}}$ is a cut-point of $\mathbf{B}$. Suppose not, so that there exists an element $a \in B$ incomparable with $1_{\mathbf{C}}$. Choose $a$ maximal among all elements of $B$ that are incomparable with $1_{\mathbf{C}}$. Then it is easily checked that $C'' = C \cup \{a, a \vee 1_{\mathbf{C}}\}$ is a sublattice of $\mathbf{B}$, as is $C' = C \cup \{a \vee 1_{\mathbf{C}}\}$. Clearly, this gives $\mathbf{C} \prec \mathbf{C}' \prec \mathbf{C}'' \le \mathbf{B}$, and here $\mathrm{ht}(\mathbf{C}'') = n+1$ so that $\mathbf{C}'' \ncong \mathbf{C} \oplus \mathbf{C}_2$. This contradicts (5).

It follows that $\mathbf{B}$ has a unique element of height $n$, namely $1_{\mathbf{C}}$. Now we have $\mathbf{A} \cong \mathbf{A}' \subseteq \mathbf{B}$, $\mathrm{ht}(\mathbf{A}') = \mathrm{ht}(\mathbf{A}) = \mathrm{ht}(\mathbf{B}) = m$. These conditions imply that an element of height $n$ in $\mathbf{A}'$ must be of height $n$ in $\mathbf{B}$. So $\mathbf{A}'$ contains the cut-point $1_{\mathbf{C}}$, which is therefore a cut-point of height $n$ in $\mathbf{A}'$. This concludes our proof. □

**Lemma 3.11.** *The following binary relation $R$ is definable in* LATT′*:*

> $(\mathbf{A}, \mathbf{B}) \in R$ *if and only if* $\mathbf{B} \cong \mathbf{C}_n$ *for an $n$ such that* $\mathbf{A}$ *has a cut-point at height $n$.*

*Proof.* The case $2 \le n < \mathrm{ht}(\mathbf{A})$ follows from 3.10. The case $n = 1$ has been handled in Lemma 3.8. The other cases are trivial. □

**Theorem 3.12.** *The following ternary relation $R$ is definable in* LATT′*:*

> $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ *if and only if* $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$.

*Proof.* Put $n = \mathrm{ht}(\mathbf{A})$ and $m = \mathrm{ht}(\mathbf{B})$. We have $\mathbf{C} \cong \mathbf{A} \oplus \mathbf{B}$ if and only if $\mathbf{C}$ has a cut-point at height $n$, $\mathbf{A}$ is, up to isomorphism, the largest element of LATT′ with $\mathrm{ht}(\mathbf{A}) = n$ and $\mathbf{A} \oplus \mathbf{C}_m \le \mathbf{C}$, and $\mathbf{B}$ is, up to isomorphism, the largest element of LATT′ with $\mathrm{ht}(\mathbf{B}) = m$ and $\mathbf{C}_n \oplus \mathbf{B} \le \mathbf{C}$. □

## 4. Definability of principal ideals and intervals

**Definition 4.1.** Let $\mathbf{A}$ be a finite lattice and $a \in A - \{1_{\mathbf{A}}\}$. We define a lattice $\mathbf{K}$ with $K = A \cup \{i, b\}$ in such a way that $\mathbf{A}$ is a sublattice of $\mathbf{K}$, $i = 1_{\mathbf{K}}$, $x \leq b$ for $x \in A$ if and only if $x \leq a$, and $x \geq b$ for $x \in A$ never happens. This lattice $\mathbf{K}$ will be denoted by $\mathbf{V}_a(\mathbf{A})$. By a *V-extension* of $\mathbf{A}$ we mean a lattice isomorphic to $\mathbf{V}_a(\mathbf{A})$ for some $a \in A - \{1_{\mathbf{A}}\}$. We say that $\mathbf{B}$ *is a V-extension of* $\mathbf{A}$ *with bottom* $\mathbf{C}$ if for some $a \in A - \{1_{\mathbf{A}}\}$, $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A})$ and $\mathbf{C}$ is isomorphic with the interval $I[0_{\mathbf{A}}, a]$ in $\mathbf{A}$.

**Lemma 4.2.** *Let* $\mathbf{A}, \mathbf{B} \in \text{LATT}'$. *Then* $\mathbf{B}$ *is a V-extension of* $\mathbf{A}$ *if and only if* $\mathbf{B}$ *is a cover of* $\mathbf{A} \oplus \mathbf{C}_1$ *such that* $\mathbf{B}$ *has more than one coatom. Consequently, the binary relation 'is a V-extension of' on* $\text{LATT}'$ *is definable.*

*Proof.* The direct implication is clear. Let $\mathbf{B}$ be a cover of $\mathbf{D} = \mathbf{A} \oplus \mathbf{C}_1$ with more than one coatom. We can assume that $\mathbf{D}$ is a proper sublattice of $\mathbf{B}$. If $\mathbf{B}$ contains some elements not in $D$ and below $1_{\mathbf{A}}$ then the sublattice $D \cup \{x \in B : x \leq 1_{\mathbf{A}}\}$ of $\mathbf{B}$ equals $\mathbf{B}$ and $\mathbf{B}$ has only one coatom, a contradiction. Thus all elements of $\mathbf{B}$ below $1_{\mathbf{A}}$ belong to $\mathbf{A}$. If some element $e$ of $B - D$ is larger than $1_{\mathbf{D}}$ then $B = D \cup \{e\}$ has only one coatom, a contradiction. Thus $1_{\mathbf{B}} = 1_{\mathbf{D}}$. If there is an element $e \in B$ with $1_{\mathbf{A}} < e < 1_{\mathbf{B}}$, we get a contradiction in the same way. Thus $1_{\mathbf{A}}$ is a coatom of $\mathbf{B}$. There exists a coatom $b$ of $\mathbf{B}$ other than $1_{\mathbf{A}}$. Put $a = 1_{\mathbf{A}} \wedge b$, so that $a \in A - \{1_{\mathbf{A}}\}$. It is easy to see that $D \cup \{b\}$ is a sublattice of $\mathbf{B}$. Thus $B = D \cup \{b\}$ and $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A})$. □

Let $\mathbf{A}$ be a finite lattice. Denote by $\mathbf{A}^+$ the lattice $\mathbf{A} \oplus \mathbf{M}_n$ where $n$ is the least number such that $n \geq 3$ and $\mathbf{M}_n \not\leq \mathbf{A}$. By results we have already proved, the mapping $\mathbf{A} \mapsto \mathbf{A}^+$ is definable.

**Lemma 4.3.** *The following ternary relation* $R$ *on* $\text{LATT}$ *is definable in* $\text{LATT}'$*:* $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ *iff for some* $a \in A$, $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A}^+)$ *and* $\mathbf{C} \cong I[0_{\mathbf{A}}, a]$.

*Proof.* We have $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in R$ if and only if $\mathbf{B}$ is a V-extension of $\mathbf{A}^+ = \mathbf{A} \oplus \mathbf{M}_n$, $\mathbf{A} \oplus \mathbf{M}_n^2 \not\cong \mathbf{B}$ and one of the following two cases takes place:

(i) $\mathbf{A} \oplus \mathbf{M}_n^4 \cong \mathbf{B}$ and $\mathbf{C} \cong \mathbf{A}$;

(ii) $\mathbf{A} \oplus \mathbf{M}_n^4 \not\cong \mathbf{B}$ and $\mathbf{C}$ is up to isomorphism the largest element of $\text{LATT}'$ such that $\mathbf{C} \oplus \mathbf{M}_n^5 \leq \mathbf{B}$. □

**Theorem 4.4.** *The binary relations 'is isomorphic to a principal ideal of', 'is isomorphic to a principal filter of' and 'is isomorphic to an interval of' on* $\text{LATT}'$ *are definable.*

The proof follows from 4.3.

## 5. Individual definability and automorphisms

Let $\mathbf{A}$ be a finite lattice and $s = \langle a_1, \ldots, a_k \rangle$ be a nonempty simple sequence of elements of $\mathbf{A}$. (The sequence is called simple if $a_i \neq a_j$ whenever $i \neq j$.)

We define $\mathbf{H}_s(\mathbf{A}) = \mathbf{V}_{a_k} \cdots V_{a_1}(\mathbf{A}^+)$. The lattices $\mathbf{H}_s(\mathbf{A})$ will be called simple sequential extensions of $\mathbf{A}$.

Observe that $\mathbf{H}_s(\mathbf{A})$ has only one sublattice isomorphic to $\mathbf{M}_n$ (where $n$ is such that $\mathbf{A}^+ = \mathbf{A} \oplus \mathbf{M}_n$).

Clearly, $\mathbf{H}_s(A)$ with $s = \langle a_1, \dots, a_k \rangle$ is a maximal simple sequential extension of $\mathbf{A}$ if and only if $k = |\mathbf{A}|$.

For example, a maximal simple sequential extension of the pentagon is pictured in Figure 3.

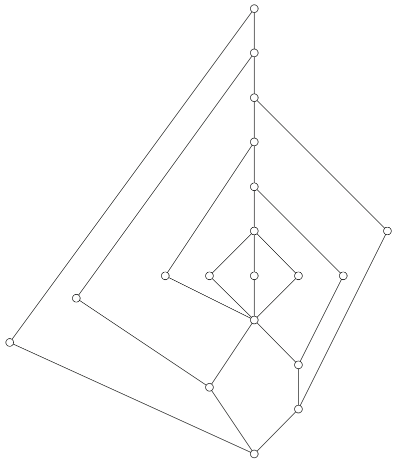

FIGURE 3

**Lemma 5.1.** *Let* $\mathbf{A}$ *be a finite lattice and let* $1 \le k \le |\mathbf{A}|$*; let* $\mathbf{A}^+ = \mathbf{A} \oplus \mathbf{M}_n$*. A finite lattice* $\mathbf{B}$ *is isomorphic to a simple sequential extension of* $\mathbf{A}$ *by a simple sequence of* $k$ *elements if and only if the following conditions are satisfied:*

(1) $\mathbf{B} \ge \mathbf{A}^+ \oplus \mathbf{C}_k$*;*
(2) $\mathrm{ht}(\mathbf{B}) = \mathrm{ht}(\mathbf{A}) + 2 + k$*;*
(3) $\mathbf{M}_n^2 \not\le \mathbf{B}$*;*
(4) $\mathbf{M}_n^6 \not\le \mathbf{B}$ *and* $\mathbf{M}_n^7 \not\le \mathbf{B}$*;*
(5) *every principal ideal of* $\mathbf{B}$ *of any height* $h > \mathrm{ht}(\mathbf{A}) + 2$ *is a V-extension of a principal ideal of* $\mathbf{B}$ *of height* $h - 1$*, with the bottom of height at most* $\mathrm{ht}(\mathbf{A})$ *(bottom as defined above for* $V$*-extensions);*
(6) *if* $\mathbf{D}$ *is a V-extension of either* $\mathbf{M}_n^4$ *or* $\mathbf{M}_n^5$ *with bottom isomorphic to* $\mathbf{C}_0$*, then* $\mathbf{D} \not\le \mathbf{B}$.

*Proof.* The direct implication is easy. Let the six conditions be satisfied. We can suppose that $\mathbf{A}^+ \oplus \mathbf{C}_k = \mathbf{A} \oplus \mathbf{M}_n \oplus \mathbf{C}_k$ is a sublattice of $\mathbf{B}$. It follows from (2) and (4) that $I[1_{\mathbf{A}^+}, 1_{\mathbf{B}}]$ is a chain of height $k$; denote its elements by $1_{\mathbf{A}^+} = b_0 \prec b_1 \prec \cdots \prec b_k = 1_{\mathbf{B}}$.

Let $1 \le i \le k$. Since $I[0_{\mathbf{B}}, b_i]$ is a principal ideal of $\mathbf{B}$ of height at least $\mathrm{ht}(\mathbf{A}) + 2 + i > \mathrm{ht}(\mathbf{A}) + 2$, by (5) this ideal has precisely two coatoms, only one of which is at height $\ge \mathrm{ht}(\mathbf{A}) + 2$; this coatom must be the element $b_{i-1}$. Denote by $c_i$ the other coatom. We have $I[0_{\mathbf{B}}, b_i] = I[0_{\mathbf{B}}, b_{i-1}] \cup \{b_i, c_i\}$. Since $(b_0]_B = A^+$, we get $B = A^+ \cup \{b_1, \dots, b_k\} \cup \{c_1, \dots, c_k\}$. Clearly, $c_i \ne b_j$ for all $i, j$ and $c_1, \dots, c_k$ are pairwise different.

For $1 \le i \le k$ denote by $a_i$ the only lower cover of $c_i$. It follows from (3) and (4) that $a_i \notin \{b_1, \dots, b_k\}$ and if $a_i \in A^+$ then $a_i \in A$.

Suppose that $a_i = c_j$ for some $i$ and $j$. Let $i$ be the least index such that $a_i = c_j$ for some $j$. Clearly, $i < j$. By the minimality of $i$, $a_i \in A$. Then $\mathbf{M}_n \cup \{a_i, b_i, b_j, c_i, c_j\}$ is a sublattice of $B$ and this sublattice is a V-extension of either $\mathbf{M}_n^4$ or $\mathbf{M}_n^5$, a contradiction with (6).

Thus $a_i \notin \{c_1, \dots, c_k\}$ and it follows that $a_i \in A$ for all $i$.

If $a_i = a_j$ for some $i \ne j$, we also get a contradiction by (6). Thus $s = \langle a_1, \dots, a_k \rangle$ is a simple sequence of elements of $\mathbf{A}$ and $\mathbf{B} \cong \mathbf{H}_s(\mathbf{A})$. □

**Lemma 5.2.** *The following binary relation $R$ on* LATT′ *is definable:* $(\mathbf{A}, \mathbf{B}) \in R$ *iff* $\mathbf{A}, \mathbf{B} \in$ LATT′ *and* $\mathbf{B}$ *is a chain of height* $|\mathbf{A}|$.

*Proof.* It follows from 5.1: for a chain $\mathbf{C}_m$ we have $(\mathbf{A}, \mathbf{C}_m) \in R$ if and only if $\mathbf{A} \oplus \mathbf{C}_2 \oplus \mathbf{C}_m$ is of the same height as any maximal simple sequential extension of $\mathbf{A}$. □

A finite lattice will be called *wide* if it is not a chain and contains an element that is both an atom and a coatom.

**Lemma 5.3.** *The set of wide elements of* LATT′ *is definable.*

*Proof.* Suppose that $\mathbf{A} \in$ LATT is not a chain and $n_0$ is the least integer $n$ such that $n \ge 3$ and $\mathbf{M}_n \not\le \mathbf{A}$. Then $\mathbf{A}$ is wide iff $\mathbf{A}^+$ has a V-extension $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A}^+)$ where $a \in A$ and $I[0_{\mathbf{A}}, a] \cong \mathbf{C}_1$ with the property: if $\mathbf{D} \in$ LATT is such that $\mathbf{M}_{n_0}^5 \le \mathbf{D} \le \mathbf{B}$ and $\mathbf{D} \ge \mathbf{E} \oplus \mathbf{M}_{n_0}^5$ holds only for a one-element lattice $\mathbf{E}$, then $\mathbf{D} \cong \mathbf{M}_{n_0}^5$.

To see that this is equivalent to a first-order definition, see Lemma 4.3. To see that the formulated properties capture the concept of wide lattice, suppose first that $\mathbf{A}$ is wide. Choosing $a$ to be an element of $\mathbf{A}$ that is both maximal and minimal in $\mathbf{A}$, it should be clear that the V-extension $\mathbf{V}_a(\mathbf{A}^+)$ fulfills the conditions.

Next, suppose that $\mathbf{B} \cong \mathbf{V}_a(\mathbf{A}^+)$ fulfills the conditions. Then obviously, the element $a$ is an atom of $\mathbf{A}$. If there exists $b \in A$ with $a < b < 1_{\mathbf{A}}$, then we have a sublattice $\mathbf{D}$ of $\mathbf{B}$ isomorphic to $(\mathbf{C}_2 \oplus \mathbf{M}_{n_0} \oplus \mathbf{C}_1) + \mathbf{C}_2$, with least element $a$. Here $\mathbf{D} > \mathbf{M}_{n_0}^5$ and $\mathbf{D} \ge \mathbf{E} \oplus \mathbf{M}_{n_0}^5$ holds only for the one-element lattice $\mathbf{E}$. □

Let $\mathbf{B} = \mathbf{H}_s(\mathbf{A})$ for a finite lattice $\mathbf{A}$ and a maximal simple sequence $s = \langle a_1, \dots, a_k \rangle$ of elements of $\mathbf{A}$ (so that $k = |\mathbf{A}|$). Denote by $b_0 \prec \cdots \prec b_k$ the elements of the chain $I[1_{\mathbf{A}^+}, 1_{\mathbf{B}}]$ in $\mathbf{B}$. For every $i = 0, \dots, k$, $b_i$ is the only element of $\mathbf{A}$ of height $\mathrm{ht}(A)+2+i$ and $I[0_{\mathbf{B}}, b_i]$ is the only ideal of $\mathbf{B}$ of that height. For $i = 1, \dots, k$ denote by $c_i$ the only coatom of $I[0_{\mathbf{B}}, b_i]$ different from $b_{i-1}$, so that $a_i = b_{i-1} \wedge c_i$. It is easy to see that the filter $I[a_i, 1_{\mathbf{B}}]$ of $\mathbf{B}$ is not isomorphic to any other filter of $\mathbf{B}$. This filter is the only filter $D$ of $\mathbf{B}$ with the property that its only ideal containing $\mathbf{M}_n$ and of height $\mathrm{ht}(D)+i-k$ (i.e., the interval $I[a_i, b_i]$ of $\mathbf{B}$) is a wide lattice. $D$ will be called the $i$-th essential filter of $\mathbf{B}$. We get:

**Lemma 5.4.** *The following quaternary relation $R$ on* LATT$'$ *is definable:*

*$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in R$ iff $\mathbf{B} \cong \mathbf{H}_s(\mathbf{A})$ for a maximal simple sequence $s = \langle a_1, \dots, a_k \rangle$ of elements of $\mathbf{A}$, $\mathbf{C}$ is a chain of height $i$ with $1 \le i \le k$ and $\mathbf{D}$ is isomorphic to the $i$-th essential filter of $\mathbf{B}$.*

**Lemma 5.5.** *The following quaternary relation $R$ on* LATT$'$ *is definable:*

*$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \in R$ iff $\mathbf{B} \cong \mathbf{H}_s(\mathbf{A})$ for a maximal simple sequence $s = \langle a_1, \dots, a_k \rangle$ of elements of $\mathbf{A}$, $\mathbf{C}$ is a chain of height $i$, $\mathbf{D}$ is a chain of height $j$, $1 \le i, j \le k$ and $a_i \le a_j$ in $\mathbf{A}$.*

*Proof.* We have $a_i \le a_j$ if and only if the $j$-th essential filter of $\mathbf{B}$ is a filter of the $i$-th essential filter of $\mathbf{B}$. □

**Theorem 5.6.** *Every element of* LATT$'$ *is definable.*

The proof follows from 5.5.

**Theorem 5.7.** *The ordered set of isomorphism types of finite lattices with respect to embeddability has only two automorphisms: the identity and the opposite map. The isomorphism type of any finite lattice is definable in this ordered set up to the two automorphisms.*

The proof follows from 5.6.

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