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Lattice theory and metric geometry

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Abstract. K. Menger and G. Birkhoff recognized 70 years ago that lattice theory provides a framework for the development of incidence geometry (affine and projective geometry). We show in this article that lattice theory also provides a framework for the development of metric geometry (including the euclidean and classical non-euclidean geometries which were first discovered by A. Cayley and F. Klein). To this end we introduce and study the concept of a Cayley–Klein lattice. A detailed investigation of the groups of automorphisms and an algebraic characterization of Cayley–Klein lattices are included.

1. Introduction

K. Menger [23] and G. Birkhoff [5] were the first mathematicians who recognised that lattice theory provides a framework for the development of projective and affine geometry (incidence geometry).

It is the aim of this paper to show that lattice theory also provides an appropriate framework for the development of metric geometry including the classical euclidean and non-euclidean geometries (such as elliptic and hyperbolic geometry, and the space-times of Galilei, Minkowski, de Sitter and Newton–Hooke). These geometries were first discovered by A. Cayley [8] and F. Klein [20]. For this reason we call the underlying lattice theoretical structures Cayley–Klein lattices. We introduce this concept in Section 2.

Section 3 contains basic examples of Cayley–Klein lattices and their groups of automorphisms which have importance in various branches of mathematics and physics. We discuss contractions of Cayley–Klein lattices (in the sense of E. Inönü and E. P. Wigner [18]) and show that for Cayley–Klein lattices the principle of duality is valid. It is worth noting that the lattice theoretical approach allows the formulation of this principle — which is well known in lattice theory (see G. Grätzer [11, §I.1]) and in incidence geometry (see D. R. Hughes and F. C. Piper [17, Theorem 3.2]) — for the first time for metric geometry.

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In Section 4 we develop some basic ideas of a general theory of Cayley–Klein lattices including metric concepts such as the notion of a polar and the orthogonality of subspaces.

Automorphisms of Cayley–Klein lattices are considered in Section 5. We analyse the structure of the automorphism group and represent automorphisms by matrices. These investigations are performed in the more general framework of universal flag lattices, a concept defined in this section.

The paper closes with an algebraic characterization of Cayley–Klein lattices as finite dimensional vector spaces endowed with a sequence of forms, each defined on the radical of the preceding one.

2. Basic notions

In this section we introduce the notion of a Cayley–Klein lattice. For the lattice theoretical terminology used in this article we refer to G. Grätzer [11] and C.-A. Faure and A. Frölicher [9]. We denote the partial ordering of a lattice $\mathcal{L}$ with $\leq$ and infimum, resp. supremum, with $\wedge$, resp. $\vee$. In this article we deal with geometric lattices and therefore often use a geometric terminology. Thus we call the elements of $\mathcal{L}$ *flats* or *subspaces*. For subspaces α and β we write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. The universal bounds are denoted by 0 and 1. *Polarities* of a projective lattice are projective correlations (anti-automorphisms) of order 2. For the notion of a projective correlation (collineation) see D. R. Hughes and F. C. Piper [17, II.5]. A correlation (resp. collineation) of a projective lattice of dimension < 2 is projective if it can be extended to a projective correlation (resp. collineation) of a projective lattice of dimension ≥ 2 which contains $\mathcal{L}$ as a subspace.

We mention explicitly that in this paper the *dimension* d of a projective lattice $\mathcal{L}$ is defined as the dimension of the associated projective geometry (see C.-A. Faure and A. Frölicher [9, 4.6.3]), that is, $d+1$ is the length of $\mathcal{L}$ (see G. Grätzer [11]). We call the subspaces of dimensions $0, 1, 2, n-1$ points, lines, planes, and hyperplanes respectively.

Definition 2.1. $(\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ with $r \geq 0$ is a *Cayley–Klein lattice of dimension* $n \geq 0$ if the following assumptions hold:

(1) $\mathcal{L}$ is a projective lattice of finite dimension n.
(2) $\epsilon_0, \epsilon_1, \ldots, \epsilon_{r+1}$ are subspaces of $\mathcal{L}$ with $0 = \epsilon_0 < \epsilon_1 \cdots < \epsilon_{r+1} = 1$.
(3) π_k (with $1 \leq k \leq r+1$) is a polarity on the interval $[\epsilon_{k-1}, \epsilon_k]$ that is:
   (a) $\alpha = (\alpha\pi_k)\pi_k$ for every $\alpha \in [\epsilon_{k-1}, \epsilon_k]$
   (b) $\alpha \leq \beta\pi_k$ implies $\beta \leq \alpha\pi_k$ for $\alpha, \beta \in [\epsilon_{k-1}, \epsilon_k]$.

For convenience we sometimes denote a Cayley–Klein lattice by $\mathcal{C}(\epsilon_0, \ldots, \epsilon_{r+1})$, if the underlying lattice $\mathcal{L}$ and the polarities π_k are of no special concern.

From conditions (a) and (b) follow immediately $\alpha \leq \beta\pi_k$ iff $\beta \leq \alpha\pi_k$ and $\alpha \leq \beta$ iff $\beta\pi_k \leq \alpha\pi_k$.

Definition 2.2. An *isomorphism* ϕ of Cayley–Klein lattices

$$\mathcal{C} = (\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$$

and

$$\bar{\mathcal{C}} = (\bar{\mathcal{L}}, (([\bar{\epsilon}_0, \bar{\epsilon}_1], \bar{\pi}_1), \ldots, ([\bar{\epsilon}_r, \bar{\epsilon}_{r+1}], \bar{\pi}_{r+1})))$$

is an isomorphism ϕ of the lattices $\mathcal{L}$ and $\bar{\mathcal{L}}$, which maps the intervals of $\mathcal{C}$ onto the intervals of $\bar{\mathcal{C}}$ (i.e., $r = \bar{r}$ and $\epsilon_k\phi = \bar{\epsilon}_k$ for $0 \leq k \leq r+1$) such that ϕ commutes with the polarities on the intervals of $\mathcal{C}$ and $\bar{\mathcal{C}}$ (i.e., $\pi_k\phi_k = \phi_k\bar{\pi}_k$ where ϕ_k is the restriction of ϕ to $[\epsilon_{k-1}, \epsilon_k]$ for $1 \leq k \leq r+1$). When $\mathcal{C} = \bar{\mathcal{C}}$ one says that ϕ is an *automorphism*.

Definition 2.1 leads immediately to:

Proposition 2.3. *Let $\mathcal{C}(\epsilon_0, \ldots, \epsilon_{r+1})$ be a Cayley–Klein lattice. Then $\mathcal{C}(\epsilon_i, \ldots, \epsilon_k)$ is a Cayley–Klein lattice (for $0 \leq i < k \leq r+1$).*

Definition 2.4. A Cayley-Klein lattice is called *irreducible* if the associated projective lattice is irreducible. An irreducible Cayley-Klein lattice of dimension ≥ 2 is called *arguesian* (*pappian*) if the associated projective lattice is arguesian (pappian). An irreducible Cayley-Klein lattice of dimension < 2 is called *arguesian* (*pappian*) if it is a subspace of an irreducible arguesian (pappian) Cayley-Klein lattice of dimension 2.

If a Cayley-Klein lattice is irreducible, this means geometrically that every line of the associated projective geometry contains at least three points (see C. A. Faure and A. Frölicher [9; 2.7]). If a Cayley-Klein lattice is arguesian (pappian) this means, that the theorem of Desargues (Pappus) holds in the associated projective geometry (see G. Grätzer [11, IV.4 and IV.5] and C. A. Faure and A. Frölicher [9, 8.4 and 9.6]).

An irreducible arguesian Cayley–Klein lattice admits homogeneous coordinates i.e., it can be represented as the lattice of subspaces of a vector space over a skew field (often called division ring). Any irreducible Cayley–Klein lattice of dimension ≥ 3 is arguesian (see G. Grätzer [11, IV.5, Theorem 13 and Theorem 15]). The theorem of Pappus holds if and only if the field of coordinates is commutative.

Definition 2.5. An irreducible arguesian Cayley–Klein lattice is of *characteristic* 2, if in the associated projective geometry the diagonal points of every complete quadrangle are collinear (i.e., iff the associated coordinate field is of characteristic 2; see C.-A. Faure and A. Frölicher [9, §9.7.2]).

Remark 2.6. The notion of a Cayley–Klein lattice of finite dimension given in Definition 2.1 can be generalized to arbitrary dimensions. The theory of projective lattices of infinite dimensions differs however in many aspects from the theory of finite dimensional projective lattices (see e.g. the work of H. Gross [13], [14], who made a deep study of infinite dimensional vector spaces equipped with a single form).

3. Cayley–Klein lattices: duality, contractions and basic examples

Interesting relationships between Cayley–Klein lattices can be described by the notions of duality and contraction. Before discussing these topics we give some basic examples of Cayley–Klein lattices.

Examples. We distinguish between the following classes of Cayley–Klein lattices:

Definition 3.1. An irreducible arguesian Cayley–Klein lattice is called

- *orthogonal* if π_k is an orthogonal polarity for $1 \leq k \leq r+1$ (i.e., a projective correlation of order 2 which is not null; algebraically π_k can be described by a non-degenerate symmetric bilinear form);
- *symplectic* if π_k is a symplectic (or null) polarity for $1 \leq k \leq r+1$ (i.e., a projective correlation of order 2 which is null; algebraically π_k can be described by a non-degenerate alternating bilinear form);
- *unitary* if π_k is an unitary (or Hermitian) polarity for $1 \leq k \leq r+1$ (i.e., a correlation of order 2 which is not projective; algebraically π_k can be described by a non-degenerate Hermitian form which is not a symmetric bilinear form);
- *of mixed type* if the polarities π_k are of different types (orthogonal, symplectic or unitary).

We give some examples for Cayley–Klein lattices of these kinds:

- The lattice of subspaces of a finite dimensional vector space endowed with a sequence of forms, each defined on the radical of the preceding one (see Definition 6.1), is a Cayley–Klein lattice (see Proposition 6.2). In Section 6 we show that every irreducible Cayley–Klein lattice of dimension ≥ 3 can be represented in this way (see Proposition 6.3).
- Pappian orthogonal Cayley–Klein lattices are projective spaces with a Cayley–Klein metric (see O. Giering [10] and H. Struve and R. Struve [33], [32]) among which are the elliptic, hyperbolic, euclidean, galilean and minkowskian metrics studied by A. Cayley [8], F. Klein [20], I. M. Yaglom [35], [36] and B. A. Rosenfeld [26].

- Finite orthogonal Cayley–Klein lattices are associated with important configurations. So, for instance, the Desargues-, Pappus-, resp. Petersen configuration is an 'Eigentlichkeitsbereich' (a term introduced by F. Klein [20]) of the elliptic plane over the field GF(5) of five elements, of the Galilean plane over GF(3), resp. of the hyperbolic plane over GF(5) (see H. Struve and R. Struve [31]).
- The classical groups (other than the exceptional ones) form four families, known as A_m, B_m, C_m, D_m (for $m = 1, 2, 3, \dots$), which are groups of automorphisms of Cayley–Klein lattices (see B. A. Rosenfeld [28] and R. Penrose [25, Section 13], who discusses their role in physics).
- If $\mathcal{C}$ is a Cayley–Klein lattice with $\epsilon_1 = 1$, then $\mathcal{L}$ together with the operator $\alpha \to \alpha\pi_1$ is an ortholattice (in the sense of C.-A. Faure and A. Frölicher [9]).

Starting from a synthetic definition of a non-euclidean geometry one arrives at a Cayley–Klein lattice by (i) embedding the non-euclidean geometry into a projective space by adding 'ideal' elements (or 'subspaces at infinity') and (ii) extending the metric of the geometry to a metric of the projective space (which can be described by a sequence of forms). This procedure is called "Begründung" of a geometry (see F. Bachmann [1, §6] and J. Hjelmslev [16]).

Duality. The dual of an interval $[\epsilon_{k-1}, \epsilon_k]$ of $\mathcal{L}$ is an interval of the dual projective lattice $\mathcal{L}^*$. Since the analogous statement holds for polarities it follows that the dual of a Cayley–Klein lattice is a Cayley–Klein lattice.

Proposition 3.2. *Let* $(\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), \dots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ *be a Cayley–Klein lattice. Then* $(\mathcal{L}^*, (([\epsilon_{r+1}, \epsilon_r], \pi_{r+1}), \dots, ([\epsilon_1, \epsilon_0], \pi_1)))$ *is a Cayley–Klein lattice.*

Thus the *principle of duality* can be extended from incidence geometry (projective geometry) to metric geometry (Cayley–Klein lattices).

Of special interest are self-dual Cayley–Klein lattices (which are isomorphic to their dual structure).

Proposition 3.3. $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ *is self-dual if and only if the Cayley–Klein lattices* $\mathcal{C}(\epsilon_k, \epsilon_{k+1})$ *and* $\mathcal{C}(\epsilon_{r-k}, \epsilon_{r+1-k})$ *are isomorphic for* $0 \leq k \leq r$.

Examples are the Cayley–Klein lattices associated to the Galilean plane, the n-dimensional elliptic geometry and the configurations of Desargues and Pappus.

Contractions. The concept of a contraction arose in the study of the relation between relativistic and classical mechanics. As is well known, when the velocity of light goes to infinity the relativistic space-time leads formally to their non-relativistic limit. To make these ideas more precise, E. Inönü and E. P. Wigner [18] introduced the concept of a contraction of a Lie group. A contraction of a Lie group starts from its Lie algebra and creates by help of a contraction parameter which tends to zero a new Lie algebra and thus a new Lie group.

Well-known examples in physics — already discussed by Inönü and Wigner — are the contraction of the inhomogeneous Lorentz group (also called Poincaré group, cf. E. Inönü and E. P. Wigner [18], or Minkowski group, cf. F. Bachmann [1]) to the Galilean group and the contraction of the group of rotations of the three-dimensional Euclidean space (the group of motions of the two-dimensional elliptic geometry) to the group of motions of the Euclidean plane.

In our lattice theoretical approach contractions can be defined for arbitrary Cayley–Klein lattices without considering contraction parameters or limits (as in E. Inönü and E. P. Wigner [18]).

Proposition 3.4. *Let* $\mathcal{C} = (\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), \dots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ *be a Cayley–Klein lattice and* $(([\epsilon_{i-1}, \dot\epsilon], \dot\pi), ([\dot\epsilon, \epsilon_i], \ddot\pi))$ *a splitting of the interval* $([\epsilon_{i-1}, \epsilon_i], \pi_i)$*, that is:*

(1) $\epsilon_{i-1} < \dot\epsilon < \epsilon_i$*;*
(2) $\dot\epsilon\pi_i \wedge \dot\epsilon = \epsilon_{i-1}$ *(i.e.,* $\dot\epsilon$ *is a non self-conjugate subspace of* $([\epsilon_{i-1}, \epsilon_i], \pi_i)$*);*
(3) $\dot\pi : [\epsilon_{i-1}, \dot\epsilon] \to [\epsilon_{i-1}, \dot\epsilon]; \alpha \to \alpha\pi_i \wedge \dot\epsilon$ *and* $\ddot\pi : [\dot\epsilon, \epsilon_i] \to [\dot\epsilon, \epsilon_i]; \alpha \to \alpha\pi_i \vee \dot\epsilon$*.*

Then $\dot{\mathcal{C}} = (\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), \dots, ([\epsilon_{i-1}, \dot\epsilon], \dot\pi), ([\dot\epsilon, \epsilon_i], \ddot\pi), \dots, ([\epsilon_r, \epsilon_{r+1}], \pi_{r+1})))$ *is as well a Cayley–Klein lattice.*

Proof. We show that $\dot\pi$ is a polarity. The corresponding proof for $\ddot\pi$ is analogous.

(1) $\dot\pi$ has order 2 (and thus $\dot\pi$ is bijective): Let α be an element of $[\epsilon_{i-1}, \dot\epsilon]$. Using the modularity of $\mathcal{L}$ we get $(\alpha\dot\pi)\,\dot\pi = (\alpha\pi_i \wedge \dot\epsilon)\pi_i \wedge \dot\epsilon = (\alpha \vee \dot\epsilon\pi_i) \wedge \dot\epsilon = \alpha \vee (\dot\epsilon\pi_i \wedge \dot\epsilon) = \alpha$. Since $\epsilon_{i-1}\dot\pi = \dot\epsilon$, the mapping $\dot\pi$ is not the identity.

(2) $\dot\pi$ is a correlation: Let α and β be elements of $[\epsilon_{i-1}, \dot\epsilon]$ with $\alpha < \beta$. Since π_i is a correlation we have $\beta\pi_i < \alpha\pi_i$, and hence $\beta\pi_i \wedge \dot\epsilon \leq \alpha\pi_i \wedge \dot\epsilon$ and $\beta\dot\pi \leq \alpha\dot\pi$. Since $\dot\pi$ is injective and $\alpha < \beta$, we get $\beta\dot\pi < \alpha\dot\pi$. □

Definition 3.5. Let $\mathcal{C}$ and $\dot{\mathcal{C}}$ be Cayley–Klein lattices. If $\dot{\mathcal{C}}$ can be constructed from $\mathcal{C}$ by splitting one or more intervals of $\mathcal{C}$, we call $\dot{\mathcal{C}}$ a *contraction* of $\mathcal{C}$.

It is easily seen that if $\dot{\mathcal{C}}$ is a contraction of $\mathcal{C}$ then the dual lattice $\dot{\mathcal{C}}^*$ of $\dot{\mathcal{C}}$ is a contraction of the dual lattice $\mathcal{C}^*$ of $\mathcal{C}$.

Example. The inhomogeneous Lorentz group is the group of projective automorphisms of the real 4-dimensional Cayley–Klein lattice

$$\mathcal{C}_{\text{Mink}} = (\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), ([\epsilon_1, \epsilon_2], \pi_2)))$$

with a subspace ϵ_1 of dimension 3 and polarities π_1, resp. π_2, which are hyperbolic, resp. elliptic. The Inönü-Wigner contraction of this group to the Galilean group corresponds to the contraction of $\mathcal{C}_{\text{Mink}}$ by splitting the interval $([\epsilon_0, \epsilon_1], \pi_1)$ with a 2-dimensional subspace $\dot\epsilon$.

4. Metric concepts

In this section we develop some basic ideas of a general theory of Cayley–Klein lattices. We introduce the concept of a polar and define an orthogonality relation between arbitrary elements of a Cayley–Klein lattice.

It is well known that the polar of a subspace α of a Cayley–Klein lattice $(\mathcal{L}(([0,1],\pi)))$ is the subspace $\alpha\pi$. In order to extend the notion of a polar to arbitrary Cayley–Klein lattices, we consider the following mapping, which is a projection from $\mathcal{L}$ into the interval $[\epsilon_{k-1},\epsilon_k]$:

Definition 4.1. For a Cayley–Klein lattice $(\mathcal{L},(([\epsilon_0,\epsilon_1],\pi_1),\ldots,([\epsilon_r,\epsilon_{r+1}],\pi_{r+1})))$ let φ_k be the mapping from $\mathcal{L}$ onto $[\epsilon_{k-1},\epsilon_k]$ with $\alpha\varphi_k = (\alpha\wedge\epsilon_k)\vee\epsilon_{k-1}$ (for $1\le k\le r+1$).

The main properties of φ_k are summarized in the following:

Proposition 4.2. *Let $\mathcal{C}(\epsilon_0,\ldots,\epsilon_{r+1})$ be a Cayley–Klein lattice.*

(a) $(\alpha\wedge\epsilon_k)\vee\epsilon_{k-1} = (\alpha\vee\epsilon_{k-1})\wedge\epsilon_k$.

(b) If $\alpha\le\beta$ then $\alpha\varphi_k\le\beta\varphi_k$ (i.e., φ_k is a $\le$-homomorphism from $\mathcal{L}$ onto the interval $[\epsilon_{k-1},\epsilon_k]$).

(c) $(\alpha\varphi_k)\varphi_k=\alpha\varphi_k$ (i.e., φ_k is a projection).

(d) The mapping φ_k of $\mathcal{C}$ induces in $\mathcal{C}^*$ the mapping $\varphi_{(r+1)-(k-1)}$.

Proof. (a), (b) and (d) follow immediately from Definition 4.1 and the modularity of the lattice $\mathcal{L}$. Furthermore we have

$$\begin{aligned}(\alpha\varphi_k)\varphi_k &= (((\alpha\vee\epsilon_{k-1})\wedge\epsilon_k)\wedge\epsilon_k)\vee\epsilon_{k-1}\\ &= ((\alpha\vee\epsilon_{k-1})\wedge\epsilon_k)\vee\epsilon_{k-1}\\ &= ((\alpha\wedge\epsilon_k)\vee\epsilon_{k-1})\vee\epsilon_{k-1}\\ &= (\alpha\wedge\epsilon_k)\vee\epsilon_{k-1}=\alpha\varphi_k,\end{aligned}$$

which proves (c). □

We now define the concept of a polar of a subspace α:

Definition 4.3. Let α be a subspace of a Cayley–Klein lattice

$$(\mathcal{L},(([\epsilon_0,\epsilon_1],\pi_1),\ldots,([\epsilon_r,\epsilon_{r+1}],\pi_{r+1}))).$$

A subspace β is called a *polar* of α if $\beta\varphi_k=(\alpha\varphi_k)\pi_k$ holds for $1\le k\le r+1$.

An immediate consequence is:

Proposition 4.4. (1) *If β is a polar of α, then α is a polar of β.*

(2) *If β is a polar of α in $\mathcal{C}$ then β is a polar of α in $\mathcal{C}^*$.*

Proposition 4.5. *For subspaces α, β of a Cayley–Klein lattice $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ the following conditions are equivalent:*

(1) *β is a polar of α in $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$.*
(2) *$\beta \wedge \epsilon_m$ is a polar of $\alpha \wedge \epsilon_m$ in $\mathcal{C}(\epsilon_0, \dots, \epsilon_m)$ for all $m \in \{1, \dots, r+1\}$.*
(3) *$\beta \vee \epsilon_m$ is a polar of $\alpha \vee \epsilon_m$ in $\mathcal{C}(\epsilon_m, \dots, \epsilon_{r+1})$ for all $m \in \{0, \dots, r\}$.*
(4) *$\beta\varphi_k$ is a polar of $\alpha\varphi_k$ in $[\epsilon_{k-1}, \epsilon_k]$ for all $k \in \{1, \dots, r+1\}$.*

Proof. (1) $\leftrightarrow$ (2): Let β be a polar of α in $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ and $m \in \{1, \dots, r+1\}$ and $k \le m$. Then $(\alpha \wedge \epsilon_m)\varphi_k = ((\alpha \wedge \epsilon_m) \wedge \epsilon_k) \vee \epsilon_{k-1} = (\alpha \wedge \epsilon_k) \vee \epsilon_{k-1} = (\alpha\varphi_k)\pi_k$ and likewise $(\beta \wedge \epsilon_m)\varphi_k = \beta\varphi_k$. Since $\beta\varphi_k = (\alpha\varphi_k)\pi_k$ one gets $((\alpha \wedge \epsilon_m)\varphi_k)\pi_k = ((\beta \wedge \epsilon_m)\varphi_k)\pi_k$ which proves (2). - The other direction is obvious.

(1) $\leftrightarrow$ (3): This is the dual statement of the equivalence of (1) and (2).

(1) $\leftrightarrow$ (4): This holds because φ_k is a projection: $\beta\varphi_k = (\alpha\varphi_k)\pi_k$ for $1 \le k \le r+1$ is equivalent to $\beta\varphi_k\varphi_k = (\alpha\varphi_k\varphi_k)\pi_k$ for $1 \le k \le r+1$. □

It is well known that in an n-dimensional projective-metric space with an elliptic or hyperbolic projective polarity the sum of the dimensions of a subspace and its polar is equal to $n-1$. The next proposition shows that this statement holds in all Cayley–Klein lattices.

Proposition 4.6. *Let $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ be a Cayley–Klein lattice of dimension n and β a polar of α. Then $\dim(\alpha) + \dim(\beta) = n - 1$.*

Proof. According to the dimension theorem of projective geometry (see C.-A. Faure and A. Frölicher [9, Theorem 4.6.5]) one gets $\sum_1^{r+1}(\dim(\alpha\varphi_k) - \dim(\epsilon_{k-1})) = \dim(\alpha)$. From this and Proposition 4.5 (4) the assertion follows. □

With regard to the existence of a polar we show:

Proposition 4.7. *Each subspace of a Cayley–Klein lattice has at least one polar.*

Proof. The proof is by induction on r; we write $\beta = \sigma \dot{\vee} \tau$ if $\beta = \sigma \vee \tau$ and $\sigma \wedge \tau = 0$. It is well known that the proposition holds for $r = 0$. Let $r \ge 1$ and β a polar of $\alpha \wedge \epsilon_r$ in $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ and $(\alpha\varphi_{r+1})\pi_{r+1} = (\alpha \vee \epsilon_r)\pi_{r+1} = \epsilon_r \dot{\vee} \tau$ Then $\beta \dot{\vee} \tau$ is a polar of α in $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ since $(\alpha\varphi_k)\pi_k = ((\alpha \wedge \epsilon_r)\varphi_k)\pi_k = \beta\varphi_k = (\beta \vee \tau)\varphi_k$ for $1 \le k \le r$ and $(\alpha\varphi_{r+1})\pi_{r+1} = \epsilon_r \vee \tau = (\beta \vee \tau) \vee \epsilon_r = (\beta \vee \tau)\varphi_{r+1}$ (since $\beta \le \epsilon_r$). □

With regard to the uniqueness of a polar we prove:

Proposition 4.8. *Let $\mathcal{C}(\epsilon_0, \dots, \epsilon_{r+1})$ be an irreducible Cayley–Klein lattice. Then the following statements are equivalent:*

(a) *α has one and only one polar β.*
(b) *α has a polar β with $\beta \in [\epsilon_{k-1}, \epsilon_k]$ for an integer k with $1 \le k \le r+1$.*
(c) *$\alpha \wedge \epsilon_{k-1} = 0$ and $\alpha \vee \epsilon_k = 1$ for an integer k with $1 \le k \le r+1$.*

Proof. The proposition holds for $r = 0$. Let $r \geq 1$.

(a) $\to$ (b): Let β be a polar of α with $\beta \notin [\epsilon_{k-1}, \epsilon_k]$ for all $k \in \{1, \ldots, r+1\}$. We show that then α has a polar $\bar{\beta}$ with $\bar{\beta} \neq \beta$. Let $m \in \{1, \ldots, r\}$ with $\epsilon_{m-1} \leq \beta$ and not $\epsilon_m \leq \beta$. Because $\beta \notin [\epsilon_{m-1}, \epsilon_m]$ it is not $\beta \leq \epsilon_m$. Hence there is a point $a \leq \epsilon_m$ with $a \wedge \beta = 0$ and a point $b \leq \beta$ with $b \wedge \epsilon_m = 0$. Since $\mathcal{C}(\epsilon_0, \ldots, \epsilon_{r+1})$ is irreducible there is a point $c \neq a, b$ on the line $a \vee b$. Since a subspace contains with two points all points of the connecting line $c \wedge \epsilon_m = c \wedge \beta = 0$.

Let $\mathfrak{B}$ be a basis of $\beta \wedge \epsilon_m$ and $\mathfrak{B}'$ a basis of β with $\mathfrak{B} \subset \mathfrak{B}'$ and $b \in \mathfrak{B}'$. We consider the subspace $\bar{\beta}$ which is generated by c and all elements of $\mathfrak{B}'$ which are different from b. Then $a \leq \epsilon_m$ and $b \leq \beta$ and $c \leq \bar{\beta}$. It follows $a, b, c \leq \beta \vee \epsilon_m$ and $a, b, c \leq \bar{\beta} \vee \epsilon_m$ and hence $(*)$ $\beta \vee \epsilon_m = \bar{\beta} \vee \epsilon_m$.

It follows further $\dim(\beta \vee \epsilon_m) = \dim(\bar{\beta} \vee \epsilon_m)$ and according to the dimension theorem of projective geometry $\dim(\beta \wedge \epsilon_m) = \dim(\bar{\beta} \wedge \epsilon_m)$. Since $\beta \wedge \epsilon_m \leq \bar{\beta}$ one gets $(**)$ $\beta \wedge \epsilon_m = \bar{\beta} \wedge \epsilon_m$. Because of $(*)$ it is $\beta \wedge \epsilon_k = (\beta \wedge \epsilon_m) \wedge \epsilon_k = (\bar{\beta} \wedge \epsilon_m) \wedge \epsilon_k = \bar{\beta} \wedge \epsilon_k$ for $k \leq m$ and because of $(**)$ it is $\beta \vee \epsilon_k = \bar{\beta} \vee \epsilon_k$ for $k \geq m$. Hence $\bar{\beta}\varphi_k = \beta\varphi_k = (\alpha\varphi_k)\pi_k$ for $1 \leq k \leq r+1$ which shows that $\bar{\beta}$ is a polar of α. It is $\beta \neq \bar{\beta}$ (since c is a point of $\bar{\beta}$ but not of β).

(b) $\to$ (c): Let β be a polar of α with $\beta \in [\epsilon_{k-1}, \epsilon_k]$ for an integer k with $1 \leq k \leq r+1$. According to Proposition 4.5, $\beta \wedge \epsilon_{k-1} = \epsilon_{k-1}$ is a polar of $\alpha \wedge \epsilon_{k-1}$ in $\mathcal{C}(\epsilon_0, \ldots, \epsilon_{k-1})$, and $\beta \vee \epsilon_k = \epsilon_k$ is a polar of $\alpha \vee \epsilon_k$ in $\mathcal{C}(\epsilon_k, \ldots, \epsilon_{r+1})$. Hence $\alpha \wedge \epsilon_{k-1} = 0$ and $\alpha \vee \epsilon_k = 1$.

(c) $\to$ (a): Let $k \in \{1, \ldots, r+1\}$ and $\alpha \wedge \epsilon_{k-1} = 0$ and $\alpha \vee \epsilon_k = 1$ and β a polar of α. Then (according to Proposition 4.5) $\beta \wedge \epsilon_{k-1}$ is a polar of $\alpha \wedge \epsilon_{k-1} = 0$ in $\mathcal{C}(\epsilon_0, \ldots, \epsilon_{k-1})$ and $\beta \vee \epsilon_k$ is a polar of $\alpha \vee \epsilon_k = 1$ in $\mathcal{C}(\epsilon_k, \ldots, \epsilon_{r+1})$. Hence $\beta \wedge \epsilon_{k-1} = \epsilon_{k-1}$ and $\beta \vee \epsilon_k = \epsilon_k$ and $\epsilon_{k-1} \leq \beta \leq \epsilon_k$ and $\beta = (\beta \vee \epsilon_{k-1}) \wedge \epsilon_k = \beta\varphi_k = (\alpha\varphi_k)\pi_k$. □

In view of Proposition 4.8 we define:

Definition 4.9. A subspace α is called *regular* if $\alpha \wedge \epsilon_{k-1} = 0$ and $\alpha \vee \epsilon_k = 1$ for an integer k with $1 \leq k \leq r+1$.

The concept of a polar allows to introduce further metric concepts, such as the orthogonality of subspaces. Thus the different introductions of orthogonality relations which can be found for special classes of Cayley–Klein lattices in the literature are unified.

Definition 4.10. Subspaces α and $\bar{\alpha}$ are *orthogonal* if there are polars β and $\bar{\beta}$ (of α and $\bar{\alpha}$ respectively) with $\alpha \wedge \bar{\beta} \neq 0$ and $\bar{\alpha} \wedge \beta \neq 0$.

A further development of metric concepts of Cayley–Klein lattices is beyond the scope of this paper.

5. Automorphisms of Cayley–Klein lattices

We start our investigations in a more general framework (with references to notions and concepts of universal algebra; see G. Grätzer [12]).

Definition 5.1. $\mathcal{F} = (\mathcal{L}, (([\epsilon_0, \epsilon_1], \mathfrak{R}_1), \ldots, ([\epsilon_r, \epsilon_{r+1}], \mathfrak{R}_{r+1})))$ with $r \geq 0$ is a *universal flag lattice* if the following assumptions hold:

(1) $\mathcal{L}$ is an irreducible arguesian projective lattice of finite dimension.
(2) $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{r+1})$ is a flag, i.e., a chain of subspaces of $\mathcal{L}$ with $0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{r+1} = 1$.
(3) $\mathfrak{R}_k$ (with $1 \leq k \leq r+1$) is a finite set of finitary relations on $[\epsilon_{k-1}, \epsilon_k]$.

Adhering to the Erlangen Program of F. Klein condition (3) can be replaced by allowing on each interval $[\epsilon_{k-1}, \epsilon_k]$ only automorphisms from a prescribed subgroup of the collineation group of the interval.

We give some basic examples of universal flag lattices:

- Every irreducible arguesian Cayley–Klein lattice is an universal flag lattice (with $\mathfrak{R}_k = \{\pi_k\}$ and a polarity π_k for $1 \leq k \leq r+1$).
- Every irreducible arguesian projective lattice is an universal flag lattice (with $r = 0$ and $\mathfrak{R}_1 = \{\,\}$).
- The projective closure of an arguesian affine space $\mathfrak{A}$ can be considered as an universal flag lattice $(\mathcal{L}, (([0, \epsilon_1], \{\,\}), ([\epsilon_1, 1], \{\,\})$ with a hyperplane ϵ_1.
- Further examples are generalized affine spaces, i.e., universal flag lattices with subspaces ϵ_k of arbitrary dimensions (which are called the subspaces 'at infinity') and $\mathfrak{R}_k = \{\,\}$. For a more detailed example we refer to R. Lingenberg [22, §2.3]).

The examples show that the concept of an universal flag lattice covers not only Cayley–Klein lattices and the corresponding metric geometries but also incidence geometry (affine and projective geometry).

The aim of this section is to describe the automorphisms of a flag lattice $\mathcal{F}$ and to analyse the structure of the group of automorphisms. We are especially interested in the group of projective automorphisms of $\mathcal{F}$ which can be represented as a group of linear transformations (just as the classical groups) and which are defined in the following manner:

Definition 5.2. If φ is a collineation of $\mathcal{L}$ with $(\epsilon_j)\varphi = \epsilon_j$ (for $0 \leq j \leq r+1$), then φ is called an *automorphism* of $\mathcal{F}$ if $\rho(\alpha_0, \ldots, \alpha_m)$ with $\alpha_0, \ldots, \alpha_m \in [\epsilon_{k-1}, \epsilon_k]$ imply that $\rho(\alpha_0\varphi, \ldots, \alpha_m\varphi)$ for every m-ary relation $\rho \in \mathfrak{R}_k$ (for $1 \leq k \leq r+1$), see G. Grätzer [11, §36]. We denote the group of automorphisms of $\mathcal{F}$ by $Aut(\mathcal{F})$. We call $\varphi \in Aut(\mathcal{F})$ a *projective automorphism* of $\mathcal{F}$ if φ is a projective collineation of $\mathcal{L}$ and denote the group of projective automorphisms by $PAut(\mathcal{F})$.

φ is a *perspective collineation* of $\mathcal{L}$ with axis α and center β, if α is a hyperplane and β a point and if all points on α and all hyperplanes through β are fixed by φ. A perspective collineation is called an *elation* if α and β are incident and a *homology* otherwise. The perspective collineations of $\mathcal{L}$ generate the group of *projective collineations* (or *projectivities*). A collineation is projective if and only if it can be described in homogeneous coordinates by a linear mapping.

We start with a matrix representation of $PAut(\mathcal{F})$. As mentioned in Section 2 we can represent $\mathcal{L}$ as the lattice of subspaces of a left vector space V (of dimension n) over a division ring D. We write the elements of V as row vectors. A projectivity $\varphi \in PAut(\mathcal{F})$ can be described in homogeneous coordinates by a linear mapping (an automorphism) of V (see C.-A. Faure and A. Frölicher [9, p. 232]), which is unique up to a non-zero constant factor that lies in the center $Z(D)$ of D.

The linear mapping associated to φ can be represented by an element A of the group $GL(n, D)$ of invertible $(n \times n)$ matrices with entries in D, acting on V by right matrix multiplication. In the non-commutative case it is crucial that V be treated as row space under right multiplication by $GL(n, D)$. Treating $v \in V$ as a column vector implies to represent $\varphi \in PAut(\mathcal{F})$ by an element of $GL(n, D^*)$ (where D^* denotes the opposite ring of D) acting on V by left multiplication (see D. G. Northcott [24]). Since A and λA with $\lambda \neq 0$ and $\lambda \in Z(D)$ induce the same projective collineation, $PAut(\mathcal{F})$ can be represented as a subgroup of the projective general linear group $PGL(n, D)$ which is isomorphic to $GL(n, D)/Z(n, D)$ where $Z(n, D) = \{\lambda I : \lambda \in Z(D)\}$ is the center of $GL(n, D)$.

We now determine the special form of matrices which represent elements of $PAut(\mathcal{F})$. Let $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{r+1})$ be the flag associated to $\mathcal{F}$. We choose a fixed *reference system* $\kappa = \{\kappa_k\colon\ \kappa_k$ is a complement of ϵ_{k-1} relative to ϵ_k for $1 \leq k \leq r+1\}$. Let Y_i be subspaces of V which represent κ_i (for $1 \leq i \leq r+1$). Then $V = Y_1 \oplus \cdots \oplus Y_{r+1}$ (direct sum) and we can choose an ordered basis $(e_1, \ldots, e_n)$ of V which is the concatenation of ordered basis of $Y_1, \ldots, Y_{r+1}$.

Let $A \in PGL(n, D)$ be a matrix (with respect to the given basis) which represents $\varphi \in PAut(\mathcal{F})$. The choice of the basis $(e_1, \ldots, e_n)$ shows, that the matrix A admits a block decomposition

$$\begin{bmatrix} A_1 & * & * & \ldots & * \\ 0 & A_2 & * & \ldots & * \\ 0 & 0 & A_3 & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & A_{r+1} \end{bmatrix} \tag{$\spadesuit$}$$

with $(n_i - n_{i-1}) \times (n_i - n_{i-1})$ matrices A_i which represent the group of projective automorphisms of $\mathcal{F}$ which leave Y_i invariant. We denote a matrix of the form (♠) with matrices $A_1, \dots, A_{r+1}$ on the main diagonal and zeros elsewhere by $\mathrm{diag}(A_1, \dots, A_{r+1})$.

The matrix representation is useful to analyse the group structure and to classify the elements of $PAut(\mathcal{F})$. To this end we define:

Definition 5.3. $\varphi \in PAut(\mathcal{F})$ is called a *dilatation* of $\mathcal{F}$ if the restriction of φ to $[\epsilon_i, \epsilon_{i+1}]$ is the identity (for $0 \le i \le r$).

Proposition 5.4. *The group $D(\mathcal{F})$ of dilatations of $\mathcal{F}$ is a normal subgroup of $Aut(\mathcal{F})$.*

Proof. Let $\varphi \in D(\mathcal{F})$ and $\sigma \in Aut(\mathcal{F})$. Since φ is the identity on $[\epsilon_i, \epsilon_{i+1}]$ (for $0 \le i \le r$) $\sigma^{-1}\varphi\sigma$ is the identity on these intervals and $D(\mathcal{F})$ is a normal subgroup of $Aut(\mathcal{F})$. □

We define two types of dilatations (translations and stretchings).

Proposition 5.5. *Let φ be an elation of $\mathcal{L}$ (which is not the identity) with axis α and center β. Then $\varphi \in D(\mathcal{F})$ if and only if there exists a number $k \in \{1, \dots, r\}$ with $\epsilon_k \le \alpha$ and $\beta \le \epsilon_k$.*

Proof. Let $\varphi \in D(\mathcal{F})$ be a perspective collineation with axis α and center β which is not the identity. A subspace is fixed under φ if and only if it is contained in the hyperplane α or if it contains the point β. Let k be the maximal number with $\epsilon_k \le \alpha$. Then $k \ge 1$ (since each point of $[\epsilon_0, \epsilon_1]$ is fixed under φ) and $\beta \le \epsilon_k$ (since each point of $[\epsilon_k, \epsilon_{k+1}]$ is fixed under φ). □

Definition 5.6. $\varphi \in D(\mathcal{F})$ is called a *translation* of $\mathcal{F}$ if φ is an elation.

Proposition 5.7. *The group $T(\mathcal{F})$ of products of translations of $\mathcal{F}$ is a normal subgroup of $Aut(\mathcal{F})$ and $D(\mathcal{F})$.*

Proof. A translation τ is an elation which fixes all intervals $[\epsilon_i, \epsilon_{i+1}]$ of $\mathcal{F}$. If $\sigma \in Aut(\mathcal{F})$, then $\sigma^{-1}\tau\sigma$ is also an elation which fixes all intervals. Hence $T(\mathcal{F})$ is a normal subgroup of $Aut(\mathcal{F})$. □

A projective collineation φ of $\mathcal{L}$ is called a collineation with axes α and β if α and β are complementary subspaces and if φ fixes each point of α and each point of β. If A is a point not on α and β then there is one and only one line through A, which intersects α and β (see H. Lenz [21, p. 231]) and the projective collineation φ is uniquely determined by α, β and the pair of points $(A, A\varphi)$.

Proposition 5.8. *Let φ be a projective collineation of $\mathcal{L}$ with axes α and β and $\alpha = \epsilon_k$ for a number $k \in \{1, \dots, r\}$. Then $\varphi \in D(\mathcal{F})$.*

Proof. Let φ be a projective collineation with axes $\alpha = \epsilon_k$ and β. It is sufficient to show, that φ is the identity on $[\alpha, 1] = [\epsilon_k, 1]$. Let δ be a point of $[\alpha, 1]$. Then $\delta \geq \alpha$ and since $\alpha \vee \beta = 1$ one gets by modularity $\delta = \delta \wedge (\alpha \vee \beta) = (\delta \wedge \beta) \vee \alpha$. Since β is an axis of φ, we have $\gamma\varphi = \gamma$ for all subspaces $\gamma \leq \beta$. Hence $(\delta \wedge \beta)\varphi = \delta \wedge \beta$ and since $\alpha\varphi = \alpha$ we get $\delta\varphi = \delta$. □

Definition 5.9. Let $\varphi \in PAut(\mathcal{F})$ and $m \in \{1, \dots, r+1\}$. We call φ a *stretching* of $\mathcal{F}$ with axes ϵ_m and β if φ is a collineation with axes ϵ_m and β. The group of stretchings of $\mathcal{F}$ with axis ϵ_m is denoted by $S_m(\mathcal{F})$.

Proposition 5.10. *$S_m(\mathcal{F})$ is a normal subgroup of $Aut(\mathcal{F})$ and $D(\mathcal{F})$ for $m \in \{1, \dots, r+1\}$.*

Proof. Let $\varphi \in S_m(\mathcal{F})$ and $\sigma \in Aut(\mathcal{F})$. Then $\sigma^{-1}\varphi\sigma$ is the identity on ϵ_m and hence an element of $S_m(\mathcal{F})$. With Proposition 5.8 we get $S_m(\mathcal{F}) \subseteq D(\mathcal{F})$. □

Definition 5.11. Let κ be a reference system and $m \in \{1, \dots, r\}$. We denote the group $\{\varphi \in PAut(\mathcal{F})$: φ is a stretching with axes ϵ_m and $\kappa_{m+1} \vee \cdots \vee \kappa_{r+1}\}$ by $S_m(\mathcal{F}, \kappa)$ and the group which is generated by $S_m(\mathcal{F}, \kappa)$ by $S(\mathcal{F}, \kappa)$.

We now want to represent dilatations by matrices. Let $\varphi \in D(\mathcal{F})$. A computation using the matrix representation (♠) shows:

Proposition 5.12. *Let $\varphi \in D(\mathcal{F})$. With respect to the the chosen basis $(e_1, \dots, e_n)$ of V the following statements hold (for $1 \leq k \leq r+1$):*

(1) *Dilatations can be represented by matrices of the form (♠) with matrices $A_k = \lambda_k I_k$ which are non-zero multiples of the $(n_k - n_{k-1}) \times (n_k - n_{k-1})$ identity matrix I_k (with $\lambda_k \in D$ and $\lambda_k \neq 0$).*
(2) *Elements of $T(\mathcal{F})$ can be represented by matrices of the form (♠) with $A_k = I_k$.*
(3) *Elements of $S_k(\mathcal{F}, \kappa)$ can be represented by matrices*
$$\operatorname{diag}(\lambda I_1, \dots, \lambda I_k, I_{k+1}, \dots, I_{r+1})$$
with $\lambda \in D$ and $\lambda \neq 0$.
(4) *Elements of $S(\mathcal{F}, \kappa)$ can be represented by matrices*
$$\operatorname{diag}(\lambda_1 I_1, \lambda_2 I_2, \dots, \lambda_{r+1} I_{r+1})$$
with $\lambda_k \in D$ and $\lambda_k \neq 0$.

An immediate consequence of Proposition 5.12 is that a dilatation of $\mathcal{F}$ is a product of translations and stretchings. Letting $\bullet$ denote semi-direct product:

Proposition 5.13. *$D(\mathcal{F}) = S(\mathcal{F}, \kappa) \bullet T(\mathcal{F})$.*

The following proposition describes the role of dilatations with respect to the group of all projective automorphisms of $\mathcal{F}$.

Proposition 5.14. *Let $\varphi \in PAut(\mathcal{F})$. Then φ is up to a dilatation uniquely determined by its operation on $[\epsilon_k, \epsilon_{k+1}]$ (with $0 \leq k \leq r$).*

Proof. Let $\varphi, \gamma \in PAut(\mathcal{F})$ and let φ_k (resp. γ_k) be the restriction of φ (resp. γ) to $[\epsilon_k, \epsilon_{k+1}]$ for $k \in \{0, \dots, r\}$. If $\varphi_k = \gamma_k$ for all k with $0 \leq k \leq r$ then $\varphi\gamma^{-1} = \delta \in D(\mathcal{F})$ (according to Definition 5.3) and $\varphi = \delta\gamma$. □

We now study the operation of $\varphi \in PAut(\mathcal{F})$ on an interval $[\epsilon_{k-1}, \epsilon_k]$. Let A be the matrix of the form (♠) associated to φ. Since $V = Y_1 \oplus \cdots \oplus Y_{r+1}$ the matrix A_k on the main diagonal of A represents an endomorphism η_k of Y_k with $\eta_k \in PAut([\epsilon_{k-1}, \epsilon_k], \mathfrak{R}_k)$.

Conversely an arbitrary element $\eta_k \in PAut([\epsilon_{k-1}, \epsilon_k], \mathfrak{R}_k)$ (which can be represented by a matrix G_k) induces an automorphism $\grave{\eta}_k \in PAut(\mathcal{F})$ with an associated matrix of the form $\operatorname{diag}(I_1, \dots, I_{k-1}, G_k, I_{k+1}, \dots, I_{r+1})$. We call $\grave{\eta}_k$ an automorphism of $\mathcal{F}$ which is induced by an automorphism of the interval $([\epsilon_{k-1}, \epsilon_k], \mathfrak{R}_k)$.

Let $G_k(\mathcal{F})$ be the subgroup of $PAut(\mathcal{F})$ generated by $\grave{\eta}_k \in PAut(\mathcal{F})$ with $\eta_k \in PAut([\epsilon_{k-1}, \epsilon_k], \mathfrak{R}_k)$ and $G(\mathcal{F})$ the group generated by $G_1(\mathcal{F}), \dots, G_{r+1}(\mathcal{F})$. Then $G(\mathcal{F})$ is the direct product of $G_1(\mathcal{F}), \dots, G_{r+1}(\mathcal{F})$.

Using these definitions we get (with Propositions 5.12, 5.13 and 5.14) that every projective automorphism of $\mathcal{F}$ is the product of translations of $\mathcal{F}$, of stretchings of $\mathcal{F}$, and of automorphisms induced by automorphisms of an interval of $\mathcal{F}$.

Proposition 5.15. $PAut(\mathcal{F}) = G(\mathcal{F}) \bullet D(\mathcal{F})$

Let $PAut(\mathcal{F}, \kappa) = \{\varphi \in PAut(\mathcal{F}) : \kappa_k\varphi = \kappa_k \text{ for } 0 \leq k \leq r+1\}$, i.e., the group of elements of $PAut(\mathcal{F})$ which can be represented by matrices of the form $\operatorname{diag}(A_1, \dots, A_{r+1})$. Then $PAut(\mathcal{F}, \kappa) = G(\mathcal{F}) \bullet S(\mathcal{F}, \kappa)$. With Propositions 5.13 and 5.15 we get the following:

Proposition 5.16. *$PAut(\mathcal{F}) = PAut(\mathcal{F}, \kappa) \bullet T(\mathcal{F})$.*

Remarks 5.17. (a) If Proposition 5.16 is specialized to affine geometry or to classical metric affine geometry (i.e., euclidean, minkowskian and galilean geometry; see H. Struve and R. Struve [33, §5]) one gets well-known representation theorems (see F. Buekenhout [7, Theorem 3.4] and F. Bachmann [1], [2], [3, §3]).

(b) The concept of a dilatation introduced in the theory of universal flag lattices generalizes the notion of a dilatation which is given in incidence geometry (as a transformation preserving direction; see E. Snapper and R. J. Troyer [29, p.37]) and in similarity geometry (as a transformation preserving circles or angular measure; see W. Benz [4, §5.2]).

6. Cayley–Klein vector spaces

Let V be a left vector space of finite dimension over a skew field K of arbitrary characteristic and f a non-singular alternating bilinear form or a non-singular σ-Hermitian form with an anti-isomorphism σ of K of order 2. If σ is the identity then f is a symmetric bilinear form and K commutative (see H. Lenz [21, §IV, 8] and G. Birkhoff and J. von Neumann [6]).

It is well known that f induces a polarity on the projective lattice of subspaces of V and that conversely any polarity on V is induced by such a form (see C.-A. Faure and A. Frölicher [9, Proposition 11.5.6 and Remark 14.1.9] and Definition 2.4 for projective lattices of dimension < 2).

Using these facts it follows that Cayley–Klein lattices can be described by vector spaces in which a metric structure is given by one or more bilinear forms (resp. Hermitian forms). We call these mathematical structures Cayley–Klein vector spaces:

Definition 6.1. $(V, f_1, \ldots, f_r)$ is a *Cayley–Klein vector space* if the following assumptions hold:

(a) V is a left vector space of finite dimension n over a skew field.
(b) f_i (with $i \in \{1, \ldots, r\}$) is an alternating bilinear form or a σ-Hermitian form with $\text{rank}(f_i) \geq 1$ on R_i, with $R_i := \text{rad}(R_{i-1}, f_i)$ and $R_0 := V$
(c) $\text{rank}(f_1) + \cdots + \text{rank}(f_r) = n$.

Proposition 6.2. *Let $(V, f_1, \ldots, f_r)$ be a Cayley–Klein vector space and*

- *$\mathcal{L}$ the lattice of subspaces of V;*
- *$\epsilon_0 = 0$ and $\epsilon_j := R_{r-j}$ (for $j \in \{1, \ldots, r\}$);*
- *π_j the polarity on $[\epsilon_{j-1}, \epsilon_j]$ induced by f_{r-j+1} on $[R_{r-j+1}, R_{r-j}]$.*

Then $(\mathcal{L}, (([\epsilon_0, \epsilon_1], \pi_1), \ldots, ([\epsilon_{r-1}, \epsilon_r], \pi_r)))$ is an irreducible arguesian Cayley–Klein lattice, which we call the associated Cayley–Klein lattice.

Proposition 6.3. *Let $\mathcal{C}$ be an irreducible Cayley–Klein lattice of dimension ≥ 3. Then there exists a Cayley–Klein vector space, whose associated Cayley–Klein lattice is isomorphic to $\mathcal{C}$.*

Proof of Propositions 6.2 *and* 6.3. The proofs are straightforward using the remarks at the beginning of this section and the fact that any irreducible Cayley–Klein lattice of dimension ≥ 3 is arguesian. □

Remark 6.4. As a consequence of Proposition 6.3 we get a representation of the group of projective automorphisms of an irreducible Cayley–Klein lattice $\mathcal{C}$ of dimension ≥ 3. According to Section 5 we can represent $\varphi \in PAut(C)$ by a matrix A of the form (♠) with matrices A_k on the main diagonal of A which are (up to a multiple of the identity matrix I_k) orthogonal, unitary or symplectic matrices (representing projective automorphisms of the intervals $([\epsilon_{k-1}, \epsilon_k], \pi_k)$ of $\mathcal{C}$).

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