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Many-valued relation algebras

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Dedicated to the Memory of Wim Blok

Abstract. We introduce *MV-relation algebras* (MVRAs) and *distributive MV-relation algebras* (DMVRAs), many-valued generalizations of classical relation algebras and study some of their arithmetical properties. We provide corresponding notions of *group relation algebra* and *complex algebra* and generalize some results about them from the classical case. For this, we work with more general structures than MVRAs and DMVRAs, by replacing the MV part with a BL-algebra, obtaining what we call *fuzzy relation algebras* and *distributive fuzzy relation algebras*.

1. Introduction

Relation algebras were introduced in [12] as abstract algebraic models for the study of binary relations, purified of the set theoretical structure of relations. The theory of relation algebras has a wide range of applications, from the fundamentals of mathematics to the development of programming languages. In [4], it is mentioned that "in principle, the whole of mathematical research can be carried out by studying identities in the arithmetic of relation algebras".

The canonical example of relation algebra, which acts as the starting point and the motivation of the whole algebraic theory, is the concrete algebra of binary relations over a set X, equipped with union, intersection, complement, empty relation, full relation, relational composition, converse and diagonal.

A relation algebra (RA) is a structure $\mathcal{A} = (A, +, \cdot, ^-, 0, 1, ;, ^{\smile}, \Delta)$, with binary operations $+$, $\cdot$, $;$, unary operations $^-$, $^{\smile}$ and constant 0, 1, Δ. There are two main (equivalent) sets of postulates for relation algebras; let us refer to them as RA(1) and RA(2). Both of them require that $(A, +, \cdot, ^-, 0, 1)$ be a Boolean algebra and $(A, ;, \Delta)$ be a monoid ($;$ stands for composition and Δ for diagonal). RA(1) asks one more thing: that the algebra $\mathcal{A}$ satisfies

$$(x\,;y) \odot z = 0 \text{ iff } (x^{\smile}\,;z) \odot y = 0 \text{ iff } (z\,;y^{\smile}) \odot x = 0 \ ,$$

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while RA(2) states instead five axioms:

$$(x + y) \mathbin{;} z = (x \mathbin{;} z) + (y \mathbin{;} z) \ ,$$

$$(x + y)^{\smile} = x^{\smile} + y^{\smile} \ ,$$

$$x^{\smile\smile} = x \ ,$$

$$(x \mathbin{;} y)^{\smile} = y^{\smile} \mathbin{;} x^{\smile} \ ,$$

$$\overline{y} = \overline{y} + (x^{\smile} \mathbin{;} \overline{x \mathbin{;} y}) \ .$$

RA(1) is more compact and elegant, while RA(2), which is very close to the original definition, has equational axioms. Among the above five axioms, the last one "has a somewhat more involved structure and a less clear algebraic content than the remaining postulates", [4].

Concerning relations between real objects, the fuzzy approach turns out to be a better approximation of reality than the classical one (take, for instance, the famous relation "to have" between persons and diverse attributes, like "intelligent", "tall", "old" etc.). Thus, a fuzzy framework appears to be a natural step to take in the study of relations. Some interesting fuzzy generalizations of results about relations and concepts can be found in [1] and [2], but the discussion is about concrete fuzzy relations. The only abstract fuzzy relation algebra study that we are aware of is [6], where the fuzzy part appears "externally", as set of scalars from the $[0, 1]$ interval, acting upon relations. Our approach, on the other hand, is "internal", working with a not necessarily idempotent conjunction. Yet, there exist some similarities between our fuzzy relation algebras and what are called in [6] *relation algebras* (structures that are not yet fuzzy, but general enough to prepare the field for dealing with fuzzy scalars), which shall be discussed in the concluding section.

The inspiring model for our generalization of relation algebra is the structure of fuzzy relations on a set X, which are functions P from $X \times X$ to $[0, 1]$ measuring, for each pair $(x, y) \in X \times X$, how much is x in relation to y, i.e., the truth degree of xPy. The $[0, 1]$ interval has to be equipped with a (not necessarily idempotent) conjunction $\odot$ in order to be able to express, given two fuzzy relations P and Q, for each (x, y), the truth degree of the sentence, xPy & xQy. This actually defines, pointwise, the fuzzy intersection. If we also have a disjunction on $[0, 1]$, we can similarly define fuzzy union. The structure on $[0, 1]$ could be a complete MV-algebra or BL-algebra, or even a complete residuated lattice, where the lattice is that given by the usual order — all these are well-known fuzzy truth value structures. Composition of fuzzy relations is defined by translating the one from the crisp case, only in terms of conjunction and existential quantification (the last being simulated by the lattice suprema): the truth degree of $x(P \mathbin{;} Q)y$ is that of

$\exists z (xPz \,\text{and}\, zQy)$, which actually means

$$(P\,;Q)(x,y) = \bigvee_{z\in X} (P(x,z) \odot Q(z,y))\ .$$

Going to an abstract level and "looking" at the above concrete fuzzy structure, we shall define diverse forms of fuzzy relation algebras. We chose to take first an MV-algebra structure — the good property of negation being idempotent and the resulting duality between conjunction and disjunction make MV-algebras "reasonable" (cf. [11]) fuzzy structures.[1] Many arithmetical results are recovered from the Boolean case. However, as shown by Sections 4 and 5, when considering some significant classes of models for fuzzy relation algebras, the MV assumptions can be weakened. In fact, only the *group relation algebras* from Section 4 have good properties with respect to this weakening — the *complex algebras* from Section 5, although constructible in this more general framework, point out that the MV assumption is, in some sense, minimal for an "iff" generalization of the classical result to hold.

The paper is structured as follows. In Section 2, we introduce MV-relation algebras and distributive MV-relation algebras, discuss some arithmetical properties and provide examples. Section 3 introduces fuzzy relation algebras and distributive fuzzy relation algebras, by considering, instead of MV, a BL structure. The next two sections deal with two significant classes of fuzzy relation algebras — group relation algebras and complex algebras. Some results are generalized from the classical case. Section 6 draws conclusions and discusses some topics for future research.

2. Arithmetic

Definition 2.1. An *MV-algebra* [3] is a structure $\mathcal{A} = (A, \oplus, \odot, ^-, 0, 1)$, where $\oplus$ and $\odot$ are binary operations, $^-$ is unary and 0, 1 are constants, satisfying the following axioms:

(M1) $(A, \oplus, 0)$ and $(A, \odot, 1)$ are commutative monoids,
(M2) $x \odot 0 = 0$ and $x \oplus 1 = 1$,
(M3) $\overline{\overline{x}} = x$,
(M4) $\overline{x \oplus y} = \overline{x} \odot \overline{y}$,
(M5) $(x \odot \overline{y}) \oplus y = (y \odot \overline{x}) \oplus x$.

[1] In this paper, we do not take any position towards the problem of distinguishing between "fuzzy" and "many-valued", but simply follow the current terminology, which calls MV-algebras by their name, "many-valued structures", and BL-algebras, "fuzzy structures" — although MV-algebras are, roughly and somehow anachronistically speaking, symmetrical BL-algebras.

Remark 2.2. The axioms are universally quantified; this will be the case throughout the paper.

Usually, MV-algebras are defined only in terms of $\oplus$, $^{-}$ and 0. Yet, in order to point out a certain symmetry, we prefer to work with the above (redundant) definition. We shall consider that $^{-}$ binds stronger than $\odot$, which in turn binds stronger than $\oplus$.

The standard MV-algebra is $L_{[0,1]} = ([0,1], \oplus, \odot, ^{-}, 0, 1)$, where $\oplus$, $\odot$ and $^{-}$ are defined by:

$$x \oplus y = \min(x + y, 1) \ \ , \ \ x \odot y = \max(0, x + y - 1) \ \ , \ \ \overline{x} = 1 - x \ ,$$

for all $x, y \in [0,1]$.

Let $\mathcal{A} = (A, \oplus, \odot, ^{-}, 0, 1)$ be an algebra of MV type. Define a binary relation $\leq$ and two binary operations $\vee$ and $\wedge$ on A by:

$$x \leq y \text{ iff } x \odot \overline{y} = 0 \ ,$$
$$x \vee y = x \oplus (\overline{x} \odot y) \ ,$$
$$x \wedge y = x \odot (\overline{x} \oplus y) \ ,$$

for all $x, y \in A$.

We list here some well-known MV-identities, that will often be used without mention.

Lemma 2.3. *The following hold in any MV-algebra:*

(1) $\leq$ *is a lattice order on* A, *with* $\sup = \vee$ *and* $\inf = \wedge$*;*
(2) $x \leq y$ *iff* $\overline{x} \oplus y = 1$*;*
(3) $x \oplus \overline{x} = 1$*;*
(4) $x \odot \overline{x} = 0$*;*
(5) $x \leq y$ *iff* $\overline{y} \leq \overline{x}$*;*
(6) $x \odot y \leq z$ *iff* $x \leq \overline{y} \oplus z$*;*
(7) $z \leq x \oplus y$ *iff* $z \odot \overline{x} \leq y$*;*
(8) $x \odot \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \odot y_i)$*;*
(9) $x \oplus \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \oplus y_i)$.

Definition 2.4. Let $\mathcal{A}$ be an MV-algebra, $\sigma \colon A \longrightarrow A$, $\tau \colon A \longrightarrow A$ two functions. We say that (σ, τ) is a *conjugate pair* if, for all $x, y \in A$, it holds

$$\sigma(x) \odot y = 0 \text{ iff } x \odot \tau(y) = 0.$$

Proposition 2.5. *If* (σ, τ) *is a conjugate pair, then:*

(a) (τ, σ) *is a conjugate pair;*
(b) $\sigma(0) = \tau(0) = 0$*;*
(c) $\sigma\left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} \sigma(x_i)$ *whenever the suprema exist;*
(d) $x \leq y$ *implies* $\sigma(x) \leq \sigma(y)$ *for all* $x, y \in A$,

(e) *If* (σ, τ') *is also a conjugate pair, then* $\tau = \tau'$.

Proof. (a): Obvious, since the definition is symmetrical.

(b): $\sigma(0) \odot 1 = 0$ iff $0 \odot \tau(1) = 0$, so $\sigma(0) = 0$.

(c): $\sigma\left(\bigvee_{i\in I} x_i\right) \leq t$ iff $\sigma\left(\bigvee_{i\in I} x_i\right) \odot \overline{t} = 0$ iff $\bigvee_{i\in I}(x_i \odot \tau(\overline{t})) = 0$ iff $\forall i \in I, x_i \odot \tau(\overline{t}) = 0$ iff $\forall i \in I, \sigma(x_i) \odot \overline{t} = 0$ iff $\forall i \in I, \sigma(x_i) \leq t$ iff $\bigvee_{i\in I} \sigma(x_i) \leq t$. Thus, t being arbitrarily taken, we get the desired equality.

(d): Follows immediately from c).

(e): $\tau(x) \leq t$ iff $\tau(x) \odot \overline{t} = 0$ iff $x \odot \sigma(\overline{t}) = 0$ iff $\tau'(x) \odot \overline{t} = 0$ iff $\tau'(x) \leq t$. Thus, $\tau(x) = \tau'(x)$. □

Proposition 2.6. *Let* $\sigma, \tau \colon A \longrightarrow A$. *The following are equivalent:*

(1) (σ, τ) *is a conjugate pair;*
(2) $\sigma(\overline{\tau(x)}) \odot x = 0$, $\tau(\overline{\sigma(x)}) \odot x = 0$ *for all* $x \in A$ *and* σ, τ *are isotone functions;*
(3) $\sigma(\overline{\tau(x)}) \odot x = 0$, $\tau(\overline{\sigma(x)}) \odot x = 0$, $\sigma(x \vee y) = \tau(x) \vee \sigma(y)$, $\tau(x \vee y) = \tau(x) \vee \tau(y)$ *for all* $x, y \in A$.

Proof. (1) implies (2) and (1) implies (3): These follow easily from Proposition 2.5 and from the properly applied definition of conjugate pair. For instance, we have that $\sigma(\overline{\tau(x)}) \odot x = 0$ iff $\overline{\tau(x)} \odot \tau(x) = 0$, which is true.

(3) implies (2): This is obvious.

(2) implies (1): Suppose $\sigma(x) \odot y = 0$. Then $y \leq \overline{\sigma(x)}$, thus $x \odot \tau(y) \leq x \odot \tau(\overline{\sigma(x)}) = 0$, so $x \odot \tau(y) = 0$. Analogously, we get $x \odot \tau(y) = 0$ implies $\sigma(x) \odot y = 0$. □

Remark 2.7. A moment's reflection shows that (σ, τ) is a conjugate pair iff $(x \mapsto \overline{\sigma(x)}, y \mapsto \overline{\tau(y)})$ is an adjunction between $(A, \leq)$ and $(A, \geq)$, when these two partial ordered sets are seen as categories.

In what follows, we shall deal with structures of the form

$$(A, \oplus, \odot, {}^-, 0, 1, \,;\,, {}^\smile, \Delta),$$

where $(A, \oplus, \odot, {}^-, 0, 1)$ is of MV-algebra type, $;$ is a binary operation, ${}^\smile$ unary and Δ constant. Besides the already stated priorities for $\oplus$, $*$ and ${}^-$, we assume that unary operators always bind stronger than binary ones.

Consider, for this type of algebras, the following axioms (or sets of axioms):

(A0) $(A, \oplus, \odot, {}^-, 0, 1)$ is an MV-algebra,
(A1) $(A, \,;\,, \Delta)$ is a monoid,
(A2) $(x \,;\, y) \odot z = 0$ iff $(x^\smile \,;\, z) \odot y = 0$ iff $(z \,;\, y^\smile) \odot x = 0$,
(A3) $(x \oplus y)^\smile = x^\smile \oplus y^\smile$.

Definition 2.8. An algebra satisfying (A0)–(A2) is called an *MV-relation algebra* (MVRA for short), while an algebra satisfying (A0)–(A3) is called a *distributive*

MV-relation algebra (DMVRA for short). The corresponding classes of algebras shall be denoted by $\mathcal{MVRA}$ and $\mathcal{DMVRA}$.

Following the classical (Boolean) theory of relation algebras, we call Δ *the diagonal element* and $;$ *the relative composition* (or *relative multiplication*).

Remark 2.9. Thus, we have $\mathcal{DMVRA} \subseteq \mathcal{MVRA}$ (as we shall see in Section 5, we have $\mathcal{DMVRA} \neq \mathcal{MVRA}$, encountering a first significant difference from the Boolean case, where, if we interpret $\odot$ and $\oplus$ as $\wedge$ and $\vee$, (A3) follows from (A0)–(A2) — see [18]).

As a consequence of the properties of residuated pairs, we get:

Proposition 2.10. *$\mathcal{MVRA}$ and $\mathcal{DMVRA}$ are varieties.*

Proof. In order to show that the two classes are varieties, it suffices to express axiom (A2) by some equations. But (A2) just says that, for each $x \in A$, the functions $y \mapsto x\,;y$ and $y \mapsto x^{\smile}\,;y$ form a conjugate pair and so do $y \mapsto y\,;x$ and $y \mapsto y\,;x^{\smile}$. It remains to apply Proposition 2.6 ("(1) iff (3)") to get the desired expression for (A2). $\square$

Example 2.11. Consider the classical MV-algebra structure on $[0,1]^{X\times X}$, namely $([0,1]^{X\times X}, \oplus, \odot, ^-, 0, 1)$, where 0 and 1 are the constant functions and $\oplus$, $\odot$ and $^-$ are the pointwise operations from $L_{[0,1]}$. Now define $;$, $^{\smile}$ and Δ as follows:

$$P\,;Q\ (x,y) = \bigvee_{z\in X} P(x,z)\odot Q(z,y)\,,$$
$$P^{\smile}(x,y) = P(y,x)\,,$$
$$\Delta(x,y) = \begin{cases} 0 & \text{if } x \neq y\,, \\ 1 & \text{if } x = y\,, \end{cases}$$

whenever $P, Q \in [0,1]^{X\times X}$ and $x, y \in X$. Denote this structure with $MVRel(X)$. Let us show that it is a DMVRA. We know that (A0) holds. Now,

$$\begin{aligned}(P\,;Q)\,;R\,(x,y) &= \bigvee_{z\in X} (P\,;Q)(x,z)\odot R(z,y)\\ &= \bigvee_{z\in X}\left(\bigvee_{v\in X} P(x,v)\odot Q(v,z)\right)\odot R(z,y)\\ &= \bigvee_{z,v\in X} P(x,v)\odot Q(v,z)\odot R(z,y).\end{aligned}$$

In a similar way, the last expression turns out to be equal to $P\,;(Q\,;R)\,(x,y)$. Thus, $;$ is associative. Further,

$$P\,;\Delta\ (x,y) = \bigvee_{z\in X} P(x,z)\odot\Delta(z,y) = P(x,y)\odot 1 = P(x,y),$$

and the same goes for $\Delta\,;P$. We have proved (A1). In order to prove (A2), notice that $(P\,;Q)\,;R = 0$ means that, for all $x, y, z \in X$, $P(x,y) \odot Q(y,z) \odot R(x,z) = 0$. Then the desired equivalences follow immediately, using that $P^{\smile}(x,y) = P(y,x)$. (A3) is a trivial consequence of the fact that $\oplus$ is defined pointwise.

Example 2.12. Define the structure as in the previous example, except that we consider, instead of $L_{[0,1]}$ as starting point, any MV-algebra which is complete as a lattice. Since completeness was the only additional property that we used above, this structure also turns out to be a DMVRA.

Example 2.13 (Relativization)**.** Fix an element $e \in A$. Define the structure

$$\mathcal{A}_{\|e} = (A_{\|e}, \oplus_e, \odot_e, ;_e, {}^{-e}, {}^{\smile e}, 0_e, 1_e, \Delta_e)$$

as follows:

$A_{\|e} = \{x \in A \mid x \leq e\}$,
$x \oplus_e y = (x \oplus y) \odot e$, $\overline{x}^e = \overline{x} \odot e$, $x \odot_e y = x \odot y$,
$0_e = 0$, $1_e = e$,
$x ;_e y = x\,;y$, $x^{\smile e} = x^{\smile}$, $\Delta_e = \Delta$.

Notice that because e is an equivalence, the operations $;_e$, ${}^{\smile e}$ and Δ_e are well-defined, and that because e is crisp, the operation $\odot_e$ is well-defined.

Proposition 2.14.
(1) *If A is an MVRA, then $A_{\|e}$ is an MVRA.*
(2) *If A is an DMVRA, then $A_{\|e}$ is an DMVRA.*

Proof. (1): We already know that $(A_{\|e}, \oplus_e, \odot_e, {}^{-e}, 0_e, 1_e)$ is an MV-algebra from the theory of MV-algebras. Notice also that the order $\leq$ on $A_{\|e}$ is the one induced by the order on A. So (A1) and (A2) immediately hold for $A_{\|e}$, since they only involve operations and relations induced by those from A.

(2): To check (A3), let $x, y \leq e$. Then

$$(x \oplus_e y)^{\smile e} = ((x \oplus y) \wedge e)^{\smile} = (x \oplus y)^{\smile} \wedge e^{\smile} = (x^{\smile} \oplus y^{\smile}) \wedge e = x^{\smile e} \oplus_e y^{\smile e} .$$

□

Example 2.15. Any (Boolean) RA is also a DMVRA, if we let $\oplus = \vee$ and $\odot = \wedge$.

Example 2.16. Consider an MV-algebra $(A, \oplus, \odot, {}^{-}, 0, 1)$ and define $\Delta = 1$, $; = \odot$ and $x^{\smile} = x$. It is easy to see that we obtain a DMVRA, that we call a *thin* MVRA. What we call *thin* corresponds to what is called *Boolean* in the classical, crisp case. We keep the word *Boolean* to refer the classical case, as in the previous example.

Example 2.17. Let $\mathcal{M}$ be an arbitrary complete MV-algebra and X a set. Consider the structure on $M^{X\times X}$ from Example 2.12, except that we define $^{\smile}$ and $;$ in a different way:

$$P^{\smile}(x,y) = P(x,y) \odot \alpha \,,$$
$$P\,;Q\,(x,y) = \bigvee_{z\in X} P(x,z) \odot Q(z,y) \odot \alpha \,,$$

for each $P \in M^{X\times X}$, $x, y \in X$, where α is a fixed idempotent element of M. It is easy to check (as with at Example 2.11) that we obtain a DMVRA.

In the next three examples, the MV part shall be L_{n+1} (with n being 4, 4 and 3 respectively), the sub-MV-algebra of $L_{[0,1]}$ with support $\{0, 1/n, \ldots, (n-1)/n, 1\}$. The operation $^{\smile}$ is the identity in all the three cases. At $;$, the first argument is always displayed vertically and the second horizontally. All these examples are DMVRAs.

Example 2.18.

;	**0**	**1/4**	**2/4**	**3/4**	**1**
0	0	0	0	0	0
1/4	0	1/4	2/4	3/4	1
2/4	0	2/4	2/4	3/4	1
3/4	0	3/4	1	1	1
1	0	1	1	1	1

$\Delta = 1/4$

Example 2.19.

;	**0**	**1/4**	**2/4**	**3/4**	**1**
0	0	0	0	0	0
1/4	0	0	0	0	1/4
2/4	0	0	0	1/4	2/4
3/4	0	0	0	1/4	3/4
1	0	1/4	2/4	3/4	1

$\Delta = 1$

Example 2.20.

;	**0**	**1/3**	**2/3**	**1**
0	0	0	0	0
1/3	0	1/3	1/3	1
2/3	0	1/3	2/3	1
1	0	1	1	1

$\Delta = 2/3$

Proposition 2.21. *The following hold in any MVRA (whenever we use suprema or infima, we assume that they exist):*

(1) $x^{\smile\smile} = x$,

(2) $\left(\bigvee_{i\in I} x_i\right) ; y = \bigvee_{i\in y}(x_i \,; y)$, $\quad y\,; \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in y}(y\,; x_i)$,

(3) *$x_1 \leq x_2$ and $y_1 \leq y_2$ imply $x_1 \,;\, y_1 \leq x_2 \,;\, y_2$*
(in particular, $x_1 \leq x_2$ implies $x_1 \,;\, y \leq x_2 \,;\, y$ and $y \,;\, x_1 \leq y \,;\, x_2$),
(4) *$x \leq x \,;\, 1$ and $x \leq 1 \,;\, x$,*
(5) *$x^{\smile} \odot y = 0$ iff $x \odot y^{\smile} = 0$,*
(6) *$\left(\bigvee_{i \in I} x_i\right)^{\smile} = \bigvee_{i \in I} {x_i}^{\smile}$,*
(7) *$0^{\smile} = 0$,*
(8) *$1^{\smile} = 1$,*
(9) *$\overline{x}^{\smile} \leq \overline{x^{\smile}}$,*
(10) *$x^{\smile} \leq y^{\smile}$ iff $x \leq y$,*
(11) *$(x \,;\, y)^{\smile} = y^{\smile} \,;\, x^{\smile}$,*
(12) *$\Delta^{\smile} = \Delta$,*
(13) *$1 = 1 \,;\, 1$,*
(14) *$x^{\smile} = y^{\smile}$ iff $x = y$,*
(15) *$x \leq 1 \,;\, x \,;\, 1$,*
(16) *$x^{\smile} \,;\, \overline{x \,;\, y} \leq \overline{y}$,*
(17) *$x \,;\, 0 = 0$,*
(18) *$0 \,;\, x = 0$,*
(19) *$x = 0$ iff $x \,;\, 1 = 0$ iff $1 \,;\, x = 0$,*
(20) *$\overline{y \,;\, x} \,;\, x^{\smile} \leq \overline{y}$,*
(21) *$\overline{x}^{\smile} = \overline{x^{\smile}}$,*

Proof. (1): Let $t \in A$. We have the equivalences: $x \leq t$ iff $x \odot \overline{t} = 0$ iff $(x;\Delta) \odot \overline{t} = 0$ iff $(x^{\smile} \,;\, \overline{t}) \odot \Delta = 0$ iff $(x^{\smile\smile} \,;\, \Delta) \odot \overline{t} = 0$ iff $x^{\smile\smile} \odot \overline{t} = 0$ iff $x^{\smile\smile} \leq t$ (we applied (A2)). Thus $x = x^{\smile\smile}$.

(2): Let us prove the first equality. We know that, in any MV-algebra, whenever the suprema exist, $\left(\bigvee_{i \in I} a_i\right) \odot b = \bigvee_{i \in I}(a_i \odot b)$. We apply this, together with (A2), (1), the definition of suprema and some properties of order in MV-algebras: $\left(\bigvee_{i \in I} x_i\right) ; y \leq t$ iff $((\bigvee_{i \in I} x_i) \,;\, y) \odot \overline{t} = 0$.

$$\begin{aligned}(\overline{t} \,;\, y^{\smile}) \odot \Big(\bigvee_{i \in I} x_i\Big) = 0 \text{ iff } &\bigvee_{i \in I}(\overline{t} \,;\, y^{\smile}) \odot x_i = 0 \text{ iff } \forall i \in I,\ (\overline{t} \,;\, y^{\smile}) \odot x_i = 0 \\ &\text{iff } \forall i \in I,\ (x_i \,;\, y^{\smile\smile}) \odot \overline{t} = 0 \text{ iff } \forall i \in I,\ (x_i \,;\, y) \odot \overline{t} = 0 \\ &\text{iff } \forall i \in I,\ (x_i \,;\, y) \leq t \text{ iff } \bigvee_{i \in I}(x_i \,;\, y) \leq t.\end{aligned}$$

Thus $\left(\bigvee_{i \in I} x_i\right) ; y = \bigvee_{i \in I}(x_i \,;\, y)$. The other equality follows in the same manner.

(3): An easy consequence of (2) is that the operation ; is isotone in both arguments. Hence, we get: $x_1 \leq x_2$ and $y_1 \leq y_2$ implies $x_1 \,;\, y_1 \leq x_2 \,;\, y_1 \leq x_2 \,;\, y_2$.

(4): Immediately from (3) and $\Delta \leq 1$.

(5): We apply (A2) and (1): $x \odot y^{\smile} = 0$ iff $(x;\Delta) \odot y^{\smile} = 0$ iff $(x^{\smile};y^{\smile}) \odot \Delta = 0$ iff $(\Delta \,;\, y^{\smile\smile}) \odot x^{\smile} = 0$ iff $y \odot x^{\smile} = 0$.

(6): We use (5). Take $s \in A$; then:

$$\Big(\bigvee_{i\in I} x_i\Big)^{\smile} \leq s \text{ iff } \Big(\bigvee_{i\in I} x_i\Big)^{\smile} \odot \overline{s} = 0 \text{ iff } \Big(\bigvee_{i\in I} x_i\Big) \odot \overline{s}^{\smile} = 0$$
$$\text{iff } \bigvee_{i\in I}(x_i \odot \overline{s}^{\smile}) = 0 \text{ iff } \forall i \in I,\ x_i \odot \overline{s}^{\smile} = 0 \text{ iff } \forall i \in I,\ {x_i}^{\smile} \odot \overline{s} = 0$$
$$\text{iff } \forall i \in I,\ {x_i}^{\smile} \leq s \text{ iff } \bigvee_{i\in I} {x_i}^{\smile} \leq s.$$

Thus $\big(\bigvee_{i\in I} x_i\big)^{\smile} = \bigvee_{i\in I} {x_i}^{\smile}$.

(7): The desired equality follows from (5): $0^{\smile} = 0$ iff $0^{\smile} \odot 1 = 0$ iff $0 \odot 1^{\smile} = 0$ iff $0 = 0$. Thus $0^{\smile} = 0$.

(8): Applying (1) and (6), we obtain:

$$1 = 1 \vee 1^{\smile} = 1^{\smile\smile} \vee 1^{\smile} = (1^{\smile} \vee 1)^{\smile} = 1^{\smile}.$$

(9): Because $\overline{x} \odot x = 0$, it follows from point (1) that $\overline{x} \odot x^{\smile\smile} = 0$. Furthermore, from point (5), we get $\overline{x}^{\smile} \odot x^{\smile} = 0$, hence $\overline{x}^{\smile} \leq \overline{x^{\smile}}$.

(10): For the "only if" part, we apply points (9),(5) and (1): $x^{\smile} \leq y^{\smile}$ implies $x^{\smile} \odot \overline{y}^{\smile} = 0$ implies $x^{\smile\smile} \odot \overline{y} = 0$ implies $x \odot \overline{y} = 0$ implies $x \leq y$. Conversely, if $x \leq y$, then, according to point (1), $x^{\smile\smile} \leq y^{\smile\smile}$, so, as we just proved, $x^{\smile} \leq y^{\smile}$.

(11): From (A2), (5) and (1), we get: $(x\,;y)^{\smile} \leq s$ iff $(x\,;y)^{\smile} \odot \overline{s} = 0$ iff $(x\,;y) \odot \overline{s}^{\smile} = 0$ iff $(x^{\smile}\,;\overline{s}^{\smile}) \odot y = 0$ iff $(y\,;\overline{s}^{\smile\smile}) \odot x^{\smile} = 0$ iff $(y\,;\overline{s}) \odot x^{\smile} = 0$ iff $(y^{\smile}\,;x^{\smile}) \odot \overline{s} = 0$ iff $y^{\smile}\,;x^{\smile} \leq s$. So $(x\,;y)^{\smile} = y^{\smile}\,;x^{\smile}$.

(12): Using (1) and (11), we obtain:

$$\Delta^{\smile} = \Delta^{\smile}\,;\Delta = \Delta^{\smile}\,;\Delta^{\smile\smile} = (\Delta^{\smile}\,;\Delta)^{\smile} = \Delta^{\smile\smile} = \Delta\,.$$

(13): $1 = 1\,;\Delta \leq 1\,;1 \leq 1$. Hence $1\,;1 = 1$.

(14): Immediately from (10).

(15): We apply (4) twice: $x \leq x\,;1 \leq 1\,;x\,;1$.

(16): We apply (A2): $x^{\smile}\,;\overline{x\,;y} \leq \overline{y}$ iff $(x^{\smile}\,;\overline{x\,;y}) \odot y = 0$ iff $(x\,;y) \odot \overline{(x\,;y)}$, which is true.

(17): $x\,;0 = 0$ iff $(x\,;0) \odot 1 = 0$ iff $(x^{\smile}\,;1) \odot 0 = 0$, the last being true.

(18): Similar to (17).

(19): We know, from (4), that $x \leq x\,;1$ and $x \leq 1\,;x$. On the other hand, (17) and (18) tell us that $x = 0$ implies $x\,;1 = 0$ and $1\,;x = 0$.

(20): Similar to (16).

(21): Consider an arbitrary element t. We apply MV properties and points $t \leq \overline{x^{\smile}}$ iff $t \odot x^{\smile} = 0$ iff $t^{\smile} \odot x$ iff $t^{\smile} \odot x = 0$ iff $t^{\smile} \leq \overline{x}$ iff $t^{\smile\smile} \leq \overline{x}^{\smile}$ iff $t \leq \overline{x}^{\smile}$. So $\overline{x}^{\smile} = \overline{x^{\smile}}$. □

Proposition 2.22. *Every totally ordered MVRA has* $^{\smile}$ *the identity function, hence is a DMVRA.*

Proof. This is because, according to Proposition 2.21.(10), $^\smile$ is an increasing bijection, and also an involution by 2.21.(1). □

Proposition 2.23. *The following are equivalent for an MVRA $\mathcal{A}$:*
(a) *$\mathcal{A}$ is a DMVRA (i.e., it satisfies* (A3)*).*
(b) *For all $x, y \in A$, $(x \odot y)^\smile = x^\smile \odot y^\smile$.*

Proof. The desired equivalence follows at once from Proposition 2.21.(21) and the MV facts $\overline{\overline{x}} = x$ and $x \oplus y = \overline{\overline{x} \odot \overline{y}}$. □

Remark 2.24. Our definition of MVRA generalizes directly the Boolean definition RA(1). As consequences of the MVRA axioms, we get (of course, in an MV framework) all the axioms of the alternative Boolean definition RA(2), except two: $(x + y) \,;\, z = (x \,;\, z) + (y \,;\, z)$ and $(x + y)^\smile = x^\smile + y^\smile$. While the first of these is not even satisfied by the basic Example 2.11 and hence we do not take it into consideration (at least not with respect to the MV addition), the second one became an axiom for what we called DMVRA.

On an arbitrary MVRA, define the following derived operations:

$$\Box = \overline{\Delta}, \text{ called the } \textit{diversity element};$$

$$x \dagger y = \overline{\overline{x} \,;\, \overline{y}}, \text{ called } \textit{relative addition}.$$

Proposition 2.25. *The following hold in any MVRA:*
(1) $x^\smile \,;\, (\overline{x} \dagger 0) = 0$,
(2) $x^\smile \,;\, (\overline{x} \dagger y) \leq y$,
(3) $x^\smile \,;\, \overline{x} \leq \Box$,
(4) $\overline{x} \,;\, x^\smile \leq \Box$,
(5) $\overline{x}^\smile \,;\, x \leq \Box$,
(6) $x \,;\, \overline{x}^\smile \leq \Box$,
(7) $x \dagger 1 = 1$,
(8) $1 \dagger x = 1$.

Proof. (1): We have that $x^\smile \,;\, (\overline{x} \dagger 0) = x^\smile \,;\, \overline{\overline{\overline{x}} \,;\, \overline{0}} = x^\smile \,;\, \overline{x \,;\, 1}$. Further, $x^\smile \,;\, \overline{x \,;\, 1} = 0$ iff $(x^\smile \,;\, \overline{x \,;\, 1}) \odot 1 = 0$ iff $(x^{\smile\smile} \,;\, 1) \odot \overline{x \,;\, 1} = 0$ iff $(x \,;\, 1) \odot (\overline{x \,;\, 1}) = 0$, which is true. We applied Proposition 2.21.(1) and (A2).

(2): $x^\smile \,;\, (\overline{x} \dagger y) \leq y$ iff $(x^\smile \,;\, \overline{x \,;\, \overline{y}}) \odot \overline{y} = 0$ iff $(x^{\smile\smile} \,;\, \overline{y}) \odot \overline{(x \,;\, \overline{y})} = 0$ if $(x \,;\, \overline{y}) \odot \overline{(x \,;\, \overline{y})} = 0$, which is true.

(3): $x^\smile \,;\, \overline{x} \leq \Box$ iff $(x^\smile \,;\, \overline{x}) \odot \Delta = 0$ iff $(x^{\smile\smile} \,;\, \Delta) \odot \overline{x} = 0$ iff $x \odot \overline{x} = 0$, true.

(4): Similar to (3).

(5): $\overline{x}^\smile \,;\, x \leq \Box$ iff $(\overline{x}^\smile \,;\, x) \odot \Delta = 0$ iff $(\overline{x}^{\smile\smile} \,;\, \Delta) \odot x = 0$ iff $\overline{x} \odot x = 0$, true.

(6): Similar to (5).

(7): $x \dagger 1 = \overline{\overline{x} \,;\, \overline{1}} = \overline{\overline{x} \,;\, 0} = \overline{0} = 1$. We applied Proposition 2.21.(17).

(8): Similar to (7). □

Proposition 2.26. *The following hold in any DMVRA:*

(1) $(x \dagger y)^{\smile} = y^{\smile} \dagger x^{\smile}$,
(2) $\Box^{\smile} = \Box$,
(3) $\overline{x \dagger \overline{x}^{\smile}} \leq \Box$,
(4) $\overline{\overline{x} \dagger x^{\smile}} \leq \Box$,
(5) $\overline{x^{\smile} \dagger \overline{x}} \leq \Box$,
(6) $\overline{\overline{x}^{\smile} \dagger x} \leq \Box$,
(7) $x \dagger \Box = \Box \dagger x = x$,
(8) $\dagger$ *is associative,*
(9) $(x \dagger y) \oplus z = 1$ *iff* $(x^{\smile} \dagger z) \oplus y = 1$ *iff* $(z \dagger y^{\smile}) \oplus x = 1$.

Proof. (1): $(x \dagger y)^{\smile} = \big(\overline{\overline{x}\,;\overline{y}}\big)^{\smile} = \overline{(\overline{x}\,;\overline{y})^{\smile}} = \overline{(\overline{y}^{\smile}\,;\overline{x}^{\smile})} = \overline{(\overline{y^{\smile}}\,;\overline{x^{\smile}})} = y^{\smile} \dagger x^{\smile}$. We applied Propositions 2.23.(1) and 2.21.(11). (2): $\Box^{\smile} = (\overline{\Delta})^{\smile} = \overline{\Delta^{\smile}} = \overline{\Delta} = \Box$.

(3)–(6): The proofs are all similar, so we only prove (3). $\overline{x \dagger \overline{x}^{\smile}} \leq \Box$ iff $\overline{x \dagger \overline{x}^{\smile}} \odot \Delta = 0$ iff $(\overline{x}\,; x^{\smile}) \odot \Delta = 0$ iff $(\Delta\,; x^{\smile\smile}) \odot \overline{x} = 0$ iff $x \odot \overline{x} = 0$, which is true.

(7): $x \dagger \Box = \overline{\overline{x}\,;\Delta} = \overline{\overline{x}} = x$; $\quad \Box \dagger x = \overline{\overline{x}\,;\Delta} = \overline{\overline{x}} = x$.

(8): $(x \dagger y) \dagger z = (\overline{\overline{x}\,;\overline{y}}) \dagger z = \overline{\overline{\overline{\overline{x}\,;\overline{y}}}\,;\overline{z}} = \overline{(\overline{x}\,;\overline{y})\,;\overline{z}} = \overline{\overline{x}\,;(\overline{y}\,;\overline{z})} = \cdots = x \dagger (y \dagger z)$.

(9): $(x \dagger y) \oplus z = 1$ iff $\overline{\overline{x}\,;\overline{y}} \oplus z = 1$ iff $\overline{\overline{\overline{\overline{x}\,;\overline{y}}} \odot \overline{z}} = 1$ iff $(\overline{x}\,;\overline{y}) \odot \overline{z} = 0$ iff $\cdots$

Here we have two cases: Case (i): $\cdots$ iff $(\overline{x}^{\smile}\,;\overline{z}) \odot \overline{y} = 0$ iff $(\overline{x^{\smile}}\,;\overline{z}) \odot \overline{y} = 0$ iff $\overline{x^{\smile} \dagger z} \odot \overline{y} = 0$ iff $\overline{\overline{x^{\smile} \dagger z \odot \overline{y}}} = 1$ iff $(x^{\smile} \dagger z) \oplus y = 1$. Case (ii): $\cdots$ iff $(\overline{z};\overline{y}^{\smile}) \odot \overline{x} = o$ iff $(\overline{z}\,;\overline{y^{\smile}}) \odot \overline{x} = 0$ iff $\overline{\overline{z \dagger y^{\smile} \odot \overline{x}}} = 1$ iff $(z \dagger y^{\smile}) \oplus x = 1$.
We used (A2) and Proposition 2.23. $\Box$

An important feature of MVRAs is that $^{\smile}$ can be expressed in terms of the other operations and suprema (that we know exist for this case). This means that $^{\smile}$ is totally determined by the other operations, as in the Boolean case [4].

Proposition 2.27. *Let $\mathcal{A}$ be a MVRA and $a \in A$. Then:*

(1) $a^{\smile}$ *is the largest* $x \in A$ *such that* $x\,;\overline{a} \leq \Box$,
(2) $a^{\smile} = \bigvee_{x;\overline{a} \leq \Box} x$.

Proof. (1): That $a^{\smile}\,;\overline{a} \leq \Box$ we know from Proposition 2.25.(3). Now, $x\,;\overline{a} \leq \Box$ implies $(x\,;\overline{a}) \odot \Delta = 0$ implies $(x^{\smile}\,;\Delta) \odot \overline{a} = 0$ implies $x^{\smile} \leq a$ implies $x^{\smile\smile} \leq a^{\smile}$ implies $x \leq a^{\smile}$. We applied (A2) and Proposition 2.21.(1).

(2): Immediate from 1). $\Box$

As already remarked, MVRAs are straight generalizations of relation algebras defined by RA(1). However, only the DMVRAs display the natural duality induced by the operator $^{\smile}$.

Proposition 2.28. *Let* $\mathcal{A} = (A, \oplus, \odot, ^-, 0, 1, \,;\, , ^\smile, \Delta)$ *be an MVRA.*

(a) *Define* $\circ\colon A \times A \longrightarrow A$ *by* $x \circ y = y \,;\, x$. *Then* $\mathcal{A} = (A, \oplus, \odot, ^-, 0, 1, \circ, ^\smile, \Delta)$ *is also a MVRA. If, in addition,* $\mathcal{A}$ *is a DMVRA, then so is* $\mathcal{A}'$ *and* $\mathcal{A}$, $\mathcal{A}'$ *are isomorphic as algebras.*

(b) *Suppose that* $\mathcal{A}$ *is a DMVRA. Take* $\mathcal{A}'' = (A, \odot, \oplus, ^-, 1, 0, \dagger, ^\smile, \Box)$. *Then* $\mathcal{A}''$ *is also a DMVRA and* $\mathcal{A}$, $\mathcal{A}''$ *are isomorphic as algebras.*

(c) *Suppose again that* $\mathcal{A}$ *is a DMVRA. Then* $\mathcal{A}''' = (A, \odot, \oplus, ^-, 1, 0, \bullet, ^\smile, \Box)$, *where* $\bullet\colon A \times A \longrightarrow A$ *is defined by* $x \bullet y = y \dagger x$, *is also a DMVRA and* $\mathcal{A}$, $\mathcal{A}'''$ *are isomorphic as algebras.*

Proof. (a): The fact that $\circ$ is associative and has Δ as identity element is trivial. Also, because of the symmetry of the axiom (A2) with respect to the two arguments of $\,;\,$, one can immediately see that (A2) also holds for $\circ$. Thus, $\mathcal{A}'$ is an MVRA. Suppose now that $\mathcal{A}$ is also a DMVRA. The axiom (A3), involving only $\oplus$ and $^\smile$, also holds for $\mathcal{A}'$. Define now $f\colon A \longrightarrow A$ by $f(a) = a^\smile$. From Proposition 2.21.(1), we know that f is a bijection. Also,

$$f(0) = 0^\smile = 0\ ,\ f(1) = 1^\smile = 1\ ,\ f(\Delta) = \Delta^\smile = \Delta\ ,$$

$$f(x \oplus y) = (x \oplus y)^\smile = x^\smile \oplus y^\smile = f(x) \oplus f(y)\ ,$$

$$f(x \odot y) = (x \odot y)^\smile = x^\smile \odot y^\smile = f(x) \odot f(y)\ ,$$

$$f(x^\smile) = x^{\smile\smile} = (f(x))^\smile \quad ,\quad f(\overline{x}) = \overline{x}^\smile = \overline{x^\smile} = \overline{f(x)}\ ,$$

$$f(x \,;\, y) = (x \,;\, y)^\smile = y^\smile \,;\, x^\smile = x^\smile \circ y^\smile = f(x) \circ f(y)\ ,$$

so f is a morphism. We used Proposition 2.21.(7,8,12), Proposition 2.23 and (A3).

(b): That $(A, \odot, \oplus, 1, 0)$ is an MV-algebra is well-known. Further, the axioms (A1)–(A3) are satisfied by $\mathcal{A}''$ according to Propositions 2.26.(7,8,9) and 2.23.(2). So $\mathcal{A}''$ is a DMVRA. Define $g\colon A \longrightarrow A$ by $g(x) = \overline{x}$. We know, from MV-algebras, that g is a bijection and an isomorphism between the MV parts of $\mathcal{A}$ and $\mathcal{A}''$. The fact that g commutes with the other operations is an easy consequence of the definitions of $\dagger$ and $\Box$ and of $\overline{x^\smile} = \overline{x}^\smile$.

(c): Immediate from a) and b), since $\mathcal{A}''' = (\mathcal{A}'')'$. $\Box$

Remark 2.29. Regarding the last proposition, we can further say that the map $\mathcal{A} \mapsto \mathcal{A}'$ is a permutation of $\mathcal{MVRA}$. This gives a way, once a theorem has been proved about an arbitrary MVRA, to get another theorem, called its *converse*, by simply interchanging everywhere the arguments of $\,;\,$ (we suppose that the theorem, expressed in the first-order language of MVRA, does not use symbols of derived operations).

A more interesting case is that of DMVRAs, where one can also extend the MV duality (which in turn extended the order duality). All three maps $\mathcal{A} \mapsto \mathcal{A}'$, $\mathcal{A} \mapsto \mathcal{A}''$ and $\mathcal{A} \mapsto \mathcal{A}'''$ are inner bijections of $\mathcal{DMVRA}$, the last being actually the composition of the first two. We obtain, for a theorem, its *converse*, its *dual* and what we might call its *dual converse*, which would be the same as *converse dual*, since $\overline{x^{\smile}} = \overline{x}^{\smile}$. The usual classical term for what we called "dual" is "converse dual", which we use for the composition of dualities. The three maps above, together with the identity on DMVRA, form a group isomorphic to the Klein 4-group, just as in the Boolean case [18].

The next proposition lists some properties of relation algebras with a certain intuitive and even "visual" meaning, that also hold for MVRA. Their proofs are easy applications of (A2) together with Proposition 2.21, so we skip them. The main point of listing these properties is that, in order for them to hold, we do not need a Boolean algebra structure, an MV-algebra doing the job.

Proposition 2.30. *The following hold in any MVRA:*

(1) *(De Morgan's equivalences)* $x\,;y \le z$ *iff* $x^{\smile}\,;\overline{z} \le \overline{y}$ *iff* $\overline{z}\,;y^{\smile} \le \overline{x}$,
((2)–(4) are Tarski's equivalences)
(2) $x\,;y \le \overline{z}$ *iff* $x^{\smile}\,;z \le \overline{y}$ *iff* $z\,;y^{\smile} \le \overline{x}$ *iff* $z^{\smile}\,;x \le \overline{y^{\smile}}$ *iff* $y\,;z^{\smile} \le \overline{x^{\smile}}$ *iff* $y^{\smile}\,;x^{\smile} \le \overline{z^{\smile}}$,
(3) $(x\,;y) \le \overline{z^{\smile}}$ *iff* $z\,;x \le \overline{y^{\smile}}$ *iff* $y\,;z \le \overline{x^{\smile}}$,
(4) $(x\,;y) \odot z^{\smile} = 0$ *iff* $(z\,;x) \odot y^{\smile} = 0$ *iff* $(y\,;z) \odot x^{\smile} = 0$,
(5) *(The cycle law)* $(x^{\smile}\,;z) \odot y = 0$ *iff* $(x\,;y) \odot z = 0$ *iff* $(z\,;y^{\smile}) \odot x = 0$ *iff* $(z^{\smile}\,;x) \odot y^{\smile} = 0$ *iff* $(y^{\smile}\,;x^{\smile}) \odot z^{\smile} = 0$ *iff* $(y\,;z^{\smile}) \odot x^{\smile} = 0$.
(6) *(Peirce's equivalences)*

$$\begin{aligned} x\,;y \le z \text{ iff } y\,;\overline{z}^{\smile} \le \overline{x}^{\smile} \text{ iff } \overline{z}^{\smile}\,;x \le \overline{y}^{\smile} \text{ iff } \overline{z} \le \overline{x} \dagger \overline{y} \text{ iff } x^{\smile} \le \overline{y} \dagger z^{\smile} \\ \text{iff } y^{\smile} \le z^{\smile} \dagger \overline{x} \text{ iff } y^{\smile}\,;x^{\smile} \le z^{\smile} \text{ iff } \overline{z}\,;y^{\smile} \le \overline{x} \text{ iff } x^{\smile}\,;\overline{z} \le \overline{y} \\ \text{iff } \overline{z}^{\smile} \le \overline{y}^{\smile} \dagger \overline{x}^{\smile} \text{ iff } x \le z \dagger \overline{y}^{\smile} \text{ iff } y \le \overline{x}^{\smile} \dagger z. \end{aligned}$$

Remark 2.31. As one might expect, there are a few arithmetical results which do not carry over from the classical case. One of them is the following: in a RA, if $x, y \le \Delta$, then $x\,;y = x \odot y$ (see [4], [14]). At the proof theoretical level, this is a consequence of the fact that for each x, we have $x \le x\,;x^{\smile}\,;x$, which in turn follows from the idempotency of the conjunction, so we depart here from the many-valued route. Still, interestingly enough,

$$x, y \le \Delta \text{ implies } x\,;y = x \odot y$$

holds in our canonical Example 2.11 (but, by Examples 2.18, 2.19 and 2.20, this property is independent from (A0)–(A3) — in fact, these examples show that for subdiagonal elements, the operation $;$ need not be comparable to $\odot$ with respect to the order $\le$). Another property satisfied by the model from Example 2.11 is that the diagonal is a Boolean element in the MV-algebra part. Neither does this follow from the DMVRA axioms, as shown by Example 2.20. A place where these properties are axiomatically considered is the paper [21], which is a further development of the present work.

We are now interested in giving a characterization of MVRAs that is quite similar to RA(2).

Consider the following properties for algebras of MVRA type:

(B1) $;$ is isotone in the first argument.
(B2) $;$ is isotone in the second argument.
(B3) $;$ is isotone in both arguments.
(B4) $(x \vee y)\,;z = (x\,;z) \vee (y\,;z)$.
(B5) $z\,;(x \vee y) = (z\,;x) \vee (z\,;y)$.
(B6) $^{\smile}$ is isotone.
(B7) $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$.
(B8) $(x \odot y)^{\smile} = x^{\smile} \odot y^{\smile}$.
(B9) $(x\,;y)^{\smile} = y^{\smile}\,;x^{\smile}$.
(B10) $\overline{x}^{\smile} = \overline{x^{\smile}}$.
(B11) $(x^{\smile}\,;\overline{x\,;y}) \vee \overline{y} = \overline{y}$.
(B12) $(\overline{x\,;y}\,;y^{\smile}) \vee \overline{x} = \overline{x}$.
(B13) $x^{\smile\smile} = x$.

Remark 2.32. In (B1)–(B3) and (B6), increasingness is assumed with respect to the relation $\leq$ defined by $x \leq y$ iff $x \odot \overline{y} = 0$. Also, in (B4), (B5), (B7), (B11) and (B12), $\vee$ is defined by $x \vee y = x \oplus (\overline{x} \odot y)$. We shall only work with algebras of MVRA type for which the MV type part is an MV-algebra, thus $\leq$ will be a lattice order and $\vee$ the supremum operation; consequently, (B11) and (B12) will actually mean $(x^{\smile}\,;\overline{x\,;y}) \leq \overline{y}$ and $(\overline{x\,;y}\,;y^{\smile}) \leq \overline{x}$.

Lemma 2.33. *Let $\mathcal{A} = (A, \oplus, \odot, ^-, 0, 1, ;, ^{\smile}, \Delta)$ be an algebra of MVRA type such that $(A, \oplus, \odot, ^-, 0, 1)$ is an MV-algebra. Then the following are true in $\mathcal{A}$:*

(1) *If* (B6), (B9) *and* (B13) *hold, then* (B1), (B2) *and* (B3) *are equivalent.*
(2) *If* (B6), (B9), (B10) *and* (B13) *hold, then* (B11) *and* (B12) *are equivalent.*

Proof. (1): It is sufficient to prove "(B1) iff (B2)". Assume (B1) holds and let $x, y \in A$ such that $x \leq y$. Then also $x^{\smile} \leq y^{\smile}$ by (B6). Further, $z\,;x = (z^{\smile\smile}\,; x^{\smile\smile}) = (x^{\smile}\,;z^{\smile})^{\smile}$ by (B13) and (B9). But, by (B1), $x^{\smile}\,;z^{\smile} \leq y^{\smile}\,;z^{\smile}$ and, applying (B6) and (B9), $(x^{\smile}\,;z^{\smile})^{\smile} \leq (y^{\smile}\,;z^{\smile})^{\smile} = (z\,;y)^{\smile\smile} = z\,;y$. Thus, $z\,;x \leq z\,;y$. The fact that (B2) implies (B1) follows analogously.

(2): Let us assume (B11) and prove, for arbitrary x and y, $(\overline{x\,;y};y^{\smile}) \leq \overline{x}$. Using (B13), (B9), (B10), (B9), (B11), (B10) and (B13),we get

$$(\overline{x\,;y}\,;y^{\smile}) = (\overline{x\,;y}\,;y^{\smile})^{\smile\smile} = (y^{\smile\smile}\,;\overline{x\,;y}^{\smile})^{\smile} = (y^{\smile\smile}\,;\overline{(x\,;y)^{\smile}})^{\smile} =$$
$$= ((y^{\smile})^{\smile}\,;\overline{(y^{\smile}\,;x^{\smile})})^{\smile} \leq \overline{x^{\smile}}^{\smile} = \overline{x}^{\smile\smile} = \overline{x}\,.$$

That (B12) implies (B11) can be proved similarly. $\square$

Proposition 2.34. *The following are equivalent for an algebra $\mathcal{A}$ of MVRA type:*

(a) *$\mathcal{A}$ is a DMVRA;*

(b) *$\mathcal{A}$ satisfies* (A0) *and* (A1) *and* ([B1) *or* (B2) *or* (B3) *or* (B4) *or* (B5)] *and* [(B6) *or* (B7)] *and* [(A3) *or* (B8)] *and* (B9) *and* (B10) *and* [(B11) *or* (B12)] *and* (B13).

Proof. (a) implies (b): This was proved, piecemeal, previously.

(b) implies (a): Because of (A0), each one of (B1), (B2) and (B3) is weaker then each one of (B4) and (B5). Also, (B6) is weaker than (B7). Thus, using the previous lemma, it is enough to show that (A0), (A1), (B2), (B6), (B8), (B9), (B10), (B11) and (B13) are sufficient for $\mathcal{A}$ to be a DMVRA. We need to prove (A3) and (A2). For (A3), we have

$$(x \oplus y)^\smile = (\overline{\overline{x} \odot \overline{y}})^\smile = \overline{\overline{x}^\smile \odot \overline{y}^\smile} = \overline{\overline{x^\smile} \odot \overline{y^\smile}} = x^\smile \oplus y^\smile$$

(using (B8) and (B10)).

Let us now prove (A2). Suppose first that $(x\,;y) \odot z = 0$. Then $z \leq \overline{x\,;y}$ so, by (B2), $x^\smile\,;z \leq x^\smile\,;\overline{x\,;y}$. But $x^\smile\,;\overline{x\,;y} \leq \overline{y}$ according to (B11). Thus, $x^\smile\,;z \leq \overline{y}$, which mens $(x^\smile\,;z) \odot y = 0$.

Assume now that $(x^\smile;z)\odot y = 0$. We get $y \leq \overline{x^\smile\,;z}$. By (B6), $y^\smile \leq (\overline{x^\smile\,;z})^\smile$. But, according to (B10), (B9) and (B13), $(\overline{x^\smile\,;z})^\smile = \overline{(x^\smile\,;z)^\smile} = \overline{z^\smile\,;x^{\smile\smile}} = \overline{z^\smile\,;x}$. So $y^\smile \leq \overline{z^\smile\,;x}$. Now, applying (B2), $z\,;y^\smile \leq z\,;\overline{z^\smile\,;x}$. On the other hand, by (B11) and (B13), $z\,;\overline{z^\smile\,;x} = (z^\smile)^\smile\,;\overline{z^\smile\,;x} \leq \overline{x}$. We get $z\,;y^\smile \leq \overline{x}$, which means $(z\,;y^\smile) \odot x = 0$.

Finally, we want to prove that $(z\,;y^\smile) \odot x = 0$ implies $(x\,;y) \odot z = 0$. Assume $(z;y^\smile)\odot x = 0$. Then $x \leq \overline{z\,;y^\smile}$, so, using (B1) (implied by (B2)), (B13) and (B12) (implied by (B11)), $x;y \leq \overline{z\,;y^\smile};y = \overline{z\,;y^\smile};(y^\smile)^\smile \leq \overline{z}$; hence, $(x;y)\odot z = 0$. $\square$

The last proposition gives alternative sets of axioms for DMVRA, making the connection with RA(2). The use of $\vee$ in the axioms is not such a strange presence in the MV framework, since one of the MV algebra axioms says that $x \vee y = y \vee x$. Both MVRAs and DMVRAs are now seen to be quite direct generalizations of RAs; it only depends which of the definitions RA(1) or RA(2) we consider. This observation is based on the fact that, since a Boolean algebra is an MV algebra in which $\oplus = \vee$ (denoted $+$) and $\odot = \wedge$ (denoted $\cdot$), an axiom of RA involving, say, $+$, can be "copied" in the MV framework in two different ways: replacing $+$ either with $\oplus$, or with $\vee$ (and similarly for $\cdot$). Table 1 compares the Boolean definitions with the fuzzy ones.

Thus, both MVRAs and DMVRAs generalize RAs, but DMVRAs are connected with both alternative definitions of RAs, while MVRAs are far from satisfying

Boolean case	**Fuzzy case**
RA(1)	**MVRA**
(A0′) $(A, +, \cdot, \bar{\ }, 0, 1)$ Boolean algebra	(A0) $(A, \oplus, \odot, \bar{\ }, 0, 1)$ MV algebra
(A1) $(A, ;, {}^{\smile})$ monoid	(A1) $(A, ;, {}^{\smile})$ monoid
(A2′) $(x ; y) \cdot z = 0$ iff $(x^{\smile} ; z) \cdot y = 0$ iff $(z ; y^{\smile}) \cdot x = 0$	(A2) $(x ; y) \odot z = 0$ iff $(x^{\smile} ; z) \odot y = 0$ iff $(z ; y^{\smile}) \odot x = 0$

RA(2)	**DMVRA (alternative definition)**
(A0′) ...	(A0) ...
(A1) ...	(A1) ...
(A3′) $(x + y)^{\smile} = x^{\smile} + y^{\smile}$	(A3) $(x \oplus y)^{\smile} = x^{\smile} \oplus y^{\smile}$ (A3b) $(x \vee y)^{\smile} = x \vee y^{\smile}$
(D1′) $(x + y) ; z = (x ; z) + (y ; z)$	(D1) $(x \vee y) ; z = (x ; z) \vee (y ; z)$
(D2) $x^{\smile\smile} = x$	(D2) $x^{\smile\smile} = x$
(D3) $(x ; y)^{\smile} = y^{\smile} ; x^{\smile}$	(D3) $(x ; y)^{\smile} = y^{\smile} ; x^{\smile}$
(D4′) $\overline{y} = \overline{y} + (x^{\smile} ; \overline{x ; y})$	(D4) $\overline{y} = \overline{y} \vee (x^{\smile} ; \overline{x ; y})$
	(D5) $\overline{x^{\smile}} = \overline{x}^{\smile}$

Table 1

properties similar to those from RA(2); (A3) and (D4) are not necessarily true in MVRAs. For now, DMVRAs seem to be a more suitable fuzzy generalization of RAs, knowing in addition that they also recover the double duality of RAs (according to Proposition 2.28.(b,c)). Yet, as we shall see in the next sections, some generalizations of theorems about RAs have "fuzzier" versions for MVRAs than for DMVRAs.

3. Fuzzy relation algebras

In this section, we shall define fuzzy relational structures that are more general than MVRAs and DMVRAs, by replacing the MV-algebra part with a BL-algebra one. Because BL-algebras are fairly general fuzzy structures (see [7]), we call these *fuzzy relation algebras* (FRAs), respectively *distributive fuzzy relation algebras* (DFRAs).

Definition 3.1. A *BL-algebra* [7] is a structure $(A, \vee, \wedge, \odot, \to, 0, 1)$ such that:

(BL0) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice.

(BL1) The binary operators $\odot$ and $\to$ form a residuated pair, that is

$$x \odot y \leq z \text{ iff } x \leq y \to z .$$

(BL2) $x \wedge y = x \odot (x \to y)$.
(BL3) $(x \to y) \vee (y \to x) = 1$.

In an arbitrary BL-algebra, define the derived operations $^{-}$ and $\leftrightarrow$ by

$$\overline{x} = x \to 0 \ , \ x \leftrightarrow y = (x \to y) \odot (y \to x) .$$

In [7], P. Hájek introduced BL-algebras as structures corresponding to a general fuzzy logic (called Basic Logic), that takes as standard truth values structure the unit interval equipped with any *continuous t-norm* ("t" comes from "triangular"), i.e., a binary continuous operation $\odot\colon [0,1] \times [0,1] \longrightarrow [0,1]$ that is associative, commutative, isotone in both arguments, has 1 as identity element and 0 as annihilator. If we define its residua, the binary operation $\to$, by $x \to y = \max\{z \in [0,1] \mid x \odot z \leq y\}$, then $([0,1], \max, \min, \odot, \to, 0, 1)$ becomes a BL-algebra. Basic Logic is the tightest generalization of Gödel, Łukasiewicz and Product Logics, which come from the three most important examples of continuous t-norms on the unit interval (the Łukasiewicz t-norm is the operation $\odot$ from $L_{[0,1]}$). That BL generalizes Łukasiewicz Logic is reflected, at the algebraic level, by the fact that MV-algebras are particular cases of BL-algebras. Indeed, any MV-algebra becomes a BL-algebra with $\vee$, $\wedge$ and $\to$ defined in the usual way (remember that $x \vee y = x \oplus (y \odot \overline{x})$, $x \wedge y = x \odot (y \oplus \overline{x})$, $x \to y = \overline{x} \oplus y = \overline{x \odot \overline{y}}$). This correspondence, together with an identity on morphisms, is an isomorphism of categories between MV-algebras and BL-algebras satisfying $x = (x \to 0) \to 0$. The inverse functor sends a BL-algebra satisfying the above property into an MV-algebra, where $^{-}$ is the above derived operator and $x \oplus y = ((x \to 0) \odot (y \to 0)) \to 0$, and is also identity on morphisms.

Definition 3.2. An FRA is a structure $\mathcal{A} = (A, \vee, \wedge, \odot, \to, 0, 1, ;, ^{\smile}, \Delta)$ that satisfies the following axioms:

(AA0) $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra,
(A1) $(A, ^{\smile}, \Delta)$ is a monoid,
(A2) $(x ; y) \odot z = 0$ iff $(x^{\smile} ; z) \odot y = 0$ iff $(z ; y^{\smile}) \odot x = 0$.

If, in addition, it satisfies

(AA3) $(x \odot y)^{\smile} = x^{\smile} \odot y^{\smile}$ and $(x \to y)^{\smile} = x^{\smile} \to y^{\smile}$,

then it is called a DFRA. The corresponding classes of algebras are denoted $\mathcal{F}RA$ and $\mathcal{D}FRA$.

If, as above, we identify MV-algebras with BL-algebras satisfying double negation ($x = (x \to 0) \to 0$), we can also identify MVRAs with FRAs satisfying double negation. Furthermore, one can easily see that DMVRAs are precisely the DFRAs that satisfy double negation.

From the Examples 2.11–2.20, the only one that really depends on the specificity of MV axioms is Example 2.13, the others having immediate generalizations to FRAs and DFRAs. In Example 2.11, we replace the Łukasiewicz t-norm by an arbitrary continuous t-norm and define the BL operations point-wise — the DFRA obtained shall be denoted with $\mathcal{F}Rel_{\odot}(X)$ (or $\mathcal{F}Rel(X)$ if $\odot$ is understood.

For the remaining two sections, we make the following denotational convention: if X is a set and $x \in X$, let λ_x denote the funcion $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ if $y \neq x$.

4. Group relation algebras

In the classical case, any algebra $\mathcal{G} = (G, *, \ ^{-1}, e)$ of group type induces an RA-type structure $Gra(\mathcal{G})$ which is an RA (and called the *group relation algebra* of $\mathcal{G}$) iff the starting algebra is actually a group. This RA is representable and there are some connections between the subgroups of $\mathcal{G}$ and the equivalence elements of $Gra(\mathcal{G})$, while relativisation in $Gra(G)$ consists of taking the group relation algebra of a subgroup (see [4], [14]). We generalize these results in our fuzzy framework. Some will hold for any continuous t-norm on $[0,1]$, others only for the Łukasiewicz t-norm.

Consider $\odot$ a continuous t-norm on $[0,1]$. Let $\mathcal{G} = (G, *, \ ^{-1}, e)$ be an algebra of group type. Define, on $[0,1]^G$, an algebra of FRA-type as follows:

The BL part has all the operations taken pointwise from $[0,1]$;

The operations Δ, $^{\smile}$ and $;$ are defined as follows:

$$\Delta = \lambda_g\ , \quad P^{\smile}(g) = P(g^{-1})\ ,$$

$$P\,;Q(g) = \bigvee_{g_1 * g_2 = g} P(g_1) \odot Q(g_2) \text{ for all } P, Q \in [0,1]^G \text{ and } g \in G.$$

Denote this structure $Gra(\mathcal{G})$.

Proposition 4.1.

(1) $Gra(\mathcal{G})$ *satisfies* (A3)*;*

(2) $;$ *is associative iff* $*$ *is associative;*

(3) Δ *is a left identity element for* $;$ *iff* e *is a left identity element for* $*$*;*

(4) Δ *is a right identity element for* $;$ *iff* e *is a right identity element for* $*$*;*

(5) $Gra(\mathcal{G})$ *satisfies* $(x\,;y) \odot z = 0$ *iff* $(x^{\smile}\,;z) \odot y = 0$ *if and only if* $\mathcal{G}$ *satisfies* $x^{-1} * (x * y) = y$ *and the function* $f\colon G \times G \longrightarrow G \times G$, $f(x,y) = (x^{-1}, x * y)$ *is surjective;*

(6) $Gra(\mathcal{G})$ *satisfies* $(x\,;y) \odot z = 0$ *iff* $(y\,;z^{\smile}) \odot x = 0$ *if and only if* $\mathcal{G}$ *satisfies* $(y * x) * x^{-1} = y$ *and the function* $h\colon G \times G \longrightarrow G \times G$, $h(x,y) = (x^{-1}, y * x)$ *is surjective;*

(7) *$Gra(\mathcal{G})$ is a DFRA iff $\mathcal{G}$ is a group.*

Proof. (1): This is immediate from the point-wise definition of $\odot$ and $\to$ on $[0,1]^G$. For instance,

$$\begin{aligned}(P \to Q)^{\smile}(g) &= (P \to Q)(g^{-1}) = P(g^{-1}) \to Q(g^{-1}) \\ &= P^{\smile}(g) \to Q^{\smile}(g) = (P^{\smile} \to Q^{\smile})(g)\ .\end{aligned}$$

(2): Suppose first that $*$ is associative. We have

$$\begin{aligned}(P\,;Q)\,;R\,(g) &= \bigvee_{g_1*g_2=g} (P\,;Q)(g_1) \odot R(g_2) \\ &= \bigvee_{g_1*g_2=g} \Big(\bigvee_{h_1*h_2=g_1} P(h_1) \odot Q(h_2)\Big) \odot R(g_2) \\ &= \bigvee_{\substack{g_1*g_2=g \\ h_1*h_2=g_1}} P(h_1) \odot Q(h_2) \odot R(g_2) \\ &= \bigvee_{(h_1*h_2)*g_2=g_1} P(h_1) \odot Q(h_2) \odot R(g_2) \\ &= \bigvee_{h_1*(h_2*g_2)=g_1} P(h_1) \odot Q(h_2) \odot R(g_2) \\ &= \bigvee_{\substack{h_1*h=g \\ h_2*g_2=h}} P(h_1) \odot Q(h_2) \odot R(g_2) \\ &= \bigvee_{h_1*h=g} P(h_1) \odot \Big(\bigvee_{h_2*g_2=h} Q(h_2) \odot R(g_2)\Big) \\ &= \bigvee_{h_1*h=g} P(h_1) \odot (Q\,;R)(h) = P\,;(Q\,;R)(g)\ .\end{aligned}$$

Conversely, assume that ; is associative and take $g_1, g_2, g_3 \in G$. Let $P = \lambda_{g_1}$, $Q = \lambda_{g_2}$ and $R = \lambda_{g_3}$. We know that

$$(P\,;Q)\,;R\,((g_1 * g_2) * g_3) = P\,;(Q\,;R)((g_1 * g_2) * g_3).$$

But

$$\begin{aligned}(P\,;Q)\,;R\,((g_1 * g_2) * g_3) &= \bigvee_{(x*y)*z=(g_1*g_2)*g_3} P(x) \odot Q(y) \odot R(z) \\ &= P(g_1) \odot Q(g_2) \odot R(g_3) = 1\ .\end{aligned}$$

So

$$1 = P\,;(Q\,;R)\,((g_1 * g_2) * g_3) = \bigvee_{x*(y*z)=(g_1*g_2)*g_3} P(x) \odot Q(y) \odot R(z)\ .$$

Since P, Q, R are crisp, we have that

$$\exists x, y, z \in G,\ P(x) \odot Q(y) \odot R(z) = 1 \text{ and } x * (y * z) = (g_1 * g_2) * g_3\ .$$

The only possibility is $x = g_1$, $y = g_2$ and $z = g_3$; thus

$$g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3\ .$$

(3): Suppose first that e is a left identity for $*$ in $\mathcal{G}$ and let $P \in [0,1]^G$, $g \in G$. Then

$$\begin{aligned} \Delta\,; P\ (g) &= \bigvee_{g_1 * g_2 = g} \Delta(g_1) \odot P(g_2) = \bigvee_{e * g_2 = g} \Delta(e) \odot P(g_2) \\ &= \bigvee_{g_2 = g} P(g_2) = P(g)\ . \end{aligned}$$

Now, suppose Δ is a left identity for $\,;$. Let $g \in G$ and take $P = \lambda_g$. We have that $P(g) = \Delta\,; P\ (g)$, that is

$$1 = \bigvee_{g_1 * g_2 = g} \Delta(g_1) * P(g_2) = \bigvee_{g_1 * g = g} \Delta(g_1) * P(g) = \bigvee_{g_1 * g = g} \Delta(g_1)\ .$$

The only possibility is $e * g = g$.

(4): Similar to (3).

(5): Let us assume that $\mathcal{G}$ satisfies the stated properties. Let $P, Q, R \in [0,1]^G$. The fact that $(P\,; Q) \odot R = 0$ is consecutively equivalent to:

$$\forall g \in G,\ \Big(\bigvee_{g_1 * g_2 = g} P(g_1) \odot Q(g_2)\Big) \odot R(g) = 0\ ,$$

$$\forall g, g_1, g_2 \in G,\quad g = g_1 * g_2 \text{ implies } P(g_1) \odot Q(g_2) \odot R(g) = 0\ ,$$

$$\forall g_1, g_2 \in G,\ \ P(g_1) \odot Q(g_2) \odot R(g_1 * g_2) = 0\ .$$

Using the surjectivity of f, this is further equivalent to

$$\forall g_3, g_4 \in G,\ P(g_3^{-1}) \odot Q(g_3 * g_4) \odot R(g_3^{-1} * (g_3 * g_4)) = 0\ ,$$

which means, via the other hypothesis,

$$\forall g_3, g_4 \in G,\ \ P(g_3^{-1}) \odot Q(g_3 * g_4) \odot R(g_4) = 0\ .$$

The last is consecutively equivalent to

$$\forall g_3, g_4 \in G,\ \ P^{\smile}(g_3) \odot Q(g_3 * g_4) \odot R(g_4) = 0\ ,$$

$$\forall g_3, g_4, g \in G,\ \ g = g_3 * g_4 \text{ implies } P^{\smile}(g_3) \odot R(g_4) \odot Q(g) = 0\ ,$$

$$\forall g \in G,\ \ \bigvee_{g = g_3 * g_4} P^{\smile}(g_3) \odot R(g_4) \odot Q(g) = 0\ ,$$

$$\forall g \in G,\ \Big(\bigvee_{g_3 * g_4 = g} P^{\smile}(g_3) \odot R(g_4)\Big) \odot Q(g) = 0\ ,$$

$$\forall g \in G,\ (P^{\smile}\,; R) \odot Q(g) = 0\ .$$

But this means $(P^{\smile}\,;R)\odot Q=0$.

For the converse, suppose that $Gra(\mathcal{G})$ satisfies the stated property. Then, for any $P,Q,R\in[0,1]^G$, we have the equivalence:

$$\forall g_1,g_2\in G,\ P(g_1)\odot Q\ (g_2)\odot R(g_3)=0$$

iff

$$\forall g_3,g_4\in G,\ P(g_3^{-1})\odot Q(g_3*g_4)\odot R(g_4)=0\ .$$

If f is not surjective, then there exist $g_1,g_2\in G$ such that, for all $g_3,g_4\in G$, $g_1\neq g_3^{-1}$ or $g_2\neq g_3*g_4$. Define R to be the constant 1 function. Define P,Q by $P(g_1)=Q(g_2)=1$, with P and Q being 0, otherwise.

This choice of P,Q,R contradicts the above equivalence, since, for all $g_3,g_4\in G$, $P(g_3^{-1})\odot Q(g_3*g_4)\odot R(g_4)=0$, but there exist g_1,g_2 such that $P(g_1)\odot Q(g_2)\odot R(g_1*g_2)\neq 0$. So f must be surjective.

Now, we want to prove that $\mathcal{G}$ satisfies $x^{-1}*(x*y)=y$. Take $g_3,g_4\in G$. Let $P=\lambda_{g_3^{-1}}$, $Q=\lambda_{g_3*g_4}$ and $R=\lambda_{g_4}$. We have that $P(g_3^{-1})\odot Q(g_3*g_4)\odot R(g_4)=1\neq 0$, so there exist $g_1,g_2\in G$ such that $P(g_1)\odot Q(g_2)\odot R(g_1*g_2)\neq 0$. The only chance is that $g_1=g_3^{-1}$, $g_2=g_3*g_4$ and $g_1*g_2=g_4$. So $g_3^{-1}*(g_3*g_4)=g_4$.

(6): Similar to (5).

(7): Suppose $\mathcal{G}$ is a group. Then all those properties about $\mathcal{G}$ from (2)–(6) are obviously satisfied. Thus, using (1)–(6), we obtain that $Gra(\mathcal{G})$ is a DFRA.

Conversely, suppose $Gra(\mathcal{G})$ is a DFRA. Then, by (2)–(4), $(G,*,e)$ is a monoid. Using (5) and (6) and taking $y=e$, we get that $\mathcal{G}$ satisfies $x^{-1}*x=e$ and $x*x^{-1}=e$. Hence $\mathcal{G}$ is a group. □

Proposition 4.2. *Suppose $\mathcal{G}=(G,*,\ ^{-1},e)$ is a group. Then $Gra(\mathcal{G})$ has an embedding (as a DFRA) into $\mathcal{F}Rel(G)$.*

Proof. Define $h\colon[0,1]^G\longrightarrow[0,1]^{G\times G}$ by $h(P)(g_1,g_2)=P(g_1^{-1}*g_2)$, for all $P\in[0,1]^G$ and $(g_1,g_2)\in G\times G$. Then h is a one-to-one mapping because, if $h(P)=h(Q)$, then, for all $g\in G$,

$$P(g)=P(e^{-1}*g)=h(P)(e,g)=h(Q)(e,g)=Q(g)\ .$$

It is easy to check that h is a DFRA morphism. For the not entirely trivial part, notice that

$$h(P\,;Q)(g_1,g_2)=(P\,;Q)(g_1^{-1}*g_2)=\bigvee_{h_1*h_2=g_1^{-1}*g_2}P(h_1)\odot Q(h_2)\ ,$$

while

$$(h(P);h(Q))(g_1,g_2)=\bigvee_{g\in G}h(P)(g_1,g)\odot h(Q)(g,g_2)=\bigvee_{g\in G}P(g_1^{-1}*g)\odot Q(g^{-1}*g_2)\ .$$

But, in a group, when g_1 and g_2 are fixed, saying that $h_1 * h_2 = g_1^{-1} * g_2$ is the same as saying that there exists $g \in G$ such that $h_1 = g_1^{-1} * g$ and $h_2 = g^{-1} * g_2$ (actually, g would be exactly $g_1 * h_1$). Hence the two suprema coincide, so

$$h(P\,;Q) = h(P)\,;h(Q).$$

Thus h is an injective DFRA morphism, which is a categorical embedding (i.e., monomorphism), since $\mathcal{DFRA}$ is a quasi-variety. $\square$

Because MV-algebras have the good property, shared with Boolean algebras, of relativisation (to a crisp element) — and, as we have seen in Example 2.13, if the element is an equivalence, MVRAs and DMVRAs can also be relativized — we are able to generalize some properties regarding subgroups of classical group relation algebras to the MV case.

Until the end of this section, we shall consider that $\odot$ is the Łukasiewicz t-norm on $[0,1]$. A fuzzy subgroup of a group $\mathcal{G} = (G, {}^{-1}, e)$ is a function $H\colon G \longrightarrow [0,1]$ (i.e., a fuzzy subset of G) such that $H(x) \odot H(y) \leq H(x * y)$, $H(x) \leq H(x^{-1})$ and $H(e) = 1$ for all $x, y \in G$. Notice that classical subgroups of $\mathcal{G}$ can be identified with the fuzzy subgroups for which the image is included $\{0,1\}$. Also, when H is a subgroup of $\mathcal{G}$, we shall denote by $\mathcal{H}$ the subgroup H when viewed with the group structure inherited from $\mathcal{G}$.

Proposition 4.3. *Let $\mathcal{G} = (G, *, {}^{-1}, e)$ be a group and $H \in [0,1]^G$. Then*

(1) *$H \in Eq(Gra(\mathcal{G}))$ iff H is a fuzzy subgroup of $\mathcal{G}$;*
(2) *$H \in Eq(Gra(\mathcal{G})) \cap B(Gra(\mathcal{G}))$ iff H is a subgroup of $\mathcal{G}$;*
(3) *If H is a subgroup of $\mathcal{G}$, then $Gra(\mathcal{G})_{\|H}$ (i.e., $Gra(\mathcal{G})$ relativized to H) and $Gra(\mathcal{H})$ are isomorphic as DMVRAs.*

Proof. (1): Firstly, assume that H is a fuzzy subgroup of $\mathcal{G}$. Since $H(e) = 1$, we have that $\Delta \leq H$. Moreover, for each $g \in G$, $H^{\smile}(g) = H(g^{-1}) \leq H((g^{-1})^{-1}) = H(g)$, so $H^{\smile} \leq H$. Finally, for each $g \in G$,

$$H\,;H\ (g) = \bigvee_{g_1 * g_2 = g} H(g_1) * H(g_2) \leq \bigvee_{g_1 * g_2 = g} H(g_1 * g_2) = H(g)\ .$$

Thus $H\,;H \leq H$.

Conversely, suppose that $H \in Eq(Gra(\mathcal{G})$. From $\Delta \leq H$ and $H^{\smile} \leq H$, it follows immediately that $H(e) = 1$ and $H(x) \leq H(x^{-1})$ for each $x \in G$. Now, take $g_1, g_2 \in G$. We have

$$H(g_1) \odot H(g_2) \leq \bigvee_{h_1 * h_2 = g_1 * g_2} H(h_1) \odot H(h_2) = H\,;H\ (g_1 * g_2) \leq H(g_1 * g_2)\ .$$

(2): Follows at once from (1) and the fact that the crisp subsets of G are precisely the elments from $B(Gra(\mathcal{G}))$.

(3): The underlying set of $Gra(\mathcal{G})_{\|H}$ is

$$\Theta = \{k \colon G \longrightarrow [0,1] \mid k(x) = 0, \ \forall x \notin H\}.$$

Define $f \colon \Theta \longrightarrow [0,1]^H$ by $f(k) = k_{|H}$ (the restriction of k to the set H) for all $k \in \Theta$. f is obviously a bijection. Also, it is easy to check that it commutes with $\oplus$ and $^{\smile}$, that $f(0) = 0$ and $f(1) = H$. Further, for any k,

$$f(\overline{k}^H) = f(\overline{k} \odot H) = (\overline{k} \wedge H)_{|H} = \overline{k_{|H}} = \overline{f(k)} \ ,$$
$$f(\Delta_H) = (\Delta_H)_{|H} = \Delta \ .$$

Let now $k_1, k_2 \in \Theta$ and $x \in H$.

$$f(k_1 \,;\, k_2)(x) = (k_1 \,;\, k_2)_{|H}(x) = \bigvee_{x_1 * x_2 = x} k_1(x_1) * k_2(x_2)$$
$$= \bigvee_{\substack{x_1 * x_2 = x \\ x_1, x_2 \in H}} k_1(x_1) * k_2(x_2)$$

(because, if $x_1 \notin H$ or $x_2 \notin H$, $k_1(x_1) * k_2(x_2) = 0$)

$$= (k_{1|H} \,;\, k_{2|H})(x) = (f(k_1) \,;\, f(k_2))(x) \ .$$

So f is a bijective morphism and hence, since DMVRA is a variety, an isomorphism. □

5. Complex algebras

The classical algebra of binary relations over a set (which was the starting point of RAs), has a double set theoretical structure: first, the relations are sets of elements and second, the elements are pairs. As a consequence, the relative composition displays a two-level dependency on the inner structure of its arguments: it is defined as a resultant of point-wise (pair-wise) gluing, while the pair gluing itself is determined by the structure of the pairs (the two components rule the left and right behaviors of each pair). Perhaps it is this encapsulation of concreteness that made impossible to abstractly reach this algebra of relations in a natural fashion, as happened for instance with groups of permutations or algebras of sets. Hence the idea of taking "decapsulated" structures, that are still concrete (i.e., whose elements have set theoretical structure that is relevant for the operations), but lose the second dependency. Complex algebras are defined following this idea. Instead of considering relations as containing pairs of elements from a set (that would already force, or rather suggest, how pairs should be composed), one takes "pairs" (we still call them this way) as just elements from an arbitrary set, providing in addition, *from outside*, a way to compose them — this is done by a ternary relation, with some properties. Not very surprisingly, one has a representation theorem saying that any RA is, up to an isomorphism, a subalgebra of a complex algebra (a Stone like theorem) [14]. This is why complex algebras form an important class of RAs.

Regarding the construction of complex algebras, there is a very nice theorem that gives, in terms of the "composition guide" ternary relation, a necessary and sufficient condition for having an RA. This result, after going through a "fuzzification" process, also holds if we take as starting point a continuous t-norm with additional properties (in particular, the Łukasiwicz t-norm), obtaining the corresponding classes of FRAs (MVRAs) and DFRAs (DMVRAs). This is what we intend to prove in this section, after we briefly review the situation from the classical case. Also, we shall get group relation algebras as particular cases of complex algebras.

Let U be a set of elements called *pairs* and T a ternary relation on U (i.e., $T \in \{0,1\}^{U\times U\times U}$). T is meant to show the way in which pairs can be combined (glued together): $(x,y,z) \in T$ means z is one possible result of gluing x with y. Using T, one defines:

– the binary relation S, the *reverse relation* by $(x,y) \in S$ if and only if for all $a,b \in U$,

$$[(x,a,b) \in T \text{ iff } (y,b,a) \in T] \text{ and } [(a,x,b) \in T \text{ iff } (b,y,a) \in T]$$

$(x,y) \in S$, i.e., "x is reverse to y", means "x acts, with respect to gluing as does y, only in the opposite direction");

– the set (unary relation) I of *identity pairs* by $x \in I$ if and only if for all $a,b \in U$ with $a \neq b$,

$$[\text{neither } (x,a,b) \in T \text{ nor } (a,x,b) \in T]$$

($x \in I$ means "x never modifies the pair it glues with, either left, or right).

Consider the following conditions:

(C1′) $(\forall a \in U)(\exists b \in U)\ (a,b) \in S$
(that is, every pair has at least a converse).

(C2′) $(\forall a \in U)(\exists i \in U)\ (i \in I \text{ and } (i,a,a) \in T)$ and
$(\forall a \in U)(\exists i \in U)\ (i \in I \text{ and } (a,i,a) \in T)$
(that is, every pair has an identity pair to which it can glue).

(C3′) $\forall x,y,z,r \in U\ ,\ [(\exists a \in U)\ (x,y,a) \in T \text{ and } (a,z,r) \in T]$ iff
$[(\exists b \in U)\ (y,z,b) \in T \text{ and } (x,b,r) \in T]$
(that is, the pairs that can be reached by gluing x, y and z do not depend on the order in which they are grouped for gluing).

Remark 5.1. In [10], these conditions are actually put in a weakened form — (C2′) loses its second part and (C3′) states only the left to right implication, written in the equivalent form: for all $x,y,z,r,a \in U$; ; ,

$$[(x,y,a) \in T \text{ and } (a,z,r) \in T] \text{ implies } [(\exists b \in U)\ (y,z,b) \in T \text{ and } (x,b,r) \in T].$$

We preferred to use these symmetrical versions for our fuzzy generalizations. Whether the above refinement of the conditions can also be carried out in the fuzzy case remains a problem.

On $\{0,1\}^U$, define an algebra of RA type (*the complex algebra of* T) by letting the Boolean part be the canonical one and by defining Δ, $\smile$ and ; as follows:

$\Delta = I$,

$x \in P^{\smile}$ iff $(\exists y)$ $(y \in P$ and $(y,x) \in P)$,

$z \in P\,;Q$ iff $(\exists x,y)$ $(x \in P$ and $y \in Q$ and $(x,y,z) \in T)$

(notice that the composition is defined "along" T).

Proposition 5.2. [10] *The above defined structure is an RA iff conditions* (C1′)–(C3′) *hold.*

It is easy to see that the algebra of relations on $X \times X$ is a particular case of a complex algebra, with $U = X \times X$ and $T = \{((a,b),(b,c),(a,c)) \mid a,b,c \in X\}$.

Coming now to the fuzzy case, we translate all the definitions and properties from above.

Fix $\odot$ a continuous t-norm on $[0,1]$, together with $\rightarrow$, $^{-}$ and $\leftrightarrow$ being the usual derived operators. Let U be a set and $T \in [0,1]^{U\times U\times U}$ a ternary fuzzy relation. Define the binary fuzzy relation $S \in [0,1]^{U\times U}$ and the fuzzy subset $I \in [0,1]^U$ by:

$$S(x,y) = \bigwedge_{a,b\in U} (T(x,a,b) \leftrightarrow T(y,b,a)) \odot (T(a,x,b) \leftrightarrow T(b,y,a))$$

$$I(x) = \bigwedge_{a,b\in U,\ a\neq b} \overline{T(x,a,b)} \odot \overline{T(a,x,b)}\ ,$$

for all $x,y \in U$.

The conditions (C1′)–(C3′) become:

(C1) $\bigwedge_{a\in U} \bigvee_{b\in U} S(a,b) = 1$;

(C2) $\bigwedge_{a\in U} \bigvee_{i\in U} (I(i) \odot T(i,a,a)) = 1$ and $\bigwedge_{a\in U} \bigvee_{i\in U} (I(i) \odot T(a,i,a)) = 1$;

(C3) $\forall x,y,z,r \in U$, $\bigvee_{a\in U} T(x,y,a) \odot T(a,z,r) = \bigvee_{b\in U} T(y,z,b) \odot T(x,b,r)$.

Define the FRA type structure on $[0,1]^U$, denoted $Cm(T)$ (*the complex algebra of* T) as follows:

– the BL part is constructed with operations from $[0,1]$ taken pointwise;

– the operations Δ, $\smile$ and ; are given by:

$$\Delta = I\ ; \quad P^{\smile}(z) = \bigvee_{y\in U} P(y) \odot S(y,z)\ ;$$

$$P\,;Q\ (z) = \bigvee_{x,y\in U} P(x) \odot Q(y) \odot T(x,y,z)$$

for all $P,Q \in [0,1]^U$ and $z \in U$.

As in the classical case, the canonical examples of DMVRAs (DFRAs), namely $MVRel(X)$ ($\mathcal{F}Rel_{\odot}(X)$) are complex algebras if we take T to be the crisp relation $\{((x,y),(y,z),(x,z)) \mid x,y,z \in X\}$.

Lemma 5.3. *Let $\odot$ be a continuous t-norm and $T \in [0,1]^{U\times U\times U}$.*

I *The following hold:*
  (1) *[; is associative in $Cm(T)$] iff* (C3),
  (2) *[I is an identity element for ; in $Cm(T)$] iff* (C2),
  (3) *[$Cm(T)$ satisfies* (A2) *] if* (C1),
  (4) *If S is crisp and for all $a,b,c \in U$, $b \neq c$ implies $S(a,b) \odot S(a,c) = 0$, then $Cm(T)$ satisfies* (A3).

II *We assume that the mapping $\alpha \mapsto \overline{\alpha}$ is injective.*[2] *If [$Cm(T)$ satisfies* (A2) *], then* (C2) *implies* (C1).

III *We assume that $\odot$ does not have idempotent elements x (i.e., such that $x \odot x = x$) other than 0 and 1.*[3] *If $Cm(T)$ satisfies* (A3)*, then S is crisp and for all $a,b,c \in U$, $b \neq c$ implies $S(a,b) \odot S(a,c) = 0$.*

Proof. I(1): Suppose first that (C3) holds. Then

$$
\begin{aligned}
(P\,;Q)\,;R\,(r) &= \bigvee_{a,z\in U} (P\,;Q)(a) \odot R(z) \odot T(a,z,r) \\
&= \bigvee_{a,z\in U} \Big(\bigvee_{x,y\in U} P(x) \odot Q(y) \odot T(x,y,a) \Big) \odot R(z) \odot T(a,z,r) \\
&= \bigvee_{a,z,x,y\in U} P(x) \odot Q(y) \odot T(x,y,a) \odot R(z) \odot T(a,z,r) \\
&= \bigvee_{x,y,z\in U} P(x) \odot Q(y) \odot R(z) \odot \Big(\bigvee_{a\in U} T(x,y,a) \odot T(a,z,r) \Big) \\
&= \bigvee_{x,y,z\in U} P(x) \odot Q(y) \odot R(z) \odot \Big(\bigvee_{b\in U} T(y,z,b) \odot T(x,b,r) \Big) \\
&= \bigvee_{x,y,z,b\in U} P(x) \odot Q(y) \odot R(z) \odot T(y,z,b) \odot T(x,b,r) \\
&= \bigvee_{x,b\in U} P(x) \odot \Big(\bigvee_{y,z\in U} Q(y) \odot R(z) \odot T(y,z,b) \Big) \odot T(x,b,r) \\
&= \bigvee_{x,b\in U} P(x) \odot (Q\,;R)(b) \odot T(x,b,r) = P\,;(Q\,;R)\,(r)\,.
\end{aligned}
$$

---

[2]It is known that the only continuous t-norm satisfying this property is isomorphic to the Łukasiewicz one.

[3]Only two continuous t-norms (up to an isomorphism) have this property: Łukasiewicz and Product t-norms.

Conversely, suppose ; is associative and let $x, y, z, r \in U$. Take $P = \lambda_x$, $Q = \lambda_y$ and $R = \lambda_z$. From $(P\,;Q)\,;R\ (r) = P\,;(Q\,;R)\ (r)$, we get

$$\bigvee_{a\in U} T(x,y,a)\odot T(a,z,r) = \bigvee_{b\in U} T(y,z,b)\odot T(x,b,r)\ .$$

I(2): Let us assume that (C2) holds and let $z \in U$. Then

$$\begin{aligned} P\,;I(z) &= \bigvee_{x,y\in U} P(x)\odot I(y)\odot T(x,y,z) \\ &= \bigvee_{x,y\in U} P(x)\odot\Big(\bigwedge_{a,b\in U,a\neq b}\overline{T(y,a,b)}\odot\overline{T(a,y,b)}\Big)\odot T(x,y,z) \\ &= \bigvee_{x,y\in U} P(x)\odot\Big(\bigwedge_{a,b\in U,a\neq b}\overline{T(y,a,b)}\odot\overline{T(a,y,b)}\odot T(x,y,z)\Big)\ . \end{aligned}$$

If $x \neq z$, then

$$\bigwedge_{a,b\in U,a\neq b}\overline{T(y,a,b)}\odot\overline{T(a,y,b)}\odot T(z,y,z)\leq\overline{T(y,x,z)}\odot\overline{T(x,y,z)}\odot T(x,y,z)=0\ ,$$

so

$$\begin{aligned} &\bigvee_{x,y\in U} P(x)\odot\Big(\bigwedge_{a,b\in U,a\neq b}\overline{T(y,a,b)}\odot\overline{T(a,y,b)}\odot T(x,y,z)\Big) \\ &\qquad= \bigvee_{y\in U} P(z)\odot\Big(\bigwedge_{a,b\in U,a\neq b}\overline{T(y,a,b)}\odot\overline{T(a,y,b)}\odot T(z,y,z)\Big) \\ &\qquad= P(z)\odot\bigvee_{y\in U}\Big(\bigwedge_{a,b\in U,a\neq b}\overline{T(y,a,b)}\odot\overline{T(a,y,b)}\Big)\odot T(z,y,z) \\ &\qquad= P(z)\odot\bigvee_{y\in U} I(y)\odot T(z,y,z) = P(z)\odot 1 = P(z)\ . \end{aligned}$$

Thus $P\,;I = P$. The fact that $I\,;P = P$ follows similarly.

Now, let us suppose that I is an identity element for ;. Let $x \in U$. Take $P = \lambda_x$. From $P\,;I(x) = P(x) = 1$, we get $\bigvee_{i\in U} I(i)\odot T(x,i,x) = 1$. The fact that $\bigwedge_{x\in U}\bigvee_{i\in U} I(i)\odot T(x,i,x) = 1$ follows similarly.

I(3): Assume (C1) holds. We have that

$$(P\,;Q)\odot R\ (z) = 0 \quad\text{iff}\quad \Big(\bigvee_{x,y\in U} P(x)\odot Q(y)\odot T(x,y,z)\Big)\odot R(z) = 0$$

iff

$$\forall x,y,z\in U,\ P(x)\odot Q(y)\odot R(z)\odot T(x,y,z) = 0\ .$$

On the other hand,

$$(P^{\smile}\,;R)\odot Q = 0 \quad\text{iff}\quad \forall x,y,z\in U,\ P^{\smile}(x)\odot R(y)\odot Q(z)\odot T(x,y,z) = 0$$

iff

$$\forall x, y, z \in U,\ \Big(\bigvee_{u \in U} P(u) \odot S(u, x)\Big) \odot R(y) \odot Q(z) \odot T(x, y, z) = 0$$

iff (via a renaming of variables)

$$\forall x, y, z, u \in U,\ P(x) \odot Q(y) \odot R(z) \odot S(x, u) \odot T(u, z, y) = 0$$

iff

$$\forall x, y, z \in U,\ P(x) \odot Q(y) \odot R(z) \odot \bigvee_{u \in U} (S(x, u) \odot T(u, z, y)) = 0\ .$$

Hence, in order to prove that $(P\,;Q) \odot R = 0$ iff $(P^{\smile}\,;R) \odot Q = 0$, it suffices to show

$$\forall x, y, z \in U,\ T(x, y, z) = \bigvee_{u \in U} (S(x, u) \odot T(u, z, y))\ .$$

For this, we fix $x, y, z \in U$. For each $u \in U$,

$$\begin{aligned} S(x, u) &= \bigwedge_{a,b \in U} \big(T(x, a, b) \leftrightarrow T(u, b, a)\big) \odot \big(T(a, x, b) \leftrightarrow T(b, u, a)\big) \\ &\leq T(u, z, y) \rightarrow T(x, y, z)\ . \end{aligned}$$

This implies

$$\forall u \in U,\ S(x, u) \odot T(u, z, y) \leq T(x, y, z)\ ,$$

which means

$$\bigvee_{u \in U} S(x, u) \odot T(u, z, y) \leq T(x, y, z)\ .$$

For the converse inequality, notice that, by (C1), $\bigvee_{u \in U} S(x, u) = 1$, which means

$$\bigvee_{u \in U} \bigwedge_{a,b \in U} (T(x, a, b) \leftrightarrow T(u, b, a)) \odot (T(a, x, b) \leftrightarrow T(b, u, a)) = 1\ .$$

Further, we have that, for every $\delta < 1$, there exists $u \in U$ such that

$$S(x, u) = \bigwedge_{a,b \in U} (T(x, a, b) \leftrightarrow T(u, b, a)) \odot (T(a, x, b) \leftrightarrow T(b, u, a)) \geq \delta\ .$$

In particular,

$$T(x, y, z) \rightarrow T(u, z, y) \geq \delta\ ,$$

that is

$$\delta \odot T(x, y, z) \leq T(u, z, y)\ .$$

Hence, we get

$$\delta \odot \delta \odot T(x, y, z) \leq S(x, u) \odot T(u, z, y) \leq \bigvee_{v \in U} S(x, v) \odot T(v, z, y)\ .$$

Applying the continuity of $\odot$,

$$\lim_{\delta \longrightarrow 1} \delta \odot \delta \odot T(x, y, z) = (\lim_{\delta \longrightarrow 1} \delta \odot \delta) \odot T(x, y, z) = T(x, y, z)\ .$$

We obtain

$$T(x,y,z) \leq \bigvee_{v \in U} S(x,v) \odot T(v,z,y) .$$

The fact that $(P\,;Q) \odot R = 0$ iff $(R\,;Q^{\smile}) \odot P = 0$ follows analogously.

I(4): Let $z \in U$. We know that there exists $y \in U$ such that, for all $x \neq y$, $S(z,x) = 0$ (because S is crisp and there exists at most one element u such that $S(z,u) = 1$, $S(z,w)$ being 0 otherwise). This immediately gives $P^{\smile}(z) = P(y)$ for all $P \in [0,1]^U$. Now, (A3) follows immediately.

II: Suppose that $Cm(T)$ satisfies (A2) and (C2) holds. $(P,Q) \odot R = 0$ iff $(P^{\smile}\,; R) \odot Q = 0$ for all $P, Q, R \in [0,1]^U$ means:

$$\forall a,b,c \in U,\ P(a) \odot Q(b) \odot T(a,b,c) \odot R(c) = 0$$

iff

$$\forall a,b,c,d \in U,\ P(a) \odot S(a,d) \odot R(c) \odot T(d,c,b) \odot Q(b) = 0 .$$

From this, we obtain that for all $x, y, z \in U$, and for all $\alpha \in [0,1]$,

$$\alpha \odot T(x,y,z) = 0 \quad \text{iff} \quad \alpha \odot \bigvee_{u \in U} (S(x,u) \odot T(u,z,y)) = 0$$

(we took $P = \lambda_x$, $Q = \lambda_y$ and $R = \alpha \odot \lambda_z$). This means that $x, y, z \in U$, and for all $\alpha \in [0,1]$,

$$\alpha \leq \overline{T(x,y,z)} \quad \text{iff} \quad \alpha \leq \overline{\bigvee_{u \in U} (S(x,u) \odot T(u,z,y))};$$

that is, for all $x, y, z \in U$,

$$\overline{T(x,y,z)} = \overline{\bigvee_{u \in U} (S(x,u) \odot T(u,z,y))} .$$

According to our assumption about the t-norm $\odot$, we get that for all $x, y, z \in U$,

$$\begin{aligned} T(x,y,z) &= \bigvee_{u \in U} (S(x,u) \odot T(u,z,y)) \\ &\leq \Big(\bigvee_{u \in U} S(x,u)\Big) \odot \Big(\bigvee_{u \in U} T(u,z,y)\Big) \leq \bigvee_{u \in U} T(u,z,y) . \end{aligned}$$

Further, for all $y, z \in U$,

$$\bigvee_{x \in U} T(x,y,z) \leq \bigvee_{u \in U} T(u,z,y) ,$$

which, by symmetry, actually means that for all $y, z \in U$,

$$\bigvee_{u \in U} T(u,y,z) = \bigvee_{u \in U} T(u,z,y) .$$

Now, let $x \in U$. We have for all $y \in U$,

$$T(x, y, x) \leq \bigvee_{u \in U} (S(x, u) \odot T(u, x, y)) = \bigvee_{u \in U} (S(x, u) \odot T(u, y, x)); \qquad (*)$$

hence,

$$\bigvee_{y \in U} T(x, y, x) \leq \bigvee_{u, y \in U} (S(x, u) \odot T(u, y, x)) \leq \Big(\bigvee_{u, y \in U} S(x, u) \Big) \odot (T(u, y, x)) \ .$$

According to (C2),

$$\bigvee_{y \in U} T(x, y, x) \geq \bigvee_{y \in U} I(y) \odot T(x, y, x) = 1, \qquad (**)$$

and further,

$$\bigvee_{u, y} T(u, y, x) \geq \bigvee_{y \in U} T(x, y, x) = 1. \qquad (***)$$

We obtain, using $(**)$ and $(***)$ in $(*)$,

$$1 \leq \Big(\bigvee_{u \in U} S(x, u) \Big) \odot 1 \ ;$$

hence,

$$\bigvee_{u \in U} S(x, u) = 1 \ .$$

Since x was arbitrary, we get (C1).

III: Let $a, b \in U$. Taking $P = Q = \lambda_a$, $(P \odot Q)^{\smile}(b) = (P^{\smile} \odot Q^{\smile})(b)$ means $S(b, a) = S(b, a) \odot S(b, a)$, hence, being idempotent, $S(a, b) \in \{0, 1\}$. So S is crisp. Now, take $a, b, c \in U$ such that $b \neq c$. Suppose that $S(a, b) \odot S(a, c) \neq 0$. Since S is crisp, this means $S(a, b) = S(a, c) = 1$. If we take $P = \lambda_b$ and $Q = \lambda_c$, $(P \odot Q)^{\smile}(a) = (P^{\smile} \odot Q^{\smile})(a)$ means $0 \vee 0 = 1 \odot 1$, a contradiction. □

A binary relation $R \subseteq U \times U$ is called *functional* if it is the graph of a partial function from U to U.

Theorem 5.4. *Let $\odot$ be a continuous t-norm on $[0,1]$ and $T \subseteq [0,1]^{U \times U \times U}$. Then:*

(1) *$Cm(T)$ is a FRA if* (C1)–(C3) *hold;*

(2) *$Cm(T)$ is a DFRA if [*(C1)–(C3) *hold and S is a crisp functional relation (or, in other words, if* (C2), (C3) *hold and S if the graph of a (crisp) total function from U to U)];*

(3) *Suppose T is crisp. Then $Cm(T)$ is a DFRA if* (C1)–(C3) *hold.*

For points (4)–(6)*, we assume $\odot$ to be the Łukasiewicz t-norm.*

(4) *$Cm(T)$ is an MVRA iff* (C1)–(C3) *hold;*

(5) *$Cm(T)$ is a DMVRA iff [*(C1)–(C3) *hold and S is a crisp functional relation];*

(6) *Suppose T is crisp. Then $Cm(T)$ is a DMVRA iff* (C1)–(C3) *hold.*

Proof. (1) and (2): These are immediate consequences of point (1) of the previous lemma.

(3): All we need to show is that, if T is crisp and (C1)–(C3) hold, then S is functional. This was actually proved in the classical case [10]. Suppose that (a, b), $(a, c) \in S$. From (C3) (which now became (C3′)), we know there exists $i \in I$ such that $(i, a, a) \in T$. Further, since $(a, b) \in S$, from the definition of S, we get $(a, b, i) \in T$. Now, from $(a, c) \in S$, we get, using again the definition of S, $(a, i, b) \in T$. But $i \in I$, so, from the definition of I, $c = b$.

(4): This follows from the previous lemma, points I(1–3) and II.

(5): This follows from the previous lemma.

(6): This is a consequence of (5) if we take into account that [if T is crisp and (C1)–(C3) hold, then S is functional], which was proved above. □

Points (4) and (6) of the above proposition give us two MV generalizations of the crisp theorem, the one for MVRAs having a fuzzier flavor than that for DMVRAs. These points also show that $\mathcal{DMVRA} \neq \mathcal{MVRA}$ — just take the complex algebra of a suitable fuzzy relation $T \subseteq [0, 1]^{U\times U\times U}$ such that S is not crisp.

Notice that, in order to obtain an "iff" theorem on complex algebras, that is, a full characterization in terms of the starting relation T, we had to assume that the negation of $\odot$ is injective, which forced $\odot$ to be the Łukasiewicz t-norm, leading to complex algebras of MVRAs and DMVRAs.

One can see that, when T is crisp, S and I are actually defined crisply, as in the classical case, while the composition "along T" and the converse operation can be written with a less fuzzy look:

$$P\,;Q(z) = \bigvee_{(x,y,z)\in T} P(x) \odot Q(y)\ ;$$
$$P^{\smile}(z) = \bigvee_{(x,z)\in S} P(x)\ .$$

Because, when T is crisp, (C1)–(C3) become (C1′)–(C3′), we have:

Corollary 5.5. *If $\odot$ is the Łukasiewicz t-norm and $T \in \{0, 1\}^{U\times U\times U}$, then $Cm(T)$ is an MVRA iff it is a DMVRA.*

Notice also that, because the MV-operation $\oplus$ is dual to the conjunction $\odot$, we can faithfully translate variations of the conditions and definitions from the classical case. For instance, the crisp definition of I could be expressed:

$$x \in I \quad \text{iff} \quad \forall a, b \in U,\ [(x, a, b) \in T \text{ or } (a, x, b) \in T \text{ implies } a = b]\ .$$

For the fuzzy translation of this, we consider equality a fuzzy relation Eql, which is in fact the diagonal function. Then we define

$$I(x) = \bigvee_{a,b\in U} (T(x,a,b) \oplus T(a,x,b)) \rightarrow Eql(a,b)\ ,$$

a definition that one can easily see that is equivalent to the original fuzzy one.

Example 5.6. Let $U = \{1,2\}$ and $T \in [0,1]^{U\times U\times U}$ be defined by:

$$T(1,1,1) = 1, T(1,2,2) = 1, T(2,1,2) = 1, T(2,2,1) = 1, T(2,2,2) = \delta\ ,$$

where $\delta \in [0,1]$. On the rest of the domain, T is 0. If we consider the Łukasiewicz t-norm on $[0,1]$, we have $S = \lambda_1 \vee \lambda_2$ and $I = \lambda_1$. Since the needed properties are satisfied, $Cm(T)$ is a DMVRA. In fact, this structure can be identified with the one having $[0,1]^2$ as its support, the component-wise MV operation from $L_{[0,1]}$, $^{\smile}$ being the identity function, $\Delta = (1,0)$ and $;$ being defined by

$$(x,y)\,;(x',y') = (x \odot x' \vee y \odot y', x \odot y' \vee y \odot x' \vee y \odot y' \odot \delta)\ .$$

If we take a number n and force δ to be in $\{0, 1/n, \ldots, (n-1)/n, 1\}$, we obtain a DMVRA on L^2_{n+1} which is only trivial with respect to $^{\smile}$.

In the next proposition, we get group relation algebras as special cases of complex algebras.

Proposition 5.7. *Let $\odot$ be a continuous t-norm. Let $\mathcal{G} = (G, *, {}^{-1}, e)$ be a group and define the crisp ternary relation $T \in \{0,1\}^G$ by*

$$(g_1, g_2, g_3) \in T \quad \textit{iff} \quad g_1 * g_2 = g_3\ ,$$

for all $g_1, g_2, g_3 \in G$. Then $Gra(\mathcal{G}) = Cm(T)$.

Proof. Notice first that $Gra(\mathcal{G})$ and $Cm(T)$ have the same underlying set and the same constants 0 and 1. Denote by $\oplus$, $\odot$, $;$, $^{\smile}$, Δ the operations from $Cm(T)$ and by $\oplus'$, $\odot'$, $;'$, $^{\smile\prime}$, Δ' the ones from $Gra(\mathcal{G})$. We need to show that they coincide. Let $P, Q \in [0,1]^G$ and $g \in G$.

$$P\,;Q(g) = \bigvee_{(g_1,g_2,g_3)\in T} P(g_1) \odot Q(g_2) = \bigvee_{g_1 * g_2 = g} P(g_1) * Q(g_2) = P\,;' Q(g)\ .$$

Now,

$$P^{\smile}(g) = \bigvee_{(g,h)\in S} P(h)\ .$$

But $(g,h) \in S$ means that for all $x, y \in G$,

$$[(g,x,y) \in T \text{ iff } (h,y,x) \in T] \;\text{ and }\; [(x,g,y) \in T \text{ iff } (y,h,x) \in T]\ ;$$

that is, for all $x, y \in G$,

$$[g * x = y \text{ iff } h * y = x] \;\text{ and }\; [x * g = y \text{ iff } y * h = x]\ .$$

The last is equivalent to $h = g^{-1}$. Thus,

$$P^{\smile}(g) = \bigvee_{(g,h)\in S} P(h) = P(g^{-1}) = P^{\smile\prime}(g) \ .$$

Finally,

$$g \in \Delta \quad \text{iff} \quad \forall (x, y \in G),\ [(g, x, y) \in T \text{ or } (x, g, y) \in T] \text{ implies } x = y \ ;$$

that is,

$$g \in \Delta \quad \text{iff} \quad \forall (x, y \in G),\ [g * x = y \text{ or } x * g = y] \text{ implies } x = y \ .$$

But this means $g \in \Delta$ iff $g = e$ (for the nontrivial implication, take $x = e$ and $y = g$ to get $e = g$). So $\Delta = \{e\} = \lambda_e = \Delta'$. □

Remark 5.8. We used very freely the identification of fuzzy sets (relations) with crisp ones, whenever their images were included in $\{0, 1\}$.

6. Concluding remarks

As already mentioned, complex algebras are a significant class of RAs because they provide a representation theorem — every RA is, up to an isomorphism, the subalgebra of a complex algebra. This theorem is obtained using the Stone Theorem for Boolean algebras and Proposition 5.2 that gives necessary and sufficient conditions for a ternary relation to induce a complex algebra. That the last proved to have a faithful fuzzy generalization to the MV case (Theorem 5.4 ((1) and (5)) might be a step towards a similar representation for MVRAs or DMVRAs (at least for the ones with semi-simple MV part).

The paper [6] also deals with a notion of fuzzy relation algebra and has as starting point Example 2.11, but with the Gödel t-norm. Fuzzy relation algebras are seen there as an amalgamation of two concepts: *fuzzy algebra* and *relation algebra*. While a fuzzy algebra is a complete distributive lattice equipped with a semi-scalar multiplication with scalars from $[0, 1]$, what kept our attention was the relation algebra part, which is a generalization of the RAs of Jónsson and Tarski; it is a complete distributive lattice that keeps some of the axioms from RAs (namely, the ones asserting that $;$, $^{\smile}$, Δ and 0 give an involutive monoid structure), replaces axiom (A2$'$) by the so called *Dedekind Formula* and states residuation for the relative composition $;$. In the case of MVRAs, because of the double negation property, residuation of $;$ is assured by the cycle law, so we do not need to state it explicitly — the two residua (left and right) of $;$ are in fact polynomially expressed here:

$$x \to y = \overline{\overline{x} \, ; y^{\smile}} \quad \text{and} \quad x \Rightarrow y = \overline{y^{\smile} \, ; \overline{x}} \ .$$

Another main difference between the relation algebras of [6] and the ones discussed in our paper, besides the choice of relational axioms, is that we work with a non-idempotent conjunction, $\odot$.

By abstractly capturing, besides the basic properties of collections of objects, also the interconnection between objects, RA's are a step beyond Boolean algebras in modeling the mathematical universe. And they are a sufficient step to take for dealing, in an elegant, algebraic approach, with foundational issues, as shown by [24], where there are given, in a RA style, formalizations of set theory and number theory that are equational and do not use any variables. On the other hand, there has been done some work lately on axiomatizing fuzzy set theory [8, 9, 23]. It might be the case that a similar approach as the one from [24], using a form of fuzzy (or many-valued) relation algebra, would be suitable for this.

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