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Standard topological algebras: syntactic and principal congruences and profiniteness

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In Celebration of the Sixtieth Birthday of Ralph N. McKenzie

Abstract. A topological quasi-variety $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ generated by a finite algebra $\underset{\sim}{\mathbf{M}}$ with the discrete topology is said to be *standard* if it admits a canonical axiomatic description. Drawing on the formal language notion of syntactic congruences, we prove that $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard provided that the algebraic quasi-variety generated by $\underset{\sim}{\mathbf{M}}$ is a variety, and that syntactic congruences in that variety are determined by a finite set of terms. We give equivalent semantic and syntactic conditions for a variety to have Finitely Determined Syntactic Congruences (FDSC), show that FDSC is equivalent to a natural generalisation of Definable Principle Congruences (DPC) which we call Term Finite Principle Congruences (TFPC), and exhibit many familiar algebras $\underset{\sim}{\mathbf{M}}$ that our method reveals to be standard. As an application of our results we show, for example, that every Boolean topological lattice belonging to a finitely generated variety of lattices is profinite and that every Boolean topological group, semigroup, and ring is profinite. While the latter results are well known, the result on lattices was previously known only in the distributive case.

1. Background, motivation and overview of results

An algebra $\underline{\mathbf{M}} = \langle M; F\rangle$ with finite underlying set M and operations F generates an (algebraic) *quasi-variety* $\mathfrak{Q}(\underline{\mathbf{M}}) := \mathbb{ISP}\,\underline{\mathbf{M}}$ consisting of all isomorphic copies of subalgebras of direct powers of $\underline{\mathbf{M}}$. Similarly a structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ with finite underlying set M, operations G, partial operations H, relations R and discrete topology $\mathcal{T}$ generates a *topological quasi-variety* $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ consisting of all isomorphic copies of topologically closed substructures of non-zero direct powers, with the product topology, of $\underset{\sim}{\mathbf{M}}$. Interest in topological quasi-varieties stems from the fact that they arise as the duals to algebraic quasi-varieties under natural dualities. The general theory of natural dualities provides methods to

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produce, from the algebra $\underline{\mathbf{M}}$, a structure $\underset{\sim}{\mathbf{M}}$ that will yield a natural duality on $\mathcal{Q}(\underline{\mathbf{M}})$. (See Clark and Davey [10].)

To maximise the usefulness of a natural duality, it is necessary to find a simple description of the members of the topological quasi-variety $\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Attempts to do this have drawn on Mal'cev's description of the members of a finitely generated algebraic quasi-variety $\mathcal{Q}(\underline{\mathbf{M}})$. Mal'cev [30] discovered that $\mathcal{Q}(\underline{\mathbf{M}})$ consists exactly of the models of the quasi-equations (equations and implications) that hold in $\underline{\mathbf{M}}$; in symbols,

$$\mathcal{Q}(\underline{\mathbf{M}}) = \mathrm{Mod}(\mathrm{Th}_{\mathrm{qe}}(\underline{\mathbf{M}})).$$

Often a small set $\Sigma \subseteq \mathrm{Th}_{\mathrm{qe}}(\underline{\mathbf{M}})$ of *axioms* for $\mathrm{Th}_{\mathrm{qe}}(\underline{\mathbf{M}})$ provides a simple description of the members of $\mathcal{Q}(\underline{\mathbf{M}})$ as $\mathcal{Q}(\underline{\mathbf{M}}) = \mathrm{Mod}(\Sigma)$.

The presence of partial operations, relations and topology in $\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ makes an analogous description considerably more elusive. By a *Boolean topological structure* (of type $\langle G, H, R\rangle$) we mean a structure $\underset{\sim}{\mathbf{X}} = \langle X; G^{\underset{\sim}{\mathbf{X}}}, H^{\underset{\sim}{\mathbf{X}}}, R^{\underset{\sim}{\mathbf{X}}}, \mathcal{T}^{\underset{\sim}{\mathbf{X}}}\rangle$ such that

(i) $\langle X; \mathcal{T}^{\underset{\sim}{\mathbf{X}}}\rangle$ is a Boolean (i.e., compact, totally disconnected) space,

(ii) if $h \in G \cup H$ is n–ary, then the domain $\mathrm{dom}(h^{\underset{\sim}{\mathbf{X}}})$ is a closed subset of X^n and $h^{\underset{\sim}{\mathbf{X}}}\colon \mathrm{dom}(h^{\underset{\sim}{\mathbf{X}}}) \to X$ is continuous, and

(iii) if $r \in R$ is n–ary, then $r^{\underset{\sim}{\mathbf{X}}}$ is a closed subset of X^n.

A *universal Horn formula* in the first-order language of $\underset{\sim}{\mathbf{M}}$ is an expression of one of the forms

$$\chi \quad \text{or} \quad \bigvee_{i\in I} \neg\psi_i \quad \text{or} \quad \bigwedge_{i\in I} \psi_i \Rightarrow \chi \qquad (*)$$

where χ and each ψ_i are atomic formulæ and I is a finite set. We denote by $\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})$ the set of all universal Horn formulæ that hold in $\underset{\sim}{\mathbf{M}}$. If Σ is a set of universal Horn formulæ, $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ denotes the class of all Boolean topological structures that satisfy each universal Horn formula in Σ. Necessary conditions for a structure $\underset{\sim}{\mathbf{X}}$ to be in $\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ are given by the Preservation Theorem (Clark and Davey [10, 1.4.3]), which states that

$$\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) \subseteq \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})).$$

In [11], Clark, Davey, Haviar, Pitkethly and Talukder define $\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ to be a *standard topological quasi-variety*, or $\underset{\sim}{\mathbf{M}}$ to be *standard*, if $\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is exactly the class of all Boolean topological models of the universal Horn theory of $\underset{\sim}{\mathbf{M}}$; in symbols,

$$\mathcal{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})).$$

This is not the case for every choice of $\underset{\sim}{\mathbf{M}}$. A classic example of Stralka [40] shows that the ordered space $\underset{\sim}{\mathbf{M}} = \langle\{0,1\}; \leq, \mathcal{T}\rangle$, which generates the category of Priestley spaces, is not standard.

A subset $\Sigma \subseteq \mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})$ is said to *axiomatise* $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ if $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\Sigma)$. In [11] we find many examples in which $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ has a simple finite axiomatisation, and is therefore standard. We also find that knowing *in advance* that $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard greatly simplifies the process of finding an axiomatisation by making it an essentially finitary process:

Proposition 1.1. ([11, 1.4]) *Assume that* $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ *is standard. Then* $\Sigma \subseteq \mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}})$ *axiomatises* $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ *provided every model of* Σ *is locally finite and each finite model of* Σ *is in* $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$.

In this paper we restrict our attention to the case in which $\underset{\sim}{\mathbf{M}}$ has neither partial operations nor relations. Thus we assume that $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ is a *Boolean topological algebra* with the discrete topology $\mathcal{T}$. Algebras $\underline{\mathbf{M}}$ whose generated quasi-variety admits a strong duality via a total algebra are described by the Two-for-One Strong Duality Theorem ([10, 3.3.2]). However, we note here that the theory of natural dualities will play no further role in this study. We mention it only to provide motivation for the investigation of the question as to when the topological quasi-variety $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard.

Serendipitously, it turns out that the notion of syntactic congruence from the theory of formal languages leads to a congruence condition that provides a simple criterion for a topological algebra $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ to be standard. This condition, called finitely determined syntactic congruences (FDSC), is introduced in the next section. We show (Lemma 2.3) that FDSC is equivalent to term finite principal congruences (TFPC), a congruence condition which is a natural generalization of definable principal congruences (Proposition 2.8). In Section 2 we also define semantic shadowing, which can be used to describe FDSC (Theorem 2.5), and use it to show that both the variety of monoids and any variety generated by a finite unary algebra have FDSC. In Section 3 we introduce the notion of syntactic shadowing and prove (Completeness Theorem 3.2) that, in an $\mathbb{ISP}$-closed class, it is equivalent to the semantic notion introduced earlier. In general, shadowing can be a complex process. Nevertheless, it turns out that in many cases a variant, known as simple shadowing, will suffice (Simple Shadowing Theorem 3.4).

Section 4 contains our main result on standardness. Let $\underset{\sim}{\mathbf{M}}$ be a finite topological algebra and let $\mathbf{M}$ be its underlying algebra. The FDSC-HSP Theorem 4.3 says that if $\mathbb{HSP}\,\mathbf{M}$ has FDSC and $\mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M}$, then $\underset{\sim}{\mathbf{M}}$ generates a standard topological quasi-variety. Thus two natural, purely algebraic conditions on $\mathbf{M}$ lead to a topological conclusion on $\underset{\sim}{\mathbf{M}}$. The proof of this theorem depends upon a fundamental lemma (the Clopen Equivalence Lemma 4.2) that has been been rediscovered several times, in varying levels of generality, since 1957.

Sections 5 and 6 present examples which can be shown to have FDSC via simple shadowing (for example, distributive lattices, vector spaces, Boolean algebras, groups, semigroups, rings) and via general shadowing (the primary example being any finitely generated variety of lattices). By applying the FDSC-HSP Theorem 4.3, these results yield a host of examples of finite topological algebras that generate standard topological quasi-varieties. Section 6 also gives an example of a four-element algebra that generates a variety with FDSC, proved via general shadowing, but for which simple shadowing will not suffice (Example 6.2).

Examples of varieties of algebras which do not have FDSC are given in Section 7. Our main proofs are based on the observation that, if a variety $\mathcal{V}$ contains an algebra which admits a compatible Boolean topology with respect to which it is not topologically residually finite, and in particular if $\mathcal{V}$ contains a non-residually finite algebra or infinite subdirectly irreducible algebra which admits a compatible Boolean topology, then $\mathcal{V}$ does not have FDSC (Lemma 7.1 and Theorem 7.3). For example, this is used to show that the varieties of Ockham algebras and modular lattices do not have FDSC. The section concludes with a discussion of the algebra $\mathbf{A}(\mathcal{T})$, built from a Turing machine $\mathcal{T}$, that played the fundamental role in the undecidability results in McKenzie [34]. We prove that, if the Turing machine $\mathcal{T}$ does not halt, then the variety generated by $\mathbf{A}(\mathcal{T})$ does not have FDSC (Example 7.7). Because of the importance of this example, we give—in addition to a topological proof—a purely algebraic proof of this fact. The theme of topological residual finiteness is continued in Section 8. One of the consequences of the main theorem of the section (Theorem 8.1) is that if $\mathcal{V}$ is a variety with FDSC and $\underset{\sim}{\mathbf{X}}$ is a Boolean topological algebra whose underlying algebra $\mathbf{X}$ is in $\mathcal{V}$, then $\underset{\sim}{\mathbf{X}}$ is profinite (that is, $\underset{\sim}{\mathbf{X}}$ is an inverse limit of finite topological algebras from $\mathcal{V}$). This yields a number of known results as special cases: groups, semigroups and rings, for example. Since we proved in Section 6 that every finitely generated variety of lattices had FDSC, it now follows that the Boolean topological lattices in every finitely generated variety of lattices are profinite. This was previously known only in the case of distributive lattices.

The paper concludes in Section 9 with a number of open problems.

2. Two congruence conditions: FDSC and TFPC

Given a finite alphabet A, let $\mathbf{A}^*$ denote the free monoid of words over A. Then a subset $L \subseteq A^*$ is called a *language* over A. An important class of languages, the *regular languages*, have a number of equivalent characterisations. Primarily, they are the languages accepted by finite state acceptors, the languages generated by regular grammars, and the languages denoted by regular expressions. One characterisation is purely algebraic. A language L over A determines an equivalence relation θ_L on A^* with the two classes, L and $A^*\backslash L$. Let $\mathrm{Syn}(L)$ denote the set of all $(a,b) \in A^* \times A^*$ such that $(cad, cbd) \in \theta_L$ for all $c, d \in A^*$. Then $\mathrm{Syn}(L)$ is a congruence on $\mathbf{A}^*$ called the *syntactic congruence* of L, and the language L is regular if and only if $\mathrm{Syn}(L)$ has finite index in $\mathbf{A}^*$. (See, for example, Eilenberg [19].)

The notion of syntactic congruence has a natural extension to arbitrary algebras. To see this, let $\mathbf{X} = \langle X; G^{\mathbf{X}} \rangle$ be an algebra and let θ be an equivalence relation on X. Let T_x be the set of all terms with operation symbols G in the countable sequence of variables $x, z_1, z_2, z_3, \dots$, and let F be a subset of T_x. We define an equivalence θ_F on X by $(a,b) \in \theta_F$ if and only if, for all $f(x, z_1, z_2, \dots) \in F$,

$$(f(a, c_1, c_2, \dots), f(b, c_1, c_2, \dots)) \in \theta \quad \text{for all} \quad c_1, c_2, \dots \in X. \tag{$*$}$$

Finally, we define $\mathrm{Syn}(\theta) = \theta_{T_x}$. The following facts are easy to check.

Lemma 2.1. *For every algebra* $\mathbf{X}$ *and every equivalence relation* θ *on* X*,*

(i) $\mathrm{Syn}(\theta)$ *is a congruence on* $\mathbf{X}$*,*
(ii) $\mathrm{Syn}(\theta) \subseteq \theta$ *(since* $x \in T_x$*), and*
(iii) $\psi \subseteq \mathrm{Syn}(\theta)$ *if* ψ *is a congruence on* $\mathbf{X}$ *and* $\psi \subseteq \theta$*.*

Thus $\mathrm{Syn}(\theta)$ *is the largest congruence on* $\mathbf{X}$ *that is contained in* θ*.*

We shall refer to $\mathrm{Syn}(\theta)$ as the *syntactic congruence* of θ. This formulation of syntactic congruences appears to have been observed by a number of authors, amongst the first being Słomiński [39] (in the context of formal language theory), Choe [9] and Day [17] (in the context of topological algebras). For a detailed discussion of the role of syntactic congruences in the study of pseudo-varieties, see Chapter 3 of Almeida [1].

Our present study begins with the observation that it is never necessary to test *all* terms $t \in T_x$ in order to establish membership in $\mathrm{Syn}(\theta)$. For example, if $\mathbf{X}$ is a ring and $(*)$ is true for $f = z_1 x + z_2$, then $(*)$ is certainly also true for $f = z_3 x + z_4$ and for $f = (z_2 - z_3 z_5)x$. Thus $\mathrm{Syn}(\theta)$ may be more efficiently defined as being θ_F for some proper subset $F \subseteq T_x$. We will say that a subset $F \subseteq T_x$ of terms *determines syntactic congruences* in a class $\mathcal{M}$ of algebras if $\mathrm{Syn}(\theta) = \theta_F$ for every $\mathbf{X} \in \mathcal{M}$ and every equivalence relation θ on X. In Section 4 we will see that $\mathrm{Syn}(\theta)$ will be useful to us provided that it is determined by some *finite* set of terms. Accordingly we say that a class $\mathcal{M}$ of algebras has *finitely determined syntactic congruences*, abbreviated *FDSC*, if there is a finite set $F \subseteq T_x$ that determines syntactic congruences in $\mathcal{M}$.

It turns out that the notion of FDSC can be formulated in a different way which is a natural generalisation of a familiar notion. Both formulations will prove useful in our subsequent work. Recall that, for an algebra $\mathbf{X}$ and $a, b, c, d \in X$, the pair (c, d) is in the principal congruence $\mathrm{Cg}^{\mathbf{X}}(a, b)$ generated by (a, b) if and only if there are terms $f_1, f_2, \dots, f_k \in T_x$ and elements $e_{i,j} \in X$ such that

$$\begin{aligned} c &= f_1(d_1, e_{1,1}, \dots, e_{1,m}), \\ f_1(d'_1, e_{1,1}, \dots, e_{1,m}) &= f_2(d_2, e_{2,1}, \dots, e_{2,m}), \\ f_2(d'_2, e_{2,1}, \dots, e_{2,m}) &= f_3(d_3, e_{3,1}, \dots, e_{3,m}), \\ &\vdots \\ f_k(d'_k, e_{k,1}, \dots, e_{k,m}) &= d, \end{aligned} \tag{1}$$

where $\{d_i, d'_i\} = \{a, b\}$ for $i = 1, 2, \dots, k$. Let $\mathrm{C}^{\mathbf{X}}_F(a, b)$ denote the equivalence on X consisting of the set of pairs $(c, d) \in X^2$ such that (1) holds for some choice of $f_1, f_2, \dots, f_k \in F$ and $e_{i,j} \in X$. Then $\mathrm{C}^{\mathbf{X}}_F(a, b) \subseteq \mathrm{Cg}^{\mathbf{X}}(a, b)$ for every $F \subseteq T_x$. We say that a subset $F \subseteq T_x$ of terms *determines principal congruences* in a class $\mathcal{M}$ of algebras if $\mathrm{C}^{\mathbf{X}}_F(a, b) = \mathrm{Cg}^{\mathbf{X}}(a, b)$ for all $\mathbf{X} \in \mathcal{M}$ and $a, b \in X$. We say that a class $\mathcal{M}$ has *term finite principal congruences*, abbreviated *TFPC*, if there is a finite set $F \subseteq T_x$ that determines principal congruences in $\mathcal{M}$.

Lemma 2.2. *For an algebra $\mathbf{X}$, an equivalence θ on X, and elements $a, b \in X$, we have $(a, b) \in \mathrm{Syn}(\theta)$ if and only if $\mathrm{Cg}^{\mathbf{X}}(a, b) \subseteq \theta$.*

Proof. If $(a, b) \in \mathrm{Syn}(\theta)$, then $\mathrm{Cg}^{\mathbf{X}}(a, b) \subseteq \mathrm{Syn}(\theta) \subseteq \theta$. Conversely, assume $\mathrm{Cg}^{\mathbf{X}}(a, b) \subseteq \theta$. Then we have $(f(a, c_1, c_2, \dots), f(b, c_1, c_2, \dots)) \in \mathrm{Cg}^{\mathbf{X}}(a, b) \subseteq \theta$, for all $f \in T_x$ and all $c_1, c_2, \dots \in X$, and therefore $(a, b) \in \mathrm{Syn}(\theta)$. □

Lemma 2.3. *Let $\mathbf{X}$ be an algebra and let $F \subseteq T_x$. Then F determines syntactic congruences on $\mathbf{X}$ if and only if F determines principal congruences on $\mathbf{X}$.*

Proof. Assume that F determines syntactic congruences on $\mathbf{X}$ and consider the equivalence $\theta := \mathrm{C}^{\mathbf{X}}_F(a, b)$. We have $(f(a, c_1, c_2, \dots), f(b, c_1, c_2, \dots)) \in \mathrm{C}^{\mathbf{X}}_F(a, b)$, for all $f \in F$ and $c_1, c_2, \dots \in X$, and therefore $(a, b) \in (\mathrm{C}^{\mathbf{X}}_F(a, b))_F = \mathrm{Syn}(\mathrm{C}^{\mathbf{X}}_F(a, b))$. By Lemma 2.2 we conclude that $\mathrm{Cg}^{\mathbf{X}}(a, b) \subseteq \mathrm{C}^{\mathbf{X}}_F(a, b)$ and therefore $\mathrm{Cg}^{\mathbf{X}}(a, b) = \mathrm{C}^{\mathbf{X}}_F(a, b)$.

Now assume that F determines principal congruences on $\mathbf{X}$ and let θ be an equivalence on X. We must prove that $\theta_F \subseteq \mathrm{Syn}(\theta)$ as the reverse inclusion always holds. Let $(a, b) \in \theta_F$. Then $(f(a, c_1, c_2, \dots), f(b, c_1, c_2, \dots)) \in \theta$ for all $f \in F$ and $c_1, c_2, \dots \in X$. It follows that $\mathrm{C}^{\mathbf{X}}_F(a, b) \subseteq \theta$. Hence $\mathrm{Cg}^{\mathbf{X}}_F(a, b) = \mathrm{C}^{\mathbf{X}}_F(a, b) \subseteq \theta$, whence $(a, b) \in \mathrm{Syn}(\theta)$ by Lemma 2.2. □

Corollary 2.4. *A class $\mathcal{M}$ of algebras has* FDSC *if and only if it has* TFPC.

In the remainder of this section we will develop a practical method to verify that a set of terms determines syntactic congruences in a class of algebras. For this purpose we fix a type G and a class $\mathcal{M}$ of algebras of type G. For a set $F \subseteq T_x$ of terms and a single term $t \in T_x$, we write

$$F \models t \text{ in } \mathcal{M} \quad (F \textit{ semantically shadows } t \text{ in } \mathcal{M})$$

if, for each $\mathbf{X} \in \mathcal{M}$ and each equivalence θ on X, we have $\theta_F \subseteq \theta_t$. Thus $F \models t$ in $\mathcal{M}$ if t is redundant in the presence of F for the determination of syntactic congruences on members of $\mathcal{M}$. We say that $F \models F'$ in $\mathcal{M}$ if $F \models t$ in $\mathcal{M}$ for each $t \in F'$. Equivalently, $F \models F'$ in $\mathcal{M}$ if $\theta_F \subseteq \theta_{F'}$ for every $\mathbf{X} \in \mathcal{M}$ and every equivalence θ on $\mathbf{X}$. If $t(x, z_1, z_2, \dots)$ does not depend upon x in $\mathcal{M}$, that is, if $\mathcal{M}$ satisfies the identity

$$t(x, z_1, z_2, \dots) \approx t(y, z_1, z_2, \dots)$$

(in particular, if t is a nullary term), then $\theta_t = X \times X$, for each $\mathbf{X} \in \mathcal{M}$, and hence $F \models t$, for every subset F of T_x. Notice that $\models$ is a transitive relation on the subsets of T_x.

The following theorem gives five criteria for a set F to determine syntactic congruences. Let T_{1x} denote the set of all terms in T_x that have exactly one occurrence of the variable x. For example, x occurs three times in the group term $z^{-1}x^3y$ but exactly once in $x^{-1}z^3y$. Criterion (ii) shows that (i) is equivalent to semantic shadowing. Criterion (iii), which implies that T_{1x} always determines syntactic congruences, will be needed later in our applications of the Shadowing Theorems 3.3 and 3.4. Criterion (iv) implies that shadowing can be established by checking finitely many instances in case F and G are both finite. The two examples that follow will illustrate criteria (iv) and (v). Note that $F \models x$ in $\mathcal{M}$ if and only if $\theta_F \subseteq \theta$ for every $\mathbf{X} \in \mathcal{M}$ and every equivalence θ on X, since we always have $\theta_x = \theta$.

Theorem 2.5. *For a set $F \subseteq T_x$ of terms and a class $\mathcal{M}$ of algebras, the following are equivalent:*

(i) *F determines syntactic congruences in $\mathcal{M}$,*
(ii) *$F \models T_x$ in $\mathcal{M}$,*
(iii) *$F \models T_{1x}$ in $\mathcal{M}$,*
(iv) *$F \models x$ and, for each $f(x, z_{m+1}, \dots, z_{m+n}) \in F$, each m–ary fundamental operation $g \in G$ and each i with $1 \le i \le m$, we have*

$$F \models f(g(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_m), z_{m+1}, \dots, z_{m+n}) \quad \textit{in } \mathcal{M},$$

(v) *$F \models x$ and θ_F is a congruence for each $\mathbf{X} \in \mathcal{M}$ and each equivalence θ on X.*

Proof. (i)⇒(ii) Let $t \in T_x$ and let θ be an equivalence on X where $\mathbf{X} \in \mathcal{M}$. Then $\theta_F = \mathrm{Syn}(\theta) = \theta_{T_x} \subseteq \theta_t$ so $F \models t$.

(ii)⇒(iv), trivially.

(iv)⇒(v) Let θ be an equivalence on X with $\mathbf{X} \in \mathcal{M}$. Let $(a, b) \in \theta_F$, let $g \in G$ be m–ary, let $1 \le i \le m$ and let $c_1, \dots, c_m \in X$. To see that θ_F is a congruence it suffices to show that

$$(g(c_1, \dots, c_{i-1}, a, c_{i+1}, \dots, c_m), g(c_1, \dots, c_{i-1}, b, c_{i+1}, \dots, c_m)) \in \theta_F.$$

Assume that $f(x, z_{m+1}, \dots, z_{m+n}) \in F$ and $c_{m+1}, \dots, c_{m+n} \in X$. We define

$$t(x, z_1, \dots, z_{m+n}) = f(g(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_m), z_{m+1}, \dots, z_{m+n}).$$

By (iv) we have $F \models t$, so $(a, b) \in \theta_t$. Thus,

$$(t(a, c_1, \dots, c_{n+m}), t(b, c_1, \dots, c_{n+m})) \in \theta,$$

and hence $(g(c_1, \dots, c_{i-1}, a, \dots, c_m), g(c_1, \dots, c_{i-1}, b, \dots, c_m)) \in \theta_F$, as required.

(v)⇒(i) Take $\mathbf{X} \in \mathcal{M}$ and let θ be an equivalence on X. Then $\theta_F \subseteq \theta_x = \theta$ so $\theta_F \subseteq \mathrm{Syn}(\theta)$, the largest congruence contained in θ, and $\mathrm{Syn}(\theta) = \theta_{T_x} \subseteq \theta_F$ since $F \subseteq T_x$. Thus $\theta_F = \mathrm{Syn}(\theta)$.

(ii)⇒(iii)⇒(i) The implication (ii)⇒(iii) is trivial, so assume (iii) to prove (i). Let $\mathbf{X} \in \mathcal{M}$ and let θ be an equivalence on X. By (iii) we have $\mathrm{Syn}(\theta) \subseteq \theta_F \subseteq \theta_{T_{1x}} \subseteq \theta$. Now it is straightforward to check that $\theta_{T_{1x}}$ is a congruence. By Lemma 2.1(iii) we have $\theta_{T_{1x}} \subseteq \mathrm{Syn}(\theta)$. Thus $\theta_F = \mathrm{Syn}(\theta)$, showing that F determines syntactic congruences in $\mathcal{M}$. □

Our first example shows that our notion of syntactic congruence is consistent with the formal language theoretic notion.

Example 2.6. The single term $F = \{z_1 x z_2\}$ determines syntactic congruences in the variety of monoids, which therefore has FDSC.

Proof. We verify Theorem 2.5(v). Let $\mathbf{X}$ be a monoid, let θ be an equivalence on X and assume $(a, b) \in \theta_F$. Then $(a, b) = (1a1, 1b1) \in \theta = \theta_x$, so $F \models x$. Now choose $c, d, e \in X$. Then

$$(d \cdot ac \cdot e, d \cdot bc \cdot e) = (d \cdot a \cdot ce, d \cdot b \cdot ce) \in \theta,$$

so $(ac, bc) \in \theta_F$. Similarly, $(ca, cb) \in \theta_F$, showing that θ_F is a congruence. □

The algebra $\mathbf{M}$ is *unary* if each of its operations is unary. Natural dualities given by unary topological algebras $\underset{\sim}{\mathbf{M}}$ are of particular interest since they tend to give accessible dual categories. Algebras $\underline{\mathbf{M}}$ strongly dualised by a unary topological algebra are characterised in [10, 6.4.5]. In [11] it is shown that every finite unary topological algebra with a single unary operation is standard. Our second example will be used in Example 5.8 to exhibit many more unary algebras that generate a standard topological quasi-variety. This example should be contrasted with Example 7.2.

Example 2.7. Every variety generated by a finite unary algebra $\mathbf{M} = \langle M; G\rangle$ has FDSC.

Proof. Since $\mathbf{M}$ is finite, there is a finite set F of unary terms in the variable x such that $x \in F$ and $\{ f^{\mathbf{M}} \mid f(x) \in F \}$ is the submonoid of M^M generated by G. We verify condition (iv) of Theorem 2.5. Let $f(x) \in F$ and $g \in G$. Then there is a term $t(x) \in F$ such that $f^{\mathbf{M}} g^{\mathbf{M}} = t^{\mathbf{M}}$, so $f(g(x)) \approx t(x)$ holds in $\mathbf{M}$. Thus $F \models f(g(x))$. □

Looking at TFPC instead of FDSC immediately yields many additional examples. Recall that a variety $\mathcal{V}$ is said to have *definable principal congruences* (*DPC*) if there is a fixed finite collection Π of finite sequences $(f_1, f_2, \ldots, f_k)$ from T_x such that, for $\mathbf{X} \in \mathcal{V}$ and $a, b, c, d \in X$, we have that $(c, d) \in \mathrm{Cg}^{\mathbf{X}}(a, b)$ if and only if equations (1) hold for some sequence in Π. If $\mathcal{V}$ has DPC, then principal congruences are *definable* in the sense that there is a first-order formula $\pi(x, y, u, v)$ such that, for $a, b, c, d \in \mathbf{X} \in \mathcal{V}$, the statement $\pi(a, b, c, d)$ is true in $\mathbf{X}$ if and only if $(c, d) \in \mathrm{Cg}^{\mathbf{X}}(a, b)$. A variety with DPC clearly has TFPC, as we see by taking as F all the terms in the sequences witnessing DPC. But TFPC imposes no limit on the length of the chains of those terms connecting c and d in equations (1).

Proposition 2.8. *If a variety has definable principal congruences* (DPC)*, then it also has term finite principal congruences* (TFPC) *and therefore finitely determined syntactic congruences* (FDSC)*.*

The notion of DPC has played an important role in the study of many varieties. (See Blok and Pigozzi [5], Burris and Lawrence [7] and Fried, Grätzer and Quackenbush [21]). The converse of Proposition 2.8 is not true; for example, in [7], Burris and Lawrence found finite groups and rings that generate varieties without DPC, whereas we shall see in Section 5 that all varieties of groups and rings have FDSC. We will present specific applications of Proposition 2.8 to standardness at the end of Section 4.

3. Syntactic shadowing and the Completeness Theorem

Condition (ii) of Theorem 2.5 suggests a quite different method for establishing that F determines syntactic congruences. We would like a purely syntactic procedure, that refers only to terms and not to models and congruences, to produce t from F when $F \models t$. A simple example suggests how this might be done. According to Example 2.6, the term $z_1 x z_2$ determines syntactic congruences in the variety of monoids. By Theorem 2.5(ii) it must be true that $z_1 x z_2$ semantically shadows, for example, the term x^3. To verify this fact directly, let θ be an equivalence on a monoid $\mathbf{X}$, let $a, b \in X$ with $(a, b) \in \theta_{z_1 x z_2}$. To see that $(a, b) \in \theta_{x^3}$, we have

$$
\begin{aligned}
a^3 &= 1 \cdot a \cdot aa \\
&\quad \theta\ 1 \cdot b \cdot aa = b \cdot a \cdot a \\
&\qquad\qquad \theta\ b \cdot b \cdot a = bb \cdot a \cdot 1 \\
&\qquad\qquad\qquad \theta\ bb \cdot b \cdot 1 = b^3,
\end{aligned}
$$

showing that $(a^3, b^3) \in \theta$. Notice that, starting from a^3, we move right using monoid identities and we move down using the fact that $(a, b) \in \theta_{z_1xz_2}$. Based on this and many other examples, we propose the following definition relative to the fixed type G and class $\mathcal{M}$ of algebras of type G. First, define T_{xy} be the set of all terms with operation symbols G in the countable sequence of variables $x, y, z_1, z_2, z_3, \ldots$. For a set $F \subseteq T_x$ of terms and a single term $t \in T_x$, we write

$$F \vdash t \text{ in } \mathcal{M} \quad (F \textit{ syntactically shadows } t \text{ in } \mathcal{M})$$

provided that there exists $k \geq 0$, terms $f_1(x, z_1, \ldots, z_m), \ldots, f_k(x, z_1, \ldots, z_m)$ in F and mk terms $w_{i,j}(x, y, z_1, \ldots, z_n) \in T_{xy}$, for $1 \leq i \leq k$ and $1 \leq j \leq m$, such that each of the following identities is satisfied by each member of $\mathcal{M}$:

$$
\begin{aligned}
t(x, z_1, \ldots, z_n) &\approx f_1(v_1, w_{1,1}, \ldots, w_{1,m}), \\
f_i(v_i', w_{i,1}, \ldots, w_{i,m}) &\approx f_{i+1}(v_{i+1}, w_{i+1,1}, \ldots, w_{i+1,m}), \\
f_k(v_k', w_{k,1}, \ldots, w_{k,m}) &\approx t(y, z_1, \ldots, z_n),
\end{aligned}
$$

where $\{v_i, v_i'\} = \{x, y\}$. (When $k = 0$, this family of identities reduces to the single identity

$$t(x, z_1, \ldots, z_n) \approx t(y, z_1, \ldots, z_n),$$

that is, to the statement that, in $\mathcal{M}$, the term $t(x, z_1, \ldots, z_n)$ does not depend upon the variable x.) For example, z_1xz_2 syntactically shadows x^3 in the variety of monoids as we have

$$
\begin{aligned}
x^3 &\approx 1 \cdot x \cdot xx, \\
&\quad 1 \cdot y \cdot xx \approx y \cdot x \cdot x, \\
&\qquad\qquad y \cdot y \cdot x \approx yy \cdot x \cdot 1, \\
&\qquad\qquad\qquad yy \cdot y \cdot 1 \approx y^3.
\end{aligned}
$$

In order to justify our definition of syntactic shadowing, we need to know that F syntactically shadows only what it should and everything that it should.

Soundness Theorem 3.1. If $F \vdash t$ in $\mathcal{M}$, then $F \models t$ in $\mathcal{M}$ for every class $\mathcal{M}$ of algebras.

Proof. Assume that $F \vdash t$ as described above. Let θ be an equivalence on an algebra $\mathbf{X} \in \mathcal{M}$ and let $a, b \in X$ with $(a, b) \in \theta_F$. To see that $(a, b) \in \theta_t$, choose $c_1, \ldots, c_n \in X$. For $1 \leq i \leq k$ and $1 \leq j \leq m$, let $e_{i,j} := w_{i,j}(a, b, c_1, \ldots, c_n)$. Then we have

$$\begin{aligned} t(a, c_1, \dots, c_n) &= f_1(d_1, e_{1,1}, \dots, e_{1,m}) \\ &\theta \quad f_1(d_1', e_{1,1}, \dots, e_{1,m}) = f_2(d_2, e_{2,1}, \dots, e_{2,m}) \\ &\qquad \theta \quad f_2(d_2', e_{2,2}, \dots, e_{2,m}) = f_3(d_3, e_{3,1}, \dots, e_{3,m}) \\ &\qquad\qquad \vdots \\ &\qquad\qquad \theta \quad f_k(d_k', e_{k,1}, \dots, e_{k,m}) = t(b, c_1, \dots, c_n), \end{aligned}$$

where $(d_i, d_i') = (a, b)$ if $(v_i, v_i') = (x, y)$ and $(d_i, d_i') = (b, a)$ if $(v_i, v_i') = (y, x)$ for $i = 1, 2, \dots, k$. Thus $(t(a, c_1, \dots, c_n), t(b, c_1, \dots, c_n)) \in \theta$ and hence $(a, b) \in \theta_t$. □

Completeness Theorem 3.2. Assume that $\mathcal{M}$ contains a free countably infinitely generated algebra. If $F \models t$ in $\mathcal{M}$, then $F \vdash t$ in $\mathcal{M}$.

Proof. Assume $F \models t$ in $\mathcal{M}$. Let $\mathbf{F}_{xy}$ be the free countably infinitely generated algebra in $\mathcal{M}$ with its generators labelled $\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \dots$. Let S be the set of all

$$(f(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \dots), f(\mathbf{y}, \mathbf{u}_1, \mathbf{u}_2, \dots)) \quad \text{and}$$

$$(f(\mathbf{y}, \mathbf{u}_1, \mathbf{u}_2, \dots), f(\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \dots)),$$

where $f(x, z_1, z_2, \dots) \in F$ and $\mathbf{u}_1, \mathbf{u}_2, \dots \in F_{xy}$, together with the identity relation on F_{xy}. Then the transitive closure of S is an equivalence relation θ on F_{xy} for which we have $(\mathbf{x}, \mathbf{y}) \in \theta_F$. Consequently $(\mathbf{x}, \mathbf{y}) \in \theta_t$. In particular,

$$(t(\mathbf{x}, \mathbf{z}_1, \mathbf{z}_2, \dots), t(\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \dots)) \in \theta.$$

This says that $t(\mathbf{x}, \mathbf{z}_1, \mathbf{z}_2, \dots)$ and $t(\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \dots)$ are connected via the transitive closure of S. Hence there exists $k \geq 0$, terms $f_i(y, z_1, z_2, \dots)$ in F, for $i = 1, \dots, k$, choices of v_i, v_i' so that $\{v_i, v_i'\} = \{x, y\}$ and terms $w_{i,j} \in T_{xy}$, for $1 \leq i \leq k$ and $1 \leq j \leq m$, such that

$$\begin{aligned} t(\mathbf{x}, \mathbf{z}_1, \mathbf{z}_2, \dots) &= f_1(\mathbf{v}_1, w_{1,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), w_{1,2}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), \dots), \\ f_i(\mathbf{v}_i', w_{i,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), w_{i,2}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), \dots) &= \\ &f_{i+1}(\mathbf{v}_{i+1}, w_{i+1,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), w_{i+1,2}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), \dots), \\ f_k(\mathbf{v}_k', w_{k,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), w_{k,2}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots), \dots) &= t(\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \dots). \end{aligned}$$

Since $\mathbf{F}_{xy}$ is $\mathcal{M}$-freely generated by $\{\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \dots\}$, we see that the equations

$$\begin{aligned} t(x, z_1, z_2, \dots) &\approx f_1(v_1, w_{1,1}, w_{1,2}, \dots), \\ f_i(v_i', w_{i,1}, w_{i,2}, \dots) &\approx f_{i+1}(v_{i+1}, w_{i+1,1}, w_{i+1,2}, \dots), \\ f_k(v_k', w_{k,1}, w_{k,2}, \dots) &\approx t(y, z_1, z_2, \dots) \end{aligned}$$

all hold in $\mathcal{M}$. Thus $F \vdash t$ in $\mathcal{M}$. □

Together these two theorems tell us that in an $\mathbb{ISP}$-closed class, and in particular in a *variety* $\mathcal{M}$, the two notions of shadowing are identical. Thus we say that F *shadows* t in an $\mathbb{ISP}$-closed class $\mathcal{M}$ if $F \models t$ and/or $F \vdash t$ in $\mathcal{M}$, and F *shadows* F' in $\mathcal{M}$ provided that F shadows t in $\mathcal{M}$ for each $t \in F'$. It will be useful to reiterate the equivalence of (i) and (iii) of Theorem 2.5 in this context.

General Shadowing Theorem 3.3. A subset $F \subseteq T_x$ determines syntactic congruences in an $\mathbb{ISP}$-closed class $\mathcal{M}$ of algebras if and only if F shadows T_{1x} in $\mathcal{M}$.

It turns out that a very simple version of shadowing is sufficient for a large number of applications. Relative to a class $\mathcal{M}$ of algebras, we say that F *simply shadows* a term $t \in T_x$ if there are terms $w_1, w_2, w_3, \dots$ in variables $\{z_1, z_2, z_3, \dots\}$, and a term $f(x, z_1, z_2, \dots) \in F$ and such that the identity

$$t(x, z_1, z_2, \dots) \approx f(x, w_1, w_2, \dots) \tag{$*$}$$

holds in $\mathcal{M}$. We say that F *simply shadows* F' in $\mathcal{M}$ if F simply shadows each term of F' in $\mathcal{M}$.

Simple Shadowing Theorem 3.4. A subset $F \subseteq T_x$ determines syntactic congruences in a class $\mathcal{M}$ of algebras provided that F simply shadows T_{1x} in $\mathcal{M}$.

Proof. Let $t \in T_{1x}$. Since F simply shadows t, there are terms $w_1, w_2, w_3, \dots$ in variables $z_1, z_2, z_3, \dots$ and a term $f(x, z_1, z_2, \dots)$ in F such that the identity $(*)$ holds in $\mathcal{M}$. Taking $k = 1$, $f_1 = f$, $v_1 = x$, ${v_1}' = y$, we obtain $F \vdash t$. Thus $F \vdash T_{1x}$ and so $F \models T_{1x}$ in $\mathcal{M}$. The conclusion follows from (iii)→(i) of Theorem 2.5. □

The reader can check that Examples 2.6 and 2.7 can both be established as immediate consequences of the Simple Shadowing Theorem. Note that an easy inductive argument shows that if Theorem 2.5(iv) holds via simple shadowing, for some set $F \subseteq T_x$, then F simply shadows all terms in T_{1x}.

A congruence coinciding with the syntactic congruence is presented in Johnstone [28] for much the same purpose as we have it here. A form of shadowing for a set of words is also given there ('completeness' of sets of words [28, Definition 2.8]). Johnstone's notion is in fact equivalent to the *simple shadowing* version of condition (iv) in Theorem 2.5. There are many instances where simple shadowing is properly weaker than general shadowing (see Examples 6.1 and 6.2) and so 'complete' in the sense of [28, Definition 2.8] should not be confused with that in the Completeness Theorem 3.2.

Since T_{1x} determines syntactic congruences in any class $\mathcal{M}$ of algebras, we know that $T_{1x} \models t$ in $\mathcal{M}$, for all $t \in T_x$. The following lemma is a local version of the corresponding result for syntactic shadowing. This, along with the Soundness Theorem 3.1, provides a direct proof of (iii)→(ii) in Theorem 2.5. If we replace some or all occurrences of variables in a term t by other variables, then we say that the resulting term t' is obtained from t by *variable replacement*. For example, the term $t' = ((x \wedge z_1) \vee (z_2 \wedge z_3)) \wedge z_4$ may be obtained by variable replacement from $t = ((y \wedge z_2) \vee (x \wedge z_3)) \wedge x$. The significance of variable replacement is that it does not alter the shape and therefore the complexity of a term.

Lemma 3.5. *Let t be a term in which the variable x occurs $k > 0$ times. Then there are terms $t_1, \dots, t_k \in T_{1x}$ such that each t_i is obtained from t by variable replacement and $\{t_1, \dots, t_k\} \vdash t$.*

Proof. Let t' be obtained from t be replacing the k occurrences of x with new variables $x_1, \dots, x_k$. Let t_i be obtained from t' by replacing x_i with x. It is straightforward to show that $\{t_1, \dots, t_k\} \vdash t$. □

4. FDSC-HSP Theorem

We will now apply the work of the previous two sections to give an efficient method to verify that a topological quasi-variety of algebras is standard. Given a topological algebra $\underset{\sim}{\mathbf{X}} := \langle X; G, \mathcal{T} \rangle$, we shall denote the underlying algebra of $\underset{\sim}{\mathbf{X}}$ by $\mathbf{X} := \langle X; G \rangle$. Our central result, the FDSC-HSP Theorem, will say that a finite topological algebra $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T} \rangle$ generates a standard topological quasi-variety provided that the algebraic quasi-variety generated by $\mathbf{M}$ is a variety and that this variety has FDSC. We use the basic characterisation of topological quasi-varieties, which we restate here for total algebras.

Separation Theorem 4.1. ([10, 1.4.4]) Let $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T} \rangle$ be a finite discrete topological algebra, let $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$, and let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra of the same type as $\underset{\sim}{\mathbf{M}}$. Then $\underset{\sim}{\mathbf{X}} \in \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ if and only if either $\underset{\sim}{\mathbf{X}}$ has only one element and $\underset{\sim}{\mathbf{M}}$ has a one-element subalgebra, or $\underset{\sim}{\mathbf{X}}$ has more than one element and for all $a, b \in X$ with $a \neq b$, there is a continuous homomorphism $\alpha\colon \underset{\sim}{\mathbf{X}} \to \underset{\sim}{\mathbf{M}}$ such that $\alpha(a) \neq \alpha(b)$.

Let $\underset{\sim}{\mathbf{X}} = \langle X; G, \mathcal{T} \rangle$ be a Boolean topological algebra and let θ be an equivalence relation on X. We say that θ is a *clopen equivalence relation* on $\underset{\sim}{\mathbf{X}}$ if each θ class is clopen; equivalently, if θ is a clopen subset of $X \times X$. It is the application of the following fundamental lemma that leads us to require syntactic congruences be finitely determined. The history of this lemma goes back at least to Numakura [38], who proved that if θ is a clopen partition on a Boolean topological semigroup, then the syntactic congruence of θ is also clopen. The first proof in the general setting appears to be that of Day [17] (see also a more detailed presentation in Johnstone [28, Lemma VI.2.7]). Because this result is so basic to the concerns of this paper, we present its proof in full.

Clopen Equivalence Lemma 4.2. Let $\underset{\sim}{\mathbf{X}} = \langle X; G, \mathcal{T} \rangle$ be a Boolean topological algebra and let θ be a clopen equivalence relation on X. If $F \subseteq T_x$ is a finite set of terms of $\mathbf{X}$, then θ_F is also a clopen equivalence relation on X.

Proof. Let $a \in X$. Because the equivalence classes of θ_F form a cover for $\underset{\sim}{\mathbf{X}}$, it suffices (by compactness) to show that a/θ_F is open. We construct a clopen subset W of a/θ_F containing a. Fix a term $f(x, z_1, \ldots, z_n) \in F$ and $\mathbf{c} := (c_1, \ldots, c_n) \in X^n$. Since f is continuous at $(a, \mathbf{c})$ and $f(a, \mathbf{c})$ is in the clopen set $f(a, \mathbf{c})/\theta$, there are clopen sets $U_{\mathbf{c}} \subseteq X$ containing a and $V_{\mathbf{c}} \subseteq X^n$ containing $\mathbf{c}$ such that $f(U_{\mathbf{c}}, V_{\mathbf{c}}) \subseteq f(a, \mathbf{c})/\theta$. Since $\underset{\sim}{\mathbf{X}}^n$ is compact, there is a finite set $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\} \subseteq X^n$ such that $V_{\mathbf{c}_1}, V_{\mathbf{c}_2}, \ldots, V_{\mathbf{c}_k}$ cover X^n. Define

$$W_f := U_{\mathbf{c}_1} \cap \cdots \cap U_{\mathbf{c}_k}.$$

Then W_f is a clopen subset of X containing a. We now define

$$W := \bigcap \{\, W_f \mid f(x, z_1, \ldots, z_n) \in F \,\}.$$

Then W contains a, and W is clopen since F is finite.

To see that $W \subseteq a/\theta_F$, choose $b \in W$. Let $f(x, z_1, \ldots, z_n) \in F$ and $\mathbf{c} \in X^n$. Then $\mathbf{c} \in V_{\mathbf{c}_j}$ for some $j \leq k$. Since $b \in W \subseteq W_f \subseteq U_{\mathbf{c}_j}$, we have $f(b, \mathbf{c}) \in f(a, \mathbf{c}_j)/\theta$. Since $a \in U_{\mathbf{c}_j}$, we also have $f(a, \mathbf{c}) \in f(a, \mathbf{c}_j)/\theta$. Consequently $(f(a, \mathbf{c}), f(b, \mathbf{c})) \in \theta$. Since $f(x, z_1, \ldots, z_n) \in F$ and $\mathbf{c} \in X^n$ were arbitrary, we conclude that $(a, b) \in \theta_F$. Thus $W \subseteq a/\theta_F$. □

We can now put the pieces together, giving a general condition which shows how FDSC leads to standardness.

FDSC-HSP Theorem 4.3. Let $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T} \rangle$ be a finite discrete topological algebra. If $\mathbb{HSP}\,\mathbf{M}$ has FDSC, and $\mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M}$, then $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ is standard.

Proof. Let $\underset{\sim}{\mathbf{X}} \in \operatorname{Mod}_{\mathcal{T}}(\operatorname{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$. To show that $\underset{\sim}{\mathbf{X}} \in \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$, we consider two cases. First assume that $\underset{\sim}{\mathbf{X}}$ is a one-element algebra. In this case, to show that $\underset{\sim}{\mathbf{X}} \in \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ it suffices to show that $\underset{\sim}{\mathbf{M}}$ has a one-element subalgebra. Suppose that $\underset{\sim}{\mathbf{M}}$ has no one-element subalgebra. Then, for all $a \in M$, there exists $g_a \in G$ such that $g_a(a, \ldots, a) \neq a$. Thus

$$\underset{\sim}{\mathbf{M}} \models \bigvee_{a \in M} g_a(x, \ldots, x) \not\approx x$$

and hence $\operatorname{Mod}_{\mathcal{T}}(\operatorname{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$ contains no one-element algebras. Since $|X| = 1$ and $\underset{\sim}{\mathbf{X}}$ belongs to $\operatorname{Mod}_{\mathcal{T}}(\operatorname{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$, it follows that $\underset{\sim}{\mathbf{M}}$ has a one-element subalgebra, as required.

Now assume that $\underset{\sim}{\mathbf{X}}$ has more than one element. We apply the Separation Theorem 4.1. Let $a, b \in X$ with $a \neq b$. Let Y be a clopen subset of X containing a

but not b, and let θ be the equivalence relation with two classes Y and $X\backslash Y$. Choose a finite set F of terms that determine syntactic congruences in $\mathbb{HSP}\,\mathbf{M}$. By the Clopen Equivalence Lemma, each θ_F class is clopen. If we give $\mathbf{X}/\theta_F$ the discrete topology, then the quotient homomorphism $\alpha\colon \underset{\sim}{\mathbf{X}} \to \underset{\sim}{\mathbf{X}}/\theta_F$ is continuous and separates a and b since $\theta_F \subseteq \theta$. Since $\underset{\sim}{\mathbf{X}} \in \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{uH}}(\underset{\sim}{\mathbf{M}}))$, we have $\mathbf{X} \in \mathrm{Mod}(\mathrm{Th}_{\mathrm{eq}}(\mathbf{M})) = \mathbb{HSP}\,\mathbf{M}$, and hence $\mathbf{X}/\theta_F \in \mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M}$. Since $\alpha(a) \neq \alpha(b)$ in $\mathbf{X}/\theta_F$, there is a homomorphism $\beta\colon \mathbf{X}/\theta_F \to \mathbf{M}$ that separates $\alpha(a)$ and $\alpha(b)$. As $\underset{\sim}{\mathbf{X}}/\theta$ and $\underset{\sim}{\mathbf{M}}$ are both discrete, β is continuous. Thus the map $\beta \circ \alpha\colon \underset{\sim}{\mathbf{X}} \to \underset{\sim}{\mathbf{M}}$ is a continuous homomorphism that separates a and b. □

Corollary 4.4. *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ *be a finite discrete topological algebra and assume that* $\mathbb{ISP}\,\mathbf{M} = \mathbb{HSP}\,\mathbf{M}$*. If* $\mathbb{HSP}\,\mathbf{M}$ *has* DPC*, then* $\mathfrak{Q}^{+}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$ *is standard.*

This corollary yields many examples of finite topological algebras that generate standard topological quasi-varieties. For example, the variety $\mathfrak{I}$ of *implication algebras* $\mathbf{I} = \langle I; \to, 1\rangle$ has DPC (see [5]). Since $\mathfrak{I} = \mathbb{ISP}\,\underline{\mathbf{2}}$ (see [36, 16]), we conclude that $\underset{\sim}{\mathbf{2}} = \langle 2; \to, 1, \mathcal{T}\rangle$ is standard. McKenzie [31] found that every variety that is directly representable by a finite set of finite algebras has DPC. This fact applies to the variety generated by any *para primal algebra* (Clark and Krauss [12]). While not all para primal algebras generate a quasi-variety that is a variety, a method to recognise those that do is given in [12]. Among those that do are all quasi-primal algebras, as we will see directly in Example 6.3.

5. Examples of simple shadowing

We will now exhibit a large assortment of varieties of algebras that can be shown to have FDSC using the Simple Shadowing Theorem 3.4. Many of our examples generate a quasi-variety that is also a variety. The FDSC-HSP Theorem tells us that each of these examples, when endowed with the discrete topology, generates a standard topological quasi-variety.

The condition of the Simple Shadowing Theorem 3.4 says, loosely speaking, that F determines syntactic congruences on $\mathfrak{M}$ if every term with a single x can be rewritten, modulo the equational theory of $\mathfrak{M}$, in the form of a member of F. For example, let $\mathfrak{M}$ be the variety of distributive lattices $\mathbf{L} = \langle L; \vee, \wedge\rangle$ and define

$$F := \{x,\, x \vee z_2,\, z_1 \wedge x,\, (z_1 \wedge x) \vee z_2\}.$$

If $t(x, z_1, z_2, \dots) \in T_{1x}$, then it can be rewritten as $(w_1 \wedge x) \vee w_2$, or as $x \vee w_2$ or $w_1 \wedge x$ or x (if either w_1 or w_2 or both are missing) where w_1 and w_2 are terms in $z_1, z_2, \dots$.

Example 5.1. Let $\mathcal{D}$ be the category of Boolean topological distributive lattices, that is, Boolean spaces with continuous distributive lattice operations and let $\underset{\sim}{\mathbf{M}} = \langle M; \vee, \wedge, \mathcal{T} \rangle$ be any non-trivial finite member of $\mathcal{D}$. Let $\mathcal{M}$ be the variety of *distributive lattices*.

(i) $F = \{x,\, x \vee z_2,\, z_1 \wedge x,\, (z_1 \wedge x) \vee z_2\}$ determines syntactic congruences in $\mathcal{M}$.
(ii) $\mathcal{D} = \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard.

Proof. The Simple Shadowing Theorem 3.4 implies (i) by the argument above. Since $\mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M} = \mathcal{M}$, the FDSC-HSP Theorem tell us that $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard. We know that $\mathrm{Th}_{\mathrm{qe}}(\underset{\sim}{\mathbf{M}})$ is the quasi-equational theory of distributive lattices. Since $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard, we have $\mathcal{D} := \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{\mathrm{qe}}(\underset{\sim}{\mathbf{M}})) = \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. □

For *bounded distributive lattices* $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$, a similar assertion is true: the first three terms of F are now redundant in the presence of the fourth. This illustrates a recurrent theme. Having distinguished identity elements for binary operations generally serves to reduce the number of terms needed to define syntactic congruences. In fact, the term $(z_1 \wedge x) \vee z_2$ determines syntactic congruences even in the unbounded case by using a slightly longer calculation—see Lemma 6.4.

We give several more examples whose proofs, which we omit, are easy and similar to the above argument for distributive lattices.

Example 5.2. Let K be a finite field. Let $\mathcal{V}$ be the category of Boolean topological vector spaces over K, that is, Boolean spaces with continuous vector space operations, and let $\underset{\sim}{\mathbf{M}} = \langle M; +, \alpha, 0, \mathcal{T}\rangle_{\alpha \in K}$ be any non-trivial finite member of $\mathcal{V}$. Let $\mathcal{M}$ be the variety of *vector spaces* $\mathbf{V} = \langle V; +, \alpha, 0\rangle_{\alpha\in K}$ over K.

(i) $F = \{\, z_1 + \alpha x \mid \alpha \in K \,\}$ determines syntactic congruences in $\mathcal{M}$.
(ii) $\mathcal{V} = \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard.

Example 5.3. Let $\mathcal{B}$ be the category of non-trivial Boolean topological Boolean algebras, that is, Boolean spaces with continuous Boolean algebra operations, and let $\underset{\sim}{\mathbf{M}} = \langle M; \vee, \wedge, ', 0, 1, \mathcal{T}\rangle$ be any non-trivial finite member of $\mathcal{B}$. Let $\mathcal{M}$ be the variety of *Boolean algebras* $\mathbf{B} = \langle B; \vee, \wedge, ', 0, 1\rangle$.

(i) $F = \{(z_1 \wedge x) \vee z_2,\, (z_1 \wedge x') \vee z_2\}$ determines syntactic congruences in $\mathcal{M}$.
(ii) $\mathcal{B} = \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard.

Example 5.4. Let $\mathcal{A}$ be the category of Boolean topological abelian groups, that is, Boolean spaces with continuous abelian group operations. Let $\underset{\sim}{\mathbf{M}} = \langle M; +, -, 0, \mathcal{T}\rangle$ be any non-trivial finite member of $\mathcal{A}$ and let $\mathcal{M}$ be the variety of *abelian groups* $\mathbf{A} = \langle A; +, -, 0\rangle$.

(i) $F = \{z_1 + x,\, z_1 - x\}$ determines syntactic congruences in $\mathcal{M}$.
(ii) $\mathcal{A} = \mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard.

Part (ii) of the previous three examples was established in Davey and Werner [16], where it is shown that $\boldsymbol{\mathcal{V}}$ is dual to the variety of all vector spaces over K, that $\boldsymbol{\mathcal{B}}$ is dual to the variety of non-empty sets, and that $\boldsymbol{\mathcal{A}}$ is dual to the variety of all abelian groups satisfying $mx \approx 0$ for some integer m.

We exhibit several other familiar varieties that can be easily shown to have FDSC using the Simple Shadowing Theorem 3.4. In these examples it is not true that every finite member generates a quasi-variety that is a variety, but many finite members do. In those cases we can apply the FDSC-HSP Theorem to show that they generate a standard topological quasi-variety.

Example 5.5. Syntactic congruences in the variety of *groups* $\mathbf{G} = \langle G; \cdot, {}^{-1}, 1\rangle$ are determined by the finite set

$$F = \{z_1 x z_2, z_1 x^{-1} z_2\}.$$

Example 5.6. Syntactic congruences in the variety of *semigroups* $\mathbf{S} = \langle S; \cdot \rangle$ are determined by the finite set

$$F = \{x,\ z_1 x,\ x z_2,\ z_1 x z_2\}.$$

Comparing semigroups and monoids, we again see the need for additional terms when the identity element is missing.

Example 5.7. Syntactic congruences in the variety of *rings* $\mathbf{R} = \langle R; +, \cdot, -, 0\rangle$ are determined by the finite set

$$F = \{z_1 + x,\ z_1 - x,\ z_1 + z_2 x,\ z_1 + x z_3,\ z_1 + z_2 x z_3\}.$$

From Example 2.7 we know that every finite unary algebra generates a variety with FDSC. The following observation, kindly contributed to this effort by Jane Pitkethly, exhibits many finite unary algebras that generate a quasi-variety that is a variety.

Example 5.8. Let $\mathbf{M}$ be a finite unary algebra that has at most one constant term function. Then there is a finite unary algebra $\mathbf{M}'$ such that $\mathbb{HSP}\,\mathbf{M} = \mathbb{HSP}\,\mathbf{M}' = \mathbb{ISP}\,\mathbf{M}'$, and consequently $\underset{\sim}{\mathbf{M}}'$ generates a standard topological quasi-variety.

Proof. Higgs [25] gave a short proof that the variety generated by a finite unary algebra has only finitely many isomorphism types of subdirectly irreducibles, all of which are finite. Let S be a set of disjoint representatives of these types. We take $\mathbf{M}'$ to be the coproduct of S, that is, $\mathbf{M}'$ is the union of S if $\mathbf{M}$ has no constant term function and is the union of S with the values of the constant term function amalgamated otherwise. Because of the restriction on constant term functions, $\mathbf{M}'$ must satisfy every equation satisfied by $\mathbf{M}$. Thus $\mathbf{M}$ and $\mathbf{M}'$ generate the same variety, which is also $\mathbb{ISP}\,\mathbf{M}'$. □

Recall that an *Ockham algebra* (see Blyth and Varlet [6]) is an algebra $\mathbf{O} = \langle O; \vee, \wedge, c, 0, 1\rangle$ where $\langle O; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the identities

$$c(0) \approx 1, \quad c(1) \approx 0, \quad c(x \vee y) \approx c(x) \wedge c(y), \quad c(x \wedge y) \approx c(x) \vee c(y)$$

are satisfied. The variety of Ockham algebras includes Boolean algebras, Kleene algebras and De Morgan algebras as subvarieties.

Example 5.9. The variety generated by a finite Ockham algebra has FDSC.

Proof. Let $\mathbf{O} = \langle O; \vee, \wedge, c, 0, 1\rangle$ be a finite Ockham algebra. Since $\mathbf{O}$ is finite, there is a positive integer m such that, for all $n > m$ there is an $i \leq m$ with $c^n(x) \approx c^i(x)$ true in $\mathbf{O}$. Define

$$F := \left\{\, f_i(x, z_1, z_2) := (z_1 \wedge c^i(x)) \vee z_2 \;\middle|\; i \leq m \,\right\}.$$

We use the Simple Shadowing Theorem to show that F determines syntactic congruences in $\mathbb{HSP}\,\mathbf{O}$. If $t(x, z_1, z_2, \dots) \in T_{1x}$, we can find terms $w_1(z_1, z_2, \dots)$ and $w_2(z_1, z_2, \dots)$ and a non-negative integer n such that $t \approx (w_1 \wedge c^n(x)) \vee w_2$ holds for all Ockham algebras. Let $i \leq m$ be such that $c^n(x) \approx c^i(x)$ holds in $\mathbf{O}$. Then $\mathbf{O}$ satisfies the identity $t(x, z_1, z_2, \dots) \approx f_i(x, w_1, w_2)$. □

Goldberg [22] gives an elementary way of deciding if the quasi-variety generated by a finite subdirectly irreducible Ockham algebra is a variety. If it is, we can conclude that the finite Ockham algebra generates a standard topological quasi-variety.

6. Examples of general shadowing

The reader may well wonder if every instance of shadowing in familiar varieties can be achieved by simple shadowing. In this section we present examples of varieties that we prove to have FDSC using applications of general shadowing. We begin with a familiar variety in which it is easy to see that simple shadowing is not adequate.

Example 6.1. Syntactic congruences in the variety of *groups* $\mathbf{G} = \langle G; \cdot, ^{-1}, 1\rangle$ are determined by the finite set

$$F = \{z_1 x z_2\},$$

but this fact can not be established by simple shadowing.

Proof. To apply the General Shadowing Theorem 3.3, let $t(x, z_1, z_2, \dots) \in T_{1x}$ be a group term. Let $t'(x, z_1, z_2, \dots)$ be obtained from t by successively replacing inverses of products with reversed products of inverses. Then $t \approx t'$ holds in every group, and there are terms $w_1(z_1, z_2, \dots)$ and $w_2(z_1, z_2, \dots)$ such that t' is either $w_1 x w_2$ or $w_1 x^{-1} w_2$. We now show that $z_1 x z_2$ syntactically shadows t.

If $t' = w_1 x w_2$, then $z_1 x z_2$ simply shadows t' and therefore also t. Assume $t' = w_1 x^{-1} w_2$. Then we have

$$t(x, z_1, z_2, \dots) \approx w_1 x^{-1} w_2 \approx w_1 x^{-1} y y^{-1} w_2$$
$$w_1 x^{-1} x y^{-1} w_2 \approx w_1 y^{-1} w_2 \approx t(y, z_1, z_2, \dots).$$

The fact that $z_1 x z_2$ shadows the term x^{-1} cannot be established by simple shadowing, as there do not exist terms $w_1(z_1, z_2, \dots)$ and $w_2(z_1, z_2, \dots)$ for which $x^{-1} \approx w_1 x w_2$ holds in every group. □

Hidden in this syntactic shadowing argument is the exploitation of a peculiar property of groups: every equivalence on a group that is preserved by the binary operation is also preserved by the inverse operation. The fact that $z_1 x z_2$ determines syntactic congruences in groups can be established directly from this property as well. Consider an equivalence θ on a group $\mathbf{G} = \langle G; \cdot, {}^{-1}, 1\rangle$. According to Example 2.6, the relation $\theta_{z_1 x z_2}$ is a congruence on the monoid $\langle G; \cdot, 1\rangle$ and is therefore also a congruence on $\mathbf{G}$. Thus $F := \{z_1 x z_2\}$ determines syntactic congruences in the variety of groups.

While Example 6.1 demonstrates that simple shadowing is weaker than general shadowing, we have also shown in Example 5.5 that there is a finite set of terms that simply shadows all group terms. The distinction between simple and general shadowing is made clearer in the next example.

Let $\mathbf{S} := \langle \{0, a, b, 1\}; \cdot, C\rangle$ denote the usual 4 element Boolean meet semilattice (where meet is written multiplicatively) with $0 < a, b < 1$, and with C the unary operation that fixes each element of $\{0, b, 1\}$ and has $C(a) = 1$.

Example 6.2. The variety generated by $\mathbf{S}$ has FDSC, but no finite set of terms simply shadows T_{1x}.

Proof. The algebra $\mathbf{S}$ is investigated in Jackson [27] as a member of the more general class of *closure semilattices* (CSLs)—algebras with operations $\cdot$ and C satisfying the semilattice axioms along with $C(C(x)) \approx C(x)$, $C(xy)C(y) \approx C(xy)$ and $xC(x) \approx x$. (A useful consequence is the identity $C(x)C(y) \approx C(C(x)C(y))$.) There a complete description of the equational theory of CSLs can be found. In particular, it is shown ([27, Proposition 5.2]) that within the variety of closure semilattices the following set, Σ, is a basis for the identities of $\mathbf{S}$:

$$C(xy) \approx C(xC(y))C(yC(x)), \quad C(xC(yz))) \approx C(xC(y))C(xC(z))C(yz),$$
$$C(xC(yC(z))) \approx C(xC(y))C(xC(z))C(yC(z)), \quad C(xyz) \approx C(xy)C(yz)C(zx).$$

A term not involving the unary operation C, will be called a *semilattice term*, while a term of the form $C(t(\vec{x}))$ will be called a *closed term*. We observe that the semilattice axioms ensure that every term t is equivalent to one for which each subterm is the product of a (possibly empty) semilattice term (the *semilattice part* of t) with a (possibly empty) collection of closed terms. For example, $C(y)xC(C(z)xC(C(w)y))w$ becomes $xwC(xC(z)C(yC(w)))C(y)$, with semilattice part xw. We will assume throughout this proof that all terms are written in this way.

We first show that no finite set of terms can simply shadow all terms in T_{1x}. With any term t we may associate a directed graph $G(t)$ whose vertices are the variables appearing in t and such that (x, y) is an edge if $x \neq y$ and there is a subterm s of t such that y appears (anywhere) in s and x appears in the semilattice part of s. While we omit the details, it follows from the proof that Σ is a basis for the identities of $\mathbf{S}$ that for closed terms u and v, we have $\mathbf{S} \models u \approx v$ if and only if $G(u) = G(v)$. (While this is not explicitly stated in [27], the proof of Proposition 5.2 there involves a reduction to normal forms that are distinct if and only if the corresponding graphs are distinct. Observe also that each identity in Σ preserves graphs of terms.)

Let $t_n(x, z_0, \dots, z_n)$ denote the term $C(z_0C(z_1C(\cdots C(z_nC(x))\cdots)))$ and let F be any finite subset of T_x. We show that, for sufficiently large n, the set F does not simply shadow t_n.

Let n be the maximum indegree of x amongst the graphs $G(s)$ for $s \in F$. Assume that F simply shadows t_n, that is, there is $s(x, \vec{z}) \in F$ and terms $p_1(\vec{z})$, $p_2(\vec{z})$, …such that $\mathbf{S} \models t_n(x, z_0, \dots, z_n) \approx s(x, p_1(\vec{z}), \dots)$. Then the CSL axiom $C(C(x)) \approx C(x)$ shows that $\mathbf{S}$ also satisfies $t_n(x, z_0, \dots, z_n) \approx C(s(x, p_1(\vec{z}), \dots))$, and so we may assume without loss of generality that s is a closed term. Now the graph $G(t_n(x, z_0, \dots, z_n))$ is antisymmetric and as the graphs $G(s(x, p_1(\vec{z}), \dots))$ and $G(t_n(x, z_0, \dots, z_n))$ are identical, the graph $G(s(x, p_1(\vec{z}), \dots))$ is also antisymmetric. Hence the semilattice part of each term $p_i(\vec{z})$ is either empty, or a single variable. But then the indegree of x in $G(s(x, p_1(\vec{z}), \dots))$ is at most that of x in $G(s(x, \vec{z}))$, which was n. This contradicts the fact that x has indegree $n + 1$ in $G(t_n(x, z_0, \dots, z_n))$. It follows that F does not simply shadow all terms from $\{t_1, t_2, \dots\} \subseteq T_{1x}$.

To complete the proof, it remains to show that the variety generated by $\mathbf{S}$ has FDSC. For this we can use the set

$$F := \{x, z_1x, C(x), C(z_1x), z_1C(x), z_1C(z_2x), C(z_1C(x)), z_1C(z_2C(x))\}.$$

We prove that Theorem 2.5(iv) holds for the term $z_1C(z_2C(x))$ and leave the remaining (easier) cases to the reader. The case where x is replaced by $C(x)$ holds trivially because $\mathbf{S} \models C(C(x)) \approx C(x)$. By commutativity, it remains to shadow (syntactically) the term $t(x, z_1, z_2, z_3) := z_1C(z_2C(z_3x))$. Using the law $C(xC(yz))) \approx C(xC(y))C(xC(z))C(yz)$ in Σ and commutativity we have the following shadowing of $t(x, z_1, z_2, z_3)$ by F:

$$\begin{aligned} &z_1C(z_2C(z_3x)) \approx [z_1C(z_2C(z_3))C(z_2C(x))]C(z_3x) \\ &\quad [z_1C(z_2C(z_3))C(z_2C(x))]C(z_3y) \approx [z_1C(z_2C(z_3))C(z_3y)]C(z_2C(x)) \\ &\qquad\qquad [z_1C(z_2C(z_3))C(z_3y)]C(z_2C(y)) \approx z_1C(z_2C(z_3y)). \end{aligned}$$

Thus the variety of $\mathbf{S}$ has FDSC. □

We saw in Example 2.7 that the variety generated by a finite unary algebra has FDSC. In order to complement this result, consider a fixed finite type of algebras with at least one non-unary operation. Murskiĭ [37] showed that, in a reasonable statistical sense, almost all finite algebras of this type are quasi-primal. Recall that $\mathbf{M}$ is *quasi-primal* if it has the ternary discriminator operation

$$t_3(x, y, z) := \begin{cases} z & \text{if } x = y; \\ x & \text{if } x \neq y. \end{cases}$$

as a term function, and therefore also has the *normal transform*

$$n(x, y, u, v) := \left.\begin{cases} u & \text{if } x = y; \\ v & \text{if } x \neq y. \end{cases}\right\} = t_3(t_3(x, y, u), t_3(x, y, v), v)$$

as a term function. It is well known that a variety generated by an algebra with a discriminator term has definable principal congruences (see Burris and Sankappanavar [8, Exercise IV.3.3] for example) and so Proposition 2.8 shows that the variety generated by a quasi-primal algebra has FDSC. We now use general shadowing to show that a single term is always sufficient.

Example 6.3. Syntactic congruences in the variety generated by a quasi-primal algebra are determined by $F = \{n(x, z_1, z_2, z_3)\}$. Consequently,

(i) the topological quasi-variety generated by a quasi-primal algebra with the discrete topology is standard, and
(ii) almost every finite algebra of finite type generates a variety with FDSC.

Proof. For an arbitrary term $t(y, z_1, z_2, \dots)$ we have

$$\begin{aligned} &t(x, z_1, z_2, \dots) \approx n(x, x, t(x, z_1, z_2, \dots), t(y, z_1, z_2, \dots)) \\ &\qquad n(y, x, t(x, z_1, z_2, \dots), t(y, z_1, z_2, \dots)) \approx t(y, z_1, z_2, \dots). \end{aligned}$$

Thus F determines syntactic congruences and we apply the FDSC-HSP Theorem to obtain (i). Then (ii) follows from (i), Example 2.7 and Murskiĭ [37]. □

We round out this section by showing that finitely generated lattice varieties have FDSC. We begin by defining lattice terms $f_0 = x$ and $f_m = f_{m-1} \wedge z_m$ if m is odd, and $f_m = f_{m-1} \vee z_m$ if m is even. The first few of these are

$$\begin{aligned} f_0 &= x \\ f_1 &= x \wedge z_1 \\ f_2 &= (x \wedge z_1) \vee z_2 \\ f_3 &= ((x \wedge z_1) \vee z_2) \wedge z_3. \end{aligned}$$

The dual of a lattice term t will be denoted by t^∂.

Lemma 6.4. *In the variety of all lattices, f_{m+1} shadows both f_m and f_m^∂.*

Proof. We first show that $f_{m+1} \vdash f_m$.

$$\begin{aligned} f_m(x, z_1, \dots, z_m) &\approx f_{m+1}(x, z_1, \dots, z_m, f_m(x, z_1, \dots, z_m)) \\ f_{m+1}(y, z_1, \dots, z_m, f_m(x, z_1, \dots, z_m)) &\approx f_{m+1}(x, z_1, \dots, z_m, f_m(y, z_1, \dots, z_m)) \\ f_{m+1}(y, z_1, \dots, z_m, f_m(y, z_1, \dots, z_m)) &\approx f_m(y, z_1, \dots, z_m) \end{aligned}$$

Thus f_{m+1} syntactically shadows f_m. Now consider f_m^∂.

$$\begin{aligned} f_m^\partial(x, z_1, \dots, z_m) &\approx f_{m+1}(x, x, z_1, \dots, z_m) \\ f_{m+1}(y, x, z_1, \dots, z_m) &\approx f_{m+1}(x, y, z_1, \dots, z_m) \\ f_{m+1}(y, y, z_1, \dots, z_m) &\approx f_m^\partial(y, z_1, \dots, z_m) \end{aligned}$$

Hence f_{m+1} syntactically shadows f_m^∂. □

We wish to induct on the complexity of bracket-reduced lattice terms in which all parentheses that are unnecessary by associativity have been omitted. We now make this precise. *Formal joins* and *formal meets* are expressions obtained by a finite number of applications of the following rules:

(i) each variable is both a formal join and a formal meet,
(ii) if $k \geq 2$ and $t_1, \dots, t_k$ are formal meets, then $(t_1 \vee \cdots \vee t_k)$ is a formal join,
(iii) if $k \geq 2$ and $t_1, \dots, t_k$ are formal joins, then $(t_1 \wedge \cdots \wedge t_k)$ is a formal meet.

An expression t is a *bracket-reduced lattice term* if it is either a formal join or a formal meet. As usual, we always omit the outer pair of parentheses. The *a-height* (short for associative height) of a bracket-reduced lattice term is defined inductively by declaring that variables have a-height 0, if $t = t_1 \vee \cdots \vee t_k$ is a formal join and the maximum a-height of $t_1, \dots, t_k$ is n, then the a-height of t is $n+1$, and similarly for formal meets. The a-height of a usual lattice term is defined to be the a-height of the corresponding bracket-reduced lattice term. Thus, for example, the a-height of the lattice term

$$z_1 \wedge ((z_1 \vee (z_2 \vee z_3)) \wedge (z_2 \vee z_4))$$

is 2 since the corresponding bracket-reduced lattice term is

$$z_1 \wedge (z_1 \vee z_2 \vee z_3) \wedge (z_2 \vee z_4).$$

Lemma 6.5. *If $t \in T_x$ is a lattice term of a-height d then, for all $m > d$, the term f_m shadows t in the variety of all lattices.*

Proof. We may assume that t involves the variable x as otherwise every term shadows t. Since variable replacement does not alter the a-height of a lattice term, by Lemma 3.5 we may assume that $t \in T_{1x}$. Let t' be the bracket-reduced lattice term corresponding to t. If t' is a formal join, then using commutativity and associativity we can write $t' = g \vee h$, where g is a formal meet involving x and h is free of x. Let $t'' = g \vee z_k$, where z_k does not occur in t. Clearly t'' shadows t. A dual reduction process produces $t'' = g \wedge z_k$ in the case that t' is a formal meet. Now we apply the same procedure to g. Inducting backwards in this way we obtain, up to variable substitution, either f_k or f_k^∂ for some $k \leq d$. By Lemma 6.4 it follows that f_m shadows t for all $m > d$. □

Corollary 6.6. *For any class $\mathfrak{L}$ of lattices, the set $\{ f_m \mid m = 0, 1, 2, \dots \}$ determines syntactic congruences in $\mathfrak{L}$. If $\mathfrak{L}$ has* FDSC*, then there exists m such that $\{f_m\}$ determines syntactic congruences in $\mathfrak{L}$.*

The key to showing that the variety $\mathbb{V}(\mathbf{L})$ generated by $\mathbf{L}$ has FDSC is to prove there is a bound d such that for every term t there is a term t' of a-height at most d such that $\mathbb{V}(\mathbf{L})$ satisfies $t \approx t'$. Let $\mathbf{F}_{\boldsymbol{\mathcal{V}}}(X)$ denote the free algebra in $\boldsymbol{\mathcal{V}}$ generated by X. We would like to thank Ralph McKenzie for a suggestion that simplified the following proof.

Theorem 6.7. *Let $\mathbf{L}$ be a finite lattice. Then there is a number d such that for each lattice term t there is a term t' of a-height at most d such that $t \approx t'$ holds in $\mathbf{L}$.*

Proof. By Jónsson's Theorem [29] we can choose n so that every subdirectly irreducible lattice in $\boldsymbol{\mathcal{V}} = \mathbb{V}(\mathbf{L})$ has at most n elements: indeed, $n := |L|$ will suffice. Let X be a set with n elements. Of course each element of $\mathbf{F}_{\boldsymbol{\mathcal{V}}}(X)$ can be represented by some term with variables from X. Since $\mathbf{F}_{\boldsymbol{\mathcal{V}}}(X)$ is finite we can choose k such that every one of these representatives has a-height at most k. We shall prove that we can choose $d := k + 2$.

Now let Y be an arbitrary finite set. Let $u = t(y_1, \dots, y_m)$ be a join-irreducible element in $\mathbf{F}_{\boldsymbol{\mathcal{V}}}(Y)$ and let u_* be its unique lower cover. There is a surjective homomorphism $\varphi \colon \mathbf{F}_{\boldsymbol{\mathcal{V}}}(Y) \twoheadrightarrow \mathbf{K}$, for some subdirectly irreducible lattice $\mathbf{K}$ in $\boldsymbol{\mathcal{V}}$, with $\varphi(u) \neq \varphi(u_*)$. Let $\Phi := \ker(\varphi)\restriction_Y$. Consider

$$v := t(\bigwedge(y_1/\Phi), \dots, \bigwedge(y_m/\Phi)).$$

Clearly $v \le u$, but $\varphi(v) = \varphi(u)$ and hence $v \nleq u_*$. It follows that $v = u$. Since $\mathbf{K}$ has at most n elements, we have $\ell := |Y/\Phi| \le n$, and hence there is a term $t'(x_1, \dots, x_\ell)$ of a-height at most k, and a partition $Y_1, \dots, Y_\ell$ of Y such that

$$u = t(y_1, \dots, y_m) = t'(\bigwedge(Y_1), \dots, \bigwedge(Y_\ell)).$$

Consequently, u has a-height at most $k + 1$. Since every element is a join of join-irreducibles, every element of $\mathbf{F}_{\mathcal{V}}(Y)$ can be represented by a term of a-height at most $d := k + 2$. □

Example 6.8. Every finitely generated lattice variety has FDSC.

Proof. Let $\mathcal{V} = \mathbb{V}(\mathbf{L})$, where $\mathbf{L}$ is a finite lattice and let m be one greater than the number d from Theorem 6.7. We claim f_m shadows every $t \in T_x$. By Theorem 6.7 we may assume that t has a-height at most d and so the result follows from Lemma 6.5. □

Remark 6.9. In retrospect we find that these ideas have arisen in the past in quite different contexts. It is relatively straightforward to see that a lattice variety $\mathcal{V}$ has TFPC if and only if there is a bound on the lengths of weak projectivities needed in Dilworth's description [18] of principal congruences on lattices in $\mathcal{V}$. (See [24] for a discussion of weak projectivities and principal congruences on lattices.) In [3, Theorem 4.1] Baker proved that a finitely generated lattice variety has such a bound, and consequently has TFPC. More generally, in [43] Wang proved that every finitely generated congruence distributive variety has a "finite principal length property" which is easily seen to be equivalent to TFPC, and therefore also to FDSC.

7. Varieties without FDSC

Not every variety has FDSC. The core idea of Theorem 4.3 provides an internal condition on an algebra that guarantees that its syntactic congruences cannot be finitely determined. We say the Boolean topological algebra $\underset{\sim}{\mathbf{X}}$ is *topologically residually finite* if, for all $a, b \in X$ with $a \neq b$, there is a finite discrete $\underset{\sim}{\mathbf{Y}}$ and a continuous homomorphism $\alpha \colon \underset{\sim}{\mathbf{X}} \to \underset{\sim}{\mathbf{Y}}$ such that $\alpha(a) \neq \alpha(b)$.

Lemma 7.1. *If $\underset{\sim}{\mathbf{X}}$ is a Boolean topological algebra which is not topologically residually finite, then no finite set of terms determines syntactic congruences on $\underset{\sim}{\mathbf{X}}$.*

Proof. Suppose that a finite set $F \subseteq T_x$ determines syntactic congruences on $\underset{\sim}{\mathbf{X}}$. Choose distinct $a, b \in X$ that cannot be separated by a continuous homomorphism into any finite discrete algebra. Let U be a clopen subset of X containing a but not b, and let θ be the equivalence relation with two classes U and $X \backslash U$. By the Clopen Equivalence Lemma 4.2, each θ_F class is clopen. Since $\underset{\sim}{\mathbf{X}}$ is compact, θ_F must have finite index in $\underset{\sim}{\mathbf{X}}$. It follows that the quotient homomorphism $\alpha\colon \mathbf{X} \to \mathbf{X}/\theta_F$ is also continuous, and it separates a and b since $\theta_F \subseteq \theta$, a contradiction. □

The first examples of algebras satisfying the conditions of Lemma 7.1 appear to be those found by Banaschewski [4] and by Taylor [41]. These examples are based on unary algebras.

Example 7.2. The variety of all unars (unary algebras with a single operation) does not have FDSC.

Proof. Let $\underset{\sim}{\mathbf{Z}}_\infty$ be the one point compactification of the integers as a loop, with $s(n) := n+1$ for $n \in \mathbb{Z}$ and $s(\infty) := \infty$. Then $\underset{\sim}{\mathbf{Z}}_\infty$ is infinite and Boolean. However no proper congruence on $\underset{\sim}{\mathbf{Z}}_\infty$ is clopen. Thus $\underset{\sim}{\mathbf{Z}}_\infty$ is not topologically residually finite whence Lemma 7.1 applies. □

Lemma 7.1 also gives a striking example of a topological condition on a variety which implies a purely algebraic condition on the variety.

Theorem 7.3. *Let $\mathcal{V}$ be a variety of algebras. If there is a non-residually finite algebra in $\mathcal{V}$ which admits a compatible Boolean topology, and in particular if $\mathcal{V}$ contains an infinite subdirectly irreducible algebra that has a compatible Boolean topology, then $\mathcal{V}$ does not have* FDSC*.*

We list a few of the many applications of this theorem that easily follow. The first two provide an interesting contrast with Example 6.8 and Example 5.9.

Example 7.4. The variety of all modular lattices, and therefore the variety of all lattices, does not have FDSC.

Proof. Let $\mathbf{L}_\omega$ be the lattice depicted in Figure 1. Then $\mathbf{L}_\omega$ is an infinite subdirectly irreducible modular lattice, and the cofinite sets containing 0 and their complements form a compatible Boolean topology. This is easily proved using the fact that the set of elements incomparable with any given element of $\mathbf{L}_\omega$ is finite. (That $\mathbf{L}_\omega$ has this compatible Boolean topology was observed by Clinkenbeard [14].)

It is also easy to give a direct algebraic proof using Corollary 6.6. □

Example 7.5. The variety $\mathcal{O}$ of all Ockham algebras does not have FDSC.

Proof. We give two examples. For the first example, we recall (see [42]) the fact that $\mathcal{O} = \mathbb{ISP}\,\mathbf{O}_1$, where $\mathbf{O}_1$ is an infinite subdirectly irreducible Ockham algebra. The underlying bounded distributive lattice of $\mathbf{O}_1$ is $\mathbf{2}^{\mathbb{N}}$ and c is defined by $c(a) := s(a)'$, where $'$ is the usual Boolean complement and $s\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the shift map given by $s(a)_n := a_{n+1}$. Clearly, the operations on $\mathbf{O}_1$ are compatible with the product topology on $2^{\mathbb{N}}$. (This algebra is the basis for a natural duality for the variety of Ockham algebras; see [22] for details.)

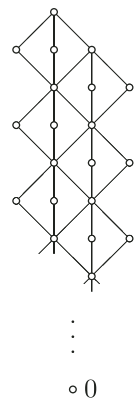

FIGURE 1. the modular lattice $\mathbf{L}_\omega$

The following simpler example was kindly contributed by David Hobby. Let

$$\mathbf{O}_2 := \langle \{ \tfrac{1}{n} \mid 0 \neq n \in \mathbb{Z} \} \cup \{0\}; \vee, \wedge, c, -1, 1 \rangle$$

where O_2 is given the usual order, $c(-1) = 1$, $c(0) = 0$, $c(1) = -1$, $c(\frac{1}{n}) = \frac{-1}{n}$ and $c(\frac{-1}{n}) = \frac{1}{n-1}$ for $n > 1$. Then $\mathbf{O}_2$ is a subdirectly irreducible Ockham algebra whose monolith identifies only -1 and $-\frac{1}{2}$. Under the usual topology $\mathbf{O}_2$ is Boolean and its operations are continuous. □

For other examples of this type the reader is directed to Johnstone [28].

The application of FDSC given in the FDSC-HSP Theorem 4.3, which provides the original motivation for our study of FDSC, concerns only varieties generated by a single finite algebra. It is therefore important to ask if there is a variety with a single finite generator that does not have FDSC. The answer is affirmative, as we now show.

A significant class of finite algebras of current interest was constructed by McKenzie [34] for the purpose of resolving a number of different algebraic decidability questions. McKenzie gave an effective procedure to construct, for each Turing machine $\mathcal{T}$, a finite algebra $\mathbf{A}(\mathcal{T})$ of finite type for which he established the following facts.

McKenzie's Theorem 7.6. [34] Let $\mathcal{T}$ be a Turing machine.

(i) If $\mathcal{T}$ eventually halts when started on a blank tape, then there is a finite bound on the sizes of the subdirectly irreducibles in the variety $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ generated by $\mathbf{A}(\mathcal{T})$.

(ii) If $\mathcal{T}$ does not halt when started on a blank tape, then the variety $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ contains an infinite subdirectly irreducible.

(iii) The algebra $\mathbf{A}(\mathcal{T})$ has a distinguished element 0 which determines a one-element subalgebra of $\mathbf{A}(\mathcal{T})$.

Thus, for example, it is recursively undecidable as to whether or not the variety generated by a finite algebra of finite type is residually finite.

Example 7.7. If the Turing machine $\mathcal{T}$ does not halt, then $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ does not have FDSC.

Proof. Let $\mathbf{Q}_\omega$ be the algebra

$$\mathbf{Q}_\omega := \langle \{\, 0, a_n, b_n \mid n < \omega \,\}; \wedge, \cdot \rangle$$

with distinct elements 0, a_n and b_n, for $n < \omega$, and binary operations defined by

$$\begin{aligned} p \wedge q &= 0 && \text{if } p \neq q \\ p \wedge p &= p && \\ a_n \cdot b_{n+1} &= b_n && \text{for all } n < \omega \\ p \cdot q &= 0 && \text{otherwise.} \end{aligned}$$

It is easy to check that $\mathbf{Q}_\omega$ is subdirectly irreducible: every non-trivial congruence includes the pair $(0, b_0)$. McKenzie showed in [34, Lemma 4.1] that $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ contains an algebra that is term equivalent to $\mathbf{Q}_\omega$. Since the topology of cofinite subsets of $\mathbf{Q}_\omega$ containing 0 and their complements is compatible with the operations and is Boolean, $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ does not have FDSC. □

So far, all the examples where FDSC fails have been proved using Lemma 7.1. However Theorem 2.5 and the Completeness Theorem 3.2 provide a second approach. Indeed, the authors initially established all of the above examples via this method. To demonstrate the technique, we now give an alternative proof of the statement in Example 7.7.

We first recall that the *height* $\mathrm{ht}(t)$ of a term t is the height of the term tree of t (note that this differs from the notion of a-height introduced in Section 6). More formally, constant symbols and variables have height 0 and if f is an k–ary fundamental operation symbol and $u_1, u_2, \dots, u_k$ are terms of maximum height n, then the term $f(u_1, u_2, \dots, u_k)$ has height $n + 1$.

Let θ be the equivalence on the set Q_ω with two classes, $\{b_0\}$ and $Q_\omega \setminus \{b_0\}$. Then $\mathrm{Syn}(\theta)$ is the identity congruence and consequently, for each $n < \omega$, we have $(0, b_n) \notin \mathrm{Syn}(\theta)$. This fact is witnessed by the height-n term

$$f_n(x, z_0, z_1, \dots, z_{n-1}) := z_0 \cdot (z_1 \cdot (z_2 \cdot (\cdots (z_{n-1} \cdot x) \cdots)))$$

since

$$f_n(0, a_0, a_1, \ldots, a_{n-1}) = a_0(a_1(a_2(\cdots(a_{n-1}0)\cdots))) = 0$$
$$\not\equiv_\theta b_0 = a_0(a_1(a_2(\cdots(a_{n-1}b_n)\cdots))) = f_n(b_n, a_0, a_1, \ldots, a_{n-1}).$$

We prove that no term of lower height witnesses $(0, b_n) \notin \mathrm{Syn}(\theta)$.

Lemma 7.8. *Let $f(x, z_0, \ldots, z_{j-1}) \in T_x$ have height $k < n$; let $c_0, \ldots, c_{j-1} \in Q_\omega$ and assume that $f(0, c_0, \ldots, c_{j-1}) \neq f(b_n, c_0, \ldots, c_{j-1})$. Then $f(0, c_0, \ldots, c_{j-1}) = 0$ and $f(b_n, c_0, \ldots, c_{j-1}) = b_m$ for some m with $m \geq n-k$. In particular, $(0, b_n) \in \theta_f$.*

Proof. The statement is clearly true if f is the height 0 term x. Assume that f has height $k+1 < n$. We consider two possibilities.

Assume that $f = p \wedge q$ where the maximum of the heights of p and q is k. Since

$$f(0, c_0, \ldots, c_{j-1}) \neq f(b_n, c_0, \ldots, c_{j-1})$$

and $\wedge$ is commutative, we may as well assume that

$$p(0, c_0, \ldots, c_{j-1}) \neq p(b_n, c_0, \ldots, c_{j-1}).$$

By induction we conclude that

$$0 = p(0, c_0, \ldots, c_{j-1}) = f(0, c_0, \ldots, c_{j-1})$$

and that

$$p(b_n, c_0, \ldots, c_{j-1}) = b_m = q(b_n, c_0, \ldots, c_{j-1}) = f(b_n, c_0, \ldots, c_{j-1})$$

for some m with $m \geq n-k > n-(k+1)$.

Now assume that $f = p \cdot q$ where again the maximum of the heights of p and q is k. Either

$$p(0, c_0, \ldots, c_{j-1}) \neq p(b_n, c_0, \ldots, c_{j-1}) \text{ or } q(0, c_0, \ldots, c_{j-1}) \neq q(b_n, c_0, \ldots, c_{j-1}).$$

By induction $f(0, c_0, \ldots, c_{j-1}) = 0$ in either case. Since $f(b_n, c_0, \ldots, c_{j-1}) \neq 0$ we have, for some m, that $p(b_n, c_0, \ldots, c_{j-1}) = a_m$, that $q(b_n, c_0, \ldots, c_{j-1}) = b_{m+1}$ and that $f(b_n, c_0, \ldots, c_{j-1}) = b_m$. As a_m is neither a product nor a proper meet, $p(x, z_0, \ldots, z_{j-1})$ must be a variable or meet of variables not involving x. Thus $p(0, c_0, \ldots, c_{j-1}) = a_m$, so

$$q(0, c_0, \ldots, c_{j-1}) \neq q(b_n, c_0, \ldots, c_{j-1}).$$

By induction, $m+1 \geq n - \mathrm{ht}(q) \geq n-k$ giving us $m \geq n-(k+1)$. □

Example 7.9. No finite set of terms determines the syntactic congruence of the equivalence θ on the algebra $\mathbf{Q}_\omega$.

Proof. If F is a finite set of terms, then we can choose an upper bound n for the heights of the terms in F. By Lemma 7.8 we have $(0, b_n) \in \bigcap\{\, \theta_f \mid f \in F \,\} = \theta_F$, and therefore θ_F is not $\mathrm{Syn}(\theta)$, the identity congruence. □

From this example it now follows that the variety $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ does not have FDSC. Indeed, $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ contains an algebra term equivalent to $\mathbf{Q}_\omega$.

We will return to $\mathbf{A}(\mathcal{T})$ and several applications of McKenzie's Theorem 7.6 in Section 9.

8. FDSC and profiniteness

The next theorem contains several nice characterisations of topological residual finiteness. The result has an extensive history, despite the fact that there appears to be no published proof in the form stated below. The semigroup version is in essence proved by Numakura [38] (see also Hunter [26]). Many of the details for the more general version can be extracted from the results of Choe [9] and Day [17] (see Johnstone [28] for the most detailed exposition, where unpublished work of G. Bergman is also cited). The result as stated is observed in Almeida and Weil [2] and was suggested to us by J. Almeida.

Theorem 8.1. *Let $\underset{\sim}{\mathbf{X}}$ be a compact topological algebra. The following conditions are related by*

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)},$$

and all are equivalent provided some finite set of terms determines syntactic congruences on $\underset{\sim}{\mathbf{X}}$.

(i) $\underset{\sim}{\mathbf{X}}$ *is topologically residually finite.*
(ii) $\underset{\sim}{\mathbf{X}}$ *is profinite, that is, $\underset{\sim}{\mathbf{X}}$ is an inverse limit of finite topological algebras.*
(iii) $\underset{\sim}{\mathbf{X}}$ *is a closed subdirect product of finite topological algebras.*
(iv) $\underset{\sim}{\mathbf{X}}$ *is a Boolean topological algebra.*

Proof. Condition (iii) easily implies (i) by using the projection homomorphisms onto the finite factors. Condition (ii) implies (iii) since the inverse limit is itself a special closed subdirect product of its finite factors. Condition (iv) follows from (iii) by the Tychonoff Product Theorem, and Lemma 7.1 tells us that (iv) implies (i) in case $\underset{\sim}{\mathbf{X}}$ has FDSC. It only remains to prove that (i) implies (ii).

Assume that $\underset{\sim}{\mathbf{X}}$ is topologically residually finite. Then there is a set $\mathcal{F}$ of congruences on $\underset{\sim}{\mathbf{X}}$ such that $\bigcap\mathcal{F}$ is the identity congruence on $\underset{\sim}{\mathbf{X}}$ and, for each $\psi \in \mathcal{F}$, each block of ψ is clopen and ψ has finite index. Let $\mathcal{S}$ be the collection of finite intersections of members of $\mathcal{F}$ ordered by reverse inclusion: $\alpha \le \beta$ if $\alpha \supseteq \beta$. For each $\alpha \in \mathcal{S}$ define $\underset{\sim}{\mathbf{X}}_\alpha := \underset{\sim}{\mathbf{X}}/\alpha$. If $\beta \ge \alpha$ in $\mathcal{S}$, we define $f_{\beta\alpha} \colon \underset{\sim}{\mathbf{X}}_\beta \to \underset{\sim}{\mathbf{X}}_\alpha$ to be the natural map given by $f_{\beta\alpha}(x/\beta) := x/\alpha$. Since $\beta \ge \alpha$, the map $f_{\beta\alpha}$ is well defined.

Let $\varprojlim\{\, \mathbf{X}_\alpha \mid \alpha \in \mathcal{S} \,\}$ be the inverse limit, consisting of all $U \in \prod\{\, \mathbf{X}_\alpha \mid \alpha \in \mathcal{S} \,\}$ such that $f_{\beta\alpha}(U(\beta)) = U(\alpha)$ whenever $\beta \ge \alpha$ in $\mathcal{S}$. Then the natural map

$$f \colon \underset{\sim}{\mathbf{X}} \to \varprojlim\{\, \mathbf{X}_\alpha \mid \alpha \in \mathcal{S} \,\}, \quad \text{given by} \quad f(x)(\alpha) := x/\alpha,$$

is a continuous, one-to-one homomorphism. It remains to prove that f is surjective.

Let $U \in \varprojlim\{\, \mathbf{X}_\alpha \mid \alpha \in \mathcal{S}\,\}$. Assume that $x \in \bigcap\{\, U(\alpha) \mid \alpha \in \mathcal{S}\,\}$. Then we would have, for each $\alpha \in \mathcal{S}$,

$$f(x)(\alpha) = x/\alpha = U(\alpha),$$

showing that $U = f(x)$. Thus, to show that f is surjective, we must establish that $\bigcap\{\, U(\alpha) \mid \alpha \in \mathcal{S}\,\} \neq \varnothing$. Since $\underset{\sim}{\mathbf{X}}$ is compact, it is sufficient to show that $U(\alpha_1) \cap U(\alpha_2) \cap \cdots \cap U(\alpha_n) \neq \varnothing$ for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{S}$. To this end, define $\beta := \alpha_1 \cap \alpha_2 \cap \cdots \cap \alpha_n \in \mathcal{S}$. Since $U(\beta)$ is a non-empty block of β, we can choose $x \in U(\beta)$. For $i = 1, 2, \ldots, n$, we have $\beta \geq \alpha_i$ and therefore

$$U(\alpha_i) = f_{\beta\alpha_i}(U(\beta)) = f_{\beta\alpha_i}(x/\beta) = x/\alpha_i.$$

Thus $x \in U(\alpha_i)$, for all i, and therefore $U(\alpha_1) \cap U(\alpha_2) \cap \cdots \cap U(\alpha_n) \neq \varnothing$. □

In a 1957 paper, Numakura showed that every Boolean topological distributive lattice is profinite [38]. As an application of our methods we give the following extension of this result.

Example 8.2. Let $\mathcal{V}$ be a finitely generated variety of lattices and $\underset{\sim}{\mathbf{X}}$ be a Boolean topological lattice with $\mathbf{X} \in \mathcal{V}$. Then $\underset{\sim}{\mathbf{X}}$ is profinite.

Proof. Example 6.8 shows that $\underset{\sim}{\mathbf{X}}$ has FDSC and then Theorem 8.1 shows that $\underset{\sim}{\mathbf{X}}$ is profinite. □

This result cannot be extended indefinitely, because the Boolean topological lattice in Example 7.4 is modular but not profinite. However other varieties we have shown to have FDSC, can be used in place of finitely generated lattice varieties.

Example 8.3. Let $\underset{\sim}{\mathbf{X}}$ be a Boolean topological algebra whose underlying algebra $\mathbf{X}$ lies in one of the following classes:

(1) groups, (2) rings, (3) semigroups, (4) a variety with DPC.

Then $\underset{\sim}{\mathbf{X}}$ is profinite.

9. Lingering questions

The work presented in this paper suggests the following question.

Problem 9.1. Is there an algorithm to decide if a given finite algebra of finite type generates a standard topological quasi-variety?

The FDSC-HSP Theorem 4.3 imposes two hypotheses on the finite algebra $\mathbf{M}$,

(i) the quasi-variety generated by $\mathbf{M}$ is a variety and
(ii) the variety generated by $\mathbf{M}$ has FDSC,

which together imply that $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ is standard. We conclude by raising several questions suggested by this theorem.

Are conditions (i) and (ii) necessary for $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ to be standard? The results of Davey and Talukder [15] show that they are not: they construct a three element unary algebra $\mathbf{M}$ for which (i) fails but $\mathfrak{Q}^+_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is nevertheless standard. This example shows that there is more to understand about standardness of algebras than the FDSC-HSP Theorem reveals. It seems unlikely that Problem 9.1 will be answered without first gaining an understanding of the extent to which standardness extends beyond applications of the FDSC-HSP Theorem.

Are conditions (i) and (ii) independent? By Example 5.6 we know that every variety of semigroups has FDSC. Yet the studies of residually finite semigroups in Golubov and Sapir [23] and residually small varieties of semigroups in McKenzie [32] provide many examples of finite semigroups $\mathbf{M}$ generating varieties that are not even residually small, and therefore certainly do not satisfy $\mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M}$. Since the submission of this paper an example has been found of a finite algebra $\mathbf{M}$ such that $\mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M}$, but this variety does not have FDSC. This resolution of the independence question will appear soon.

We can also enquire as to whether or not we can effectively determine if the FDSC-HSP Theorem is applicable.

Problem 9.2. Is there an algorithm to decide if the quasi-variety generated by a finite algebra of finite type is a variety?

Note that $\mathbb{HSP}\,\mathbf{M} = \mathbb{ISP}\,\mathbf{M}$ if and only if each subdirectly irreducible of $\mathbb{HSP}\,\mathbf{M}$ embeds into $\mathbf{M}$. Thus we can decide this equality for the algebras $\mathbf{M}$ in a given class $\mathcal{K}$ if, within $\mathcal{K}$, we have

(1) an effective method to determine if $\mathbb{HSP}\,\mathbf{M}$ is residually finite

and, in case it is,

(2) an effective method to compute a number $f(\mathbf{M}) \in \mathbb{N}$ such that $\mathbb{HSP}\,\mathbf{M}$ is residually less than $f(\mathbf{M})$.

Indeed, if $\mathbb{HSP}\,\mathbf{M}$ is residually less than $f(\mathbf{M})$, then every subdirectly irreducible of $\mathbb{HSP}\,\mathbf{M}$ is isomorphic to a quotient of the free algebra in $\mathbb{HSP}\,\mathbf{M}$ on $f(\mathbf{M})$ generators. We can then effectively (although laboriously) check if each subdirectly irreducible quotient of this finite free algebra embeds into $\mathbf{M}$.

Freese and McKenzie [20, Theorem 10.15] exhibited (1) and (2) for finite algebras $\mathbf{M}$ generating a congruence modular variety, giving f as $f(\mathbf{M}) = |M| + |M|^{|M|^{|M|+3}}$. Higgs [25] gave a short argument that if $\mathbf{M} = \langle M; F\rangle$ is a unary algebra, then $\mathbb{HSP}\,\mathbf{M}$ is residually less than 2^{n+1}, where n is the size of the one-generated free algebra in $\mathbb{HSP}\,\mathbf{M}$. Algorithms (1) and (2) can be extracted for semigroups from the papers Golubov and Sapir [23] and McKenzie [32], and for rings from McKenzie [33].

In spite of these far reaching results, a positive answer to Problem 9.2 can not be established for all finite algebras by means of (1) and (2) alone, as McKenzie's Theorem 7.6 shows that (1) does not hold in the class of all finite algebras.

Problem 9.3. Is there an algorithm to decide if the variety generated by a finite algebra of finite type has FDSC?

In view of Example 7.7 and McKenzie's Theorem 7.6, we could conclude that there was no such algorithm if we could show that the variety generated by $\mathbf{A}(\mathcal{T})$ has FDSC when $\mathcal{T}$ does halt.

In the proof of Example 5.8 we had to construct, from a finite algebra $\mathbf{M}$, a finite algebra $\mathbf{M}'$ for which $\mathbb{HSP}\,\mathbf{M} = \mathbb{HSP}\,\mathbf{M}' = \mathbb{ISP}\,\mathbf{M}'$. This construction suggests a variant of Problem 9.2 which is also answered by McKenzie's Theorem.

Proposition 9.4. *There is no algorithm to decide, given a finite algebra* $\mathbf{M}$ *of finite type, if there is a finite algebra* $\mathbf{M}'$ *such that* $\mathbb{HSP}\,\mathbf{M} = \mathbb{HSP}\,\mathbf{M}' = \mathbb{ISP}\,\mathbf{M}'$.

Proof. From McKenzie's Theorem 7.6 we see that, for a Turing machine $\mathcal{T}$, there is a finite algebra $\mathbf{A}(\mathcal{T})'$ such that $\mathbb{HSP}\,\mathbf{A}(\mathcal{T}) = \mathbb{HSP}\,\mathbf{A}(\mathcal{T})' = \mathbb{ISP}\,\mathbf{A}(\mathcal{T})'$ if and only if $\mathcal{T}$ halts; namely, take $\mathbf{A}(\mathcal{T})'$ to be the direct product of the non-isomorphic subdirectly irreducibles in $\mathbb{HSP}\,\mathbf{A}(\mathcal{T})$. □

Yet another variant of Problem 9.2 is to ask for an algorithm to decide when a given finite set of quasi-identities determines a variety. McNulty [35, Theorem 18] shows that this also is undecidable.

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