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ℓ-prime elements in multiplicative lattices

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Abstract. In this paper we study ℓ-prime elements in C-lattices and characterize Prüfer lattices, almost principal element lattices and principal element lattices in terms of ℓ-prime elements. Using these results, some new characterizations are given for general ZPI-rings and almost multiplication rings. Finally some new equivalent conditions are given for Dedekind lattices.

1. Introduction

Throughout this paper R denotes a commutative ring with identity and $L(R)$ denotes the lattice of all ideals of R. R is called a general ZPI-ring, if every ideal is a finite product of prime ideals. For various characterizations of general ZPI-rings, the reader is referred to [17], [18], [21] and [22]. It is well known that if R is a Noetherian integral domain, then R is a Dedekind domain if and only if for any maximal ideal P of R, the set of P-primary ideals of R is totally ordered by set inclusion [16, Theorem 6.20, page 137]. This result is true even for non-domains. In fact R is a general ZPI-ring if and only if R is a Noetherian ring and for any maximal ideal P of R, the set of P-primary ideals of R is totally ordered by set inclusion (see [22, Theorem 4] and [15, Theorem 3]). It is well known that R a general ZPI-ring if and only if $L(R)$ is a principal element lattice [14, Theorem 2.2] and R is a Noetherian ring if and only if $L(R)$ is a Noether lattice [7]. Therefore $L(R)$ is a principal element lattice if and only if $L(R)$ is a Noether lattice and for any maximal ideal P of R, the set of P-primary ideals of R is totally ordered by set inclusion. Our aim is to extend this result to non-modular multiplicative lattices. For various examples of non-modular multiplicative lattices, the reader is referred to [1]. It should be mentioned that there is a significant difference between our proofs and the already existing ones presented in ring-theory books. In Section 2, we give some definitions and known results that we use in this paper. In Section 3, we study ℓ-prime elements in C-lattices. In Section 4, we prove that if L is a principally generated reduced C-lattice satisfying the a.c.c. (ascending chain

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condition) for prime elements, then L is a Prüfer lattice if and only if every prime is an ℓ-prime (see Theorem 1). Next we show that if L is a principally generated reduced lattice, then every branched prime element is contained in a rank one ℓ-prime if and only if L is a Prüfer lattice and dim L ≤ 1 (see Theorem 2). Also a new characterization is given for an arithmetical ring (see Corollary 3). In Section 5, we show that if L is principally generated, then L is a principal element lattice if and only if L satisfies the a.c.c. for prime elements and every maximal prime element is a strong compact ℓ-prime element (see Theorem 3). As a consequence, it is shown that if L is principally generated, then L is a principal element lattice if and only if L is Noetherian and every maximal prime is an ℓ-prime (see Corollary 5). Further it is proved that if L is a principally generated reduced lattice, then L is an almost principal element lattice if and only if every branched prime element is contained in a non idempotent rank one ℓ-prime (see Theorem 4). Also it is shown that if L is principally generated, then L is a principal element lattice if and only if every prime element is contained in a strong compact, strong ℓ-prime element (see Theorem 5). Using these results, some new characterizations are given for general ZPI-rings and almost multiplication rings (see Theorem 6, Theorem 7 and Theorem 8). Finally in Section 6, we characterize Dedekind lattices in terms of non-minimal prime elements (see Theorem 9).

2. Basic notions

An element e of a multiplicative lattice L is said to be principal if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $(a \vee be) : e = (a : e) \vee b$. Elements satisfying (i) are called meet principal and elements satisfying (ii) are called join principal. Elements satisfying the weaker identity (iii) $a \wedge e = (a : e)e$ obtained from (i) by setting $b = 1$ are called weak meet principal. An element $a \in L$ is called invertible if a is principal and $(0 : a) = 0$. By a C-lattice we mean a (not necessarily modular) complete multiplicative lattice, with least element 0 and compact greatest element 1 (a multiplicative identity), which is generated under joins by a multiplicatively closed subset C of compact elements. A C-lattice is said to be a domain if the zero element is a prime element. In a principally generated C-lattice, principal elements are compact [2, Theorem 1.3] and a finite product of principal elements is again a principal element [7].

Throughout this paper we assume that L is a C-lattice generated by compact join principal elements. L_* denotes the set of all compact elements of L. For any $a \in L$, we denote $\theta(a) = \vee\{(x : a) \mid x \in L_* \text{ and } x \leq a\}$. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \in C \mid x \nleq p\}$. For basic properties of localization, the reader is referred to [10]. A prime element

p of L is said to be an ℓ-prime if the set of all p-primary elements of L is linearly ordered. For any prime p of L, p^{Δ} denotes the meet of all p-primary elements of L. A prime element p of L is said to be branched (unbranched) if $p > p^{\Delta}$ ($p = p^{\Delta}$). Let p, m be two prime elements of L. We say m covers p if $m > p$ and there is no prime element p_1 of L such that $m > p_1 > p$. A prime element p of L is said to be an S-element if p is an ℓ-prime, p^{Δ} is prime and p^{Δ} contains every prime element properly contained in p. L is said to be an S-lattice if every prime element is an S-element. S-lattices have been studied in [13]. Following [11], a prime element p of L is said to be a d-prime, if L_p is a discrete valuation lattice (i.e., consists just of the elements $0, 1$, and the powers of p all of which are distinct). L is said to be an almost discrete valuation lattice if L_m is a discrete valuation lattice (i.e., m is a d-prime) for every maximal prime m of L [11].

L is said to be *reduced* if 0 is the only nilpotent element of L. Principal elements were introduced into multiplicative lattices by R.P. Dilworth [7]. A multiplicative lattice L in which every element is principal is called a *principal element lattice*. Similarly, L is said to be an *almost principal element lattice*, if L_m is a principal element lattice, for every maximal element m of L. It is well known that if L is principally generated, then L is a principal element lattice if and only if every prime element is principal. For various characterizations of almost principal element lattices and principal element lattices, the reader is referred to [5], [8], [9] and [12]. L is said to be a Prüfer lattice if every compact element is principal. It is well known that a principally generated C-lattice L is a Prüfer lattice if and only if L_p is totally ordered for every prime p of L. For more information on Prüfer lattices, the reader is referred to [2, Theorem 3.4] and [19]. A reduced lattice L is said to be a Dedekind lattice if every element not contained in any minimal prime is weak meet principal. L is said to be a Baer lattice if, for any $x \in L_*$, $(0 : (0 : x)) \vee (0 : x) = 1$. For more information on Dedekind lattices and Baer lattices, the reader may consult [11]. L is a Noetherian lattice, if L satisfies the ascending chain condition. L is a locally Noetherian lattice, if L_m is a Noetherian lattice for every maximal prime element m of L. It should be mentioned that if L is a principally generated Noetherian lattice, then L is a Noether lattice (in the sense of [7]) if and only if L is a modular lattice.

A prime ideal P of R is said to be branched (unbranched, ℓ-prime) if P is the branched (unbranched, ℓ-prime) element of $L(R)$. Recall that an ideal I of R is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. R is a *multiplication ring* if every ideal is a multiplication ideal. R is an *almost multiplication ring* if R_M is a multiplication ring, for every maximal ideal M of R. Multiplication rings and almost multiplication rings have been extensively studied — for example, see [6] and [21]. An ideal M of R is called a *quasi-principal* *ideal* [16, Exercise 10, Page 147] (or a principal element of $L(R)$ [20]) if it satisfies the following identities (i) $(A\cap(B{:}M))M = AM\cap B$ and (ii) $(A+BM){:}M = (A{:}M)+B$, for all A, $B \in L(R)$. Obviously every quasi-principal ideal is a multiplication ideal. It should be mentioned that every quasi-principal ideal is finitely generated and also a finite product of quasi-principal ideals of R is again a quasi-principal ideal [16, Exercise 10, Page 147]. In fact, an ideal I of R is quasi-principal if and only if it is finitely generated and locally principal [4, Theorem 3]. It should be mentioned that every principal ideal of R is quasi-principal and hence $L(R)$, the lattice of all ideals of R, is a principally generated modular C-lattice.

For general background and terminology, the reader may consult [2], [4], [10] and [16].

3. ℓ-prime elements in C-lattices

In this section we study ℓ-prime elements in C-lattices.

We shall begin with the following lemma.

Lemma 1. *Let m be a prime element and suppose m covers p for some prime element p of L. Then p is the meet of all m-primary elements containing p.*

Proof. Let $a = \wedge\{q \in L \mid p \leq q$ and q is m-primary$\}$. Clearly $p \leq a$. Suppose $p < a$. Choose any compact join principal element $x \leq a$ such that $x \nleq p$. Then m is minimal over $p \vee x^2$ and hence by [10, Property 0.5], $(p \vee x^2)_m$ is m-primary. Again $x \leq a \leq (p\vee x^2)_m$, so $xz \leq p\vee x^2$ for some $z \nleq m$. But then $z \leq (p\vee x^2 : x) = x \vee (p : x) \leq x \vee p \leq m$, a contradiction and therefore $p = a$. □

Lemma 2. *Let L be a domain. If m is a rank one prime element, then $m^{\Delta} = 0$.*

Proof. The proof of the lemma follows from Lemma 1. □

Lemma 3. *If m is an ℓ-prime element of L and if m covers p for some prime element p of L, then $p = m^{\Delta}$.*

Proof. By Lemma 1, $m^{\Delta} \leq p$. If $m^{\Delta} < p$, then there exists an m-primary element q such that $p \nleq q$. As m is an ℓ-prime, it follows that $q \leq p$, so $m = \sqrt{q} \leq p$, a contradiction and therefore $p = m^{\Delta}$. □

Definition 1. A prime element p of L is said to be a strong ℓ-prime if the primary elements contained in p are linearly ordered.

Note that every strong ℓ-prime element is an ℓ-prime.

Lemma 4. *Let m be a non-minimal strong ℓ-prime element of L. Then m is an S-element.*

Proof. If $m = m^{\Delta}$, then we are through. Suppose $m^{\Delta} < m$. Let $p = \vee\{p_\alpha \mid p_\alpha < m$ is a prime element of $L\}$. As the primary elements contained in m are linearly ordered, it follows that $p \leq m^{\Delta}$ and p is prime. Again since m covers p, by Lemma 3, $p = m^{\Delta}$. Therefore m^{Δ} is prime and contains every prime element properly contained in m. Therefore m is an S-element. □

Lemma 5. *Let L satisfy the a.c.c. for prime elements. If p is a non-minimal ℓ-prime element, then p is a branched S-element.*

Proof. Let p be a non-minimal ℓ-prime element of L. Let $\Psi = \{q \in L \mid q < p$ is a prime element$\}$. By the a.c.c. for prime elements, Ψ contains a maximal element, say $N(p)$. By Lemma 3, $N(p) = p^{\Delta}$, so $N(p)$ is unique and hence $N(p)$ contains every prime element properly contained in p. Therefore p is a branched S-element. □

Lemma 6. *Let L be a principally generated zero dimensional quasi-local lattice with maximal element m. If m is a non-idempotent ℓ-prime, then m is principal.*

Proof. Note that every element is primary. As m is an ℓ-prime, it follows that L is totally ordered. By Lemma 7 of [8], m is principal. □

For any $a \in L$, we denote $a^{\omega} = \bigwedge_{n=1}^{\infty} a^n$. Following [12], an element $a \in L$ is said to be strong compact if both a and a^{ω} are compact elements. Strong compact elements have been studied in [12] to characterize almost principal element lattices and principal element lattices.

Lemma 7. *Let L be a principally generated quasi-local lattice satisfying the a.c.c. for prime elements. Suppose m is a strong compact, non-minimal maximal element. If m is an ℓ-prime, then L is a discrete valuation lattice.*

Proof. The proof of the lemma follows from Lemma 5 and [13, Lemma 5]. □

4. Prüfer lattices

In this section we characterize Prüfer lattices, in terms of ℓ-prime elements. Further a new characterization is given for an arithmetical ring.

Theorem 1. *Let L be a principally generated reduced lattice satisfying the a.c.c. for prime elements. Then L is a Prüfer lattice if and only if every prime is an ℓ-prime.*

Proof. Obviously, if L is a Prüfer lattice, then every prime is an ℓ-prime. Conversely, assume that every prime is an ℓ-prime. Note that in a reduced lattice, the minimal prime elements are unbranched. Therefore by Lemma 5 and [13, Theorem 4], L is a Prüfer lattice. □

Corollary 1. *Let L be a principally generated reduced lattice satisfying the a.c.c. for prime elements. If there exists a strong ℓ-prime element p such that every prime is contained in p, then L is totally ordered.*

Proof. The proof of the corollary follows from Theorem 1. □

Theorem 2. *Suppose L is a principally generated reduced lattice. Then every branched prime element is contained in a rank one ℓ-prime if and only if L is a Prüfer lattice and dim $L \leq 1$.*

Proof. Suppose every branched prime element is contained in a rank one ℓ-prime element. We show that dim $L \leq 1$. Suppose $p < m$ be any two prime elements. Choose any principal element $x \leq m$ such that $x \not\leq p$. Let $p_0 \leq m$ be a prime minimal over $p \vee x$. Then $(p \vee x^2)_{p_0}$ is p_0-primary [10, Property 0.5] and $(p \vee x^2)_{p_0} < p_0$ [9, Lemma 12] and hence p_0 is a branched prime element. Let q be a rank one ℓ-prime element such that $p_0 \leq q$. Since $p < p_0 \leq q$ and rank $q = 1$, it follows that $p_0 = q$ and hence p is minimal. This shows that dim $L \leq 1$. Again note that $p_0 = m$ and hence every non-minimal maximal prime element is a branched ℓ-prime. As L is reduced, it follows that every minimal prime is unbranched [11, Lemma 3] and so every minimal prime is an ℓ-prime. Again by Theorem 1, L is a Prüfer lattice.

Conversely, assume that dim $L \leq 1$ and L is a Prüfer lattice. As L is reduced, branched prime elements are non-minimal primes. Again by hypothesis, branched prime elements are rank one ℓ-primes. Also by Lemma 3, rank one ℓ-primes are branched ℓ-primes. This completes the proof of the theorem. □

Corollary 2. *Suppose L is a principally generated domain. Then every prime element of L is contained in a rank one ℓ-prime element if and only if L is a one dimensional Prüfer domain in which every non zero prime element is branched.*

Proof. The proof of the corollary follows from Lemma 2 and Theorem 2. □

R is called a reduced ring if the zero element is the only nilpotent element. R is an arithmetical ring if every finitely generated ideal is locally principal. Again note that R is an arithmetical ring if and only if $L(R)$ is a Prüfer lattice.

As a consequence of Theorem 1 and the fact that $L(R)$ is a principally generated C-lattice, we have the following result.

Corollary 3. *Let R be a reduced ring satisfying the a.c.c. for prime ideals. Then R is an arithmetical ring if and only if every prime ideal is an ℓ-prime ideal.*

5. Principal element lattices

In this section we characterize principal element lattices and almost principal element lattices in terms of ℓ-prime elements. Using these results, some new characterizations are given for general ZPI-rings and almost multiplication rings.

Theorem 3. *Suppose L is principally generated. Then L is a principal element lattice if and only if L satisfies the following conditions:*

(i) *Every maximal prime element is a strong compact ℓ-prime element.*
(ii) *L satisfies the a.c.c. for prime elements.*

Proof. Obviously, every principal element lattice satisfies the conditions (i) and (ii). Now assume that L satisfies the conditions (i) and (ii). Let m be a maximal prime element of L. If m is idempotent, then by [2, Theorem 1.4], L_m is a two element chain. If m is non-idempotent and non-minimal, then by Lemma 7, L_m is a discrete valuation lattice. If m is non-idempotent and minimal, then by Lemma 6, L_m is a principal element lattice. Therefore L is an almost principal element lattice. Again since L is an almost principal element lattice, it follows that dim $L \leq 1$, and also if $p < m$ are prime elements, then $p = m^{\omega}$ (see the proof of [9, Theorem 7]). So by (i), every prime element is compact. As every prime element is compact and locally principal, it follows that every prime element is principal [4, Theorem 1] and hence L is a principal element lattice. This completes the proof of the theorem. □

As a consequence, we have the following results.

Corollary 4. *Suppose L is principally generated. Then L is an almost principal element lattice if and only if L is locally Noetherian and every maximal prime is an ℓ-prime.*

Corollary 5. *Suppose L is principally generated. Then L is a principal element lattice if and only if L is Noetherian and every maximal prime is an ℓ-prime.*

Theorem 4. *Let L be a principally generated reduced lattice. Then L is an almost principal element lattice if and only if every branched prime element is contained in a non-idempotent rank one ℓ-prime.*

Proof. Suppose every branched prime element of L is contained in a non-idempotent rank one ℓ-prime. By Theorem 2, L is a Prüfer lattice and dim $L \leq 1$. Suppose m is a maximal prime element of L. If m is minimal, then m is unbranched and hence L_m is a two element chain. If m is non-minimal, then by Lemma 3, m is branched and hence by hypothesis, m is a non-idempotent rank one ℓ-prime. Therefore by Lemma 2, Theorem 1 and Theorem 2 of [11], L_m is a discrete valuation lattice. Hence L is an almost principal element lattice.

Conversely, assume that L is an almost principal element lattice. Then by Lemma 2, Theorem 1 and Theorem 5 of [9], L is a Prüfer lattice and dim $L \leq 1$. Again by [10, Lemma 5] and [9, Corollary 1], rank one ℓ-primes are non-idempotent. Now the result follows from Theorem 2. This completes the proof of the theorem. □

Theorem 5. *Let L be a principally generated lattice. Then L is a principal element lattice if and only if every prime is contained in a strong compact, strong ℓ-prime element.*

Proof. If L is a principal element lattice, then we are through. So assume that every prime is contained in a strong compact, strong ℓ-prime element. Then every maximal prime is a strong compact, strong ℓ-prime element. Let m be a maximal prime element of L. Then by Lemma 4 and [13, Lemma 5], it follows that rank $m \leq 1$. This shows that dim $L \leq 1$. As L satisfies the a.c.c. for prime elements, by Theorem 3, L is a principal element lattice and the proof is complete. □

An ideal I of R is said to be a strong finitely generated ideal (strong ℓ-prime ideal), if I is a strong compact (strong ℓ-prime) element of $L(R)$.

Observe that by [14, Theorem 2.2], R is a general ZPI-ring if and only if $L(R)$ is a principal element lattice. Also it should be mentioned that if $L = L(R)$ is the lattice of all ideals of R and if P is a prime ideal of R, then the lattice $L(R_P)$ of all ideals of R_P is naturally isomorphic to the localization L_P of the lattice $L = L(R)$. Therefore by [6, Theorem 2.0], R is an almost multiplication ring if and only if $L(R)$ is an almost principal element lattice.

We now establish some new characterizations for general ZPI-rings and almost multiplication rings.

Theorem 6. *The following statements on R are equivalent:*

(i) *R is a general ZPI-ring.*
(ii) *R satisfies the a.c.c. for prime ideals and every maximal ideal is a strong finitely generated ℓ-prime ideal.*
(iii) *Every maximal ideal is a strong finitely generated, strong ℓ-prime ideal.*

Proof. The proof of the theorem follows from Theorem 3, Theorem 5 and the fact that R is a general ZPI-ring if and only if $L(R)$ is a principal element lattice. □

Theorem 7. *R is an almost multiplication ring if and only if R satisfies the following conditions:*

(i) *Every maximal prime ideal is an ℓ-prime ideal.*
(ii) *Every maximal prime ideal is a locally strong finitely generated ideal.*
(iii) *R satisfies the a.c.c. for prime ideals.*

Proof. The proof of the theorem follows from Theorem 3 and the fact that R is an almost multiplication ring if and only if $L(R)$ is an almost principal element lattice. □

Theorem 8. *Suppose R is a reduced ring. Then R is an almost multiplication ring if and only if every branched prime ideal is contained in a non-idempotent rank one ℓ-prime ideal.*

Proof. The proof of the theorem follows from Theorem 4. □

6. Dedekind lattices

In this section, we establish some new characterizations for Dedekind lattices. We shall begin with the following lemma.

Lemma 8. *Let $a \in L$ be weak meet principal. Then a is compact if and only if $\theta(a) = 1$.*

Proof. If a is compact, then $\theta(a) \geq (a : a) = 1$, so $\theta(a) = 1$. Now assume that $\theta(a) = 1$. Since 1 is compact, it follows that $1 = \bigvee_{i=1}^{n}\{(x_i : a) \mid x_i \in L_* \text{ and } x_i \leq a\}$. So $a = (\bigvee_{i=1}^{n}(x_i : a))a = \bigvee_{i=1}^{n}(x_i : a)a \leq \bigvee_{i=1}^{n} x_i \leq a$. Therefore $a = \bigvee_{i=1}^{n} x_i$ and hence a is compact. □

Lemma 9. *Let $a \in L$ be a weak meet principal element. If a is not contained in any minimal prime element, then a is compact.*

Proof. By using Lemma 8 and by imitating the proofs of Proposition 3 and Theorem 3 of [3] we can get the result. □

Dedekind lattices have been studied in [11]. The following Theorem 9 establishes some new characterizations for Dedekind lattices.

Theorem 9. *Suppose L is a reduced lattice. Then the following statements on L are equivalent:*

(i) *L is a Dedekind lattice.*
(ii) *Every non-minimal prime is invertible.*
(iii) *Each non-minimal prime is principal.*
(iv) *Every non-minimal prime is weak meet principal.*
(v) *Every element not contained in any minimal prime is invertible.*

Proof. (i)⇒(ii). Suppose L is a Dedekind lattice. By [11, Lemma 10], L is principally generated. Let p be a non-minimal prime. By (i), p is weak meet principal. By Lemma 9, p is compact and by [4, Proposition 2(d)], p is locally principal.

Therefore by [4, Theorem 1], p is principal. As L is reduced and p is a non-minimal principal prime, it follows that $(0:p)=0$. Therefore p is invertible.

(ii) $\Rightarrow$(iii)$\Rightarrow$(iv) is obvious.

(iv)$\Rightarrow$(v). Suppose (iv) holds. We claim that L is an almost principal element lattice. Let m be a maximal prime element of L. If m is minimal, then L_m is a two element chain (as L is reduced). Suppose m is a non-minimal prime element of L. By imitating the proof of [12, Theorem 3], it can be easily shown that rank $m=1$. Again since rank $m=1$ and L is reduced, by [10, Property 0.4], L_m is a domain. Therefore by [12, Corollary 1], L_m is a principal element lattice and hence L is an almost principal element lattice. Again by imitating the proof of [11, Lemma 9], we can easily show that L is a Baer lattice and hence by Theorem 4 and Theorem 9 of [11], L is a Dedekind lattice. Let x be any element of L such that x is not contained in any minimal prime element of L. Since L is a Dedekind lattice, it follows that x is weak meet principal, so by Lemma 9, x is compact and hence principal by [4, Theorem 1]. Obviously $(0:x)=0$ and hence x is invertible. Thus (v) holds and (v)$\Rightarrow$(i) is obvious. This completes the proof of the theorem. □

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