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Quotients of partial abelian monoids and the Riesz decomposition property

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Abstract. Partial abelian monoids (PAMs) are structures $(P; \perp, \oplus, 0)$, where $\oplus$ is a partially defined binary operation with domain $\perp$, which is commutative and associative in a restricted sense, and 0 is the neutral element. PAMs with the Riesz decomposition properties and binary relations with special properties on PAMs are studied. Relations with abelian groups, dimension equivalence and K_0 for AF C*-algebras are discussed.

1. Introduction

The basic algebraic structure that is studied in this paper is a partial abelian monoid (PAM). A PAM (or a partial abelian semigroup, PAS, cf. [54]) is a structure $(P; \perp, \oplus, 0)$ where $\oplus$ is a partial binary operation with the definition domain $\perp$, which is commutative and associative in a restricted sense, and has a neutral element 0. Similar structures have already appeared in literature and found applications in several fields.

A general theory of universal (partial) algebras can be found in [8], [22, pp. 80–81], a special case of algebras with partially defined binary operation has been studied in [36].

Beginning with a PAM on the lowest level, there is a hierarchy of partial algebraic structures. On higher levels we get, successively, a cancellative PAM (c-PAM), a positive and cancellative PAM (cp-PAM), a unital cp-PAM. We note that cp-PAMs coincide with generalized effect algebras (GEA), or generalized difference posets (GDP) [27], and also with abelian RI-sets [34]. As subclasses we obtain here generalized orthomodular posets [41] and generalized orthomodular lattices [30], which play an important role as models of the sets of projections in (non-unital) rings [4], and in quantum mechanical applications.

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The class of unital cp-PAMs coincides with effect algebras (or D-posets). Effect algebras [18, 19, 23] and D-posets [35] have been introduced as abstract models for studying quantum effects, that is, self-adjoint operators between 0 and I on a Hilbert space, and unsharp quantum measurements. These structures are more general than previous models considered in the quantum logic approach to the foundations of quantum theory, namely orthomodular lattices and posets and orthoalgebras [2, 3, 9, 24, 33, 45].

A special branch of PAMs consists of PAMs with the Riesz decomposition properties, which on higher levels contain some commutative clans [55] and commutative, positive minimal clans [52], introduced as a common abstraction of boolean rings and lattice ordered groups. Some commutative BCK-algebras, and on a higher level, MV-algebras can also be included into this hierarchy. Notice that BCK-algebras originated from both set theory and propositional calculi, classical and non-classical [29, 50]. MV-algebras have been introduced by Chang [12] as an algebraic model in many-valued logic. They have been extensively studied by many authors, and have found applications in different branches of mathematical logic, functional analysis, probability theory, group theory and fuzzy-set theory.

Relations among some above mentioned partial algebraic structures can be found in [10, 47].

Relations between MV-algebras and the K_0 theory of certain AF C*-algebras have been shown in [42], where also the categorical equivalence between MV-algebras and unit intervals in lattice ordered groups with strong unit has been shown.

Effect algebras with the Riesz decomposition properties have been studied in [49], where their equivalence with unit intervals in abelian interpolation groups with strong unit has been shown, extending the result obtained in [42] for MV-algebras and lattice ordered groups. Categorical equivalence between a certain class of effect algebras with Riesz decomposition properties and unital AF C*-algebras has been shown in [48], extending the results of [42].

In the present paper, we study equivalence relations with some special properties on PAMs. In particular, we study weak congruences (in the sense of [26]) with additional properties which make the quotient satisfy the Riesz decomposition properties. We show that these additional properties are related to a dimension equivalence. We also consider direct limits in a category of cp-PAMs endowed with weak congruences, and find conditions under which the direct limit can be endowed with a weak congruence with the same properties as the members of the corresponding directed system.

We also study relations between cp-PAMs with the Riesz decomposition properties and interpolation groups as their universal groups. We show that an upper directed cp-PAM with the Riesz decomposition property is lattice ordered if and only if its universal group is lattice ordered, which extends the result of [44]. We also show that there is a one-to-one correspondence between cp-PAMs with the Riesz decomposition property which are lower semilattices and commutative BCK-algebras with a relative cancellation property [10].

We introduce some elements of a dimension theory for effect algebras, based on the results of [37, 38, 39, 51] and [32].

In the last section, some applications to K_0 theory of AF C*-algebras are mentioned.

2. Basic properties of PAMs

Definition 2.1. A *partial abelian monoid* (*PAM*, for short) is a nonempty set P endowed with a partially defined binary relation $\oplus$ with a domain $\perp \subseteq P \times P$ satisfying the following conditions.

- (PM1) (Commutativity). If $a \perp b$ then $b \perp a$ and $a \oplus b = b \oplus a$.
- (PM2) (Associativity). If $a \perp b$ and $(a \oplus b) \perp c$ then $b \perp c$ and $a \perp (b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (PM3) (Neutral element). There is an element $0 \in P$ such that $0 \perp a$ for any $a \in P$ and $a \oplus 0 = a$.

Observe that the neutral element 0 is uniquely defined. For, if 0_1 is another neutral element, then $0 = 0 \oplus 0_1 = 0_1 \oplus 0 = 0_1$.

We say that a and b are *orthogonal* if $a \perp b$. In what follows, when we write $a \oplus b$ we mean that $a \oplus b$ is defined (i.e., $a \perp b$). Owing to associativity (PM2), we may omit parentheses in $a_1 \oplus a_2 \oplus a_3$ and $a_1 \oplus a_2 \oplus \cdots \oplus a_n$, the latter term being defined by induction. We will say that the elements $a_1, \ldots, a_n$ are *summable* if the element $a_1 \oplus \cdots \oplus a_n$ exists in L. More generally, we say that $\{a_\alpha\}_\alpha$ is a *summable family* if every finite subfamily is summable.

For $a \in P$ and $n \in \mathbb{N}$, define $na = a \oplus a \oplus \cdots \oplus a$ (n-times) if the right-hand side exists, and define $\iota(a) = \max\{n : \exists na\}$, the *isotropic index* of a. We say that a has an *infinite isotropic index* if na exists for every $n \in \mathbb{N}$.

Definition 2.2. Let $(P; \perp, \oplus, 0)$ be a PAM. We say that P is:

(1) *Cancellative* if $a \oplus b = a \oplus c$ implies $b = c$.
(2) *Positive* if $a \oplus b = 0$ implies $a = b = 0$.
(3) *Unital* if it contains a *unit*, i.e., an element $u \in P$ such that for any $a \in P$ there is $b \in P$ such that $a \oplus b = u$.

The element b such that $a \oplus b = u$ for a distinguished unit u in a cancellative PAM is unique and is called the *u-supplement* of a, denoted by a'.

For every cancellative PAM, a partial binary operation $\ominus$ can be defined by the following rule :

(d) $a \oplus b = c$ iff $c \ominus b$ is defined and then $c \ominus b = a$.

By cancellativity, $\ominus$ is well-defined.

Observe that if P is cancellative and positive, it contains at most one unit. Indeed, if u and v are units, then there are u_1, v_1 such that $u \oplus u_1 = v$, $v \oplus v_1 = u$, so that $u \oplus u_1 \oplus v_1 = u = u \oplus 0$, and by cancellativity and positivity, $u_1 \oplus v_1 = 0$ and $u_1 = v_1 = 0$, whence $u = v$. In this case we denote the unit by 1, and call a 1-supplement an orthosupplement.

In what follows, we write c-PAM for a cancellative PAM and cp-PAM for a positive, cancellative PAM.

Notice that the class of unital cp-PAMs coincides with the class of effect algebras [54]. Recall that an effect algebra P becomes an orthoalgebra if and only if $a \perp a$ implies $a = 0$, and P becomes and orthomodular poset if and only if $\oplus$ coincides with the supremum for orthogonal elements, equivalently, if and only if any three pairwise orthogonal elements are summable.

On every PAM $(P; \perp, \oplus, 0)$ we can introduce a binary relation $\leq$ in the following way

$$a \leq b \text{ if there is } c \in P \text{ with } a \oplus c = b.$$

Owing to $a \oplus 0 = 0 \oplus a = a$ and associativity, $0 \leq a$, and $\leq$ is reflexive and transitive, hence a preorder. If $\leq$ is a partial order then P is positive. Indeed, if $a \oplus b = 0$, then $0 = 0 \oplus (a \oplus b) = (0 \oplus a) \oplus b$, hence $0 \leq 0 \oplus a \leq 0$ and hence $a = 0$.

Lemma 2.3. *If P is a positive PAM and for any a and x in P, $a \oplus x = a$ implies $x = 0$, then $\leq$ is a partial order.*

In particular, if P is a PAM on which for any $a \neq 0$ the isotropic index $\iota(a)$ is finite, then $\leq$ is a partial order.

Proof. If $a \leq b$ and $b \leq a$, then there are x and y such that $a \oplus x = b$ and $b \oplus y = a$, hence $a \oplus (x \oplus y) = a$. By hypothesis, $x \oplus y = 0$ and by positivity, $x = y = 0$, therefore $a = b$.

If every nonzero element in P has a finite isotropic index, then $x \oplus y = 0$ implies that $n(x \oplus y) = nx \oplus ny$ (by associativity) is defined, hence nx exists for all n. By assumptions, $x = 0$. So P is positive. If $a \oplus x = a$ then $a = a \oplus x = (a \oplus x) \oplus x$ and by induction, $a = a \oplus nx$, so nx is defined for all n, hence $x = 0$. □

The following theorem gives a necessary and sufficient condition under which $\leq$ is a partial order.

Lemma 2.4. *Let P be a PAM. The relation $\leq$ is a partial order if and only if, for any $a, x, y \in P$, $a \oplus x \oplus y = a$ implies $a \oplus x = a$.*

Proof. Assume $a \leq b$, $b \leq a$. For some x, y we have $a \oplus x = b$, $b \oplus y = a$. Then $a \oplus x \oplus y = a$, hence by assumptions, $a \oplus x = a = b$.

Conversely, assume that $\leq$ is a partial order. Then $a \leq a \oplus x \leq a \oplus x \oplus y = a$ implies $a = a \oplus x = a \oplus x \oplus y$. □

Let P be a partially ordered PAM, and $\{a_\alpha\}_\alpha$ be a summable family. We define $\bigoplus_\alpha a_\alpha := \bigvee_F \bigoplus_{\alpha \in F} a_\alpha$, where the supremum goes over all finite subfamilies F of α's, if the supremum on the right hand side exists.

We will say that a partially ordered PAM P is *m-orthocomplete* for an infinite cardinal m if every summable family of at most m elements has an $\oplus$-sum in L. Thus P is *orthocomplete* (*σ-orthocomplete*) if the $\oplus$-sum exists for any summable family (any countable summable family).

Definition 2.5. Let P and Q be PAMs. A mapping $h\colon P \to Q$ is a *morphism* if

(M1) $h(0) = 0$.

(M2) $h(a \oplus b) = h(a) \oplus h(b)$ (in the sense that if $a \oplus b$ exists, then $h(a) \oplus h(b)$ exists, and the above equality holds).

If P and Q are unital with distinguished units u_P and u_Q we also require

(M3) $h(u_P) = u_Q$.

A morphism h is called a *monomorphism* if $h(a) \perp h(b)$ implies $a \perp b$ $(a, b \in P)$.
A bijective morphism is an *isomorphism* if h^{-1} is also a morphism.

Observe that an isomorphism is the same thing as a bijective monomorphism.

Note that a monomorphism of cp-PAMs need not be injective : Let $P = \{0, a, b\}$ with $0 \oplus 0 = 0$, $0 \oplus a = a \oplus 0 = a$, $b \oplus 0 = 0 \oplus b = b$ and no other sums defined. Let $f\colon P \to P$ be given by $f(a) = f(b) = b$. Then f is a monomorphism. However, in the class of unital cp-PAMs (i.e., effect algebras) monomorphisms are injective.

If $h\colon P \to Q$ is a morphism, then $h(a) \leq h(b)$ whenever $a \leq b$. Indeed, the latter relation holds if and only if there is $x \in P$ with $a \oplus x = b$, and then $h(a \oplus x) = h(a) \oplus h(x) = h(b)$.

Definition 2.6. A morphism $h\colon P \to Q$ is *full* if $h(a) \oplus h(b) \in h(P) := \{h(a) : a \in P\}$ implies that there are $a_1, b_1 \in P$ such that $a_1 \perp b_1$ and $h(a) = h(a_1)$, $h(b) = h(b_1)$. Then $h(a) \oplus h(b) = h(a_1 \oplus b_1)$.

A morphism $h\colon P \to Q$ is *strong* (cf. [26]), if whenever $h(a) \perp h(b)$, there exists $a_1 \in P$ with $a_1 \perp b$ and $h(a) = h(a_1)$.

Observe that if $h\colon P \to Q$ is a strong morphism then $h(P)$ with the operation $\oplus$ inherited from Q is a PAM, which is cancellative (positive) if Q is cancellative (positive).

3. Congruences and quotients of cp-PAMs

Throughout this section, we assume that $(P; \perp, \oplus, 0)$ is a positive, cancellative PAM.

Consider a binary relation $\sim$ on P and the following properties.

(C1) $\sim$ is an equivalence relation.
(C2) $a \sim a_1$, $b \sim b_1$ and $a \perp b, a_1 \perp b_1$ implies $a \oplus b \sim a_1 \oplus b_1$.
(C3) $a \sim b$ and $b \perp c$ implies that there is a $d \in P$ with $c \sim d$ and $a \perp d$.
(C4) $a \sim b$ and $a \oplus c \sim b \oplus d$ implies $c \sim d$.
(C5) $a \sim b \oplus c$ implies that there are a_1, a_2 with $a = a_1 \oplus a_2$ and $a_1 \sim b$, $a_2 \sim c$.
(C6) (in an effect algebra=unital cp-PAM): $a \sim b$ implies $a' \sim b'$ (where a' and b' are orthosupplements of a and b, respectively).

In accordance with [26], the relation $\sim$ is called a *weak congruence* if it satisfies (C1) and (C2). We denote by $[a]$ the equivalence class containing a ($a \in P$), and we denote by $P/{\sim}$ the set of all equivalence classes. Owing to (C1) and (C2), $P/{\sim}$ can be endowed with a relation $\perp$ and a partially defined binary operation $\oplus$ as follows. We say the $[a] \perp [b]$ if there are $a_1 \sim a$, $b_1 \sim b$ such that $a_1 \perp b_1$, and then define $[a] \oplus [b] := [a_1 \oplus b_1]$. The operation $\oplus$ is well defined and commutative, but not necessarily associative [26].

Again in accordance with [26] and [46], we say that $\sim$ is a *congruence* if (C1), (C2) and (C3) are satisfied.

In the next theorem, we collect some basic known properties of weak congruences.

Theorem 3.1. *Let $\sim$ be a weak congruence on a cp-PAM P.*

(i) *If $\sim$ is a congruence, then $P/{\sim}$ is a PAM. Moreover, $P/{\sim}$ is cancellative if and only if* (C4) *holds.*
(ii) *If $\sim$ satisfies* (C5)*, then $P/{\sim}$ is a PAM. Moreover, $P/{\sim}$ is positive and it is cancellative if and only if* (C4) *holds.*
(iii) *If P is an effect algebra and $\sim$ is a congruence, then $P/{\sim}$ is an effect algebra. In particular, every congruence on an effect algebra satisfies* (C4).
(iv) *Let $\sim$ be a weak congruence on an effect algebra P. Then* (C4) *and* (C6) *are equivalent. Moreover,* (C3) *is equivalent to* (C5) & (C6) *or equivalently, to* (C5) & (C4).

Proof. (i) has been proved in [26]. (ii) To prove positivity, assume $0 \sim y \oplus z$. By (C5), $0 = x_1 \oplus x_2$ for some $x_1 \sim y$ and $x_2 \sim z$. By positivity of P, $x_1 = 0 = x_2$, hence $y \sim 0$, $z \sim 0$. For the rest see [26]. (iii) See [26]. (iv) (C4) implies (C6): $a \sim b$, $a \oplus a' = b \oplus b' = 1$ implies $a' \sim b'$. (C6) implies (C4): Assume $a \sim b$ and $a \oplus a_1 \sim b \oplus b_1$. There are v, w such that $a \oplus a_1 \oplus v = b \oplus b_1 \oplus w = 1$. By (C6), $v \sim w$. By (C2), $a \oplus v \sim b \oplus w$, and by (C6), $a_1 \sim b_1$. For the rest, see [13]. □

4. The Riesz decomposition properties

Recall that for a partially ordered abelian group (G, G^+) the following conditions are equivalent:

(RDP1) If $0 \leq a \leq b + c$ then there are $a_1 \leq b$, $a_2 \leq c$ such that $a = a_1 + a_2$.

(RDP2) For $a_1, a_2, b_1, b_2 \in G^+$ with $a_1+a_2 = b_1+b_2$, there are $w_{ij} \in G^+$, $i, j = 1, 2$ such that $a_i = w_{i1} + w_{i2}$, $i = 1, 2$, $b_j = w_{1j} + w_{2j}$, $j = 1, 2$.

Properties (RDP1) and (RDP2) are called the *Riesz decomposition properties.* On any partially ordered set Q, the *Riesz interpolation property* is defined as follows.

(RIP) If $a_1, a_2 \leq b_1, b_2$, then there is $c \in Q$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

We recall that property (RIP) on G, where (G, G^+) is a partially ordered abelian group, is equivalent to either of (RDP1) and (RDP2). If (G, G^+) satisfies the interpolation property, or equivalently any of the Riesz decomposition properties, it is called an *interpolation group* [20]. In analogy, we introduce the following definition.

Definition 4.1. Let P be a PAM. We say that P satisfies properties (WRDP1) or (WRDP2) if

(WRDP1) Whenever $a \leq b \oplus c$, there are $a_1, a_2 \in P$ such that $a_1 \leq b$, $a_2 \leq c$ and $a = a_1 \oplus a_2$.

(WRDP2) Whenever $a_1 \oplus a_2 = b_1 \oplus b_2$, there are w_{ij}, $i, j = 1, 2$ such that $a_i = w_{i1} \oplus w_{i2}$ $(i = 1, 2)$ and $b_j = w_{1j} \oplus w_{2j}$ $(j = 1, 2)$.

We will call properties (WRDP1) and (WRDP2) the *weak Riesz decomposition properties.*

Lemma 4.2. *Let P be a PAM.* (i) (WRDP2) *implies* (WRDP1). (ii) *If P is cancellative, then* (WRDP1) *and* (WRDP2) *are equivalent.*

Proof. (i) Assume (WRDP2) and let $a \leq b \oplus c$. Then there is $d \in P$ such that $a \oplus d = b \oplus c$. By (WRDP2), there are w_{ij} such that $a = w_{11} \oplus w_{12}$ and $b = w_{11} \oplus w_{21}$, $c = w_{12} \oplus w_{22}$. Hence $w_{11} \leq b$, $w_{12} \leq c$.

(ii) It remains to prove that (WRDP1) implies (WRDP2). Let $a_1 \oplus a_2 = b_1 \oplus b_2$. From $a_1 \leq b_1 \oplus b_2$ it follows by (WRDP1) that there are w_{11}, w_{12} such that $w_{11} \leq b_1, w_{12} \leq b_2$ and $a_1 = w_{11} \oplus w_{12}$. From the above inequalities it follows that there are w_{21} and w_{22} such that $b_1 = w_{11} \oplus w_{21}$, $b_2 = w_{12} \oplus w_{22}$. Now $w_{11} \oplus w_{12} \oplus a_2 = w_{11} \oplus w_{21} \oplus w_{12} \oplus w_{22}$. By cancellativity, $a_2 = w_{21} \oplus w_{22}$. □

If a cp-PAM P satisfies (WRDP1) or, equivalently, (WRDP2), we will say that P satisfies the *Riesz decomposition property* ((RDP) for short).

Example. It is straightforward to show that the set $\mathcal{E}(H)$ of the Hilbert space effects (that is self-adjoint operators on a Hilbert space H between 0 and I) does not satisfy the interpolation property. Indeed, take arbitrary $a, b \in \mathcal{E}(H)$ and let $L(a, b)$ denote the set of all lower bounds of a, b. Assume that the interpolation property is satisfied. Then $L(a, b)$ is upper directed, so that by the monotone convergence of self-adjoint operators (see e.g. [53, Lemma 1]), $L(a, b)$ has a supremum, which is the infimum of a, b. But the non-lattice structure of $\mathcal{E}(H)$ is well known (see e.g. [25]) (see also [11] for a more explicit proof). Since by [49], the Riesz decomposition property in effect algebras implies the interpolation property, it follows that the Riesz decomposition property is not satisfied either.

Proposition 4.3. *Let P be a PAM satisfying* (WRDP1) *[*(WRDP2)*]. If $\sim$ is a weak congruence satisfying* (C5)*, then $P/{\sim}$ also satisfies* (WRDP1) *[*(WRDP2)*].*

Proof. Let (WRDP1) hold. Assume that $[a] \leq [b] \oplus [c]$. Without loss of generality, we may assume that $b \perp c$. Let $d \in P$ be such that $[a] \oplus [d] = [b] \oplus [c] = [b \oplus c]$. There are $a_1 \sim a, d_1 \sim d$ with $a_1 \perp d_1$, and $a_1 \oplus d_1 \sim b \oplus c$. By (C5), $a_1 \oplus d_1 = b_1 \oplus c_1$, $b_1 \sim b$, $c_1 \sim c$. By (WRDP1) in P, $a_1 = b_2 \oplus c_2$, where $b_2 \leq b_1, c_2 \leq c_1$. This entails that $[a] = [b_2] \oplus [c_2]$, $[b_2] \leq [b], [c_2] \leq [c]$.

Let (WRDP2) hold. Assume that $[a] \oplus [b] = [c] \oplus [d]$. Without loss of generality we may assume that $a \perp b$ and $c \perp d$, that is, $a \oplus b \sim c \oplus d$. By (C5), there are c_1, d_1 such that $c_1 \sim c, d_1 \sim d$, and $a \oplus b = c_1 \oplus d_1$. The desired result is now easily obtained by application of (WRDP2) in P. □

Let P be a PAM and let $\sim$ be a binary relation. We will consider the following properties:

- (S1) $a \leq b \oplus c$ implies that there are a_1, a_2 such that $a = a_1 \oplus a_2$, and $a_1 \sim b_1$, $a_2 \sim c_1$ for some $b_1 \leq b$, $c_1 \leq c$.
- (S2) If $a \oplus b = c \oplus d$, then there are elements e, f, E, F in P such that $a = e \oplus E$, $b = f \oplus F$ and $c \sim e \oplus f$, $d \sim E \oplus F$.

Theorem 4.4. *Let P be a cp-PAM and let $\sim$ be a weak congruence satisfying* (C5)*.* (i) *$[a] \leq [b]$ if and only if there is $b_1 \leq b$, $b_1 \sim a$.* (ii) *$P/{\sim}$ satisfies* (WRDP1) *if and only if $\sim$ satisfies* (S1)*.* (iii) *$P/{\sim}$ satisfies* (WRDP2) *if and only if $\sim$ satisfies* (S2)*.* (iv) *If $\sim$ satisfies* (C4)*, then* (S1) *and* (S2) *are equivalent.*

Proof. (i) If $a \sim b_1$ and $b_1 \leq b$, then there is c with $b_1 \oplus c = b$. Therefore, $[a] \oplus [c] = [b_1 \oplus c] = [b]$, so that $[a] \leq [b]$.

Conversely, assume that $[a] \leq [b]$. Then there is a c with $[a] \oplus [c] = [b]$. It follows that there are $a_1 \sim a, c_1 \sim c$, $a_1 \perp c_1$ such that $b \sim a_1 \oplus c_1$. By (C5), there are b_1, b_2 such that $b = b_1 \oplus b_2$ and $b_1 \sim a_1$, $b_2 \sim c_1$. It follows that $b_1 \leq b$, and $b_1 \sim a$.

(ii) Let $\sim$ be a weak congruence satisfying (C5) and (S1). Assume that $[a] \leq [b] \oplus [c]$. Then there are $b_1 \sim b$, $c_1 \sim c$, $b_1 \perp c_1$ such that $[b] \oplus [c] = [b_1 \oplus c_1]$. Moreover, there is $d \leq b_1 \oplus c_1$ such that $a \sim d$. By (S1)), $d \sim b_2 \oplus c_2$, $b_2 \leq b_1$,

$c_2 \leq c_1$. Therefore $[a] = [d] = [b_2 \oplus c_2] = [b_2] \oplus [c_2]$, $[b_2] \leq [b]$, $[c_2] \leq [c]$, which proves (WRDP1).

Conversely, if $P/{\sim}$ satisfies (WRDP1), and $a \leq b \oplus c$, then $[a] \leq [b] \oplus [c]$. By (WRDP1) and (C5), $[a] = [a_1 \oplus a_2]$, $[a_1] \leq [b]$, $[a_2] \leq [c]$. By (i) there are $b_1 \leq b$, $c_1 \leq c$ with $a_1 \sim b_1$, $a_2 \sim c_1$, and $a \sim a_1 \oplus a_2 \sim b_1 \oplus c_1$. This proves (S1).

(iii) Let (S2) be satisfied. Assume $[a] \oplus [b] = [c] \oplus [d]$. Without loss of generality, we may assume that $a \oplus b \sim c \oplus d$. By (C5), $a \oplus b = c_1 \oplus d_1$, $c_1 \sim c$, $d_1 \sim d$. By (S2), there are e, f, E, F such that $a = e \oplus E$, $b = f \oplus F$ and $c \sim c_1 \sim e \oplus f$, $d \sim d_1 \sim E \oplus F$. It follows that $[a] = [e] \oplus [E]$, $[b] = [f] \oplus [F]$, $[c] = [e] \oplus [f]$, $[d] = [E] \oplus [F]$, so (WRDP2) is satisfied.

Conversely, assume that $P/{\sim}$ satisfies (WRDP2) and assume $a \oplus b = c \oplus d$. Then $[a] \oplus [b] = [c] \oplus [d]$, and there are $[e], [E], [f], [F]$ such that $[a] = [e] \oplus [E]$, $[b] = [f] \oplus [F]$, $[c] = [e] \oplus [f]$ and $[d] = [E] \oplus [F]$. Using (C5) and properties of $\sim$, we get that $a = e_1 \oplus E_1$, $e_1 \sim e$, $E_1 \sim E$, $b = f_1 \oplus F_1$, $f_1 \sim f$, $F_1 \sim F$, and $c = e_2 \oplus f_2$, $e_2 \sim e$, $f_2 \sim f$, $d = E_2 \oplus F_2$, $E_2 \sim E$, $F_2 \sim F$. From this we can deduce that $c \sim e_1 \oplus f_1$, $d \sim E_1 \oplus F_1$, where the existence of the sums follows by $a \perp b$. This proves (S2).

(iv) This follows by cancellativity of $P/{\sim}$. □

Sherstnev [51] introduced a *dimension OMP* as an OMP P with an equivalence relation $\sim$ satisfying the following axioms.

- (D1) $a \sim 0$ implies $a = 0$.
- (D2) If $\{a_i : i \in I\}$, $\{b_i : i \in I\}$ are summable families and $a_i \sim b_i$ for all $i \in I$, then $a = \bigoplus a_i, b = \bigoplus b_i$ exist in P and $a \sim b$.
- (D3) If $a \oplus b \sim c \oplus d$, then there are $e, f, E, F \in P$ such that $a = e \oplus E$, $b = f \oplus F$, $c \sim e \oplus f$, $d \sim E \oplus F$.
- (D4) If $a, b \in P$ and $a \not\perp b$ then there are nonzero $c, d \in P$ such that $a \geq c \sim d \leq b$.

We may conclude from (D2) and (D3) that $\sim$ is a weak congruence satisfying (C5). In fact, (D2) is much stronger than (C2), and expresses complete additivity of $\sim$. It also implies orthocompleteness of P. It can be easily seen that for a weak congruence, (D3) implies (C5).

A *dimension lattice* in the sense of Loomis [37] is an OML satisfying axioms (D1), (D2) and

- (B) If $x \sim y \oplus Y$, then there are $z, Z \in P$ with $x = z \oplus Z$, $z \sim y$, $Z \sim Y$.
- (D′) If x, y have a common complement in P then $x \sim y$.

Notice that in [37], also the weaker condition (C2) has been considered.

It has been shown that a dimension lattice in the sense of Loomis is always a dimension poset in the sense of Sherstnev (cg. [37, Lemma 43]), but there are OMLs which are dimension posets in the sense of Sherstnev, but which are not dimension lattices in the sense of Loomis [32], [3, Chap. VIII].

We will extend Sherstnev's definition as follows.

Definition 4.5. We will say that a cp-PAM P is a *dimension* cp-PAM if it is endowed with an equivalence relation $\sim$ that satisfies (D1), (C2), (D3) and (D4). The relation $\sim$ is called *dimension equivalence*.

We will sometimes make use of the stronger form (D2) of (C2).

Theorem 4.6. (a) *Let P be a dimension cp-PAM with dimension equivalence $\sim$. Then $P/\sim$ is a positive PAM satisfying* (WRDP 2). (b) *If, in addition, $(P, \sim)$ satisfies (the countable version of)* (D2)*, then $P/\sim$ is partially ordered.*

Proof. (a) This follows by Theorem 4.4. (b) The proof is in essential similar to that in [39, p. 77]. See [31, Proposition 3] for more details. □

Lemma 4.7. *Let P be a cp-PAM endowed with an equivalence relation $\sim$ satisfying* (D1)*,* (C2) *and* (D4)*. If $L/\sim$ happens to be a lattice ordered cp-PAM, then also* (D3) *is satisfied.*

Proof. By [47], in a lattice ordered cp-PAM, condition ($a \wedge b = 0$ implies $a \perp b$) is equivalent to RDP. Now condition (D4) implies that whenever $c \leq a$, $d \leq b$ and $c \sim d$, then $c = d = 0$. It follows that $[a] \wedge [b] = [0]$ implies $a \perp b$, hence $[a] \perp [b]$. □

It is worth noticing that the equality relation on any cp-PAM E with the Riesz decomposition properties satisfies (D1), (C2) and (D3). To prove (D4), we need to suppose that E is upper directed. If it is the case, then observe that (D4) means that $a \wedge b = 0$ implies $a \perp b$. Indeed, assume that $a, b \leq d$ and $a \wedge b = 0$. Then from $a \leq b \oplus (d \ominus b)$ (where $d \ominus b$ is defined by (d)) it follows by (RDP) that $a = a_1 \oplus a_2$, where $a_1 \leq b$, $a_2 \leq d \ominus b$. It follows that $a_1 = 0$, and so $a = a_2 \leq d \ominus b$, hence $a \perp b$. We will also observe that if (D2) holds then E is closed under infima. We will show it under a more general condition.

In the proofs of the next theorem, we follow the pattern of [28]. First we prove a lemma.

Lemma 4.8. *Let L be an m-orthocomplete cp-PAM, σ an ordinal number satisfying card$(\sigma) \leq m$, and $(y_\alpha : \alpha < \sigma)$ a family of elements from L satisfying*

(i) $y_0 = 0$,
(ii) $\alpha \leq \beta < \sigma \ \Rightarrow \ y_\alpha \leq y_\beta$ *(increasing),*
(iii) β *a limit ordinal* $< \sigma \ \Rightarrow \ \bigvee(y_\alpha : \alpha < \beta)$ *exists and* $= y_\beta$ *(continuous from the left).*

Then for every ordinal β satisfying $2 \leq \beta < \sigma$ we have

$$\bigvee(y_\alpha : \alpha < \beta) = \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta).$$

Proof. The join at the left-hand side of the assertion exists by assumption (iii). Indeed, assumption (iii) covers the case when β is a limit ordinal; if β is not a limit ordinal, then clearly $\bigvee(y_\alpha : \alpha < \beta) = y_{\beta-1}$. To prove that the $\oplus$-sum on the right-hand side exists, we need to prove that the family $(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta)$ is a summable family, and then use the m-orthocompleteness of L.

Define $z_\rho := y_{\rho+1} \ominus y_\rho$. Choose any finite subfamily $y_{\rho_1}, y_{\rho_2}, \ldots, y_{\rho_n}$, $\rho_i + 1 < \beta, i = 1, 2, \ldots, n$. We may assume that $\rho_1 < \rho_2 < \cdots < \rho_n$. Then we have

$$y_{\rho_1} \le y_{\rho_1+1} \le y_{\rho_2} \le y_{\rho_2+1} \le \cdots \le y_{\rho_n} \le y_{\rho_n+1}$$

and therefore

$$\begin{aligned} y_{\rho_n+1} \ominus y_{\rho_1} &= (y_{\rho_1+1} \ominus y_{\rho_1}) \oplus (y_{\rho_2} \ominus y_{\rho_1+1}) \\ &\quad \oplus (y_{\rho_2+1} \ominus y_{\rho_2}) \oplus \cdots \oplus (y_{\rho_{n+1}} \ominus y_{\rho_n}) \\ &\ge z_{\rho_1} \oplus z_{\rho_2} \oplus \cdots \oplus z_{\rho_n}. \end{aligned}$$

This proves that the family $(z_\rho : \rho + 1 < \beta)$ is a summable family. Moreover, from the above inequality we see that for any finite family $\rho_1, \ldots, \rho_n$, $z_{\rho_1} \oplus \cdots \oplus z_{\rho_n} \le y_{\rho_n+1}$, and therefore

$$\bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta) \le \bigvee(y_\alpha : \alpha < \beta).$$

We need therefore only to prove the statement $P(\beta)$: $\bigvee(y_\alpha : \alpha < \beta) \le \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta)$. P(2) is the assertion $y_1 \le y_1 \ominus y_0$, which is true because $y_0 = 0$. Assume that $P(\gamma)$ is true for all $\gamma < \beta$. If β is a limit ordinal, then for any $\alpha < \beta$, $\alpha + 1 < \beta$, and using the induction hypothesis,

$$\begin{aligned} y_\alpha = \bigvee(y_\sigma : \sigma < \alpha) &= \bigvee(y_\sigma : \sigma < \alpha + 1) \\ &\le \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \alpha + 1) \\ &\le \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta). \end{aligned}$$

Hence $\bigvee(y_\alpha : \alpha < \beta) \le \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta)$. If β is not a limit ordinal, then $\bigvee(y_\alpha : \alpha < \beta) = \bigvee(y_\alpha : \alpha \le \beta - 1)$. Now there are two possibilities: either $\beta - 1$ is a limit ordinal or it is not. If $\beta - 1$ is a limit ordinal, then by (iii) and the induction hypothesis,

$$\begin{aligned} y_{\beta-1} = \bigvee(y_\alpha : \alpha < \beta - 1) &\le \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta - 1) \\ &\le \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta) \end{aligned}$$

and we are done. If $\beta - 1$ is not a limit ordinal, then

$$\begin{aligned}\bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta) &= \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 \le \beta - 1) \\ &= (y_{\beta-1} \ominus y_{\beta-2}) \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta - 1) \\ &\ge (y_{\beta-1} \ominus y_{\beta-2}) \oplus \bigvee(y_\alpha : \alpha < \beta - 1) \\ &= (y_{\beta-1} \ominus y_{\beta-2}) \oplus y_{\beta-2} = y_{\beta-1},\end{aligned}$$

which proves $P(\beta)$. □

Theorem 4.9. *Every chain of at most m elements in an m-orthocomplete cp-PAM L has a supremum.*

Proof. Let $(x_\alpha : \alpha \in \Sigma)$ be an increasing chain in L with $\mathrm{card}\Sigma \le m$, and assume that the supremum of any chain with index set Σ' exists when $\mathrm{card}\Sigma' \le \mathrm{card}\Sigma$. Let σ be the least ordinal corresponding to $\mathrm{card}\Sigma$. We may assume that σ is infinite and replace Σ by the set $\{\alpha : \alpha < \sigma\}$. So we are dealing with an ordinal-indexed chain $(x_\alpha : \alpha < \sigma)$. By the induction hypothesis,

$$y_\gamma := \bigvee(x_\rho : \rho < \gamma)$$

exists for all $\gamma < \sigma$. This family $(y_\alpha : \alpha < \sigma)$ obviously satisfies conditions (i) and (ii) of Lemma 4.8 and (iii) follows by the fact that for β a limit ordinal $< \sigma$,

$$\begin{aligned}\bigvee(y_\alpha : \alpha < \beta) &= \bigvee_{\alpha<\beta} \bigvee(x_\rho : \rho < \alpha) \\ &= \bigvee(x_\rho : \rho < \beta) = y_\beta.\end{aligned}$$

The element $z = \bigoplus(y_{\alpha+1} \ominus y_\alpha : \alpha + 1 < \sigma)$ exists by m-orthocompleteness, and we show that z is the desired supremum $\bigvee(x_\rho : \rho < \sigma)$. If σ is not a limit ordinal, then $\bigvee(x_\rho : \rho < \sigma) = y_{\sigma-1}$, and we are done. Assume that σ is a limit ordinal. If $\beta < \sigma$, then σ being a limit ordinal, $\beta + 2 < \sigma$, hence

$$\begin{aligned}x_\beta &\le \bigvee(x_\rho : \rho < \beta + 1) = y_{\beta+1} \\ &= \bigvee(y_\alpha : \alpha \le \beta + 1) = \bigvee(y_\alpha : \alpha < \beta + 2) \\ &= \bigoplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta + 2) \le z,\end{aligned}$$

where in the second-to-the last step we used Lemma 4.8. This proves that z is an upper bound of $(x_\rho : \rho < \sigma)$. If $w \ge x_\rho$, $\forall \rho < \sigma$, then $w \ge \bigvee(x_\rho : \rho < \alpha + 1) = y_{\alpha+1} \ge y_{\alpha+1} \ominus y_\alpha$ for all $\alpha + 1 < \sigma$. For every finite set $\rho_1 < \rho_2 < \cdots < \rho_n$ with $\rho_n + 1 < \sigma$, we may complete the set to $\rho_1 \le \rho_1 + 1 \le \rho_2 \le \cdots \le \rho_{n-1} + 1 \le \rho_n \le \rho_n + 1$, and show that $(y_{\rho_1+1} \ominus y_{\rho_1}) \oplus (y_{\rho_2+1} \ominus y_{\rho_2}) \oplus \cdots \oplus (y_{\rho_n+1} \ominus y_{\rho_n}) \le y_{\rho_n+1} \ominus y_{\rho_1} \le y_{\rho_n+1} \le w$, and since z is defined as the supremum over all such finite $\oplus$-sums, it follows that $z \le w$. This proves the desired result. □

We will say that a cp-PAM is *Dedekind complete* if every upper bounded chain has a supremum. In particular, the preceding theorem implies that an orthocomplete cp-PAM is Dedekind complete.

Theorem 4.10. *Let L be a Dedekind complete cp-PAM with the (RDP). Then for any $a, b \in L$, the infimum $a \wedge b$ exists in L.*

Proof. If L has (RDP), it has the interpolation property. The proof is the same as the proof of [11, Proposition 3]. For the convenience of readers we include it here.

Assume that $a, b \leq c, d$, i.e., each of a, b is smaller then each of c, d. Then there is an $a_1 \in L$ such that $a \oplus a_1 = c$ and from $b \leq a \oplus a_1$ it follows by (RDP1) that there are $e \leq a$ and $\bar{b} \leq a_1$ such that $b = e \oplus \bar{b}$. Moreover, for some $\bar{a} \in L$ we have $a = e \oplus \bar{a}$, and since $e \leq a, b$, for some $\bar{c}, \bar{d}$ also $c = e \oplus \bar{c}$, $d = e \oplus \bar{d}$. By cancellation, we get $\bar{a}, \bar{b} \leq \bar{c}, \bar{d}$, and from $e \oplus \bar{a} \oplus \bar{b} \leq a \oplus a_1 = c = e \oplus \bar{c}$ it follows that $\bar{a} \oplus \bar{b} \leq \bar{c}$. Now choose $b_1 \in L$ such that $b_1 \oplus \bar{b} = \bar{d}$, and since $\bar{a} \leq b_1 \oplus \bar{b}$, there are $\bar{\bar{a}} \leq b_1$ and $f \leq \bar{b}$ such that $\bar{a} = \bar{\bar{a}} \oplus f$. Choose $\bar{\bar{b}} \in L$ such that $\bar{b} = \bar{\bar{b}} \oplus f$. Let $\bar{x} = \bar{\bar{a}} \oplus \bar{\bar{b}} \oplus f$. Then $\bar{a}, \bar{b} \leq \bar{x} \leq \bar{a} \oplus \bar{b} \leq \bar{c}$ as well as $\bar{x} \leq b_1 \oplus \bar{b} = \bar{d}$. So $x = e \oplus \bar{x}$ is the required interpolant to show that (RIP) holds.

Now assume that L is Dedekind complete, and let $L(a, b)$ be the set of all lower bounds of a, b. Let $\{c_i\}$ be a maximal chain in $L(a, b)$. By the Dedekind property, the supremum $c = \bigvee c_i$ exists in L, and $c \leq a, b$. If d is any lower bound of a, b such that $d \not\leq c$, then by the interpolation property there is x such that $c, d \leq x \leq a, b$. But this contradicts the maximality of $\{c_i\}$. Hence c is the infimum of a, b in L. □

5. Direct limits of cp-PAMs with binary relations

Let $\mathcal{P}$ denote the category with cp-PAMs as objects and monomorphisms as morphisms.

Definition 5.1. A *directed system* (DS) in the category $\mathcal{P}$ is a pair

$$(P_i; \{\phi^i{}_j\}_{i \leq j})_{j \in D}$$

satisfying the following conditions:

(DS1) $(D, \leq)$ is a directed set;
(DS2) P_i is a cp-PAM for every $i \in D$;
(DS3) If $i, j \in D$, $i \leq j$, then $\phi^i{}_j \colon P_i \to P_j$ is an injective monomorphism (of PAMs);
(DS4) $\phi^j{}_k \phi^i{}_j = \phi^i{}_k$ whenever $i \leq j \leq k$;
(DS5) $\phi^i{}_i = id_{P_i}$ for all $i \in D$.

Definition 5.2. A *direct limit* (DL) for a directed system is a pair

$$(P; \{\phi^i\}_i) \in \mathcal{P},$$

where P is a cp-PAM and each $\phi^i\colon P_i \to P$ is an injective monomorphism such that

(DL1) $\phi^j {\phi^i}_j = \phi^i$ for $i \le j$;

(DL2) If $\psi_i\colon P_i \to Q$, where Q is a cp-PAM and ψ_i is an injective monomorphism such that $\psi_j {\phi^i}_j = \psi_i$ for $i \le j$, then there exists a unique injective monomorphism $\psi\colon P \to Q$ such that $\psi_i = \psi\phi^i$ for all $i \in D$.

Recall that in the category of cp-PAMs, the Riesz decomposition properties (WRDP1) and (WRDP2) are equivalent (Lemma 4.2). If a cp-PAM P satisfies (WRDP1), or equivalently (WRDP2), we say that P satisfies the Riesz decomposition property.

Theorem 5.3. *A direct limit in the category $\mathcal{P}$ exists. Moreover, if every $P_i, i \in D$ satisfies the Riesz decomposition property, then the direct limit satisfies the Riesz decomposition property.*

Proof. Let $(P_i; \{{\phi^i}_j\}_{i \le j})_{i \in D}$ be a DS in the category $\mathcal{P}$. Assume $P_i \cap P_j = \emptyset$ if $i \ne j$, and put $X := \bigcup_{i \in D} P_i$. Define a binary relation $\equiv$ on X by $x \equiv y$, $x \in P_i$, $y \in P_j$ if there is $k \in D$, $k \ge i, j$ with ${\phi^i}_k(x) = {\phi^j}_k(y)$. We shall prove that $\equiv$ is an equivalence. It is clear that $\equiv$ is reflexive and symmetric. It can be verified that if $x \equiv y$, $x \in P_i$, $y \in P_j$, then for any $r \ge i, j$, ${\phi^i}_r(x) = {\phi^j}_r(y)$. Indeed, let $s \ge k, r$. Then ${\phi^i}_k(x) = {\phi^j}_k(y)$ implies ${\phi^k}_s {\phi^i}_k(x) = {\phi^k}_s {\phi^j}_k(y)$, but ${\phi^k}_s {\phi^i}_k = {\phi^i}_s = {\phi^r}_s {\phi^i}_r$. Therefore ${\phi^r}_s {\phi^i}_r(x) = {\phi^r}_s {\phi^j}_r(y)$, and since ϕ^r_s is injective, it follows ${\phi^i}_r(x) = {\phi^j}_r(y)$. In particular, if $i = j$, then $x = y$. To prove the transitivity of $\equiv$, assume $x \equiv y$, $y \equiv z$, where $x \in P_i$, $y \in P_j$, $z \in P_k$. Then there are $r, s \in D$ such that $r \ge i, j$, $\phi^i_r(x) = \phi^j_r(y)$; and $s \ge j, k$, $\phi^j_s(y) = \phi^k_s(z)$. Since D is directed, there is $t \ge r, s$ and we have $\phi^r_t(\phi^j_r(y)) = \phi^j_t(y) = \phi^s_t(\phi^j_s(y))$, hence $\phi^r_t(\phi^i_r(x)) = \phi^s_t(\phi^k_s(z))$, so that $x \equiv z$.

Put $\bar{x} := \{y \in X : y \equiv x\}$, and $P := \{\bar{x} : x \in X\}$. Define $\phi^i\colon P_i \to P$ by $\phi^i(x) = \bar{x}$. Clearly, $\phi^i(0) = \bar{0}$ and $\phi^j {\phi^i}_j(x) = \overline{{\phi^i}_j(x)} = \bar{x} = \phi^i(x)$, for any $x \in P_i$. Observe also that for every $x \in X$ and $i \in D$, $\mathrm{card}(\bar{x} \cap P_i) \le 1$, and for every $x, y \in X$ $\bar{x} = \bar{y}$ iff there is $i \in D$ with $\bar{x} \cap \bar{y} \cap P_i \ne \emptyset$.

Further, define on P a partial binary operation $\oplus$ as follows. $\bar{x} \oplus \bar{y}$ is defined if there is $i \in D$ such that $x_i \in \bar{x} \cap P_i$, $y_i \in \bar{y} \cap P_i$ and $x_i \perp y_i$. In this case put $\bar{x} \oplus \bar{y} = \overline{x_i \oplus y_i}$.

Now it is clear that ϕ^i is a monomorphism. If $\phi^i(x) = \phi^i(y)$, $x, y \in P_i$, then there is $j \ge i$ such that ${\phi^i}_j(x) = {\phi^i}_j(y)$, whence $x = y$, since ${\phi^i}_j$ is injective. So ϕ^i is injective.

Next we will prove that $(P; \perp, \oplus, 0)$ is a PAM. Commutativity of $\oplus$ is clear. To prove associativity, assume $\bar{x} \perp \bar{y}$, and $\bar{x} \oplus \bar{y} \perp \bar{z}$. Then there is $i \in D$ and $x_i \in \bar{x} \cap P_i$, $y_i \in \bar{y} \cap P_i$ with $x_i \perp y_i$, and $\bar{x} \oplus \bar{y} = \overline{x_i \oplus y_i}$. Moreover, there is $j \in D$ and $z_j \in \bar{z} \cap P_j$ with ${\phi^i}_j(x_i \oplus y_i) \perp z_j$. It then follows that $({\phi^i}_j(x_i \oplus_i y_i)) \oplus_j z_j = ({\phi^i}_j(x_i) \oplus_j {\phi_i}^j(y_i)) \oplus_j z_j = {\phi^i}_j(x_i) \oplus_j ({\phi^i}_j(y_i) \oplus_j z_j)$ (where we denoted by $\oplus_i, \oplus_j$ the operation in P_i, P_j, respectively) and this implies that $(\bar{x} \oplus \bar{y}) \oplus \bar{z} = \bar{x} \oplus (\bar{y} \oplus \bar{z})$.

It remains to prove cancellativity and positivity.

If $\bar{x} \oplus \bar{y} = \bar{x} \oplus \bar{z}$, then for a suitable $i \in D$, there are $x_i \in \bar{x} \cap P_i$, $y_i \in \bar{y} \cap P_i$, $z_i \in \bar{z} \cap P_i$, and $x_i \oplus y_i = x_i \oplus z_i$, hence $y_i = z_i$, and therefore $\bar{y} = \bar{z}$.

If $\bar{x} \oplus \bar{y} = \bar{0}$, and $x_i \in \bar{x} \cap P_i$, $y_j \in \bar{y} \cap P_j$, then for $k \geq i, j$ we have ${\phi^i}_k(x_i) \perp {\phi^j}_k(y_j)$ and ${\phi^i}_k(x_i) \oplus_k {\phi^j}_k(y_j) = 0_k$, and hence $\bar{x_i} = \bar{y_i} = \bar{0_k} = \bar{0}$.

Now assume that Q is a cp-PAM and $\psi_i\colon P_i \to Q$ is an injective monomorphism for any $i \in D$ such that $\psi_j {\phi^i}_j = \psi_i$. Define $\psi\colon P \to Q$ by $\psi(\bar{x_i}) = \psi_i(x_i)$ for $x_i \in P_i$. Suppose that $x_i \in \bar{x} \cap P_i$ and $x_j \in \bar{x} \cap P_j$. Pick $k \geq i, j$; then we have $x_k := \phi_k^i(x_i) = \phi_k^j(x_j) \in \bar{x} \cap P_k$. Thus

$$\psi_i(x_i) = \psi_k(\phi_k^i(x_i)) = \psi_k(x_k) = \psi_k(\phi_k^j(x_j)) = \psi_j(x_j).$$

This shows that ψ is well defined. It is straightforward that ψ is an injective morphism. If $\psi(\bar{x_i}) \perp \psi(\bar{y_j})$, then $\psi_i(x_i) \perp \psi_j(x_j)$, and for $k \geq i, j$, $\psi_i(x_i) = \psi_k {\phi^i}_k(x_i)$, $\psi_j(x_j) = \psi_k {\phi^j}_k(x_j)$, hence $\psi_k {\phi^i}_k(x_i) \perp \psi_k {\phi^j}_k(x_j)$, and since ψ_k is a monomorphism, ${\phi^i}_k(x_i) \perp {\phi^j}_k(x_j)$. Hence $\bar{x_i} \perp \bar{x_j}$, and

$$\begin{aligned} \psi(\bar{x_i}) \oplus \psi(\bar{x_j}) &= \psi_i(x_i) \oplus \psi_j(x_j) \\ &= \psi_k {\phi^i}_k(x_i) \oplus \psi_k {\phi^j}_k(x_j) \\ &= \psi_k({\phi^i}_k(x_i) \oplus {\phi^j}_k(x_j)) \\ &= \psi(\bar{x_i} \oplus \bar{x_j}). \end{aligned}$$

Therefore ψ is a monomorphism. It is straightforward to prove that ψ is unique.

Assume that in every P_i, $i \in D$, (WRDP2) is satisfied. Let $\bar{x} \oplus \bar{y} = \bar{u} \oplus \bar{v}$. Then for a suitable $k \in D$, there are $x_k \in \bar{x} \cap P_k$, $y_k \in \bar{y} \cap P_k$, $u_k \in \bar{u} \cap P_k$ and $v_k \in \bar{v} \cap P_k$ with $x_k \oplus y_k = u_k \oplus v_k$, and using (WRDP2) in P_k, we find appropriate elements $\bar{w_{ij}}, i, j = 1, 2$ as required by (WRDP2). □

In the next theorem, we consider a directed system of PAMs in the category $\mathcal{P}$ with a binary relation, and find a sufficient condition under which the direct limit can also be endowed with a similar binary relation.

Theorem 5.4. *Let $(P_i; \{{\phi^i}_j\}_{i \leq j})_{i \in D}$ be a directed system in the category $\mathcal{P}$. Assume that every P_i is equipped with a binary relation $\sim_i$. Let $(P; \{\phi_i\}_{i \in D})$ be its direct limit. Further, let the following condition be satisfied.*

(CDL) *If $a, b \in P_i$ and $a \sim_i b$, then for any $j \geq i$, ${\phi^i}_j(a) \sim_j {\phi^i}_j(b)$.*

Then P can be endowed with a binary relation $\sim$, extending all $\sim_i$. Moreover, if $\sim_i$ satisfy any of conditions (C1)–(C5), (S1), (S2), *then $\sim$ satisfies the same condition.*

Proof. Define a binary relation $\sim$ on P by

$$\bar{x} \sim \bar{y} \text{ if there is } i \in D \text{ and } x_i \in \bar{x} \cap P_i,\ y_i \in \bar{y} \cap P_i \text{ such that } x_i \sim_i y_i.$$

Then $\sim$ is well defined. Indeed, if $x_j \in \bar{x} \cap P_j$, then for any $k \geq i, j$,

$$\phi^j{}_k(x_j) = \phi^i{}_k(x_i) \sim_k \phi^i{}_k(y_i)$$

by the definition of $\bar{x}$ and (CDL), and hence $\bar{x_j} \sim \bar{y_i}$. Observe that if $\bar{x_i} \sim \bar{y_j}$ then for any $k \geq i, j$, $\phi^j{}_k(y_j) \sim_k \phi^i{}_k(x_i)$.

Now assume that $\sim_i$ is a weak congruence for any $i \in D$. We will prove that $\sim$ is also a weak congruence, i.e., conditions (C1) and (C2) are satisfied.

(C1) Let $\bar{x} \sim \bar{y}$ and $\bar{y} \sim \bar{z}$. Then there are $i, j \in D$ and $x_i \in \bar{x} \cap P_i$, $y_i \in \bar{y} \cap P_i$ with $x_i \sim_i y_i$, and $y_j \in \bar{y} \cap P_j$, $z_j \in \bar{z} \cap P_j$ with $y_j \sim_j z_j$. Then for any $k \geq i, j$ we have $\phi^i{}_k(x_i) \sim_k \phi^i{}_k(y_i)$ and $\phi^j{}_k(y_j) \sim_k \phi^j{}_k(z_j)$. As $y_i \equiv y_j$ and $k \geq i, j$, $\phi^i{}_k(y_i) = \phi^j{}_k(y_j)$, so that by transitivity of $\sim_k$, we get $\phi^i{}_k(x_i) \sim_k \phi^j{}_k(z_j)$, and hence $\bar{x} \sim \bar{z}$.

(C2) Assume $\bar{x} \perp \bar{y}$, $\bar{x_1} \perp \bar{y_1}$ and $\bar{x} \sim \bar{x_1}$, $\bar{y} \sim \bar{y_1}$. Then for suitable $i, j \in D$ there exists $x_i \in \bar{x} \cap P_i$, $x_{1i} \in \bar{x_1} \cap P_i$ with $x_i \sim_i x_{1i}$, and there exist $y_j \in \bar{y} \cap P_j$, $y_{1j} \cap P_j$ with $y_j \sim_j y_{1j}$. Since $\bar{x} \perp \bar{y}$, $\bar{x_1} \perp \bar{y_1}$ and (CDL) holds, we get for $k \geq i, j$, $\phi^i{}_j(x_i) \perp \phi^j{}_k(y_j)$, $\phi^i{}_k(x_{1i}) \perp \phi^j{}_k(y_{1j})$, and $\phi^i{}_k(x_i) \sim_k \phi^i{}_k(x_{1i})$, $\phi^j{}_k(y_j) \sim_k \phi^j{}_k(y_{1j})$. Therefore, by (C2) for $\sim_k$, we have $\phi^i{}_k(x_i) \oplus \phi^i{}_k(y_j) \sim_k \phi^i{}_k(x_{1i}) \oplus \phi^j{}_k(y_{1j})$ and hence $\bar{x} \oplus \bar{y} \sim \bar{x_1} \oplus \bar{y_1}$.

The proofs of other properties are analogous and we leave them to the reader. □

In the following theorem, we find a sufficient condition under which, roughly speaking, the quotient of a direct limit is a direct limit of quotients. Since the quotient of a cp-PAM with respect to a weak congruence satisfying (C5) is always positive, but not necessary cancellative, we introduce a category $\mathcal{P}_0$, where the objects are positive PAMs and morphisms are injective morphisms of PAMs.

Theorem 5.5. *Let $(P_i; \{\phi^i{}_j\}_{i \leq j})_{i \in D})$ be a directed system satisfying conditions of Theorem* 5.4.

Let $\bar{P}_i := P_i/{\sim_i}$, $i \in D$, and define, for $i \leq j$, $\bar{\phi^i}_j : \bar{P}_i \to \bar{P}_j$ by $\bar{\phi^i}_j[x]_i = [\phi^i{}_j(x)]_j$, where $[x]_i$ is the class in $\bar{P}_i$ containing $x, x \in P_i$. Then $(\bar{P}_i; \{\bar{\phi^i}_j\}_{i \leq j})$ is a DS in the category $\mathcal{P}_0$. Moreover, if $(P; \{\phi^i\}_{i \in D})$ is a DL for $(P_i; \{\phi^i{}_j\}_{i \leq j})$ in the category $\mathcal{P}$, then $(\bar{P}; \{\bar{\phi^i}\}_{i \in D})$, where $\bar{P} := P/{\sim}$, and $\bar{\phi^i} : \bar{P}^i \to \bar{P}$ is defined by $\bar{\phi^i}[x]_i = [\bar{x}]$, is the DL for $(\bar{P}_i; \{\bar{\phi^i}_j\}_{i \leq j})$ in the category $\mathcal{P}_0$.

Proof. Owing to (CDL), $\bar{\phi^i}_j$ is well defined, and it is an injective morphism.

Clearly, for any $i \in D$,

$$\bar{\phi^i}_i[x]_i = [\phi^i{}_i(x)]_i = [x]_i,$$

and for $i \le j \le k$,

$$\begin{aligned}\bar{\phi^j}_k \bar{\phi^i}_j [x]_i &= \bar{\phi^j}_k [{\phi^i}_j (x)]_j \\ &= [{\phi^j}_k {\phi^i}_j (x)]_k = [{\phi^i}_k (x)]_k \\ &= \bar{\phi^i}_k [x]_i.\end{aligned}$$

Define $\bar{\phi^i}\colon \bar{P}_i \to \bar{P}$ by $\bar{\phi^i}[x]_i = [\bar{x}]$, where $[\bar{x}]$ denotes the class containing x in $P/\sim\, = \bar{P}$. Since $x \sim_i y$, $x, y \in P_i$ implies $\bar{x} \sim \bar{y}$, $\bar{\phi^i}$ is well defined. As a composition of morphisms, $\bar{\phi^i}$ is a morphism, and it can be easily seen that it is injective.

The remaining part of the proof is similar to that of Theorem 5.3. □

6. cp-PAMs with RDP and interpolation groups

Let P be a cp-PAM, (G, G^+) an abelian partially ordered group. We shall say that G is a *universal group for P* if:

(1) There is an injective monomorphism $\gamma\colon P \to G^+$ such that $\gamma(P)$ is a convex subgroup of G^+, and for any partially ordered abelian group (H, H^+) and any morphism $\alpha\colon P \to H^+$ there is a homomorphism $\alpha^*\colon G \to H$ of partially ordered groups such that $\alpha = \alpha^* \circ \gamma$.
(2) Every element g in G^+ is of the form $g = \sum_{i-1}^n \gamma(x_i)$ for some $x_1, \dots, x_n \in P$.
(3) $G = G^+ - G^+$.

Clearly, if such a universal group exists, it is unique up to isomorphism. Using a technique similar to Baer [1], (Wyler [55] and Ravindran ([49]), we obtain the following theorem (cf. [10, Theorem 1.7.14], we note that although in the quoted theorem injectivity of γ is not mentioned, it easily follows from the construction).

Theorem 6.1. *Let P be an upper directed cp-PAM with RDP. Then P has a universal group (G, γ). Moreover, G is an interpolation group.*

The following theorem extends the result of [44].

Theorem 6.2. *Let X be a cp-PAM such that $X \subseteq G^+$, (G, G^+) is a directed abelian group, and X is a convex generating subset of G^+ (in the sense that every element $g \in G^+$ is a finite sum of elements from X). The X is linearly ordered if and only if (G, G^+) is linearly ordered.*

Proof. If (G, G^+) is linearly ordered then X is linearly ordered.

Assume that X is linearly ordered. Let $g \in G$, then $g = g^+ - g^-$. Let $x_1 \le x_2 \le \dots \le x_n$, $y_1 \le y_2 \le \dots \le y_m$ be such that $g^+ = x_1 + \dots + x_n$, $g^- = y_1 + \dots + y_m$. Without loss of generality we may assume that $m = n$. We will proceed by induction with respect to n. If $n = 1$, we get either $g \le 0$ or $g \ge 0$. If $n = 2$ then $g^+ = x_1 + x_2$,

$g^- = y_1 + y_2$ and $g = g^+ - g^- = x_1 + x_2 - (y_1 + y_2) = x_1 - y_1 + x_2 - y_2$. Now one of $\pm(x_1 - y_1)$ and one of $\pm(x_2 - y_2)$ belong to X, and linearity of X implies that $g \geq 0$ or $g \leq 0$. Consequently, if $g \in 2X - 2X$ then $g \geq 0$ or $g \leq 0$, in particular, $2X$ is linearly ordered. Now let $g = x_1 + x_2 + x_3 + x_4 - (y_1 + y_2 + y_3 + y_4) = x_1 + x_2 - (y_1 + y_2) + x_3 + x_4 - (y_3 + y_4)$. According the previous step, one of $\pm(x_1 + x_2 - (y_1 + y_2))$ and one of $\pm(x_3 + x_4 - (y_3 + y_4))$ belong to $2X$, and hence $g \geq 0$ or $g \leq 0$. So $4X$ is linearly ordered. By induction we get $2^n X$, $n = 1, 2, \ldots$, is linearly ordered. Since $G^+ = \cup_{n=1}^{\infty} nX$, we get that G^+ is linearly ordered, and hence G is linearly ordered. □

Theorem 6.3. *An upper directed cp-PAM P with RDP is lattice ordered if and only if its universal group is lattice ordered.*

Proof. Let P be an upper directed cp-PAM satisfying the Riesz decomposition property and let (G, γ) be its universal group.

If G is lattice ordered, then $\gamma(P)$, being a convex subset of G^+, is also lattice ordered.

Conversely, assume that P is lattice ordered. The result that G is lattice ordered can be obtained in the following two ways.

1. By [16], every directed interpolation group G is a subdirect product of anti-lattice ordered interpolation groups G_i (i.e., $a \wedge b$ in G_i exists iff $a \leq b$ or $b \leq a$). Clearly, an anti-lattice ordered group is lattice ordered iff it is linearly ordered. Then $G \subseteq \prod_i G_i$, each G_i being an anti-lattice ordered interpolation group. Put $X := \gamma(P) \subseteq G^+$, $X_i := \pi_i \circ \gamma(P) \subseteq G_i^+$, where π_i is the projection to G_i.

If P is lattice ordered, then X is also lattice ordered, since $\gamma\colon P \to X$ is an isomorphism. Then also X_i is lattice ordered, and hence linearly ordered for any i. Indeed, let $x_i, y_i \in X_i$. Then there are elements $x, y \in X$ such that $x_i = \pi_i(x), y_i = \pi_i(y)$. Since X is lattice ordered, $x \wedge y$ exists in X, and $\pi(x \wedge y) = x_i \wedge y_i \in X_i$. Since X_i is an antilattice, we have either $x_i \leq y_i$ or $y_i \leq x_i$. Since G is generated by X, each G_i is generated by X_i. Then X_i satisfies the assumptions of Theorem 6.2, and hence each G_i is linearly ordered.

To prove that G is a lattice, it suffices to show that G^+ is lattice ordered [17]. Let $g, h \in G^+$. Then $g = (g_i)_i$, $h = (h_i)_i$ with $g_i, h_i \in G_i^+$. Since G_i is linearly ordered, we have $g_i \wedge h_i \in G_i^+$ for all i. Put $u := (g_i \wedge h_i)_i$. Clearly, u is a lower bound of g, h in $\prod G_i$. We need to prove that $u \in G^+$. Since G^+ is generated by X, we have $g = a^1 + a^2 + \cdots + a^n$ for some $a^j \in X, j = 1, \ldots, n$. Then $0 \leq u \leq a^1 + \cdots + a^n$ in $\prod_i G_i$. Since $\prod_i G_i$ is lattice ordered, it is an interpolation group, and hence there are $0 \leq u^i \leq a^i$, $i = 1, \ldots, n$ such that $u = u^1 + \cdots + u^n$. Since X is convex, we have $u^i \in X$ $\forall i$, hence $u \in G^+$. If $z = (z_i)_i \in G^+$ is any lower bound of g, h, then $\forall i$, $z_i \leq g_i, h_i$, hence $z_i \leq g_i \wedge h_i$, and so $z \leq u$. This proves that $u = g \wedge h$ in G^+.

2. Recall that in every abelian group, the following equality holds (in the sense that if one side is defined, so is the other and they are equal):

$$(g_1 \wedge g_2) + h = (g_1 + h) \wedge (g_2 + h) \tag{1}$$

[20, Proposition 1.4].

Let $g \in G^+$, then $g = a_1 + \cdots + a_n$, $a_i \in X$, $i = 1, \ldots, n$. We may assume that this decomposition is minimal in the sense that $\forall i, j, i \neq j$, $a_i + a_j \notin X$. First we prove that for every $g \in G^+$ and $b \in X$, $g \wedge b$ exists in G^+. We proceed by induction on n. If $n = 1$, then $g = a_1 \in X$, hence $g \wedge b$ exists in X, and by convexity of X, it is also a g.l.b. of g and b in G^+. Assume that $g \wedge c$ exists in G^+ if $n < k$ and $c \in X$, and let $g_1 = a_1 + a_2 \cdots + a_k$. Put $g_2 = a_2 + \cdots + a_k$. Then by the induction hypothesis, $g_2 \wedge (b \ominus a_1 \wedge b)$ exists in G^+. Using (1), we have

$$\begin{aligned} a_1 \wedge b + g_2 \wedge (b - a_1 \wedge b) &= (a_1 \wedge b + g_2) \wedge (a_1 \wedge b + (b - a_1 \wedge b)) \\ &= (a_1 \wedge b + g_2) \wedge b = ((a_1 + g_2) \wedge (b + g_2)) \wedge b \\ &= g_1 \wedge b. \end{aligned}$$

This proves that $g_1 \wedge b$ exists in G^+.

Now let $g, h \in G^+$, and let $h = b_1 + \cdots + b_n$, where $b_i \in X$, $i = 1, \ldots, n$. We proceed by induction on n. If $n = 1$, then $h \in X$, and $g \wedge h$ exists by the previous part of the proof. Assume that $g \wedge h$ exists in G^+ if $g \in G^+$ and $n < k$. Let $h_1 = b_1 + \cdots + b_{k-1} + b_k$, and put $h_2 = b_1 + \cdots + b_{k-1}$. By the induction hypothesis, $b_k \wedge g$ and $(g - b_k \wedge g) \wedge h_2$ exist in G^+. We then have

$$\begin{aligned} ((g - b_k \wedge g) \wedge h_2) + b_k \wedge g &= ((g - b_k \wedge g) + b_k \wedge g) \wedge (h_2 + b_k \wedge g) \\ &= g \wedge (h_2 + b_k \wedge g) = g \wedge ((h_2 + b_k) \wedge (h_2 + g)) \\ &= g \wedge (h_1 \wedge (h_2 + g)) = g \wedge h_1. \end{aligned}$$

This proves that $g \wedge h_1$ exists in G^+; applying [20, Proposition 1.5], concludes the proof. □

Remark. An example of a not necessarily upper directed cp-PAM which is not a lattice but has a lattice ordered universal group can be obtained taking the universal group for a commutative BCK-algebra with the relative cancellation property.

Recall [10] that a commutative BCK-algebra A [56] is a structure $(A, *, 0)$ with a binary operation $*$ and a constant 0 such that the following properties hold for any $x, y, z \in A$:

(i) $x * (x * y) = y * (y * x)$,
(ii) $x * (y * z) = (x * z) * y$,
(iii) $x * x = 0$,
(iv) $x * 0 = x$

We note that A is partially ordered by $x \leq y$ iff $x * y = 0$, and for any $x, y \in A$, the infimum exists, namely $x \wedge y = x * (x * y)$. A BCK-algebra A has the *relative cancellation property* [10] if

(v) for any $a \leq x, y$, $x * a = y * a$ implies $x = y$.

A commutative BCK-algebra A can be endowed with a partially defined binary operation $+$ with the domain $S := \{(a, b) \in A \times A :$ there exists a $c \in A$ with $c \geq b$ and $a = c * b\}$, and for $(a, b) \in S$, we put $c = a + b$ if $c \geq b$, $a = c * b$. Due to (v), $a + b$ is well defined. From [10, Theorem 5.2.6], it can be derived that $(A, +)$ is a cp-PAM, and from [10, Theorem 5.2.8], that it satisfies the RDP. In the following proposition we prove a converse statement.

Proposition 6.4. *Every cp-PAM with RDP which is a lower semilattice can be endowed with a structure of a commutative BCK-algebra with the relative cancellation property.*

Proof. According to [50, Theorem 2.1 (ii)], it suffices to prove that for any $a \in A$ there is an operation $*$ on the set $[0, a]$ such that $([0, a], *, 0)$ is a commutative BCK-algebra consistent with $([0, a], \wedge, 0)$.

Assume that A is a cp-PAM with RDP which is a lower semilattice. For $x, y \in A$, define

$$x * y := x \ominus (x \wedge y).$$

Owing to cancellativity and positivity, the operation $*$ is well defined. We have $x * y = 0$ iff $x \ominus x \wedge y = 0$ iff $x = x \wedge y$ iff $x \leq y$, so that the order induced by $*$ coincides with the operation $\wedge$.

Observe that

$$\begin{aligned} x * (x * y) = x * (x \ominus (x \wedge y)) &= x \ominus (x \wedge (x \ominus (x \wedge y))) \\ &= x \ominus (x \ominus (x \wedge y)) \\ &= x \wedge y. \end{aligned}$$

Similarly we get that also $y * (y * x) = y \wedge x = x \wedge y$.

Any interval $[0, z]$, $z \in A$, bears a structure of an effect algebra with unit z and partial binary operation $\oplus_z$ defined as follows: $x \perp_z y$ if $x \perp y$ and $x \oplus y \leq z$, and then $x \oplus_z y = x \oplus y$. Now $x \leq y \leq z$ iff there is $v \in A$ such that $x \oplus v = y$, and since necessarily $v \leq z$, this happens if and only if $x \oplus_z v = y$. Hence the order induced by $\oplus_z$ coincides with the operation $\wedge$ in A. Moreover, $[0, z]$ can be endowed with the relative complement $x' = z \ominus x$. Observe that for $x, y \leq z$,

$$(y \ominus x \wedge y) \oplus (z \ominus y) \oplus x \wedge y = z,$$

so that $(y * x)' = y' \oplus x \wedge y$.

Now for every $x, y, z \in A$, if $x, y \le z$, then $(z \ominus x) \wedge (z \ominus y) \le z$, and $z \ominus ((z \ominus x) \wedge (z \ominus y)) = x \vee y$. Indeed, $z \ominus ((z \ominus x) \wedge (z \ominus y)) \ge x, y$, and if $d \le z$ is such that $x, y \le d$, then $z \ominus d \le z \ominus x, z \ominus y$, hence $z \ominus d \le ((z \ominus x) \wedge (z \ominus y))$. Consequently, $z \ominus ((z \ominus x) \wedge (z \ominus y)) \le d$. Therefore, the interval $[0, z]$ is a lattice, and $[0, z]$ inherits the RDP. It follows that $[0, z]$ is an MV-algebra [46], and hence a bounded commutative BCK-algebra [43]. Applying the result [50, Theorem 2.1], we get that A is a commutative BCK-algebra. The relative cancellation property (v) is straightforward. □

The universal group of a commutative BCK-algebra with the relative cancellation property A is a lattice ordered abelian group. Also, to every A there is a lattice ordered BCK-algebra $\tilde{A}$, called the BCK-hull of A, with the same universal group [10].

From Theorem 4.10 we obtain the following result.

Corollary 6.5. *A Dedekind complete cp-PAM with the RDP is a commutative BCK-algebra with the relative cancellation property.*

Recall that an abelian partially ordered group (G, G^+) is *unperforated* (or has an isolated order) if whenever $ng \ge 0$ for some $g \in G$ and $n \in \mathbb{N}$, then $g \ge 0$. An unperforated interpolation group is called a *dimension group*. According to [5, Corollary 4.6.5], an abelian ordered group G is isomorphic, as an ordered group, with a subgroup of a lattice ordered group if and only if G is unperforated.

Theorem 6.6. *Let X be an upward directed cp-PAM with the Riesz decomposition property. Then its universal group (G, G^+) is unperforated if and only if X can be embedded into the positive cone of an abelian lattice ordered group H.*

Proof. By [5], G is unperforated iff it is a subgroup of an abelian lattice ordered group. Assume first that there is an embedding $h\colon X \to H^+$, where H is an abelian lattice ordered group. Let $\gamma\colon X \to G^+$ be the universal embedding. Then there is a unique order-preserving group homomorphism $h^*\colon G \to H$ such that $h = h^* \circ \gamma$. Since h is injective and X generates G, h^* is injective. Hence G, as a subgroup of a lattice ordered group H, is unperforated.

Conversely, if G is unperforated, then G is isomorphic, as an ordered group with a subgroup of an lattice ordered group H, hence X can be embedded into H^+, which is lattice ordered. □

Corollary 6.7. *The universal group of an interval effect algebra E is unperforated if and only if E can be embedded into an MV-algebra as a sub-effect algebra. In particular, an interval effect algebra E with RDP has a dimension group as a universal group iff E can be embedded into an MV-algebra M as a sub-effect algebra.*

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Proof. Assume that E can be embedded into an MV-algebra M. The universal group $G(M)$ of M is lattice-ordered ([42, 49]) and M is isomorphic with an interval $[0, u]$ of its positive cone $G(M)^+$. By the properties of universal groups, $G(E)$ is isomorphic (as an ordered group) with a subgroup of $G(M)$. According to Theorem 6.6, the universal group $G(E)$ of E is unperforated.

Conversely, if $G(E)$ is unperforated, then there is a lattice ordered group H such that $G(E)$ is isomorphic as an ordered group with a subgroup H_1 of H. Let u be the image of the unit element 1 of E in H_1, then E is embedded into the interval $[0, u] \cap H_1$ of $H_1^+ \subseteq H^+$. Now $u \in H^+$, and the interval $[0, u]$ in H^+ is an MV-algebra, and E is isomorphic with a sub-effect algebra of it. □

7. Elements of a dimension theory for dimension effect algebras

In what follows, we will consider an effect algebra L endowed with a dimension relation $\sim$ satisfying (D1)–D(4). We already know that these conditions imply (C5).

Recall that a nonzero element a in L is called *sharp* if it is disjoint with its orthosupplement, i.e., $a \wedge a' = 0$. In an orthomodular poset (or even in an orthoalgebra) every nonzero element is sharp, but it is not the case in a general effect algebra.

Two elements $a, b \in L$ will be called *related* ($a \,\rho\, b$) if there exist nonzero elements $a_1 \leq a$ and $b_1 \leq b$ such that $a_1 \sim b_1$. If a, b are not related, we say that they are *unrelated* and write $a \,\bar{\rho}\, b$. Observe that by (D4), if the elements a, b are unrelated, then they are orthogonal and disjoint (i.e., $a \wedge b = 0$). We will write $a \preceq b$ if $a \sim b_1 \leq b$. Then $\preceq$ is a partial order by Theorem 4.6(b). An element $a \in L$ is called *finite* if $a \sim a_1 \leq a$ implies $a_1 = a$. An element a is *simple* if for any $b \leq a$, $b \,\bar{\rho}\, a \ominus b$.

We will say that L is a *factor* if any two nonzero elements in L are related. We will say that a factor L is of type I if there is an atom in L; if there are no atoms in L but there is a finite element, we will say that L is of type II, in remaining cases we say that L is of type III.

The following statements can be proved similarly as in [37], using only (D1), (D2) and (C5).

Lemma 7.1. (i) *If a is finite and $b \leq a$, then b is finite.*
(ii) *If a is finite and $b \sim a$, then b is finite.*
(iii) *If a is simple, then a is finite.*
(iv) *If a is simple and $b \leq a$, then b is simple.*
(v) *If a is simple and $b \sim a$, then b is simple.*
(vi) *If $a = \bigoplus_{n=1}^{\infty} a_n$, where the a_n are summable, equivalent and nonzero, then a is not finite. (Indeed, $a = \bigoplus_{n=1}^{\infty} a_n \sim \bigoplus_{n=2}^{\infty} a_n = a \ominus a_1$).*

Lemma 7.2. [37, Lemma 14] *Given elements a and b in L, there exist subelements $a_0 \le a$ and $b_0 \le b$ such that $a_0 \sim b_0$ and $a \ominus a_0$ is unrelated to $b \ominus b_0$.*

Proof. Let $\{a_\alpha, b_\alpha\}$ be a maximal collection of pairs such that $\{a_\alpha\}$ and $\{b_\alpha\}$ are summable families, and $\bigoplus a_\alpha \le a$, $\bigoplus b_\alpha \le b$, $a_\alpha \sim b_\alpha$. Set $a_0 := \bigoplus a_\alpha$, $b_0 := \bigoplus b_\alpha$. Then $a_0 \sim b_0$ by (D2). Moreover, $a \ominus a_0 \;\bar{\rho}\; b \ominus b_0$, otherwise the maximal family $\{a_\alpha, b_\alpha\}$ could be enlarged by adding a pair of subelements from $a \ominus a_0$ and $b \ominus b_0$, respectively. □

Directly from Lemma 7.2 we obtain the following.

Corollary 7.3. *If L is a factor and $a, b \in L$, then either $a \preceq b$ or $b \preceq a$.*

Lemma 7.4. *If L is a factor and $a \sim b$, $a_1 \sim b_1$, $a_1 \le a$, $b_1 \le b$ and all four elements are finite, then $a \ominus a_1 \sim b \ominus b_1$.*

Proof. Since L is a factor, we have either $a \ominus a_1 \preceq b \ominus b_1$, or $b \ominus b_1 \preceq a \ominus a_1$. If the first inequivalence were proper, then adding the equivalence $a_1 \sim b_1$ and using (D2), a would be equivalent to a subelement of b, and since $a \sim b$, this would contradict the finiteness of b. The second case is symmetric. □

Lemma 7.5. *If a is finite and $b \neq 0$, then there exists a unique integer n such that any maximal family of summable elements equivalent to b with the sum included in a contains n elements.*

Proof. Since a is finite, Lemma 7.1 (vi) implies that every summable family of elements $b_i \sim b$ such that $\bigoplus b_i \le a$ must be finite. Let $b_1, \dots, b_n$ and $c_1, \dots, c_m$ be any two such maximal families, and suppose, e.g., that $n < m$. Then the complement $a \ominus (\bigoplus_{i=1}^n b_i)$ includes no images of b (by the maximality of the set $\{b_j\}$), whereas $a \ominus (\bigoplus_{j=1}^n c_j)$ includes the image of c_m, and this contradicts the fact that these two complements are equivalent by Lemma 7.4. Thus all maximal sets contain the same number of elements. □

Now assume that there exists an atom p, that is, a nonzero element having no proper subelements except 0. Since p is related to all other elements, it follows that for every nonzero finite element a there exists a unique integer m such that a is exactly a sum of m summable images of p. If a and b are finite and orthogonal, then if $a = \bigoplus_{i=1}^n a_i$, $b = \bigoplus_{j=1}^m b_j$, with $a_i \sim b_j \sim p$, then by (C2), $a \oplus b \sim (\bigoplus a_i) \oplus (\bigoplus b_j)$. It follows that $a \oplus b$ can be decomposed into $m + n$ replicas of p.

If we define $\alpha(a)$ to be this unique integer, then two finite elements a and b are equivalent if and only if $\alpha(a) = \alpha(b)$. If 1 is finite, with $\alpha(1) = N$, we normalize α by dividing through by N. Values of the normalized α are the fractions $\frac{m}{N}$, $0 \le m \le N$. In this case L is said to be a *factor of type I_N*.

We will call a real-valued function D defined on the set L_k of all finite elements of L a *dimension function* if the following properties are satisfied:

(1) $D(0) = 0$, if 1 is finite then $D(1) = 1$.
(2) If $\bigoplus a_i \in L_k$, then $D(\bigoplus a_i) = \sum D(a_i)$.
(3) $a \sim b$ iff $D(a) = D(b)$.

If L is a factor of type I_N, put $\mathrm{D} \equiv \alpha/N$. It is easy to see that properties (1)–(3) above are satisfied.

Lemma 7.6. (see [37, Lemma 12]). *Let L be a factor. If $a \in L$ is infinite, then a includes a sequence of orthogonal nonzero equivalent elements.*

Proof. If a is not finite, then there exists $a_1 < a$ such that $a_1 \sim a$. We set $c_1 = a \ominus a_1$, then c_1 is not zero. From $a_1 \sim a = a_1 \oplus c$ we get by (C5) that $a_1 = a_2 \oplus c_2$ with $a_2 \sim a_1$ and $c_2 \sim c_1$. Continuing inductively, we obtain sequences $\{a_n\}$ and $\{c_n\}$ such that for every n, $a_n \perp c_n$, $a_n \bigoplus c_n = a_{n-1}$, $a_n \sim a_1 \sim a$ and $c_n \sim c_1$. The sequence $\{c_n\}$ has the desired properties. $\square$

Now we are able to prove the following statement.

Theorem 7.7. *Let L be a factor of type I. Then there exists a dimension function D on L_k such that for any $a, b \in L_k$, $a \perp b$ implies $D(a \oplus b) = D(a) + D(b)$. If L is of type I_N, then D admits values in $\{1/N, 2/N, \ldots, N-1/N, 1\}$. If 1 is not finite (i.e., L is of type I_∞), then D admits all positive integers.*

Proof. If $1 \in L_k$, define $D(a) = \alpha(a)/\alpha(1)$, $a \in L$. If $1 \notin L_k$, put $D(a) = \alpha(a)$, $a \in L$. By Lemma 7.6, $1 = \bigoplus_{i=1}^{\infty} p_i$, where p_i are equivalent images of an atom p. Now $D(\bigoplus_{i=1}^{m} p_i) = m$ for any positive integer m. $\square$

If there are no atoms, then every element a contains a smaller element $0 < a_1 \leq a$, so that $a = a_1 \oplus (a \ominus a_1)$, and since we are in a factor, either $a_1 \preceq (a \ominus a_1)$, or $a \ominus a_1 \preceq a_1$. Taking the smaller (with respect to $\preceq$) of the two elements, we see that every nonzero element includes a pair of nonzero, orthogonal equivalent elements.

In the sequel, we follow the pattern of [37].

Lemma 7.8. *If there exist no atoms, then every element can be expressed as the $\bigoplus$-sum of n-summable equivalent subelements, for any n.*

Proof. Let $\{(a_\alpha^1, \ldots, a_\alpha^n)\}_\alpha$ be a maximal family such that $a_\alpha^1 \sim a_\alpha^2 \sim \cdots \sim a_\alpha^n$ for any α, and such that the elements a_α^n are all summable and all finite sums are under a. If $a_j = \bigoplus_\alpha a_\alpha^j$, then the elements a_j are summable and equivalent, and $\bigoplus_j a_j \leq a$.

We claim that $\bigoplus_j a_j = a$; otherwise $a \ominus (\bigoplus_j a_j)$ includes a family of n summable equivalent nonzero elements by the remark preceding this lemma, and this contradicts the maximality assumption. $\square$

If 1 is a finite element, we obtain a factor of type II_1. If 1 is not finite, but there is a nonzero finite element, we obtain a factor of type II_∞.

Theorem 7.9. *Let L be a factor of type II. Then there exists a dimension function D, defined on the class L_k of finite elements. If 1 is a finite element, the range of D is the unit interval $[0,1]$ of real numbers. If 1 is not finite, the range of D is the set of all nonnegative real numbers.*

Proof. The proof is in essential the same as in [37]. We give only a sketch of it.

In a factor of type II the following function is uniquely defined.

$$\delta_b(a) := \sup\left\{\frac{m}{n} : \frac{m}{n} b \preceq a\right\}, \quad a, b \in L_k, b \neq 0,$$

where the symbol $\frac{m}{n} b \sim a$ means that $a = \bigoplus_{i=1}^{m} a_i$, $b = \bigoplus_{j=1}^{n} b_j$, $a_1 \sim \cdots \sim a_m \sim b_1 \cdots \sim b_n$. The number $\delta_b(a)$ is called a *relative dimension of a with respect to b*. We have

$$\delta_c(a) = \delta_c(b)\delta_b(a), \ a, b, c \in L_k, \quad b, c \neq 0. \tag{2}$$

In particular, if $1 \in L_k$, then $D(a) = \delta_1(a)$ is the desired dimension function. If 1 is infinite then D can be defined as δ_b by some fixed b. Owing to (2), D is defined uniquely up to a multiplicative constant. □

It is not difficult to prove that a direct product of factors is a dimension effect algebra. In fact, a more general statement is true.

Proposition 7.10. *Let $L = \prod L_i$, where each L_i is an effect algebra with dimension. Then L is an effect algebra with dimension.*

Proof. Let $a, b \in L$, $a = \prod a_i$, $b = \prod b_i$. Define $a \sim b$ if and only if $a_i \sim_i b_i$ for all i. Conditions (D1)–(D4) are clearly satisfied, so that L is an effect algebra with dimension. □

The general dimension theory for orthomodular lattices and posets is based on the theorem, that any hereditary class of elements admits a supremum, which is an invariant element [37, 51]. Then it can be proved that every dimension poset decomposes into factors of type I, II and the remaining part is of type III. Here a subclass F of L is called *hereditary* if $a \in F, b \preceq a$ implies $b \in F$, and an element $c \in L$ is *invariant* if c and $c' = 1 \ominus c$ are unrelated. So far, we have not been able to find a suitable modification of the above theorem for effect algebras. Nevertheless, we prove some general properties of invariant elements that may be of interest in this context.

Let C_0 denote the set of all invariant elements of L. We have the following characterization of invariant elements.

Lemma 7.11. *An element $c \in L$ is invariant if and only if c is sharp and $a \leq c$ whenever $a \preceq c$.*

Proof. If c is invariant and $b \leq c$, $b \leq c'$, then $c \geq b \sim b \leq c'$ implies $b = 0$, since c and c' are unrelated. Hence c is sharp. Assume $a \preceq c$. Then $a \sim c_1 \leq c$. Let $a_1 \leq a$. Since $a_1 \oplus d = a \sim c_1$, (D3) implies that there are c_2, t such that $t \oplus c_2 = c_1$ and $c_2 \sim a_1$, $t \sim d$. If $b_1 \leq c'$, $b_1 \sim a_1$, then $c \geq c_2 \sim a_1 \sim b_1 \leq c'$. Therefore $b_1 = 0$. (D4) implies that $a \perp c'$, hence $a \leq c$.

Conversely, if c is sharp and has the property that $a \preceq c$ implies $a \leq c$, then $c \geq u \sim v \leq c'$ implies $v \leq c, c'$, so that $v = 0$. Hence c is invariant. □

It is easy to see that if a and b are unrelated and $a_1 \leq a$, $b_1 \leq b$ then a_1 and b_1 are unrelated. Conversely, if a and b are related and $a \leq a_2$, $b \leq b_2$ then a_2 and b_2 are related. In particular, if $c_1, c_2 \in C_0$ and $c_1 \perp c_2$, then any $a \leq c_1$ and $b \leq c_2$ are unrelated.

Proposition 7.12. *If L is a factor, then $C_0 = \{0, 1\}$.*

Proof. Since $c \in C_0$, c, c' are unrelated; but L is a factor, so any two nonzero elements are related. Thus, one of c and c' must be 0, whence, the other is 1. □

Recall that an element $z \in L$ is *principal* if $a, b \leq z$ and $a \perp b$ implies $a \oplus b \leq z$, and $z \in L$ is *central* if z and z' are principal and every $a \in L$ has a unique decomposition $a = a_1 \oplus a_2$ with $a_1 \leq z$ and $a_2 \leq z'$. If z is central then L can be expressed as a direct product of effect algebras $[0, z] \times [0, z']$. Conversely, if $L = L_1 \times L_2$, then the elements $(0, 1)$ and $(1, 0)$ are central in L [23].

Let C denote the set of all central elements in L. Then C is a complete Boolean algebra. (The σ-complete case has been treated in [31, Proposition 5], the proof for the complete case is analogous.) In what follows, we will show that C_0 is a complete Boolean subalgebra of C.

Recall that two elements a, b in L are *Mackey compatible* if $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \oplus b_1 \oplus c$ exists. Alternatively, a and b are Mackey compatible iff $a = a_1 \oplus a_2$ with $a_1 \leq b$ and $a_2 \leq b'$ [13].

Theorem 7.13. *The set of all invariant elements of a dimension effect algebra L is a complete Boolean subalgebra of the centre C of L.*

Proof. Let $z \in C_0$. First we prove that for every $x \in L$, z and x are Mackey compatible, that is, there are elements z_1, x_1, u such that $z_1 \oplus x_1 \oplus u$ exists and $z = z_1 \oplus u$, $x = x_1 \oplus u$. From $z \oplus z' = 1 = x \oplus x'$ we obtain by (D3) that $x = a_1 \oplus a_2$, $x' = b_1 \oplus b_2$, $a_1 \oplus b_1 \sim z$, $a_2 \oplus b_2 \sim z'$. Invariance implies that $a_1 \oplus b_1 \leq z$, $a_2 \oplus b_2 \leq z'$. From $1 = a_1 \oplus b_1 \oplus a_2 \oplus b_2 = z \oplus z'$ it follows that $a_1 \oplus b_1 = z$, and moreover, that $a_1 \oplus a_2 \oplus b_1$ is defined. Putting $u = a_1$, $x_1 = a_2$ and $z_1 = b_1$, this proves our statement.

Now we prove that z is principal. Let $x, y \leq z$, $x \perp y$. By the preceding paragraph, $x \oplus y$ is compatible with z, therefore $x \oplus y = a \oplus b$, $a \leq z$, $b \leq z'$.

Applying (D3), we obtain that $x = x_1 \oplus x_2 \leq z$, $y = y_1 \oplus y_2 \leq z$ and $x_1 \oplus y_1 \sim a \leq z$, $x_2 \oplus y_2 \sim b \leq z'$. Since $z \in C_0$, $x_2, y_2 \leq z, z'$ implies $x_2 \oplus y_2 = 0$. Therefore $x = x_1$, $y = y_1$ and $x \oplus y = x_1 \oplus y_1 \sim a \leq z$, and since $z \in C_0$, $x \oplus y \leq z$.

By the compatibility property, for any $x \in L$ there are $x_1 \leq z$, $x_2 \leq z'$ such that $x = x_1 \oplus x_2$. To prove uniqueness, assume that there are $y_1 \leq z$, $y_2 \leq z'$ such that $x = y_1 \oplus y_2$. Then $x_1 \oplus x_2 = y_1 \oplus y_2$ and by (D3), $x_1 = a \oplus b \leq z$, $x_2 = c \oplus d \leq z'$ and $a \oplus c \sim y_1 \leq z$, $b \oplus d \sim y_2 \leq z'$. Then we must have $b = c = 0$, hence $x_1 = a \sim y_1$, $x_2 = d \sim y_2$. So $x_1 \sim y_1$, $x_2 \sim y_2$ and $x_1 \oplus x_2 = y_1 \oplus y_2 = x$. From $x_1 \leq z$, $x_2 \leq z'$ it follows that there are u, v such that $x_1 \oplus u = z$, $x_2 \oplus v = z'$. Then

$$1 = x_1 \oplus x_2 \oplus u \oplus v = y_1 \oplus y_2 \oplus u \oplus v = z \oplus z'$$

where $y_1 \oplus u \sim x_1 \oplus u = z$, $y_2 \oplus v \sim x_2 \oplus v = z'$.

By the invariantness of z and the above equality, $y_1 \oplus u = z$, $y_2 \oplus v = z'$. This entails that $x_1 \oplus u = y_1 \oplus u$, $x_2 \oplus v = y_2 \oplus v$, hence $x_i = y_i$, $i = 1, 2$. This proves the uniqueness of the decomposition into two parts under z and z'. Therefore $C_0 \subset C$.

It remains to prove that C_0 is a Boolean subalgebra of C. Assume that $e, f \in C_0$, $e \perp f$ and $g = e \oplus f$. Let x, y be such that $g \geq x \sim y \leq g'$. By (D3) applied to

$$x \oplus (g \ominus x) = e \oplus f$$

we obtain

$$x = x_1 \oplus x_2, \quad g \ominus x = y_1 \oplus y_2$$
$$e \sim x_1 \oplus y_1, \quad f \sim x_2 \oplus y_2.$$

As e, f are invariant, we have

$$x_1 \oplus y_1 \leq e, \ x_2 \oplus y_2 \leq f,$$

while

$$g \geq x = x_1 \oplus x_2 \sim y \leq g'.$$

By (D3) $y = u \oplus v$ where $u \sim x_1$, $v \sim x_2$. So

$$e \geq x_1 \sim u \leq e', \ f \geq x_2 \sim v \leq f'$$

and therefore $x_1 = x_2 = 0$. Hence $x = 0$ and so $f \oplus g \in C_0$.

Let $z_i \in C_0$, $z = \bigwedge z_i$. Then $x \preceq z$ implies $x \preceq z_i$, hence $x \leq z_i$ for all i, and so $x \leq z$. Since $z \in C$, z is sharp, and hence $z \in C_0$. This concludes the proof. □

For $a \in L$, let $e(a) := \wedge\{z \in C_0 : z \geq a\}$. Then $e(a) \in C_0$. Moreover, if $a \sim b$ then $e(a) = e(b)$.

Proposition 7.14. (i) *Let* $\{a_i\}$ *be a family of elements in* L. *If* $\bigvee a_i$ *exists then* $e(\vee a_i) = \bigvee e(a_i)$. (ii) *If* $a, b \in L$, $a \perp b$, *then* $e(a) \vee e(b) \leq e(a \oplus b)$. (iii) *Let* $a \in L$. *If* $z \in C_0$ *then* $e(z \wedge a) = z \wedge e(a)$.

Proof. (i) Let $\bigvee a_i$ exist. As $e(a_i) \leq e(\bigvee a_i)$, we have $\bigvee e(a_i) \leq e(\bigvee a_i)$. On the other hand, $\bigvee a_i \leq \bigvee e(a_i) \in C_0$, and so $e(\bigvee a_i) = \bigvee e(a_i)$. (ii) is obvious. (iii) We have $z \wedge a \leq z \wedge e(a) \in C_0$, so that $e(z \wedge a) \leq z \wedge e(a)$. Now $e(z \wedge a) \vee z' \in C_0$, and so $e(z \wedge a) \vee z' \geq e(a)$. From this, $z \wedge e(a) \leq z \wedge e(z \wedge a) \leq e(z \wedge a)$. Hence $e(z \wedge a) = z \wedge e(a)$. □

Proposition 7.15. *If $a \sim b$ and z is invariant then $a \wedge z \sim b \wedge z$.*

Proof. Since $a = (z \wedge a) \oplus (z' \wedge a) \sim b$, it follows by (D3) that there are b_1, b_2 such that $b = b_1 \oplus b_2$, $b_1 \sim z \wedge a$, $b_2 \sim z' \wedge a$. Then $z \in C_0$ implies $b_1 \leq z \wedge b$, $b_2 \leq z' \wedge b$. But $(z \wedge b) \oplus (z' \wedge b) = b$, hence $b_1 = z \wedge b$, $b_2 = z' \wedge b$. Therefore $z \wedge b \sim z \wedge a$. □

We will say that a dimension effect algebra $(L; \sim)$ satisfies the *general comparability condition* (GC) (cf. [4]) if for every a and b, there exists an invariant element e such that $a \wedge e \preceq b \wedge e$ and $b \wedge e' \preceq a \wedge e'$. We note that in a dimension lattice or even a dimension poset condition (GC) is satisfied [37, 51].

Proposition 7.16. *Let $(L; \sim)$ be a dimension effect algebra. If for any a, b, $e(a) \wedge e(b) \neq 0$ implies $a \, \rho \, b$, then condition* (GC) *is satisfied.*

Proof. Let $a_1 \leq a$ and $b_1 \leq b$ be such that $a_1 \sim b_1$ and $a \ominus a_1$ is unrelated to $b \ominus b_1$ (Lemma 7.2). Let $e = e(b \ominus b_1)$, so that $(a \ominus a_1) \wedge e = 0$ and $(b \ominus b_1) \wedge e' = 0$. Then $a \wedge e = a_1 \wedge e \sim b_1 \wedge e \leq b \wedge e$ and $b \wedge e' = b_1 \wedge e' \sim a_1 \wedge e' \leq a \wedge e'$. □

Proposition 7.17. *Let* (GC) *be satisfied. If $a_1 \sim a_2$, $b_1 \sim b_2$ and $a_1 \leq b_1$, $a_2 \leq b_2$ and all four elements are finite, then $b_1 \ominus a_1 \sim b_2 \ominus a_2$.*

Proof. We choose an invariant element e such that $(b_1 \ominus a_1) \wedge e \preceq (b_2 \ominus a_2) \wedge e$ and $(b_2 \ominus a_2) \wedge e' \preceq (a_2 \ominus b_2) \wedge e'$. If the first inequivalence were proper, we would get

$$b_1 \wedge e = a_1 \wedge e \oplus (b_1 \ominus a_1) \wedge e \preceq a_2 \wedge e \oplus (b_2 \ominus a_2) \wedge e = b_2 \wedge e,$$

and, since $b_1 \sim b_2$, this contradicts the finiteness of b_2. Similarly, $(b_1 \ominus a_1) \wedge e' \sim (b_2 \ominus a_2) \wedge e'$. Adding we get $b_1 \ominus a_1 \sim b_2 \ominus a_2$. □

Proposition 7.18. *Let* (GC) *be satisfied. If $a, b \in L$ are finite and $a \perp b$, then $a \oplus b$ is finite.*

Proof. Assume $a \oplus b \sim c \leq a \oplus b$. By (D3), $c = c_1 \oplus c_2$, $c_1 \sim a$, $c_2 \sim b$. Applying (D3) to the equality $a \oplus b = c \oplus ((a \oplus b) \ominus c)$, we obtain $a = \bar{a} \oplus d_1$, $b = \bar{b} \oplus d_2$, $\bar{a} \oplus \bar{b} \sim c$, $d_1 \oplus d_2 \sim (a \oplus b) \ominus c$.

Applying (D3) again to $\bar{a} \oplus \bar{b} \sim c_1 \oplus c_2$, we obtain $\bar{a} = a_1 \oplus a_2$, $\bar{b} = b_1 \oplus b_2$, $a_1 \oplus b_1 \sim c_1 \sim a$, $a_2 \oplus b_2 \sim c_2 \sim b$, and so $a = a_1 \oplus a_2 \oplus d_1$, $b = b_1 \oplus b_2 \oplus d_2$.

From this we obtain $a_1 \oplus b_1 \sim a = a_1 \oplus a_2 \oplus d_1$, and by Proposition 7.17, $b_1 \sim a_2 \oplus d_1$. Analogously $a_2 \sim b_1 \oplus d_2$. From $a_2 \sim a_2 \oplus d_1 \oplus d_2$ and finiteness of a_2 we get $d_1 \oplus d_2 \sim 0$, so that $c = a \oplus b$. □

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Recall that a subset I of an effect algebra L is an *ideal* if for arbitrary orthogonal elements a and b in L, $a \oplus b$ belongs to I if and only if a and b belong to I (see e.g. [13]). We will say that a subclass V of L is (1) *hereditary* if $a \in V$, $b \preceq a$ implies $b \in V$, and (2) *sharply dominating* if for every $a \in V$ there exists a sharp element $p \in V$ with $a \leq p$.

Lemma 7.1 and Proposition 7.18 imply the following statement.

Corollary 7.19. *If (GC) is satisfied then the finite elements form a hereditary ideal in L.*

Theorem 7.20. *Let F be a sharply dominating hereditary ideal in $(L, \sim)$. Then the supremum $w := \bigvee F$ exists in L and is invariant.*

Proof. Let $\{a_i\}$ be any maximal summable family of elements in F. By (D2), the element $w = \bigoplus a_i = \bigvee_K \bigoplus_{a_i \in K} a_i$, where K is any finite subfamily of $\{a_i\}$, exists in L.

Let $a \in F$, and assume that $a \not\leq w$, in other words, $a \not\perp w'$. By (D4) there are nonzero a_1, w_1 such that $a \geq a_1 \sim w_1 \leq w'$. Now $w_1 \in F$, $w_1 \perp w$, hence $w \perp \bigoplus_{a_i \in K} a_i$ for any finite subset K of $\{a_i\}$. This contradicts the maximality of $\{a_i\}$. Hence $F \leq w$ and $w = \bigvee F$ and $w = \bigvee F$.

Assume that $a \in F$, $a \leq w, w'$. Then there is a sharp element $p \in F$ with $a \leq p$. Therefore $a \leq p \leq w$ and $a \leq w' \leq p'$ implies $a = 0$.

Let $u \leq w$, then $w = u \oplus (w \ominus u)$. For any $f \in F$, the inequality $f \leq u \oplus (w \ominus u)$ implies that $f \sim f_1 \oplus f_2$, $f_1 \leq u$, $f_2 \leq w \ominus u$. Clearly, $f_1, f_2 \in F$.

If for every $f \in F$, $f \leq w \ominus u$ holds, then also $w \leq w \ominus u$, whence $u = 0$. Therefore for every $u \leq w$ there is $f \in F$ with $f \leq u$. Assume $w \geq x \sim y \leq w'$. Then there is $f \in F$ such that $f \leq x \sim y$. From this it follows that $f \oplus (x \ominus f) \sim y$, and by (D3), there are g, h such that $y = g \oplus h$ and $g \sim f$, $h \sim x \ominus f$. Hence $g \leq y \leq w'$. Since $g \sim f$ and $f \in F$, we have $g \in F$, hence $g \leq w$. So $g \leq w, w'$. By the preceding part of the proof, $g = 0$. It follows that $w \in C_0$. □

Example. Let us consider the standard effect algebra $\mathcal{E}(H)$, i.e., the set of all self-adjoint operators A on a Hilbert space H such that $0 \leq A \leq I$, where 0 and I are the zero and the identity operators on H. We recall that the $\oplus$-operation is defined as follows: $A \perp B$ iff $A + B \leq I$ and in this case $A \oplus B = A + B$, where $+$ means the usual sum of operators. It turns out that the partial order induced by this operation coincides with the usual partial order defined by $A \leq B$ iff $B - A \geq 0$, i.e., $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ for all vectors $x \in H$.

Let ω be a state on $\mathcal{E}(H)$, i.e., there is a trace-class operator D with unit trace such that $\omega(A) = trDA$, $A \in \mathcal{E}(H)$. By [26, Lemma 5.1], $\omega(\lambda A) = \lambda\omega(A)$ for every $\lambda \in [0, 1]$, $A \in \mathcal{E}(H)$, and the binary relation $\sim$ on $\mathcal{E}(H)$ defined by $A \sim B$ if $\omega(A) = \omega(B)$ is a congruence. The latter property implies that $\sim$ is cancellative and satisfies (C5), and the quotient is an effect algebra. Recall that a state ω is *faithful* if for any nonzero projection P, $\omega(P)$ is a positive number. We will show that, for a faithful state ω, $\sim$ satisfies properties (D1)–(D4) of the dimension equivalence.

(D1): Assume that $\omega(A) = 0$ for some $A \in \mathcal{E}(H)$. By spectral theorem, $A = \int_0^1 \lambda P_A(d\lambda)$, where P_A is the spectral measure of A. From this we obtain, using the fact that ω is faithful, that $P_A\{0\} = 1$, hence $A = 0$.

(D2): Completeness of $\mathcal{E}(H)$ follows by [53, Lemma 1], and the rest follows by complete additivity of ω.

(D3): Assume that $A \oplus B \sim C \oplus D$, that is, $\omega(A) + \omega(B) = \omega(C) + \omega(D) =: M$. We may assume that $M \neq 0$. Put

$$e = \frac{\omega(C)}{M} A, \quad E = \frac{\omega(D)}{M} A,$$
$$f = \frac{\omega(C)}{M} B, \quad F = \frac{\omega(D)}{M} B.$$

We obtain $e \oplus E = A$, $f \oplus F = B$, and $e \oplus f \sim C$, $E \oplus F \sim D$.

(D4): Let A, B be any nonzero elements. Consider the elements

$$A_1 := \frac{\omega(B)}{\omega(A) + \omega(B)} A, \quad B_1 := \frac{\omega(A)}{\omega(A) + \omega(B)} B.$$

Clearly, $0 \neq A_1 \leq A$, $0 \neq B_1 \leq B$, and $\omega(A_1) = \omega(B_1)$, hence $A_1 \sim B_1$. The proof implies that any two nonzero elements are related, so we have a factor.

Let C be an invariant element, then C and $I \ominus C$ unrelated. According the proof of (D4), this only happens if $\omega(C) = 0$ or $\omega(I \ominus C) = 0$, i.e., $C = 0$ or $C = I$. So the set of invariant elements is reduced to $\{0, I\}$.

The quotient of $\mathcal{E}(H)$ with respect to $\sim$ is the interval $[0, 1] \subseteq \mathbb{R}$. Our dimension effect algebra does not contain any atoms and all elements are finite. Indeed, assume $A \leq B$, then $B = A \oplus (B \ominus A)$. If $A \sim B$, then $\omega(B \ominus A) = 0$, hence $A = B$. So we have a factor of type II_1.

We see that every faithful state induces a dimension equivalence on $\mathcal{E}(H)$. For the projection lattice $\mathcal{P}(H)$, this is not the case. Indeed, if $\sim$ is a dimension equivalence, condition (D4) implies that for any two atoms P and Q, if $P \not\perp Q$ then $P \sim Q$. If P and Q are orthogonal, by the superposition principle, there is an atom $R \leq P \oplus Q$, different from both P and Q, which is not orthogonal to either of them. This entails that all atoms should belong to the same equivalence class. It follows that in the finite dimensional Hilbert space, the only state which induces a dimension equivalence is the tracial one, i.e., $\omega(P) = \frac{\mathrm{tr} P}{\mathrm{tr} I}$, and we get a factor of type I_N. In the infinite dimensional case, there is no such state.

8. Some applications to K_0-theory

Recall that two projections e, f of a C*-algebra are defined to be equivalent ($e \sim f$) if there exists $a \in A$ such that $a^*a = e$ and $aa^* = f$.

Elliott [15] considered the set $D(A)$ of all equivalence classes of idempotents of a ring A together with a partially defined addition, which he calls an abelian local semigroup. He has shown that the range $D(A)$ of the dimension equivalence is an invariant that can be used in the classification of AF C*-algebras (i.e., approximately finite dimensional). Originally, an AF-C*-algebra A is defined as a direct limit of a sequence

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

of finite dimensional C*-algebras A_i.

Traditionally, $D(A)$ is not studied directly, but as an upward directed convex subset of the positive cone of an interpolation group $K_0(A)$, called the K_0 group for A. Notice that any such subset is an upper directed cp-PAM with RDP.

The definition of $K_0(A)$ usually requires simultaneous consideration of all the matrix algebras over A. This procedure is called a stabilisation. Denote $M_n(A)$ the set of all $n \times n$ matrices with entries in A. Let $M_\infty(A)$ denote the algebraic direct limit of $M_n(A)$ under the embeddings $a \mapsto \operatorname{diag}(a, 0)$. $M_\infty(A)$ can be thought of as the algebra of all infinite matrices with only finitely many nonzero entries. The embeddings are isometries, so $M_\infty(A)$ has a natural norm. Let $V(A) := \operatorname{Proj}(M_\infty(A))$. If A is separable, then $V(A)$ is countable. There is a binary operation on $V(A)$: if $[e], [f] \in V(A)$, choose $e' \in [e], f' \in [f]$ with $e' \perp f'$ (this is always possible by "moving down the diagonal"), and define $[e] + [f] = [e' + f']$. This operation is well defined and makes $V(A)$ into an abelian semigroup with the identity $[0]$.

For any abelian semigroup H, there is a universal enveloping abelian group $G(H)$ called the Grothendieck group of H. $G(H)$ can be thought of as the group of all equivalence classes of $H \times H$ under the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ iff there is z with $x_1 + y_2 + z = x_2 + y_1 + z$.

If A is a unital C*-algebra, then $K_0(A)$ is defined as the Grothendieck group of $V(A)$. The embedding of $V(A)$ into $K_0(A)$ is injective iff $V(A)$ has cancellation, i.e., whenever $e, f, g, h \in V(A)$ and $e \perp g, f \perp h, e \sim f$ and $e + g \sim f + h$ then $g \sim h$.

$K_0(A)$ can be preordered by taking the image of $V(A)$ in $K_0(A)$ as $K_0(A)_+$. We also define the scale $\Sigma(A)$ to be the image of $\operatorname{Proj}(A)$ in $K_0(A)$. The triple $(K_0(A), K_0(A)_+, \Sigma(A))$ is called the scaled group of A. If A is a unital C*-algebra with cancellation, then $\Sigma(A) = [0, [1_A]]$ is the Elliott invariant. The map $A \mapsto K_0(A)$ preserves direct products and direct limits [6].

Using the "word" technique developed in [1, 49, 55], any upward directed cp-PAM with RDP can be embedded into the positive cone of an interpolation group, which is its universal group (see e.g. [10] for details). This interpolation group is a dimension group in the sense of [14] if and only if it can be embedded as a subgroup into a lattice ordered group [5]. Accordingly, a cp-PAM with RDP has a dimension group as a universal group if and only if it can be embedded into a lattice ordered cp-PAM with RDP. This suggests that there is a one-to-one correspondence between $K_0(A)$ for (unital) AF C*-algebras A and unital countable cp-PAMs $D(A)$ which can be embedded into MV-algebras. The corresponding $K_0(A)$ can be obtained directly from $D(A)$ using the word technique without stabilization.

We will concentrate on the case of an AF C*-algebra and show briefly how the theory of cp-PAMs (orthomodular lattices in this case) with dimension can be used to obtain the K_0 group. It is well known that a simple finite-dimensional complex C*-algebra is isomorphic with $M_n(\mathbb{C})$, the $n \times n$-matrices over complex numbers [21]. That is, the elements can be represented by the set of all operators on a finite dimensional Hilbert space. It then follows by [37] that the set L of projections in A is a complete orthomodular lattice (which is even modular), and endowed with the above equivalence relation satisfies axioms (D1)–(D4). Clearly, the quotient with respect to $\sim$ is equivalent to the MV-algebra $\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$, that is, a finite MV-chain, and the corresponding abelian group is the group of integers Z, where the Elliott invariant is embedded as the interval $[0, N]$, and N is a strong unit. Now every finite dimensional complex C*-algebra A is isomorphic (as a complex C*-algebra) to $M_{n(1)}(\mathbb{C}) \times \cdots \times M_{n(k)}(\mathbb{C})$ for some positive integers $n(1), \dots, n(k)$. The k-tuple $n(A) := (n(1), \dots, n(k))$ is an invariant (up to permutation) of A, and $n(A)$ classifies A up to isomorphism. The corresponding Elliott invariant $D(A)$ for A is a direct product of MV-chains.

Now an AF-C*-algebra A is a direct limit of a sequence

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

of finite dimensional C*-algebras A_i. Let $A_i \subseteq A_j$ and $p, q \in A_i$, $p \sim_i q$, that is, $p = ww^*$,$q = w^*w$ for some $w \in A_i$. Let $\phi^i_j \colon A_i \to A_j$ be a C*-algebra morphism. Then $\phi^i_j(p) = \phi^i_j(w)(\phi^i_j(w))^*$, $\phi^i_j(q) = (\phi^i_j(w))^*\phi^i_j(w)$, hence $\phi^i_j(p) \sim_j \phi^i_j(q)$. Therefore the condition (CDL) in Theorem 5.4 is satisfied. Applying Theorem 5.4, we obtain $D(A)$ as a direct limit of $D(A_i)$.

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