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Continuous solutions of a generalization of the Gołąb–Schinzel equation

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Summary. Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions, such that ϕ is continuous and $|\psi(1)| \neq 1$. We solve the functional equation

$$f(x\phi[f(y)] + y\psi[f(x)]) = f(x)f(y) \quad \text{for } x, y \in \mathbb{R}$$

in the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$.

In particular we give the forms of ϕ, ψ for which the equation has non-constant solutions.

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1. Introduction

By $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ we denote the sets of all real, integer and positive integer numbers, respectively. Moreover $\mathbb{R}_- := (-\infty, 0]$ and $\mathbb{R}_+ := [0, \infty)$. Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions. Functional equations of the form

$$f\left(x\phi\left(f(y)\right) + y\psi\left(f(x)\right)\right) = f(x)f(y) \qquad \text{for } \ x, y \in \mathbb{R}, \tag{1}$$

where the unknown function f maps $\mathbb{R}$ into itself, have been considered by many authors in several cases.

If $\phi(x) = 1$ and $\psi(x) = x$ for $x \in \mathbb{R}$, then (1) takes the form

$$f\left(x + yf(x)\right) = f(x)f(y) \quad \text{for } \ x, y \in \mathbb{R},$$

and is called the Gołąb–Schinzel functional equation (for details see [1], [2], [10], [11]). In the case $\phi(x) = x^k$ and $\psi(x) = x^l$ for $x \in \mathbb{R}$, where $k, l \in \mathbb{N}$ are arbitrarily fixed, we obtain the so-called generalized Gołąb–Schinzel equation

$$f\left(xf(y)^k + yf(x)^l\right) = f(x)f(y) \quad \text{for } \ x, y \in \mathbb{R},$$

which has been considered among others in [4]–[8], [12], [14], where in particular the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of that equation have been determined. In [3] it is proved that the cardinality of the set of discontinuous solutions $f : \mathbb{R} \to \mathbb{R}$ of this equation is $2^{\aleph}$, where $\aleph = \operatorname{card} \mathbb{R}$. Some applications of this type of functional equations can be found for example in [2], [4], [6], [8] and [12].

In this paper we present the general solution of (1) in the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$, under the assumption that ϕ is continuous and $|\psi(1)| \neq 1$. The case $|\psi(1)| = 1$ needs different methods and the results concerning it will be published separately.

2. Preliminary results

Remark 1. The only constant solutions of (1) are $f = 0$ and $f = 1$.

Lemma 1. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions and let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy* (1). *Then*

(i) $f(0) \in \{0, 1\}$*;*

(ii) *if $f(0) = 0$, then $f = 0$ or $\phi(0) = \psi(0) = 0$;*

(iii) *if $f(0) = 1$ and f is continuous at* 0*, then $f = 1$ or $|\phi(1)| = |\psi(1)| = 1$;*

(iv) *$f(\mathbb{R})$ is a multiplicative subsemigroup of* $\mathbb{R}$.

Proof. (i) It is enough to put in (1) $x = y = 0$.

(ii) Let $f(0) = 0$ and suppose for example that $\phi(0) \neq 0$. Setting in (1) $y = 0$, we have $f(x\phi(0)) = 0$ for $x \in \mathbb{R}$. Thus $f = 0$.

(iii) Assume that f is continuous at 0 and $f(0) = 1$. Let us suppose that $|\phi(1)| \neq 1$. By taking in (1) $y = 0$, we obtain $f(x\phi(1)) = f(x)$ for $x \in \mathbb{R}$. Hence $f(x\phi(1)^n) = f(x)$ for $x \in \mathbb{R}, n \in \mathbb{Z}$ and by the continuity of f at 0, we have $f(x) = f(0) = 1$ for $x \in \mathbb{R}$. The proof in the case $|\psi(1)| \neq 1$ is analogous.

(iv) This follows at once from (1). □

Corollary 1. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. If $f : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous solution of* (1)*, then*

(i) $f(0) = 0$,

(ii) $\phi(0) = \psi(0) = 0$.

Lemma 2. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of* (1)*, then f is either unbounded or constant.*

Proof. Suppose that f is a non-constant bounded continuous solution of (1) and let $M := \sup\{|f(x)| : x \in \mathbb{R}\}$. From Lemma 1(iv) it follows that $M \in (0, 1]$. We consider three cases:

1) $\psi \circ f = 0$. Since f is non-constant, in view of (1) $\phi \circ f$ cannot be constant. Then there exists an $x_1 \in \mathbb{R}$ such that $\phi(f(x_1)) \neq 0$ and $|f(x_1)| \neq 1$. Otherwise $\phi \circ f$ would take at most three values $0, \phi(1)$ and $\phi(-1)$, which is not possible. Setting in (1) $y = x_1$, we obtain $f(x\phi(f(x_1))) = f(x)f(x_1)$ for $x \in \mathbb{R}$. Then

$\{x\phi\,(f(x_1)) : x \in \mathbb{R}\} = \mathbb{R}$ and we get

$$M = \sup\{|f(x\phi(f(x_1)))| : x \in \mathbb{R}\} = \sup\{|f(x)f(x_1)| : x \in \mathbb{R}\} = |f(x_1)|M,$$

which gives a contradiction.

2) $\psi\,(f(x_2)) \neq 0$ for some $x_2 \in \mathbb{R} \setminus f^{-1}(\{-1,1\})$. Putting in (1) $x = x_2$, we have

$$f(x_2\phi\,(f(y)) + y\psi\,(f(x_2))) = f(x_2)f(y)$$

for $y \in \mathbb{R}$. Moreover, the continuity of ϕ implies that $\phi \circ f$ is a bounded function. Hence $\{x_2\phi(f(y)) + y\psi(f(x_2)) : y \in \mathbb{R}\} = \mathbb{R}$ and as above we obtain that $M = |f(x_2)|M$, which is impossible.

3) $\psi \circ f \neq 0$ and $\psi\,(f(x)) = 0$ for $x \in \mathbb{R} \setminus f^{-1}(\{-1,1\})$. Then from (1) it follows that

$$f\,(x\phi\,(f(y))) = f(x)f(y) \tag{2}$$

for $x \in \mathbb{R} \setminus f^{-1}(\{-1,1\}), y \in \mathbb{R}$. We divide the proof in this case into three steps.

Step 1. We prove that $\mathbb{R} \setminus f^{-1}(\{-1,1\})$ is an interval. From the assumptions of this case it follows that there is a $z \in \mathbb{R}$ such that $|f(z)| = 1$. In particular $M = 1$. Assume that $z > 0$ and let $z_1 := \min\{x > 0 : |f(x)| = 1\}$. We show that $|f(x)| = 1$ for $x \geq z_1$. Suppose that $|f(x_0)| < 1$ for some $x_0 > z_1$. Since $f(0) = 0$, in view of the continuity of f, there exists an $x_1 \in (0, z_1)$ such that $|f(x_1)| > |f(x_0)|$. Moreover $\phi(f(\mathbb{R}))$ is an interval and by Corollary 1, $0 \in \phi(f(\mathbb{R}))$. Thus $x_1\phi(f(\mathbb{R})) \subset x_0\phi(f(\mathbb{R}))$ and in virtue of (2), we get

$$\begin{aligned}|f(x_1)| &= \sup\{|f(x_1)f(y)| : y \in \mathbb{R}\} = \sup\{|f(x_1\phi(f(y)))| : y \in \mathbb{R}\}\\ &\leq \sup\{|f(x_0\phi(f(y)))| : y \in \mathbb{R}\} = \sup\{|f(x_0)f(y)| : y \in \mathbb{R}\} = |f(x_0)|,\end{aligned}$$

which cannot occur. Similarly, if there is a $z < 0$ such that $|f(z)| = 1$, then $|f(x)| = 1$ for $x \leq \max\{x < 0 : |f(x)| = 1\}$. Therefore we have proved that $\mathbb{R}\setminus f^{-1}(\{-1,1\})$ is an interval. Moreover if $-1 \in f(\mathbb{R})$, then in view of Lemma 1(iv), we get that $1 \in f(\mathbb{R})$ and, using the continuity of f, one can easily deduce that $\mathbb{R} \setminus f^{-1}(\{-1,1\})$ is a bounded interval.

Step 2. We consider the case, when $\mathbb{R} \setminus f^{-1}(\{-1,1\})$ is an unbounded interval. Then $f(\mathbb{R}) \subset (-1,1]$. Suppose that $\mathbb{R} \setminus f^{-1}(\{-1,1\}) = (-\infty, c)$, where $c > 0$ is a real constant. If $\mathbb{R} \setminus f^{-1}(\{-1,1\}) = (b, \infty)$ for some $b < 0$, the proof is analogous. Thus $f(x) = 1$ for $x \geq c$ and according to (1), we have

$$f(x\phi\,(f(y))) = f(x)f(y) \qquad \text{for } \ x < c, y \in \mathbb{R} \tag{3}$$

and

$$f(x\phi\,(f(y)) + y\psi(1)) = f(y) \qquad \text{for } \ x \geq c, y \in \mathbb{R}. \tag{4}$$

Since f is non-constant, in view of (3) $\phi \circ f$ is non-constant, too. Then $\phi \circ f$ is non-constant on $(-\infty, c)$, so there exists a $y_0 \neq 0$ with $\phi(f(y_0)) \neq 0$ and $|f(y_0)| < 1$. Setting in (3) and (4) $y = y_0$, we obtain

$$|f(x\phi\,(f(y_0)))| = |f(x)f(y_0)| < |f(x)| \qquad \text{for } \ x < c \tag{5}$$

and

$$|f(x\phi(f(y_0)) + y_0\psi(1))| = |f(y_0)| < 1 \qquad \text{for } x \geq c, \tag{6}$$

respectively. If $\phi(f(y_0)) < 0$, then $\frac{c}{\phi(f(y_0))} < c$ and putting in (5) $x := \frac{c}{\phi(f(y_0))}$, we have $1 = |f(c)| < |f\left(\frac{c}{\phi(f(y_0))}\right)|$, which is not possible. If $\phi(f(y_0)) > 0$, then there is a $c_1 \geq c$ such that $c_1\phi(f(y_0)) + y_0\psi(1) \geq c$. Taking in (6) $x = c_1$, we get $1 = |f(c_1\phi(f(y_0)) + y_0\psi(1))| = |f(y_0)| < 1$, which gives a contradiction.

Step 3. Assume that $\mathbb{R} \setminus f^{-1}(\{-1,1\}) = (b,c)$ is a bounded interval with $b < 0 < c$. Since $1 \in f(\mathbb{R})$, we may suppose $f^{-1}(\{1\}) = [c,\infty)$. The argument is analogous if $f^{-1}(\{1\}) = (-\infty, b]$. We have (6) for $x \geq c$. If $\phi(f(y_0)) > 0$, for $x > c$ big enough, we have $x\phi(f(y_0)) + y_0\psi(1) > c$, which with (6) leads to a contradiction. If $\phi(f(y_0)) < 0$, for $x > c$ big enough, we have $x\phi(f(y_0)) + y_0\psi(1) < b$, which again leads with (6) to a contradiction. □

Since by Lemma 1(iv) a non-zero continuous solution f of (1) satisfies $f(\mathbb{R}) \cap (0,\infty) \neq \emptyset$, we get

Corollary 2. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. If $f : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous solution of* (1)*, then $f(\mathbb{R}) \in \{\mathbb{R}_+, \mathbb{R}\}$.*

The following result will be very useful in our considerations ([9], Chapter 6 (Section 6.2), cf. also [13])

Proposition 1. *The general solution in the class of continuous functions $h : \mathbb{R} \to \mathbb{R}$ of the functional equation*

$$h(h(x)) = (\gamma+1)h(x) - \gamma x \text{ for } x \in \mathbb{R},$$

where $\gamma \neq -1$ is a fixed real number, is given by:

(A) *if $\gamma > 0$:*

(i)

$$h(x) = \begin{cases} \gamma x + (1-\gamma)a & \text{for } x \leq a \\ x & \text{for } a < x < b \\ \gamma x + (1-\gamma)b & \text{for } x \geq b \end{cases}$$

with $-\infty \leq a < b \leq \infty$,

(ii) $h(x) = \gamma x + \delta$ *with $\delta \in \mathbb{R}$;*

(B) *if $\gamma = 0$:*

(i) $h(x) = x$ *for $x \in h(\mathbb{R})$,*

(ii) $h(x) = \delta$ *with $\delta \in \mathbb{R}$;*

(C) *if $\gamma < 0$:*

(i) $h(x) = \gamma x + \delta$ *with $\delta \in \mathbb{R}$,*

(ii) $h(x) = x$.

Lemma 3. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant continuous solution of* (1)*. Then the following properties hold:*

(i) $f^{-1}(\{1\}) \neq \emptyset$,

(ii) *if $x_0 \in f^{-1}(\{1\})$ and $h(y) := x_0\phi(f(y)) + y\psi(1)$ for $y \in \mathbb{R}$, then $h(0) = 0$, $h(\mathbb{R})$ is an unbounded interval* (*containing* 0) *and*

$$h(h(y)) = (\psi(1) + 1)\, h(y) - \psi(1)y \qquad \textit{for} \ \ y \in \mathbb{R},$$

(iii) *if $x_0 \in f^{-1}(\{1\})$ and $\psi(1) \neq 0$, then*

$$\phi(f(x)) = \begin{cases} (1-\psi(1))\frac{a}{x_0} & \textit{for} \ \ x \leq a \\ (1-\psi(1))\frac{x}{x_0} & \textit{for} \ \ a < x < b \\ (1-\psi(1))\frac{b}{x_0} & \textit{for} \ \ x \geq b \end{cases} \tag{7}$$

with $-\infty \leq a < b \leq \infty$,

(iv) *if $x_0 \in f^{-1}(\{1\})$ and $\psi(1) = 0$, then $x_0\phi(f(\mathbb{R}))$ is an unbounded interval containing* 0 *and*

$$\phi(f(x)) = \frac{x}{x_0} \qquad \textit{for} \ \ x \in x_0\phi(f(\mathbb{R})). \tag{8}$$

Moreover $\phi(f(\mathbb{R})) \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$, and $\phi(f(\mathbb{R})) = \mathbb{R}$ if and only if $f(\mathbb{R}) = \mathbb{R}$.

Proof. (i) It results immediately from Corollary 2.

(ii) Let $x_0 \in f^{-1}(\{1\})$ and $h(y) := x_0\phi(f(y)) + y\psi(1)$ for $y \in \mathbb{R}$. From Corollary 1 it follows that $x_0 \neq 0$ and $h(0) = 0$. Putting in (1) $x = x_0$, we have

$$f(h(y)) = f(y) \qquad \text{for} \ \ y \in \mathbb{R}. \tag{9}$$

In view of Lemma 2, f is an unbounded continuous function, so by (9), $h(\mathbb{R})$ is an unbounded interval (containing 0). Further we have $\phi(f(y)) = \frac{1}{x_0}(h(y) - \psi(1)y)$ for $y \in \mathbb{R}$. Thus on account of (9), we obtain

$$\frac{1}{x_0}\left(h(h(y)) - \psi(1)h(y)\right) = \phi(f(h(y))) = \phi(f(y)) = \frac{1}{x_0}\left(h(y) - \psi(1)y\right)$$

for $y \in \mathbb{R}$, and so

$$h(h(y)) = (\psi(1) + 1)\, h(y) - \psi(1)y \quad \text{for} \ \ y \in \mathbb{R}.$$

(iii) Let $x_0 \in f^{-1}(\{1\})$ and $\psi(1) \neq 0$. Since $h(x) = x_0\phi(f(x)) + x\psi(1)$ and $h(0)=0$, from Proposition 1(A),(C) it can be easily deduced that either $\phi(f(x))=0$ for $x \in \mathbb{R}$ or (7) holds. We show that the first possibility cannot occur. In fact, if $\phi(f(x)) = 0$ for $x \in \mathbb{R}$, then putting in (1) $x = x_0$, we obtain $f(y\psi(1)) = f(y)$ for $y \in \mathbb{R}$. Thus the continuity of f and $|\psi(1)| \neq 1$ imply that f is a constant function, which gives a contradiction.

(iv) Let $x_0 \in f^{-1}(\{1\})$ and $\psi(1) = 0$.The first part of the statement follows immediately from (ii) and Proposition 1(B).

Now we show that $\phi(f(\mathbb{R})) = \mathbb{R}$ if and only if $f(\mathbb{R}) = \mathbb{R}$. Assume that $\phi(f(\mathbb{R})) = \mathbb{R}$. Then by (8) f is one-to-one on $\mathbb{R}$ and since $f(0) = 0$ (cf. Corollary 1(i)), so $f(\mathbb{R}) \cap (-\infty, 0) \neq \emptyset$. Thus, on account of Corollary 2, we have $f(\mathbb{R}) = \mathbb{R}$.

Conversely, if $f(\mathbb{R}) = \mathbb{R}$ then from (1) it follows that

$$f\left(x_0\phi\left(f(\mathbb{R})\right)\right) = f(\mathbb{R}) = \mathbb{R}.$$

Since f is continuous on $\mathbb{R}$ and one-to-one on $x_0\phi(f(\mathbb{R}))$, we get $x_0\phi(f(\mathbb{R})) = \mathbb{R}$ (because otherwise f would be bounded, above or below, on the topological closure of $x_0\phi\left(f(\mathbb{R})\right)$). Thus $\phi(f(\mathbb{R})) = \mathbb{R}$.

Now we will prove that $\phi(f(\mathbb{R})) \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$. We have that $\phi(f(\mathbb{R}))$ is an unbounded interval containing 0. Hence either $\mathbb{R}_+ \subset \phi(f(\mathbb{R}))$ or $\mathbb{R}_- \subset \phi(f(\mathbb{R}))$. Suppose for example that $\mathbb{R}_+ \subset \phi(f(\mathbb{R}))$ and $\phi(f(\mathbb{R}))\backslash\mathbb{R}_+ \neq \emptyset$. Since f is a continuous function, one-to-one on $x_0\phi(f(\mathbb{R}))$ and $f(0) = 0$, we have $f(\mathbb{R}) \cap (-\infty, 0) \neq \emptyset$. Hence, according to Corollary 2, $f(\mathbb{R}) = \mathbb{R}$ and consequently $\phi(f(\mathbb{R})) = \mathbb{R}$. $\square$

Lemma 4. *Let $A \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$ and α be an arbitrary non-zero real constant. The general solution of the equation*

$$f(\alpha xy) = f(x)f(y) \qquad \text{for } \ x \in A, y \in \mathbb{R}, \tag{10}$$

in the class of non-constant continuous functions $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} (\alpha x)^r & \text{for } \ x \in \alpha^{-1}\mathbb{R}_+ \\ b(-\alpha x)^r & \text{for } \ x \in \alpha^{-1}\mathbb{R}_-, \end{cases} \tag{11}$$

where r is an arbitrary positive real constant and:

$|b| = 1$, whenever $\alpha \in -A$;

b is an arbitrary real constant, otherwise.

Proof. A straightforward calculation shows that each function of the form (11) satisfies (10). Assume that $f : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous solution of (10). Inserting into (10) $x\alpha^{-1}$ and $y\alpha^{-1}$ in place of x and y, respectively, we obtain

$$g(xy) = g(x)g(y) \qquad \text{for } \ x \in \alpha A, y \in \mathbb{R}, \tag{12}$$

where $g(x) := f(x\alpha^{-1})$ for $x \in \mathbb{R}$.

Suppose that $\alpha \in -A$. Then either $\alpha A = \mathbb{R}$ or $\alpha A = \mathbb{R}_-$. If $\alpha A = \mathbb{R}$, then in view of (12), we have that either $g(x) = |x|^r$ for $x \in \mathbb{R}$ or $g(x) = |x|^r \operatorname{sgn}(x)$ for $x \in \mathbb{R}$, where r is some positive real constant. Hence we get (11) with $|b| = 1$. If $\alpha A = \mathbb{R}_-$, then inserting into (12) $-x$ in place of x, we obtain

$$G(xy) = G(x)g(y) \qquad \text{for } \ x \in \mathbb{R}_+, y \in \mathbb{R}, \tag{13}$$

where $G(x) := g(-x)$ for $x \in \mathbb{R}$. Actually (13) may be treated as a multiplicative Pexider equation on $\mathbb{R}_+$. Thus $g(x) = x^r$ for $x \in \mathbb{R}_+$ and $g(-x) = G(x) = bx^r$ for

$x \in \mathbb{R}_+$, where r is a positive real constant and b is a real constant. Therefore

$$f(x) = g(\alpha x) = \begin{cases} (\alpha x)^r & \text{for } x \in \alpha^{-1}\mathbb{R}_+ \\ b(-\alpha x)^r & \text{for } x \in \alpha^{-1}\mathbb{R}_-. \end{cases}$$

In particular $f(\alpha^{-1}) = 1$ and $f(-\alpha^{-1}) = b$. Then, setting in (10) $x = y = -\alpha^{-1} \in A = \alpha^{-1}\mathbb{R}_-$, we obtain $1 = f(\alpha^{-1}) = f(\alpha^{-1})^2 = b^2$. Hence $|b| = 1$.

Now, suppose that $\alpha \notin -A$. Then $\alpha A = \mathbb{R}_+$ and in virtue of (12), we get $g(x) = x^r$ for $x \in \mathbb{R}_+$, where r is a positive real constant. Moreover, setting in (12) $y = -1$ we obtain $g(-x) = g(-1)(x)^r$ for $x \in \mathbb{R}_+$. Thus $g(x) = g(-1)(-x)^r$ for $x \in \mathbb{R}_-$ and so we have (11) with $b := g(-1)$. □

3. Main results

Proposition 2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a given continuous function. The equation*

$$f\left(x\phi\left(f(y)\right)\right) = f(x)f(y) \qquad \text{for } x, y \in \mathbb{R} \tag{14}$$

has non-constant continuous solutions if and only if ϕ has one of the following forms:

(i)

$$\phi(x) = \begin{cases} -x^{\frac{1}{r}} & \text{for } x \in \mathbb{R}_+ \\ \phi_1(x) & \text{for } x \in \mathbb{R}_-, \end{cases} \tag{15}$$

(ii)

$$\phi(x) = \begin{cases} x^{\frac{1}{r}} & \text{for } x \in \mathbb{R}_+ \\ \phi_1(x) & \text{for } x \in \mathbb{R}_-, \end{cases} \tag{16}$$

where r is an arbitrary positive real constant and $\phi_1 : \mathbb{R}_- \to \mathbb{R}$ is an arbitrary continuous function with $\phi_1(0) = 0$.

Furthermore, whenever ϕ is of the form (15) *or* (16)*, then the general solution of* (14) *in the class of non-constant continuous functions $f : \mathbb{R} \to \mathbb{R}$ is given, respectively, by:*

(i)

$$f(x) = |\alpha x|^r \qquad \text{for } x \in \mathbb{R}, \tag{17}$$

where α is an arbitrary non-zero real constant;

(ii)

$$f(x) = \begin{cases} (\alpha x)^r & \text{for } x \in \alpha^{-1}\mathbb{R}_+ \\ b(-\alpha x)^r & \text{for } x \in \alpha^{-1}\mathbb{R}_- \end{cases} \tag{18}$$

or

$$f(x) = d|x|^r \operatorname{sgn}(x) \quad \text{for } x \in \mathbb{R}, \tag{19}$$

where α and d are arbitrary non-zero real constants and b is an arbitrary non-negative real constant, in the case when $\phi_1(x) = -(-x)^{\frac{1}{r}}$ for $x \in \mathbb{R}_-$; and by (18)*, otherwise.*

Proof. One can check that if ϕ has the form (15) or (16), then every function of the form (17), (18) or (19) satisfies (14) in the corresponding case. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous function satisfying (14). On account of Corollary 1 and Lemma 3(i),(iv), there exists an $x_0 \in \mathbb{R} \setminus \{0\}$ such that $f(x_0) = 1$, $\phi(f(\mathbb{R})) \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$ and (8) holds. In particular f is one-to-one on $x_0\phi(f(\mathbb{R}))$. We consider three cases.

Case 1) $\phi(f(\mathbb{R})) = \mathbb{R}$. Then in view of Lemma 3(iv), we get $f(\mathbb{R}) = \mathbb{R}$. From (8) and (14) it follows that $f(\frac{xy}{x_0}) = f(x)f(y)$ for $x, y \in \mathbb{R}$. Hence, on account of Lemma 4, $f(x) = |\frac{x}{x_0}|^r \operatorname{sgn}\left(\frac{x}{x_0}\right)$ for $x \in \mathbb{R}$, so f has the form (19) with $d := |x_0|^{-r} \operatorname{sgn}\left(x_0^{-1}\right)$. Therefore in view of (14), we have $|\phi(f(y))|^r \operatorname{sgn}(\phi(f(y))) = f(y)$ for $y \in \mathbb{R}$. Thus $|\phi(x)|^r \operatorname{sgn}(\phi(x)) = x$ for $x \in f(\mathbb{R}) = \mathbb{R}$ and an easy calculation shows that ϕ is of the form (16) with $\phi_1(x) = -(-x)^{\frac{1}{r}}$ for $x \in \mathbb{R}_-$.

Case 2) $\phi(f(\mathbb{R})) = \mathbb{R}_-$. Thus, in view of (8) and (14), $f(\frac{xy}{x_0}) = f(x)f(y)$ for $x \in \mathbb{R}$ and $y \in x_0\phi(f(\mathbb{R})) = x_0\mathbb{R}_-$. According to Corollary 2 and Lemma 3(iv), $f(\mathbb{R}) = \mathbb{R}_+$. Moreover $\frac{1}{x_0} \in x_0^{-1}\mathbb{R}_+ = x_0\mathbb{R}_+ = -x_0\phi(f(\mathbb{R}))$. Hence, in virtue of Lemma 4 (with $\alpha = x_0^{-1}$ and $A = x_0\phi(f(\mathbb{R}))$), we get $f(x) = |\frac{x}{x_0}|^r$ for $x \in \mathbb{R}$, where r is some positive real constant. Therefore by (14), we have $|\phi(f(y))|^r = f(y)$ for $y \in \mathbb{R}$. Then $\phi(x) = -x^{\frac{1}{r}}$ for $x \in f(\mathbb{R}) = \mathbb{R}_+$ and we have obtained (15) and (17) with $\alpha := x_0^{-1}$.

Case 3) $\phi(f(\mathbb{R})) = \mathbb{R}_+$.In view of (8) the equation (14) becomes $f(\frac{xy}{x_0}) = f(x)f(y)$ for $x \in \mathbb{R}$ and $y \in x_0\phi f(\mathbb{R})$. Furthermore, in view of Corollary 2 and Lemma 3(iv), $f(\mathbb{R}) = \mathbb{R}_+$ and it is easy to see that $\frac{1}{x_0} \notin -x_0\phi(f(\mathbb{R}))$. Using Lemma 4, similarly to the previous case, one can obtain (16) and (18) with $\alpha := x_0^{-1}$ and some $b \in \mathbb{R}$. Moreover $f(\mathbb{R}) = \mathbb{R}_+$ implies that $b \geq 0$. $\square$

Proposition 3. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. If the equation* (1) *has non-constant continuous solutions, then one of the following conditions holds:*

(i) $\phi(1) = -1$ *and* $\psi(1) = 0$*;*

(ii) $\phi(1) + \psi(1) = 1$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant continuous solution of (1). In view of Lemma 3(i) and Corollary 1(i), we get that $f(x_0) = 1$ for some $x_0 \in \mathbb{R} \setminus \{0\}$. Assume that $\psi(1) \neq 0$. From Lemma 3(iii) it follows (7). Setting in (1) $x = y = x_0$, we have $f(x_1) = 1$, where $x_1 := x_0(\phi(1) + \psi(1))$. Moreover from Corollary 1(i) it follows that $x_1 \neq 0$. Putting in (1) $x = x_1$ one obtains

$$\phi(f(x_1\phi(f(y)) + \psi(1)y)) = \phi(f(y))$$

for $y \in \mathbb{R}$. Hence, in view of (7), we get

$$\phi(f((1 - \psi(1))\frac{yx_1}{x_0} + \psi(1)y)) = \phi(f(y)) \tag{20}$$

for $y \in (a,b)$. Furthermore notice that

$$\left\{(1-\psi(1))\frac{yx_1}{x_0}+\psi(1)y : y \in (a,b)\right\} \subset (a,b). \tag{21}$$

In fact, suppose for example that $(1-\psi(1))\frac{zx_1}{x_0}+\psi(1)z < a$ for some $z \in (a,b)$. Then in virtue of (7) and (20), we have

$$(1-\psi(1))\frac{a}{x_0} = \phi(f((1-\psi(1))\frac{zx_1}{x_0}+\psi(1)z)) = \phi(f(z)) = (1-\psi(1))\frac{z}{x_0}.$$

Since $\psi(1) \neq 1$, this is impossible. Now from (7), (20) and (21) it follows that

$$(1-\psi(1))\frac{(1-\psi(1))\frac{yx_1}{x_0}+\psi(1)y}{x_0} = (1-\psi(1))\frac{y}{x_0}$$

for $y \in (a,b)$. Thus $x_0 = x_1$ and so $\phi(1)+\psi(1) = 1$.

Suppose now that $\psi(1) = 0$. Then according to Lemma 3(iv), we have (8). Setting in (1) $x = y = x_0$, we obtain $f(x_0\phi(1)) = 1$. Next, putting in (1) $x = x_0\phi(1)$ and $y = x_0$, we have $f(x_0\phi(1)^2) = 1$, which in view of Lemma 3(iv) gives

$$\phi(f(x)) = \frac{x}{x_0\phi(1)^2} \tag{22}$$

for $x \in x_0\phi(1)^2\phi(f(\mathbb{R}))$. Furthermore $x_0\phi(f(\mathbb{R}))$ and $x_0\phi(1)^2\phi(f(\mathbb{R}))$ are unbounded intervals. Since $x_0x_0\phi(1)^2 > 0$, so $x_0\phi(f(\mathbb{R})) \cap x_0\phi(1)^2\phi(f(\mathbb{R}))$ is an unbounded interval, too. Then from (8) and (22) it follows that

$$\frac{x}{x_0} = \frac{x}{x_0\phi(1)^2}$$

for $x \in x_0\phi(f(\mathbb{R})) \cap x_0\phi(1)^2\phi(f(\mathbb{R}))$. Hence $\phi(1)^2 = 1$, which gives either (i) or (ii). □

Proposition 4. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous, $\phi(1) = 1$ and $\psi(1) = 0$. Then* (1) *has non-constant continuous solutions if and only if ϕ is of the form* (16) *and*

$$\psi(x) = \begin{cases} 0 & \text{for } \ x \in \mathbb{R}_+ \\ \psi_1(x) & \text{for } \ x \in \mathbb{R}_-, \end{cases} \tag{23}$$

where $\psi_1 : \mathbb{R}_- \to \mathbb{R}$ is an arbitrary function.

Furthermore, whenever ϕ and ψ are of the form (16) *and* (23), *respectively, then the general solution of* (1) *in the class of non-constant continuous functions $f : \mathbb{R} \to \mathbb{R}$ is given by* (18) *or* (19) *in the case when $\phi_1(x) = -(-x)^{\frac{1}{r}}$ for $x \in \mathbb{R}_-$, $\psi_1(x) = 0$ for $x \in \mathbb{R}_-$; and by* (18), *otherwise.*

Proof. It is easy to check that if ϕ and ψ are of the form (16) and (23), respectively, then f given by (18) or (19) is a solution of (1) in the corresponding case. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous function satisfying (1). According to Proposition 2, it is enough to prove that $\psi(f(x)) = 0$ for $x \in \mathbb{R}$. In virtue of Corollary 2 and Lemma 3(iv), there exists an $x_0 \in \mathbb{R} \setminus \{0\}$ such that $f(x_0) = 1$, $x_0\phi(f(\mathbb{R})) \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$ and (8) holds. In particular f is one-to-one on $x_0\phi(f(\mathbb{R}))$. Setting in (1) $y = x_0$, we obtain

$$f(x + x_0\psi(f(x))) = f(x) \qquad \text{for } x \in \mathbb{R}. \tag{24}$$

From (24) it follows by induction that

$$f(x + nx_0\psi(f(x))) = f(x) \qquad \text{for } x \in \mathbb{R}, n \in \mathbb{N}. \tag{25}$$

If $f(\mathbb{R}) = \mathbb{R}$, then on account of Lemma 3(iv), $x_0\phi(f(\mathbb{R})) = \mathbb{R}$ and so f is one-to-one on $\mathbb{R}$. Then from (24) it follows immediately that $\psi(f(x)) = 0$ for $x \in \mathbb{R}$.

Next, assume that $f(\mathbb{R}) = \mathbb{R}_+$. According to Lemma 3(iv), we get that $x_0\phi(f(\mathbb{R})) \in \{\mathbb{R}_-, \mathbb{R}_+\}$. The following two cases are possible now:

1) There exists an $x_1 \in x_0\mathbb{R}_-$ such that $f(x_1) = 1$. From Corollary 1(i) it follows that $x_1 \neq 0$. Setting in (1) $x = x_1$, we obtain $f(x_1\phi(f(y))) = f(y)$ for $y \in \mathbb{R}$. Hence, according to Lemma 3(iv), we get that $\phi(f(x)) = \frac{x}{x_1}$ for $x \in x_1\phi(f(\mathbb{R})) = -x_0\phi(f(\mathbb{R}))$. Thus, in virtue of (8), $f|_{\mathbb{R}_-}$ and $f|_{\mathbb{R}_+}$ are one-to-one functions. Putting in (1) $y = x_1$, we have

$$f(x + x_1\psi(f(x))) = f(x) \qquad \text{for } x \in \mathbb{R}. \tag{26}$$

Let us fix an $x \in \mathbb{R}$. Since at least two of the following numbers: $x, x+x_0\psi(f(x)), x+x_1\psi(f(x))$ have the same sign, $x_0 \neq x_1$ and $f|_{\mathbb{R}_-}$, $f|_{\mathbb{R}_+}$ are one-to-one, so in view of (24) and (26), we get $\psi(f(x)) = 0$;

2) $f(x) \neq 1$ for $x \in x_0\mathbb{R}_-$. Since $f(0) = 0$ and $f(\mathbb{R}) = \mathbb{R}_+$, we have $f(x_0\mathbb{R}_-) \subset [0, 1)$. Moreover, setting in (1) $x = x_0$, one obtains that $f(x_0\phi(f(\mathbb{R}))) = f(\mathbb{R}) = \mathbb{R}_+$. Then $x_0\phi(f(\mathbb{R})) = x_0\mathbb{R}_+$. Fix an $x \in \mathbb{R}$. If $x \in f^{-1}(\{0\})$, then according to Corollary 1, $\psi(f(x)) = 0$. If $x \in f^{-1}([1, \infty))$, then on account of (24), we get that $x, x + x_0\psi(f(x)) \in x_0\mathbb{R}_+$. By (8) f is one-to-one on $x_0\phi(f(\mathbb{R}))$, so using (24) again, we have that $\psi(f(x)) = 0$.

Assume now, that $x \in f^{-1}((0, 1))$ and suppose that $\psi(f(x))) \neq 0$. If $\psi(f(x))) > 0$, then for $n \in \mathbb{N}$ sufficiently big, we have $x + nx_0\psi(f(x)) \in x_0\mathbb{R}_+$ and $x + (n + 1)x_0\psi(f(x)) \in x_0\mathbb{R}_+$. Thus, in view of (25), we obtain

$$f(x + nx_0\psi(f(x))) = f(x + (n + 1)x_0\psi(f(x))).$$

Since f is one-to-one on $x_0\mathbb{R}_+$, we get $\psi(f(x)) = 0$, which gives a contradiction. If $\psi(f(x))) < 0$, then there exists $m \in \mathbb{N}$ such that $z := x + mx_0\psi(f(x)) \in x_0\mathbb{R}_-$. Thus $f(z) \in [0, 1)$. Moreover, on account of (25), $\psi(f(z)) = \psi(f(x)) < 0$. Then by Corollary 1, $f(z) \neq 0$ and according to Corollary 2, there exists a $y \in x_0\mathbb{R}_+ = x_0\phi(f(\mathbb{R}))$ with $f(z)f(y) = 1$. In virtue of (1) and (8), we have $f(\frac{zy}{x_0} + y\psi(f(z))) = 1$. Then $\frac{zy}{x_0} + y\psi(f(z)) \in x_0\mathbb{R}_+$ and using again the invertibility of f on $x_0\mathbb{R}_+$, we obtain $\frac{zy}{x_0} + y\psi(f(z)) = x_0$. Hence $\psi(f(z)) = \frac{x_0}{y} - \frac{z}{x_0} > 0$. This is a contradiction. $\square$

Proposition 5. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous, $\phi(1) = -1$ and $\psi(1) = 0$. Then* (1) *has non-constant continuous solutions if and only if ϕ and ψ have the form* (15) *and* (23), *respectively.*

Furthermore, whenever ϕ and ψ have the form (15) *and* (23), *respectively, then the general solution of* (1) *in the class of non-constant continuous functions $f : \mathbb{R} \to \mathbb{R}$ is given by* (17).

Proof. Notice at first that if ϕ and ψ are of the form (15) and (23), respectively, then each function of the form (17) is a solution (1). Assume that f is a non-constant continuous function satisfying (1). On account of Proposition 2, it is enough to prove that $\psi(f(x)) = 0$ for $x \in \mathbb{R}$. According to Lemma 3(i) and Corollary 1(i), there is an $x_0 \neq 0$ such that $f(x_0) = 1$. Putting in (1) $x = y = x_0$, we obtain $f(-x_0) = 1$. Hence, in view of Lemma 3(iv), we get

$$\phi(f(y)) = \begin{cases} \frac{y}{x_0} & \text{for } \; y \in x_0\phi(f(\mathbb{R})) \\ -\frac{y}{x_0} & \text{for } \; y \in -x_0\phi(f(\mathbb{R})). \end{cases} \tag{27}$$

Putting in (1) $y = x_0$ and then $y = -x_0$, it is easy to obtain

$$\phi\left(f(-x + x_0\psi(f(x)))\right) = \phi\left(f(-x - x_0\psi(f(x)))\right) = \phi\left(f(x)\right) \tag{28}$$

for $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. According to (27) and (28), we have that either $-x + x_0\psi(f(x)) = -x - x_0\psi(f(x))$ or $-x + x_0\psi(f(x)) = -(-x - x_0\psi(f(x)))$. Hence, in virtue of Corollary 1, we get $\psi(f(x)) = 0$. □

Proposition 6. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. If $\phi(1) + \psi(1) = 1$ and $\psi(1) \neq 0$, then* (1) *has non-constant continuous solutions if and only if*

$$\begin{aligned} \phi(x) &= \begin{cases} cx^{\frac{1}{r}} & \textit{for } \; x \in \mathbb{R}_+ \\ \phi_1(x) & \textit{for } \; x \in \mathbb{R}_- \end{cases} \\ \psi(x) &= \begin{cases} (1-c)x^{\frac{1}{r}} & \textit{for } \; x \in \mathbb{R}_+ \\ \psi_1(x) & \textit{for } \; x \in \mathbb{R}_-, \end{cases} \end{aligned} \tag{29}$$

where r is an arbitrary positive real constant, and

(C_1) *$c \in \mathbb{R} \setminus \{0, 1, 2\}$, $\phi_1(x) = -c(-x)^{\frac{1}{r}}$ for $x \in \mathbb{R}_-$ and $\psi_1(x) = -(1-c)(-x)^{\frac{1}{r}}$ for $x \in \mathbb{R}_-$*

or

(C_2) *$c \in (0, 1)$ and $\phi_1, \psi_1 : \mathbb{R}_- \to \mathbb{R}$ are arbitrary functions such that ϕ_1 is continuous and $\phi_1(0) = 0$.*

Furthermore, whenever ϕ and ψ have the form (29), *then the general solution of* (1) *in the class of non-constant continuous functions $f : \mathbb{R} \to \mathbb{R}$ is given by* (19) *or*

$$f(x) = \begin{cases} p|x|^r & \textit{for } \; x \in D \\ 0 & \textit{for } \; x \in \mathbb{R} \setminus D, \end{cases} \tag{30}$$

where p is an arbitrary positive real constant and $D \in \{\mathbb{R}_-, \mathbb{R}_+\}$, in the case when

(C_1) and (C_2) hold; by (19) when (C_1) holds and (C_2) does not hold; and by (30) when (C_2) holds and (C_1) does not hold.

Proof. Notice that if ϕ and ψ have the form (29), then every function of the form (19) or (30) satisfies (1) in the corresponding cases. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant continuous solution of (1). According to Corollary 1(i) and Lemma 3(i), there is an $x_0 \neq 0$ such that $f(x_0) = 1$. Assume that $\phi(1)+\psi(1) = 1$ and $\psi(1) \neq 0$. Then, in view of Lemma 3(iii), we have

$$\phi\left(f(x)\right) = \begin{cases} \phi(1)\frac{a}{x_0} & \text{for } \ x \leq a \\ \phi(1)\frac{x}{x_0} & \text{for } \ a < x < b \\ \phi(1)\frac{b}{x_0} & \text{for } \ x \geq b \end{cases} \tag{31}$$

Step 1. We show that $a > -\infty$ implies $f(y) = f(a)$ for $y \leq a$. Assume that $a > -\infty$. Setting in (1) $x = x_0$, in view of (31) we obtain

$$f(a\phi(1) + \psi(1)y) = f(y) \qquad \text{for } \ y \leq a. \tag{32}$$

Let us fix a $y \leq a$ and consider the following cases:

1) $\psi(1) > 1$. Then $\frac{y-a\phi(1)}{\psi(1)} \leq a$ and inserting into (32) $\frac{y-a\phi(1)}{\psi(1)}$ in place of y, we get $f(y) = f(\frac{y-a\phi(1)}{\psi(1)})$. Hence, by easy induction we obtain

$$f(y) = f\left(y\psi(1)^{-n} - a\phi(1)\sum_{i=1}^{n}\psi(1)^{-i}\right) \qquad \text{for } \ n \in \mathbb{N}$$

and using the continuity of f, we have

$$f(y) = f\left(\lim_{n\to\infty} y\psi(1)^{-n} - a\phi(1)\sum_{i=1}^{\infty}\psi(1)^{-i}\right) = \\ f\left(-a\phi(1)\frac{1}{\psi(1)-1}\right) = f(a);$$

2) $\psi(1) \in (0,1)$. Then $a\phi(1)+y\psi(1) = a(1-\psi(1))+y\psi(1) = a+\psi(1)(y-a) \leq a$. Inserting into (32) $a\phi(1) + y\psi(1)$ in place of y, one obtains

$$f(y) = f\left(a\phi(1)(1+\psi(1)) + y\psi(1)^2\right).$$

Thus by induction, we get

$$f(y) = f\left(a\phi(1)\sum_{i=0}^{n-1}\psi(1)^i + y\psi(1)^n\right) \qquad \text{for } \ n \in \mathbb{N}.$$

Hence

$$f(y) = f\left(a\phi(1)\sum_{i=0}^{\infty}\psi(1)^i + \lim_{n\to\infty} y\psi(1)^n\right) = \\ f\left(a\phi(1)\frac{1}{1-\psi(1)}\right) = f(a);$$

3) $\psi(1) < 0$. Then $\phi(1) > 1$, $\frac{y - a\psi(1)}{\phi(1)} \le a$ and as above we have

$$\begin{aligned} f(y) &= f\left(a\varphi(1) + \lim_{n\to\infty} y\psi(1)\phi(1)^{-n} - a\psi(1)^2 \sum_{i=1}^{\infty} \phi(1)^{-i}\right) \\ &= f\left(a\varphi(1) - a\psi(1)^2 \frac{1}{\phi(1) - 1}\right) = f(a). \end{aligned}$$

Furthermore, one can analogously show that if $b < \infty$, then $f(x) = f(b)$ for $x \ge b$. In particular, since f is unbounded, so a and b cannot be both finite.

Step 2. We distinguish three cases.

Case 1) $a = -\infty$ and $b = \infty$. Then on account of (31), we have that $\phi(f(x)) = \phi(1)\frac{x}{x_0}$ for $x \in \mathbb{R}$. Moreover $\phi(1) + \psi(1) = 1$ and $|\psi(1)| \ne 1$ imply that $\phi(1) \ne 0$. Then by (31), f is one-to-one on $\mathbb{R}$ and putting in (1) $y = x_0$, we obtain $x\phi(1) + x_0\psi(f(x)) = x$. Hence $\psi(f(x)) = \psi(1)\frac{x}{x_0}$ for $x \in \mathbb{R}$ and the equation (1) becomes $f(\frac{xy}{x_0}) = f(x)f(y)$ for $x, y \in \mathbb{R}$. According to Lemma 4, there exists a positive real constant r such that $f(x) = |\frac{x}{x_0}|^r \operatorname{sgn}\left(\frac{x}{x_0}\right)$ for $x \in \mathbb{R}$. Let $u \in \mathbb{R}_+$ and $x := x_0 u^{\frac{1}{r}}$. Then in view of (31), we get

$$\phi(1)u^{\frac{1}{r}} = \phi(1)\frac{x}{x_0} = \phi(f(x)) = \phi\left(\left|\frac{x}{x_0}\right|^r \operatorname{sgn}\left(\tfrac{x}{x_0}\right)\right) = \phi\left(\left(\frac{x}{x_0}\right)^r\right) = \phi(u).$$

Now, let $u < 0$ and $x := -x_0(-u)^{\frac{1}{r}}$. Then

$$\begin{aligned} -\phi(1)(-u)^{\frac{1}{r}} = \phi(1)\frac{x}{x_0} &= \phi(f(x)) \\ &= \phi\left(\left|\frac{x}{x_0}\right|^r \operatorname{sgn}\left(\tfrac{x}{x_0}\right)\right) = \phi\left(-\left(-\frac{x}{x_0}\right)^r\right) = \phi(u). \end{aligned}$$

Thus $\phi(x) = \phi(1)|x|^{\frac{1}{r}}\operatorname{sgn}(x)$ for $x \in \mathbb{R}$. Similarly one can prove that $\psi(x) = \psi(1)|x|^{\frac{1}{r}}\operatorname{sgn}(x)$ for $x \in \mathbb{R}$. Moreover since $\phi(1)+\psi(1) = 1$, $|\psi(1)| \ne 1$ and $\psi(1) \ne 0$, so $\phi(1) \notin \{0, 1, 2\}$. Then ϕ and ψ have the form (29) with $c := \phi(1)$, (C_1) holds and f is of the form (19) with $d := |x_0|^{-r}\operatorname{sgn}\left(x_0^{-1}\right)$.

Case 2) $a > -\infty$ and $b = \infty$. Then on account of (31), we have that $f(x) = f(a)$ for $x \le a$ and f is one-to-one on $[a, \infty)$. Thus either $f(\mathbb{R}) = [f(a), \infty)$ or $f(\mathbb{R}) = (-\infty, f(a)]$. From Corollary 2 it follows that $\mathbb{R}_+ = f(\mathbb{R}) = [f(a), \infty)$. Hence $f(a) = 0 = f(0)$ and so $a \ge 0$. Suppose that $a > 0$. Then by (31), $\phi(f(0)) = \phi(1)\frac{a}{x_0} \ne 0$, which contradicts Corollary 1. Therefore $a = 0$ and $f^{-1}(\{0\}) = \mathbb{R}_-$. In particular $x_0 > 0$. Furthermore, as in the previous case we have

$$\phi(f(x)) = \phi(1)\frac{x}{x_0} \qquad \text{for } x \in \mathbb{R}_+ \tag{33}$$

and

$$\psi(f(x)) = \psi(1)\frac{x}{x_0} \qquad \text{for } x \in \mathbb{R}_+, \tag{34}$$

so in virtue of (1), we get $f(\frac{xy}{x_0}) = f(x)f(y)$ for $x, y \in \mathbb{R}_+$. Then there exists a positive real constant r such that $f(x) = (\frac{x}{x_0})^r$ for $x \in \mathbb{R}_+$. Fix a $u \in \mathbb{R}_+$ and let $x := x_0 u^{\frac{1}{r}}$. Hence $f(x) = u$ and by(31), we have

$$\phi(u) = \phi(f(x)) = \phi(1)\frac{x}{x_0} = \phi(1)u^{\frac{1}{r}}$$

and

$$\psi(u) = \psi(f(x)) = \psi(1)\frac{x}{x_0} = \psi(1)u^{\frac{1}{r}} = (1 - \phi(1))u^{\frac{1}{r}}.$$

Moreover let $\alpha < 0$ and $\beta > 0$ be fixed. Setting in (1) $x = \alpha, y = \beta$ and then $x = \beta, y = \alpha$, we obtain

$$f(\alpha\phi(f(\beta)) + \beta\psi(0)) = 0$$

and

$$f(\beta\phi(0) + \alpha\psi(f(\beta))) = 0,$$

respectively. Thus, according to (33), (34) and Corollary 1, we get

$$f\left(\phi(1)\frac{\alpha\beta}{x_0}\right) = f\left((1 - \phi(1))\frac{\alpha\beta}{x_0}\right) = 0.$$

Further $\frac{\alpha\beta}{x_0} < 0$ and $f^{-1}(\{0\}) = \mathbb{R}_-$, so $\phi(1) \in (0, 1)$. Then ϕ and ψ have the form (29) with $c := \phi(1)$, (C_2) holds and f is of the form (30) with $p := x_0^{-r}$ and $D := \mathbb{R}_+$;

Case 3) $a = -\infty$ and $b < \infty$. Similarly to the previous case one can obtain that ϕ and ψ are of the form (29) with $c := \phi(1)$, (C_2) holds and f has the form (30) with $p := |x_0|^{-r}$ and $D := \mathbb{R}_-$. □

Let us summarize our consideration in the following

Theorem 1. *Let $\phi, \psi : \mathbb{R} \to \mathbb{R}$ be given functions such that ϕ is continuous and $|\psi(1)| \neq 1$. Then $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of* (1) *if and only if one of the following conditions holds:*

1) $f = 0$ *or* $f = 1$,

2) ϕ *and* ψ *have one of the forms:*

(i)

$$\phi(x) = \begin{cases} -x^{\frac{1}{r}} & \text{for } x \in \mathbb{R}_+ \\ \phi_1(x) & \text{for } x \in \mathbb{R}_- \end{cases}$$

$$\psi(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R}_+ \\ \psi_1(x) & \text{for } x \in \mathbb{R}_-, \end{cases}$$

where r is an arbitrary positive real constant and $\phi_1, \psi_1 : \mathbb{R}_- \to \mathbb{R}$ are arbitrary functions such that ϕ_1 is continuous and $\phi_1(0) = 0$,

(ii)

$$\phi(x) = \begin{cases} x^{\frac{1}{r}} & \text{for } x \in \mathbb{R}_+ \\ \phi_1(x) & \text{for } x \in \mathbb{R}_- \end{cases}$$

$$\psi(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R}_+ \\ \psi_1(x) & \text{for } x \in \mathbb{R}_-, \end{cases}$$

where r is an arbitrary positive real constant and $\phi_1, \psi_1 : \mathbb{R}_- \to \mathbb{R}$ are arbitrary functions such that ϕ_1 is continuous and $\phi_1(0) = 0$,

(iii)

$$\phi(x) = \begin{cases} cx^{\frac{1}{r}} & \text{for } x \in \mathbb{R}_+ \\ \phi_1(x) & \text{for } x \in \mathbb{R}_- \end{cases}$$

$$\psi(x) = \begin{cases} (1-c)x^{\frac{1}{r}} & \text{for } x \in \mathbb{R}_+ \\ \psi_1(x) & \text{for } x \in \mathbb{R}_-, \end{cases}$$

where r is an arbitrary positive real constant and at least one of $(C_1), (C_2)$ holds.

Furthermore, whenever ϕ and ψ have the form (i), (ii) *or* (iii)*, then the general solution of* (1) *in the class of non-constant continuous functions is given, respectively, by:*

(i) (17)*;*

(ii) (18) *or* (19) *whenever $\phi_1(x) = -(-x)^{\frac{1}{r}}$ for $x \in \mathbb{R}_-$ and $\psi_1(x) = 0$ for $x \in \mathbb{R}_-$,*
(18)*, otherwise.*

(iii) (19) *whenever (C_1) holds and (C_2) does not hold,*
(30) *whenever (C_2) holds and (C_1) does not hold,*
(19) *or* (30) *whenever (C_1) and (C_2) hold.*

Remark 2. Determining the algebraic substructure of a generalization of the Clifford group, N. Brillouët-Belluot and J. Dhombres have considered the functional equation (cf. [6] p. 281)

$$g(xg(y) + yg(x)) = tg(x)g(y) \quad \text{for } x, y \in \mathbb{R}, \tag{35}$$

where t is a non-zero real constant and $g : \mathbb{R} \to \mathbb{R}$ is an unknown function. Some generalizations of (35) have been studied in [5], [6] and [8]. Notice that the equation

$$g(x\bar{\phi}(g(y)) + y\bar{\psi}(g(x))) = tg(x)g(y) \quad \text{for } x, y \in \mathbb{R}, \tag{36}$$

where t is a non-zero real constant and $\bar{\phi}, \bar{\psi} : \mathbb{R} \to \mathbb{R}$ are given functions, is equivalent to (1), with $f(x) := tg(x)$ for $x \in \mathbb{R}$, $\phi(x) := \bar{\phi}(\frac{x}{t})$ for $x \in \mathbb{R}$ and $\psi(x) := \bar{\psi}(\frac{x}{t})$ for $x \in \mathbb{R}$.

Thus, if $\bar{\phi}$ is a continuous function and $|\bar{\psi}(\frac{1}{t})| \neq 1$, then the general solution of (36) in the class of continuous functions may be easily deduced from Theorem 1.

Remark 3. The results of this paper are obtained under the assumption $|\psi(1)| \neq 1$, so they include neither the results of [5]–[9], nor the results concerning the Gołąb–Schinzel functional equation.

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References

[1] J. Aczél and J. Dhombres, *Functional equations in several variables*, Cambridge University Press, Cambridge, 1989.

[2] J. Aczél and S. Gołąb, *Remarks on one-parameter semigroups of the affine group and their homo- and isomorphisms*, Aequationes Math. *4* (1970), 1–10.

[3] W. Benz, *The cardinality of the set of discontinuous solutions of a class of functional equations*, Aequationes Math. *32* (1987), 58–62.

[4] N. Brillouët-Belluot, *On some functional equations of Gołąb–Schinzel type*, Aequationes Math. *42* (1991), 239–270.

[5] N. Brillouët-Belluot, *On the solutions of the functional equation* $f(f(y)^k x + f(x)^l y) = \lambda f(x)f(y)$, Publ. Math. Debrecen *41* (1992), 213–223.

[6] N. Brillouët-Belluot and J. Dhombres, *Equations fonctionnelles et recherche de sous-groupes*, Aequationes Math. *31* (1986), 253–293.

[7] J. Brzdęk, *On the solutions of the functional equation* $f(xf(y)^k + yf(x)^l) = tf(x)f(y)$, Publ. Math. Debrecen *39* (1991), 175–183.

[8] J. Brzdęk, *Subgroups of the group* Z_n *and a generalization of the Gołąb–Schinzel equation*, Aequationes Math. *43* (1992), 59–71.

[9] J. Dhombres, *Some aspects of functional equations*, Lecture Notes, Department of Mathematics, Chulalongkorn University, Bangkok, 1979.

[10] S. Gołąb and A. Schinzel, *Sur l'équation fonctionnelle* $f(x + yf(x)) = f(x)f(y)$, Publ. Math. Debrecen *6* (1959), 113–125.

[11] P. Javor, *Continuous solutions of the functional equation* $f(x + yf(x)) = f(x)f(y)$, in: *Proc. Internat. Sympos. on Topology and its Applications (Herceg-Novi, 1968)*, 206–209 Savez Druŝtava Mat. Fiz. i Astronom., Belgrade, 1969.

[12] S. Midura, *Sur la détermination de certains sous-groupes du groupe* L_s^1 *á l'aide d'équations fonctionnelles*, Dissertationes Math. (Rozprawy Mat.) 105, PWN Warszawa 1973, 1–38.

[13] S. Nabeya, *On the functional equation* $f(p + qx + rf(x)) = a + bx + cf(x)$, Aequationes Math. *11* (1974), 199–211.

[14] M. Sablik and P. Urban, *On the solutions of the functional equation* $f(xf(y)^k + yf(x)^l) = f(x)f(y)$, Demonstratio Math. *18* (1985), 863–867.

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