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Two-sided delay-difference equations and evolution maps

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Abstract. We establish the equivalence of hyperbolicity and of two other properties for a two-sided linear delay-difference equation and its evolution map. These two properties are the admissibility with respect to various pairs of spaces, and the Ulam–Hyers stability of the equation, again with respect to various spaces. This gives characterizations of important properties of a linear dynamical system in terms of corresponding properties of the autonomous dynamical system determined by the associated evolution map.

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1. Introduction

The main aim of this work is to show that hyperbolicity and two related properties are equivalent for a two-sided dynamical system determined by a linear delay-difference equation and for the dynamical system determined by its evolution map. The main advantage of considering evolution maps is that they always determine autonomous dynamical systems. Indeed, it is well known that it is often much simpler to establish a given property for an autonomous system than for a general nonautonomous system. The two additional properties can be described in terms of certain perturbations of the original linear dynamical system. More precisely, other than hyperbolicity, we consider:

(i) the admissibility with respect to various pairs of admissible spaces, for the perturbations of the dynamical system;
(ii) the Ulam–Hyers stability of the system, which amounts to show that there are exact solutions when there are approximate solutions.

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Our results thus give characterizations of several important stability properties related to a linear dynamical system and its perturbations, in terms of corresponding properties for the *autonomous* dynamical system determined by the associated evolution map. To a certain extent these results are motivated by corresponding results for dynamical systems without delay, although we emphasize that to obtain related results for delay-difference equations requires several nontrivial changes. In particular, this includes dealing with the localization problem for the projections of an exponential dichotomy onto higher-dimensional spaces, as well as introducing appropriate admissible spaces that are adapted to a delay-difference equation.

We briefly recall the importance of the notions considered in the paper. The introduction of hyperbolicity goes back to seminal work of Perron [23] and has many consequences, such as the construction of stable and unstable invariant manifolds, the closing and shadowing lemmas, etc. For details and further references on the notion of hyperbolicity and its consequences, we refer the reader to the books [10,12,18,25] and specifically to [9,11] for delay equations. The notion of admissibility also goes back to Perron, in the same work [23], and allows one to characterize hyperbolicity via the existence and uniqueness of solutions of the perturbations of a given linear system, taking the perturbations and the solutions in certain admissible Banach spaces. We refer the reader to the books [8,18] for details and many early references. Finally, for the Ulam–Hyers stability property we refer the reader to the book [14] for details on the origin of the notion and further references. For many developments one can see the books [7,15,26] and the references therein. In the context of differential equations Ulam–Hyers stability seems to have been first considered by Obłoza [19] and then by Alsina and Ger [1]. There are also some works for delay equations, such as [13,20,21,27].

As already noted above, our main aim is to consider each of these three properties (hyperbolicity, admissibility, and Ulam–Hyers stability) for a delay-difference equation and show that each of them is equivalent to a corresponding property of the evolution map associated to the given equation. The evolution map is defined on a certain space of sequences, and while the original dynamical system may be nonautonomous, this map always defines an *autonomous* dynamical system.

Before proceeding, we mention with more detail why the equivalence results for hyperbolicity and Ulam–Hyers stability between a nonautonomous setting and the associated autonomous setting given by an evolution map may be of interest. We first note that Ulam–Hyers stability can be described, equivalently, as a shadowing property (see the books [22,24] for details and references). We emphasize that shadowing theory was mainly motivated by hyperbolic dynamical systems. In particular, results of Anosov [2] and Bowen [6] lead to the structural stability of hyperbolic sets. These shadowing results also have important generalizations to nonuniformly hyperbolic systems (a detailed description falls out of the scope of our work). In particular, a closing lemma was first proved by Katok in [16] (see [17] for a shadowing lemma for nonuniformly hyperbolic systems). These results and their proofs are somewhat technical, and it is often convenient to use instead an autonomous setting, although at the expense of considering a higher-dimensional space. In another direction, while the notion of hyperbolicity for a *nonautonomous* linear dynamical system gives rise to a spectrum such as the Sacker–Sell spectrum, on the other hand an *autonomous* linear system defined by a single linear operator leads to the study of the spectrum of this operator.

So that we can describe rigorously how evolution maps can characterize the former properties, we first introduce several basic notions. Take $r \in \mathbb{N}$ and let $I_r = [-r, 0] \cap \mathbb{Z}$. Given a Banach space X with norm $|\cdot|$, the set Y of all functions $\varphi\colon I_r \to X$ is a Banach space when equipped with the (supremum) norm

$$\|\varphi\| = \max\{|\varphi(s)| : s \in I_r\}.$$

Now let $L_m\colon Y \to X$ be bounded linear operators for $m \in \mathbb{Z}$ such that

$$c := \sup_{m\in\mathbb{Z}} \|L_m\| < +\infty \tag{1}$$

and consider the delay-difference equation

$$x(m+1) = L_m x_m \quad \text{for}\, m \in \mathbb{Z}. \tag{2}$$

Here the function $x_m \in Y$ is defined by

$$x_m(s) = x(m+s) \quad \text{for}\, s \in I_r,$$

provided that the domain of x contains $I_r + m$. In general, equation (2) gives rise to a nonautonomous dynamical system.

Given a set $A \subset \mathbb{Z}^n$ for some $n \in \{1, 2\}$, let $\ell^\infty(A)$ be the Banach space of bounded functions $\psi\colon A \to X$ equipped with the supremum norm $\|\cdot\|_\infty$. We define a linear operator $\mathcal{L}\colon \ell^\infty(I_r \times \mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by

$$(\mathcal{L}\varphi)(k) = L_{k-1}\varphi(\cdot, k-1) \quad \text{for}\, k \in \mathbb{Z} \tag{3}$$

and we consider the *induced equation*

$$u(m+1, k) = (\mathcal{L}u_m)(k) \quad \text{for}\, m, k \in \mathbb{Z}, \tag{4}$$

where the map $u_m\colon I_r \times \mathbb{Z} \to X$ is given by

$$u_m(s, k) = u(m+s, k+s) \quad \text{for}\, (s, k) \in I_r \times \mathbb{Z}. \tag{5}$$

Since the operator $\mathcal{L}$ is (3) does not depend on m, equation (4) gives rise to an *autonomous* dynamical system.

The solutions of the equations (2) and (4) induce certain linear operators that in particular can be used to describe the hyperbolicity of the equations.

Namely, one can define linear operators $T(m,n)\colon Y \to Y$ for $m \geq n$ by requiring that

$$T(m,n)x_n = x_m \tag{6}$$

for any solution x of equation (2) and any $m,n \in \mathbb{Z}$ with $m \geq n$. Moreover, one can define a linear operator $S\colon \ell^\infty(I_r \times \mathbb{Z}) \to \ell^\infty(I_r \times \mathbb{Z})$ by requiring that

$$Su_n = u_{n+1}$$

for any solution u of equation (4) and any $n \in \mathbb{Z}$. One can show that

$$(S\varphi)(s,k) = (T(k,k-1)\varphi(\cdot,k-1))(s) \quad \text{for } (s,k) \in I_r \times \mathbb{Z}$$

(see Proposition 1). Using the equations and these operators, we show in the paper that:

(i) equation (2) has an exponential dichotomy if and only if equation (4) has an exponential dichotomy (see Theorem 2);
(ii) equation (2) satisfies an admissibility property if and only if equation (4) satisfies an analogous admissibility property (see Theorem 3);
(iii) equation (2) is Ulam–Hyers stable if and only if equation (4) is Ulam–Hyers stable (see Theorem 5).

The notion of exponential dichotomy is recalled in Sect. 3 while the notions of admissibility and Ulam–Hyers stability are recalled, respectively, in Sects. 4 and 5. For the convenience of the reader, in the following paragraphs we also describe briefly the latter property in a particular case.

Given a Banach space E of functions $x\colon \mathbb{Z} \to X$, we say that the pair (E,E) is *admissible* for equation (2) if for each $y \in E$ there exists a unique $x \in E$ satisfying

$$x(m+1) = L_m x_m + y(m+1) \quad \text{for } m \in \mathbb{Z}. \tag{7}$$

Similarly, given a Banach space F of functions $u\colon \mathbb{Z}^2 \to X$, we say that the pair (F,F) is *admissible* for equation (4) if for each $v \in F$ there exists a unique $u \in F$ satisfying

$$u(m+1) = \mathcal{L}u_m + v(m+1) \quad \text{for } m \in \mathbb{Z}. \tag{8}$$

We consider in particular the spaces of bounded sequences

$$E^\infty = \ell^\infty(\mathbb{Z}) \quad \text{and} \quad F^\infty = \ell^\infty(\mathbb{Z}^2). \tag{9}$$

Moreover, for each $p \in [1,+\infty)$ we consider Banach spaces E^p and F^p that are obtained modifying in some appropriate manner the spaces $\ell^p(\mathbb{Z})$ and $\ell^p(\mathbb{Z}^2)$ so that they are adapted to delay-difference equations (see Sect. 4 for more details). Namely, E^p is the set of all functions $x\colon \mathbb{Z} \to X$ such that

$$\sum_{m\in\mathbb{Z}} \max_{s\in I_r} \|x(m+s)\|^p < +\infty, \tag{10}$$

while F^p is the set of all functions $u\colon \mathbb{Z}^2 \to X$ such that

$$\sum_{m,k\in\mathbb{Z}} \max_{s\in I_r} \|u(m+s,k+s)\|^p < +\infty. \tag{11}$$

We say that equation (2) is *uniformly Ulam–Hyers stable* with respect to E^∞ if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and $x, y \in E^\infty$ satisfying

$$\sup_{m\in\mathbb{Z}} |x(m+1) - L_m x_m - y(m+1)| < \varepsilon$$

there exists $z \in E^\infty$ satisfying

$$z(m+1) = L_m z_m + y(m+1) \quad \text{for } m \in \mathbb{Z}$$

such that

$$\sup_{m\in\mathbb{Z}} |x(m) - z(m)| < \kappa\varepsilon.$$

Here $|\cdot|$ denotes the norm on the space X. One can define similarly the notion of uniform Ulam–Hyers stability for equation (4) with respect to the space F^∞. See Sect. 5 for corresponding notions with respect to the spaces E^p and F^p for each $p \in [1, +\infty)$.

We observe that is shown in [4] that if $\sup_{m\in\mathbb{Z}} \|L_m\| < +\infty$ (see (1)), then for each $p \in [1, +\infty]$ equation (2) has an exponential dichotomy if and only if the pair (E^p, E^p) is admissible. Together with Theorems 2 and 3 this yields the following result.

Theorem 1. *Let $L_m\colon Y \to X$, for $m \in \mathbb{Z}$, be bounded linear operators satisfying $\sup_{m\in\mathbb{Z}} \|L_m\| < +\infty$. Then the following properties are equivalent:*

(i) *equation (2) has an exponential dichotomy;*

(ii) *equation (4) has an exponential dichotomy;*

(iii) *given $p \in [1, +\infty]$, for each $y \in E^p$ there exists a unique $x \in E^p$ satisfying (7);*

(iv) *given $p \in [1, +\infty]$, for each $v \in F^p$ there exists a unique $u \in F^p$ satisfying (8).*

A simple consequence of our work is that property (iii) holds for some $p \in [1, +\infty]$ if and only if it holds for all $p \in [1, +\infty]$. A similar observation applies to property (iv). In addition, we also show that if any of the properties in Theorem 1 holds, then for each $p \in [1, +\infty]$ equation (2) is uniformly Ulam–Hyers stable with respect to E^p, and equation (4) is uniformly Ulam–Hyers stable with respect to F^p (see Theorem 4).

To some extent Theorem 2 is based on a related approach in [5] for one-sided equations, but there are many differences between the two. In particular, the two notions of exponential dichotomy are necessarily distinct (more precisely, unlike in (22) below, the evolution map S need not take the unstable space *onto* itself). Theorem 3 and its proof are inspired by related work in [3] for equations without delay, but the existence of a delay requires various nontrivial changes, starting with the choice of appropriate pairs of admissible spaces.

2. Induced equations

In this section we introduce some basic notions from the theory of delay-difference equations. Moreover, to each nonautonomous delay-difference equation we associate an autonomous delay-difference equation on a higher-dimensional space that is crucial for our approach.

2.1. Delay-difference equations

We continue to use the same notations and notions as in the introduction. Let $L_m\colon Y \to X$ be bounded linear operators for $m \in \mathbb{Z}$ satisfying (1). We consider the nonautonomous delay-difference equation in (2).

Since the space Y can be identified with X^{r+1}, one can think of each operator L_m as a row of bounded linear operators $L_m^s\colon X \to X$ for $s = -r, \ldots, 0$ that applies to the column with values $x_m(s)$ for $s = -r, \ldots, 0$. Then equation (2) is equivalent to

$$x(m+1) = \sum_{s=-r}^{0} L_m^s x_m(s) = \sum_{s=-r}^{0} L_m^s x(m+s).$$

Now we consider appropriate initial value problems and their solutions. Namely, given $n \in \mathbb{N}$ and $\varphi \in Y$, we denote by $x\colon [n-r, +\infty) \cap \mathbb{Z} \to X$ the unique solution of the problem

$$x(m+1) = L_m x_m \quad \text{for}\, m \geq n \,\text{with}\, x_n = \varphi.$$

This means that we are given the values $x(n+s) = \varphi(s)$ for $s = -r, \ldots, 0$, and that the remaining ones, that is, $x(m)$ for $m > n$, are obtained from equation (2). As already noted in the introduction, these solutions induce linear operators $T(m,n)\colon Y \to Y$ for $m \geq n$ defined by (6).

It follows readily from (2) and (6) that

$$(T(m+1,m)\varphi)(0) = L_m \varphi \quad \text{for}\, \varphi \in Y \,\text{and}\, m \in \mathbb{Z}. \tag{12}$$

Moreover, we have

$$\|x_{m+1}\| \leq \max\{\|x_m\|, |x(m+1)|\} \leq \max\{1, c\} \|x_m\|$$

and so

$$\|T(m+1,m)\| \leq \max\{1, c\} \quad \text{for}\, m \in \mathbb{Z}. \tag{13}$$

2.2. Induced equations

In this section we introduce an autonomous delay-difference equation associated to equation (2), although on a higher-dimensional space. It turns out that the new dynamical system can be used to characterize completely some properties of the original (nonautonomous) dynamical system induced by equation (2). This is the case for example of hyperbolicity, which is considered in Sect. 3.

Given a function $u\colon B \to X$ with domain $B \subset \mathbb{Z}^2$ and $m \in \mathbb{Z}$:

(i) when $B \supset I_r \times \{m\}$, we define a function $u^m \in Y$ by

$$u^m(s) = u(s,m) \quad \text{for}\, s \in I_r; \tag{14}$$

(ii) when $B \supset \{m\} \times \mathbb{Z}$, we define a function $u(m)\colon \mathbb{Z} \to X$ by

$$u(m)(k) = u(m,k) \quad \text{for}\, k \in \mathbb{Z}. \tag{15}$$

Now let $\mathcal{L}\colon \ell^\infty(I_r \times \mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be the linear operator defined by (3) or, equivalently,

$$(\mathcal{L}\varphi)(k) = L_{k-1}\varphi^{k-1} \quad \text{for}\, k \in \mathbb{Z},$$

using the notation introduced in (14). We consider the autonomous delay-difference equation

$$u(m+1) = \mathcal{L}u_m \quad \text{for}\, m \in \mathbb{Z}, \tag{16}$$

which can also be written in the equivalent form in (4). We call it the *induced equation*. Note that

$$u(m+1,k) = u(m+1)(k) = (\mathcal{L}u_m)(k) = L_{k-1}u_m^{k-1}$$

for $m, k \in \mathbb{Z}$, where

$$u_m^{k-1}(s) = (u_m)^{k-1}(s) = u_m(s,k-1) = u(m+s,k-1+s) \tag{17}$$

for $(s,k) \in I_r \times \mathbb{Z}$ (following (14) and then (5)). The next function u_{m+1} in (16) is given by

$$\begin{aligned} u_{m+1}(s,k) &= \begin{cases} u(m+1,k) & \text{if}\, s = 0, \\ u(m+1+s,s+k) & \text{if}\, s < 0 \end{cases} \\ &= \begin{cases} L_{k-1}u_m^{k-1} & \text{if}\, s = 0, \\ u_m(s+1,k-1) & \text{if,}\, s < 0. \end{cases} \end{aligned}$$

Given $n \in \mathbb{Z}$ and $\varphi \in \ell^\infty(I_r \times \mathbb{Z})$, we denote by

$$u\colon ([n-r,+\infty) \cap \mathbb{Z}) \times \mathbb{Z} \to X$$

the unique solution of the problem

$$u(m+1) = \mathcal{L}u_m \quad \text{for}\, m \geq n\, \text{with}\, u_n = \varphi. \tag{18}$$

Using these solutions, we can introduce the *evolution map* associated to equation (16). We recall that this is the linear operator $S\colon \ell^\infty(I_r\times\mathbb{Z})\to\ell^\infty(I_r\times\mathbb{Z})$ defined by

$$S\varphi = u_{n+1} \quad \text{for } \varphi\in\ell^\infty(I_r\times\mathbb{Z}),$$

where u is the unique solution of problem (18). It follows readily from the definitions that

$$(S\varphi)(0)=\mathfrak{L}\varphi \quad \text{for } \varphi\in\ell^\infty(I_r\times\mathbb{Z}).$$

We also define another linear operator $R\colon \ell^\infty(I_r\times\mathbb{Z})\to\ell^\infty(I_r\times\mathbb{Z})$ by

$$(R\varphi)(s,k)=(T(k,k-1)\varphi^{k-1})(s) \quad \text{for } (s,k)\in I_r\times\mathbb{Z}. \tag{19}$$

Proposition 1. *We have $S=R$ on $\ell^\infty(I_r\times\mathbb{Z})$.*

Proof. By (12) we obtain

$$(T(k,k-1)\varphi^{k-1})(0)=L_{k-1}\varphi^{k-1}=(\mathfrak{L}\varphi)(k).$$

On the other hand, for $s<0$ we have

$$(T(k,k-1)\varphi^{k-1})(s)=\varphi^{k-1}(s+1)$$

and writing $\varphi=u_n$ we obtain

$$\begin{aligned}\varphi^{k-1}(s+1)&=u_n^{k-1}(s+1)=u_n(s+1,k-1)\\&=u(n+s+1,k+s)=u_{n+1}(s,k).\end{aligned}$$

Therefore,

$$\begin{aligned}(R\varphi)(s,k)&=\begin{cases}(\mathfrak{L}u_n)(k) & \text{if } s=0,\\ u_{n+1}(s,k) & \text{if } s<0\end{cases}\\&=\begin{cases}u(n+1,k) & \text{if } s=0,\\ u_{n+1}(s,k) & \text{if } s<0\end{cases}\\&=u_{n+1}(s,k)=(S\varphi)(s,k),\end{aligned}$$

which shows that the operators S and R are equal. □

It follows readily from (1) and (13) that the operators $\mathfrak{L}$ and S are well defined and bounded. Moreover, it follows from (19) and Proposition 1 that

$$(S^m\varphi)(s,k)=(T(k,k-m)\varphi^{k-m})(s) \quad \text{for } m\in\mathbb{N}, \tag{20}$$

where S^m denotes the mth power of the operator S (no confusion arises with the definition of u^m in (14) since we use capitals only for linear operators). Incidentally, using the notation in (14) one can rewrite (20) in the form

$$(S^m\varphi)^k=T(k,k-m)\varphi^{k-m} \quad \text{for } m\in\mathbb{N}.$$

The constructions and arguments presented in this section are two-sided versions of corresponding constructions and arguments introduced in [5]. Nevertheless, here the operator S may be invertible along certain subspaces, while there the corresponding operator is invertible along no subspace. This corresponds to consider the space $\ell^\infty(I_r \times \mathbb{Z})$ instead of $\ell^\infty(I_r \times \mathbb{N})$.

3. Characterization of hyperbolicity

In this section we show that equation (2) has an exponential dichotomy if and only if equation (16) has an exponential dichotomy. We first recall these notions. Equation (2) is said to have an *exponential dichotomy* if there exist $\lambda, D > 0$ and projections $P_m \colon Y \to Y$ for $m \in \mathbb{Z}$ such that for each $m, n \in \mathbb{Z}$ with $m \geq n$:

(i) $P_m T(m,n) = T(m,n) P_n$;
(ii) letting $Q_m = \mathrm{Id} - P_m$, the map

$$T(m,n)|_{Q_n(Y)} \colon Q_n(Y) \to Q_m(Y) \tag{21}$$

is onto and invertible;
(iii) $\|T(m,n)P_n\| \leq De^{-\lambda(m-n)}$ and $\|T(m,n)|_{Q_n(Y)}^{-1}\| \leq De^{-\lambda(m-n)}$.

Similarly, equation (16) is said to have an *exponential dichotomy* if there exist $\lambda, D > 0$ and a projection $P \colon \ell^\infty(I_r \times \mathbb{Z}) \to \ell^\infty(I_r \times \mathbb{Z})$ such that:

(i) $PS = SP$;
(ii) letting $Q = \mathrm{Id} - P$, the map

$$S|_{Q(\ell^\infty(I_r \times \mathbb{Z}))} \colon Q(\ell^\infty(I_r \times \mathbb{Z})) \to Q(\ell^\infty(I_r \times \mathbb{Z})) \tag{22}$$

is onto and invertible;
(iii) $\|S^m P\| \leq De^{-\lambda m}$ and $\|S^m|_{Q(\ell^\infty(I_r \times \mathbb{Z}))}^{-1}\| \leq De^{-\lambda m}$ for $m \geq 0$.

Theorem 2. *Let $L_m \colon Y \to X$, for $m \in \mathbb{Z}$, be bounded linear operators satisfying (1). Then equation (2) has an exponential dichotomy if and only if equation (16) has an exponential dichotomy.*

Proof. We first assume that equation (2) has an exponential dichotomy. We define a linear operator $P \colon \ell^\infty(I_r \times \mathbb{Z}) \to \ell^\infty(I_r \times \mathbb{Z})$ by

$$(Pu)^m = P_m u^m \quad \text{for } u \in \ell^\infty(I_r \times \mathbb{Z}) \text{ and } m \in \mathbb{Z}.$$

One can easily verify that P is a projection. Moreover,

$$\begin{aligned}(SPu)^m &= T(m,m-1)(Pu)^{m-1} = T(m,m-1)P_{m-1}u^{m-1} \\ &= P_m T(m,m-1)u^{m-1} = P_m(Su)^m = (PSu)^m\end{aligned} \tag{23}$$

for each $m \in \mathbb{Z}$ and so $SP = PS$.

Now take $v \in \ell^\infty(I_r \times \mathbb{Z})$ such that $SQv = 0$, where $Q = \mathrm{Id} - P$. Proceeding as in (23) we obtain

$$0 = (SQv)^m = T(m, m-1)(Qv)^{m-1} = T(m, m-1)Q_{m-1}v^{m-1}.$$

Since the map in (21) with $n = m - 1$ is one-to-one, we have $Q_{m-1}v^{m-1} = 0$ for all $m \in \mathbb{Z}$ and so $Qv = 0$. This shows that the map $S|_{Q(\ell^\infty(I_r \times \mathbb{Z}))}$ is also one-to-one. We will show that it is onto after having exponential bounds.

To obtain the exponential bounds, take $u \in \ell^\infty(I_r \times \mathbb{Z})$ and $m, n \in \mathbb{Z}$ with $m \geq 0$. By (20) we have

$$(S^m Pu)^n = T(n, n-m)(Pu)^{n-m} = T(n, n-m)P_{n-m}u^{n-m}.$$

Hence,

$$\begin{aligned}\|S^m Pu\|_\infty &= \sup_{n\in\mathbb{Z}} \|T(n, n-m)P_{n-m}u^{n-m}\| \\ &\leq De^{-\lambda m} \sup_{n\in\mathbb{Z}} \|u^{n-m}\| = De^{-\lambda m}\|u\|_\infty\end{aligned}$$

and so $\|S^m P\| \leq De^{-\lambda m}$ for $m \geq 0$. Similarly, we have

$$(S^m Qu)^n = T(n, n-m)Q_{n-m}u^{n-m}$$

and so

$$\begin{aligned}\|S^m Qu\|_\infty &= \sup_{n\in\mathbb{Z}} \|T(n, n-m)Q_{n-m}u^{n-m}\| \\ &\geq D^{-1}e^{\lambda m} \sup_{n\in\mathbb{Z}} \|Q_{n-m}u^{n-m}\| = D^{-1}e^{\lambda m}\|Qu\|_\infty\end{aligned}$$

for $m \geq 0$. This readily yields the second exponential bound in the notion of exponential dichotomy.

Take $u \in Q(\ell^\infty(I_r \times \mathbb{Z}))$. Since the maps $T(m, n)|_{Q_n(Y)}$ in (21) are onto for $m \geq n$, there exists $w^{n-1} \in Q_{n-1}(Y)$ such that $u^n = T(n, n-1)w^{n-1}$ for each $n \in \mathbb{Z}$. These functions w^{n-1} determine a function $w\colon I_r \times \mathbb{Z} \to X$. Since

$$\|w^{n-1}\| = \|T(n, n-1)|_{Q_{n-1}(Y)}^{-1} u^n\| \leq De^{-\lambda}\|u^n\|,$$

we obtain $\|w\|_\infty \leq De^{-\lambda}\|u\|_\infty$ and $w \in \ell^\infty(I_r \times \mathbb{Z})$. Clearly, $Sw = u$ and so the map in (22) is onto. Summing up, equation (16) has an exponential dichotomy.

Now we establish the converse statement. Assume that equation (16) has an exponential dichotomy. Given a bounded sequence $\alpha = (\alpha^n)_{n\in\mathbb{Z}}$ of real numbers, for each $u \in \ell^\infty(I_r \times \mathbb{Z})$ we consider the new sequence $\alpha u \in \ell^\infty(I_r \times \mathbb{Z})$ such that

$$(\alpha u)^n = \alpha^n u^n \quad \text{for } n \in \mathbb{Z}.$$

We have

$$(S^m(\alpha Pu))^n = T(n, n-m)\alpha^{n-m}(Pu)^{n-m} = \alpha^{n-m}(S^m Pu)^{n-m}$$

and so

$$\|S^m(\alpha Pu)\|_\infty \le \|\alpha\|_\infty \|S^m Pu\|_\infty \to 0 \tag{24}$$

when $m \to \infty$. One can easily verify that

$$P(\ell^\infty(I_r \times \mathbb{Z})) = \{v \in \ell^\infty(I_r \times \mathbb{Z}) : S^m v \to 0 \text{ when } m \to \infty\}$$

and so it follows from (24) that $\alpha Pu \in P(\ell^\infty(I_r \times \mathbb{Z}))$.

Now observe that $\alpha u = \alpha Pu + \alpha Qu$, which gives

$$P(\alpha u) = P(\alpha Pu) + P(\alpha Qu) = \alpha Pu + P(\alpha Qu). \tag{25}$$

For each $m \in \mathbb{Z}$ we define a new sequence $\alpha(m)$ by $\alpha(m)^n = \alpha^{n+m}$ for each $m \in \mathbb{Z}$. Then

$$\begin{aligned}(\alpha S^m U^m Qu)^n &= \alpha^n (S^m U^m Qu)^n \\ &= \alpha^n T(n, n-m)(U^m Qu)^{n-m} \\ &= T(n, n-m)\alpha(m)^{n-m}(U^m Qu)^{n-m} \\ &= T(n, n-m)(\alpha(m)^n U^m Qu)^{n-m} \\ &= (S^m \alpha(m)^n U^m Qu)^n.\end{aligned}$$

Letting $U = S|^{-1}_{Q(\ell^\infty(I_r \times \mathbb{Z}))}$, we obtain

$$\begin{aligned}\|P(\alpha Qu)\|_\infty &= \|P(\alpha S^m U^m Qu)\|_\infty \\ &= \|P(S^m(\alpha(m)U^m Qu))\|_\infty \\ &= \|S^m P(\alpha(m)U^m Qu)\|_\infty,\end{aligned} \tag{26}$$

because S^m and P commute. Using the exponential bounds in the notion of exponential dichotomy, it follows from (26) that

$$\|P(\alpha Qu)\|_\infty \le De^{-\lambda m}\|\alpha^m U^m Qu\|_\infty \le D^2 e^{-2\lambda m}\|\alpha\|_\infty \|u\|_\infty$$

for $m \ge 0$, which implies that $P(\alpha Qu) = 0$. Hence, it follows from (25) that

$$P(\alpha u) = \alpha Pu \quad \text{for } u \in \ell^\infty(I_r \times \mathbb{Z}).$$

Now take $\alpha = (\alpha^n)_{n\in\mathbb{Z}}$ with $\alpha^m = 1$ and $\alpha^n = 0$ for $n \ne m$. For each $u \in \ell^\infty(I_r \times \mathbb{Z})$ we have

$$\begin{aligned}\|(Pu)^m\| &= \|\alpha^m (Pu)^m\| = \|(\alpha Pu)^m\| = \|\alpha Pu\|_\infty \\ &= \|P(\alpha u)\|_\infty \le \|P\| \cdot \|\alpha u\|_\infty = \|P\| \cdot \|u^m\|.\end{aligned}$$

Therefore, if $u^m = 0$, then $(Pu)^m = 0$ and so one can define a linear operators $P_m \colon Y \to Y$ by

$$P_m \varphi = (Pu)^m \quad \text{for any } u \in \ell^\infty(I_r \times \mathbb{Z}) \text{ with } u^m = \varphi. \tag{27}$$

It follows readily from (27) that P_m is onto for each $m \in \mathbb{Z}$. Moreover,

$$P_m^2 \varphi = P_m (Pu)^m = (P^2 u)^m = (Pu)^m$$

and so P_m is a projection. On the other hand, by (20) we have

$$\begin{aligned} P_mT(m,n)u_n &= (PS^{m-n}u)^m = (S^{m-n}Pu)^m \\ &= T(m,n)(Pu)^n = T(m,n)P_nu_n, \end{aligned}$$

which shows that $P_mT(m,n) = T(m,n)P_n$.

To obtain exponential bounds, take $n \in \mathbb{Z}$, $\varphi \in Y$ and $u \in \ell^\infty(I_r \times \mathbb{Z})$ with $u^n = \varphi$ and $u^m = 0$ for $m \neq n$. Note that $(S^{m-n}Pu)^k = 0$ for $m \geq n$ and $k \neq m$. Hence, by (20) we have

$$\begin{aligned} \|T(m,n)P_n\varphi\| &= \|(S^{m-n}Pu)^m\| = \|S^{m-n}Pu\|_\infty \\ &\leq De^{-\lambda(m-n)}\|u\|_\infty = De^{-\lambda(m-n)}\|\varphi\| \end{aligned}$$

for $m \geq n$. Similarly,

$$\begin{aligned} \|T(m,n)Q_n\varphi\| &= \|(S^{m-n}Qu)^m\| = \|S^{m-n}Qu\|_\infty \\ &\geq D^{-1}e^{\lambda(m-n)}\|Qu\|_\infty = D^{-1}e^{\lambda(m-n)}\|Q_n\varphi\|. \end{aligned}$$

Finally, we show that the maps in (21) are onto for $m \geq n$. For $\psi \in Q(\ell^\infty(I_r \times \mathbb{Z}))$ we have $\psi^{n-1} \in Q_{n-1}(Y)$ for all $n \in \mathbb{Z}$. Moreover,

$$(S\psi)^n = T(n,n-1)\psi^{n-1} \quad \text{for } n \in \mathbb{Z}.$$

Since the maps in (22) are onto, $T(n,n-1)\psi^{n-1}$ attains all values of $Q_n(Y)$ by taking an appropriate function ψ (with an appropriate component ψ^{n-1}). In other words, each map

$$T(n,n-1)|_{Q_{n-1}(Y)} \colon Q_{n-1}(Y) \to Q_n(Y)$$

is onto. This readily implies that each map in (21) is also onto. □

4. Admissibility properties

In this section we show that equation (2) satisfies an admissibility property if and only if equation (16) satisfies an analogous admissibility property on appropriate corresponding spaces. For the convenience of the reader, we first recall these notions. Given a Banach space E of functions $x\colon \mathbb{Z} \to X$, we say that the pair (E,E) is *admissible* for equation (2) if for each $y \in E$ there exists a unique $x \in E$ satisfying (7). Similarly, given a Banach space F of functions $u\colon \mathbb{Z}^2 \to X$, we say that the pair (F,F) is *admissible* for equation (16) if for each $v \in F$ there exists a unique $u \in F$ satisfying (8).

Now we consider several admissible spaces, such as the Banach spaces E^∞ and F^∞ in (9). In particular, F^∞ can be identified with the space of bounded sequences $\psi\colon \mathbb{Z} \to \ell^\infty(\mathbb{Z})$ equipped with the supremum norm: a function $u \in$

$\ell^\infty(\mathbb{Z}^2)$ can be identified with the sequence $(u(m))_{m\in\mathbb{Z}}$ with $u(m)$ as in (15). More precisely, for any $x \in E^\infty$ we have $\|x\|_\infty = \|x\|'_\infty$, where

$$\|x\|'_\infty = \sup_{m\in\mathbb{Z}} \|x_m\|.$$

Analogously, for any $u \in F^\infty$ we have $\|u\|_\infty = \|u\|'_\infty$, where

$$\|u\|'_\infty = \sup_{m,k\in\mathbb{Z}} \|u_m^k\|$$

with $u_m^k \in Y$ as in (17) with $k-1$ replaced by k. Thus,

$$E^\infty = (\ell^\infty(\mathbb{Z}), \|\cdot\|'_\infty) \quad \text{and} \quad F^\infty = (\ell^\infty(\mathbb{Z}^2), \|\cdot\|'_\infty).$$

Similarly, for each $p \in [1, +\infty)$ the space E^p is the set of all functions $x\colon \mathbb{Z} \to X$ such that

$$\|x\|'_p := \left(\sum_{m\in\mathbb{Z}} \|x_m\|^p\right)^{1/p} < +\infty$$

equipped with the norm $\|\cdot\|'_p$ (see (10)). Moreover, the space F^p is the set of all functions $u\colon \mathbb{Z}^2 \to X$ such that

$$\|u\|'_p := \left(\sum_{m,k\in\mathbb{Z}} \|u_m^k\|^p\right)^{1/p} < +\infty$$

equipped with the norm $\|\cdot\|'_p$ (see (11)). One can verify with standard arguments that E^p and F^p are Banach spaces for each $p \in [1, +\infty]$.

Theorem 3. *Let $L_m\colon Y \to X$, for $m \in \mathbb{Z}$, be bounded linear operators. Then for each $p \in [1, +\infty]$ the following properties are equivalent:*

(i) *the pair (E^p, E^p) is admissible, that is, for each $y \in E^p$ there exists a unique $x \in E^p$ satisfying equation (7);*

(ii) *the pair (F^p, F^p) is admissible, that is, for each $v \in F^p$ there exists a unique $u \in F^p$ satisfying equation (8).*

Proof. We first prove an auxiliary result.

Lemma 1. *Given $u, v\colon \mathbb{Z}^2 \to X$, equation (8) holds if and only if*

$$x^{[\iota]}(m+1) = L_m x_m^{[\iota]} + y^{[\iota]}(m+1) \quad \text{for } m, \iota \in \mathbb{Z},$$

where the functions $x^{[\iota]}, y^{[\iota]}\colon \mathbb{Z} \to \mathbb{Z}$ are defined for each $\iota \in \mathbb{Z}$ by

$$x^{[\iota]}(k) = u(\iota + k, k) \quad \text{and} \quad y^{[\iota]}(k) = v(\iota + k, k). \tag{28}$$

Proof of the lemma. By the definition of $\mathcal{L}$, equation (8) is equivalent to

$$u(m+1, k) = L_{k-1} u_m^{k-1} + v(m+1, k) \quad \text{for } m, k \in \mathbb{Z}. \tag{29}$$

We claim that

$$x_{k-1}^{[m+1-k]} := (x^{[m+1-k]})_{k-1} = u_m^{k-1},$$

using the notation in (28). Indeed, for $s \in I_r$ we have

$$x_{k-1}^{[m+1-k]}(s) = x^{[m+1-k]}(k-1+s) = u(m+s, k-1+s) = u_m^{k-1}(s).$$

Thus, equation (29) is equivalent to

$$x^{[m+1-k]}(k) = L_{k-1} x_{k-1}^{[m+1-k]} + y^{[m+1-k]}(k) \quad \text{for } m, k \in \mathbb{Z}. \tag{30}$$

Since m and k are arbitrary, replacing k by $k+1$ and then m by $\iota + k$, we find that equation (30) is equivalent to

$$x^{[\iota]}(k+1) = L_k x_k^{[\iota]} + x^{[\iota]}(k+1) \quad \text{for } m, k \in \mathbb{Z}.$$

This completes the proof of the lemma. □

We proceed with the proof of the theorem. We first assume that property (i) holds and we show that property (ii) holds. Take a function $v \in F^p$ for some $p \in [1, +\infty]$.

Claim: We have $y^{[\iota]} \in E^p$ for each $\iota \in \mathbb{Z}$.

Proof of the claim. The statement is clear for $p = \infty$. On the other hand, for $p < +\infty$ we have

$$v_m^k(s) = v(m+s, k+s) = y^{[m-k]}(k+s) = y_k^{[m-k]}(s)$$

for each $s \in I_r$, and so

$$\left(\|v\|_p'\right)^p = \sum_{m,k \in \mathbb{Z}} \|v_m^k\|^p = \sum_{m,k \in \mathbb{Z}} \|y_k^{[m-k]}\|^p = \sum_{\iota,k \in \mathbb{Z}} \|y_k^{[\iota]}\|^p < +\infty. \tag{31}$$

Therefore,

$$\left(\|y^{[\iota]}\|_p'\right)^p = \sum_{k \in \mathbb{Z}} \|y_k^{[\iota]}\|^p < +\infty,$$

which shows that $y^{[\iota]} \in E^p$ for each $\iota \in \mathbb{Z}$. □

By property (i), for each $\iota \in \mathbb{Z}$ there exists a unique solution $x^{(\iota)} \in E^p$ of the equation

$$x^{(\iota)}(m+1) = L_m x_m^{(\iota)} + y^{[\iota]}(m+1) \quad \text{for } m \in \mathbb{Z}. \tag{32}$$

On the other hand, by Lemma 1, if $u\colon \mathbb{Z}^2 \to X$ is a solution of equation (8), then for each $\iota \in \mathbb{Z}$ the function $x^{[\iota]}$ in (28) is a solution of equation (32) (with $x^{(\iota)}$ replaced by $x^{[\iota]}$). Since each solution $x^{(\iota)}$ is unique, necessarily $x^{[\iota]} = x^{(\iota)}$ for all $\iota \in \mathbb{Z}$, and so

$$u(m,n) = x^{(m-n)}(n) \quad \text{for } m, n \in \mathbb{Z}. \tag{33}$$

This shows that any solution of equation (8) (if it exists) is given by (33) and so, in particular, it is unique.

Given $v \in F^p$ and the corresponding functions $x^{(\iota)}$ obtained above, the former discussion leads us to introduce a function $u\colon \mathbb{Z}^2 \to X$ by (33). It follows from Lemma 1 that it satisfies equation (8) (and it follows from the discussion that u is the unique solution of that equation). Hence, to show that property (ii) holds, it remains to verify that u belongs to F^p.

Now we define a linear operator $R_p\colon \mathcal{D}(R_p) \to E^p$ by

$$(R_p x)(m) = x(m) - L_{m-1}x_{m-1} \quad \text{for}\, m \in \mathbb{Z} \tag{34}$$

on the domain formed by all sequences $x \in E^p$ such that

$$\big(x(m) - L_{m-1}x_{m-1}\big)_{m\in\mathbb{Z}} \in E^p.$$

We show that R_p is closed. Let $(x^i)_{i\in\mathbb{Z}}$ be a sequence in $\mathcal{D}(R_p)$ converging to $x \in E^p$ such that $R_p x^i$ converges to $y \in E^p$. Then

$$\begin{aligned} x(m) - L_{m-1}x_{m-1} &= \lim_{i\to\infty} \big(x^i(m) - L_{m-1}(x^i)_{m-1}\big) \\ &= \lim_{i\to\infty} (R_p x^i)(m) = y(m) \end{aligned}$$

for $m \in \mathbb{Z}$. This shows that $R_p x = y$ and so $x \in \mathcal{D}(R_p)$. Hence, the operator R_p is closed. We consider the graph norm on E^p given by

$$\|x\|' = \|x\|'_p + \|R_p x\|'_p.$$

Since R_p is closed, $(\mathcal{D}(R_p), \|\cdot\|')$ is a Banach space and the operator

$$R_p\colon (\mathcal{D}(R_p), \|\cdot\|') \to E^p$$

is bounded. By property (i), it is onto and invertible. It follows from the open mapping theorem that it has a bounded inverse

$$R_p^{-1}\colon E^p \to (\mathcal{D}(R_p), \|\cdot\|').$$

Claim: We have $u \in F^p$.

Proof of the claim. First take $p = \infty$. For each $m \in \mathbb{Z}$ we have

$$\sup_{n\in\mathbb{Z}} \|x^{(n-m)}(n)\| = \sup_{k\in\mathbb{Z}} \|x^{(k)}(m+k)\| \le \sup_{k\in\mathbb{Z}} \|x^{(k)}\|_\infty \le \sup_{k\in\mathbb{Z}} \|x^{(k)}\|'.$$

Since $R_\infty x^{(k)} = y^{[k]}$, we obtain

$$\begin{aligned} \|u\|_\infty = \sup_{m,n\in\mathbb{Z}} \|x^{(n-m)}(n)\| &\le \sup_{k\in\mathbb{Z}} \|x^{(k)}\|' \\ &\le \|R_\infty^{-1}\| \sup_{k\in\mathbb{Z}} \|y^{[k]}\|_\infty = \|R_\infty^{-1}\| \cdot \|v\|_\infty. \end{aligned}$$

This shows that $u \in E^\infty$.

Now take $p < +\infty$. In a similar manner to that in (31) we have

$$\big(\|u\|'_p\big)^p = \sum_{\iota,k\in\mathbb{Z}} \|x_k^{[\iota]}\|^p = \sum_{\iota\in\mathbb{Z}} \big(\|x^{[\iota]}\|'_p\big)^p \le \sum_{\iota\in\mathbb{Z}} \big(\|x^{[\iota]}\|'\big)^p. \tag{35}$$

Since $R_p x^{[\iota]} = y^{[\iota]}$, it follows from (31) that

$$\begin{aligned}\big(\|u\|_p'\big)^p &\le \|R_p^{-1}\|^p \sum_{\iota\in\mathbb{Z}} \big(\|y^{[\iota]}\|_p'\big)^p\\ &= \|R_p^{-1}\|^p \sum_{\iota,k\in\mathbb{Z}} \|y_k^{[\iota]}\|^p = \|R_p^{-1}\|^p \big(\|v\|_p'\big)^p.\end{aligned}$$

Therefore,

$$\|u\|_p' \le \|R_p^{-1}\| \cdot \|v\|_p' < +\infty$$

and so $u \in F^p$. □

This establishes property (ii).

Now we assume that property (ii) holds and we show that property (i) holds. Take a function $y \in F^p$ for some $p \in [1, +\infty]$. We define $v\colon \mathbb{Z}^2 \to X$ by

$$v(m,n) = \frac{y(n)}{1 + (m-n)^2/p} \quad \text{for } m,n \in \mathbb{Z}.$$

Note that when $p = \infty$ this gives

$$v(m,n) = y(n) \quad \text{for } m,n \in \mathbb{Z}.$$

Claim: We have $v \in F^p$.

Proof of the claim. The statement is clear for $p = \infty$. When $p < +\infty$ we note that

$$v_m^k(s) = v(m+s, k+s) = \frac{y(k+s)}{1+(m-k)^2/p} = \frac{y_k(s)}{1+(m-k)^2/p} \tag{36}$$

for $s \in I_r$, and so

$$v_m^k = \frac{y_k}{1+(m-k)^2/p}.$$

This gives

$$\begin{aligned}\big(\|v\|_p'\big)^p &= \sum_{m,k\in\mathbb{Z}} \|v_m^k\|^p = \sum_{m,k\in\mathbb{Z}} \frac{\|y_k\|^p}{(1+(m-k)^2/p)^p}\\ &= \sum_{k\in\mathbb{Z}} \left(\|y_k\|^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+(m-k)^2/p)^p} \right)\\ &= \sum_{k\in\mathbb{Z}} \|y_k\|^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2/p)^p}\\ &= (\|y\|_p')^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2/p)^p} < +\infty\end{aligned} \tag{37}$$

and thus $v \in F^p$. □

Hence, by property (ii) there exists a unique $u \in F^p$ satisfying equation (8). In view of Lemma 1, for each $\iota \in \mathbb{Z}$ the function $x^{[\iota]}$ in (28) satisfies the equation

$$x^{[\iota]}(m+1) = L_m x^{[\iota]}_m + y^{[\iota]}(m+1) \quad \text{for } m \in \mathbb{Z}. \tag{38}$$

Note that

$$y^{[\iota]}(k) = v(\iota + k, k) = \frac{y(k)}{1 + \iota^2/p}$$

and so it follows from (38) that

$$\hat{x}^{[\iota]} := (1 + \iota^2/p) x^{[\iota]}$$

is a solution of the equation

$$\hat{x}^{[\iota]}(m+1) = L_m \hat{x}^{[\iota]}_m + y(m+1) \quad \text{for } m \in \mathbb{Z}. \tag{39}$$

Claim: We have $\boldsymbol{x^{[\iota]} \in E^p}$ for each $\boldsymbol{\iota \in \mathbb{Z}}$.

Proof of the claim. When $p = \infty$ it follows readily from the definition in (28) that $x^{[\iota]} \in E^\infty$ for each $\iota \in \mathbb{Z}$. On the other hand, when $p < +\infty$ it follows as in (35) that

$$\big(\|u\|'_p\big)^p = \sum_{\iota \in \mathbb{Z}} \big(\|x^{[\iota]}\|'_p\big)^p < +\infty$$

and so $x^{[\iota]} \in E^p$ for each $\iota \in \mathbb{Z}$. □

It follows readily from the claim that the solutions $\hat{x}^{[\iota]}$ of equation (39) belongs to E^p for all $\iota \in \mathbb{Z}$. Moreover, there are no other solutions of equation (39) (or equivalently of equation (7)). Indeed, otherwise one could use (33) to obtain a different solution u of equation (8), which contradicts the uniqueness of the solution. Hence, to complete the proof it remains to show that $\hat{x}^{[\iota]}$ is independent of ι.

Claim: The function $\boldsymbol{\hat{x}^{[\iota]} = (1 + \iota^2/p) x^{[\iota]}}$ is independent of $\boldsymbol{\iota}$.

Proof of the claim. When $p = \infty$ we first show that the sequence $(u(m))_{m \in \mathbb{Z}}$ is constant. Note that

$$v(m)(n) = v(m, n) = y(n)$$

and so $v(m) = y$ for all $m \in \mathbb{Z}$. Therefore, equation (8) takes the form

$$u(m+1) = \mathfrak{L} u_m + y \quad \text{for } m \in \mathbb{Z}. \tag{40}$$

Now we define a new function $\bar{u} \in F^\infty$ by requiring that $\bar{u}(m) = u(m+1)$ for each $m \in \mathbb{Z}$. By (40) we have

$$\bar{u}(m+1) = \mathfrak{L} \bar{u}_m + y \quad \text{for } m \in \mathbb{Z},$$

and it follows from the uniqueness in property (ii) that $\bar{u} = u$. This readily implies that $(u(m))_{m\in\mathbb{Z}}$ is constant and so $u(m,n)$ is independent of m. Therefore, $\hat{x}^{[\iota]} = x^{[\iota]}(k) = u(\iota + k, k)$ is independent of ι.

When $p < +\infty$ we fix $q \in \mathbb{Z}$ and we consider the function $\bar{u}\colon \mathbb{Z}^2 \to X$ defined by

$$\bar{u}(m,n) = \frac{1 + q^2/p}{1 + (m-n)^2/p} x^{[q]}(n)$$

for $m, n \in \mathbb{Z}$. Note that

$$\bar{u}_m^k(s) = \bar{u}(m+s, k+s) = \frac{1 + q^2/p}{1 + (m-k)^2/p} x_k^{[q]}(s)$$

and so, in a similar manner to that in (37), we have

$$\begin{aligned}
(\|\bar{u}\|_p')^p &= \sum_{m,k\in\mathbb{Z}} \|\bar{u}_m^k\|^p \\
&= (1 + q^2/p)^p \sum_{m,k\in\mathbb{Z}} \frac{\|x_k^{[q]}\|^p}{(1 + (m-k)^2/p)^p} \\
&= (1 + q^2/p)^p (\|x^{[q]}\|_p')^p \sum_{m\in\mathbb{Z}} \frac{1}{(1 + m^2/p)^p} < +\infty.
\end{aligned}$$

This shows that $\bar{u} \in F^p$. Note that

$$\bar{x}^{[\iota]}(k) := \bar{u}(\iota + k, k) = \frac{1 + q^2/p}{1 + \iota^2/p} x^{[q]}(k) = \frac{\hat{x}^{[q]}(k)}{1 + \iota^2/p}$$

and so it follows from (39) with ι replaced by q that

$$\bar{x}^{[\iota]}(m+1) = L_m \bar{x}_m^{[\iota]} + y^{[\iota]}(m+1) \quad \text{for } m \in \mathbb{Z}.$$

Hence, by Lemma 1, $\bar{u}$ is a solution of equation (8) and it follows from the uniqueness of the solution that $\bar{u} = u$. Therefore,

$$x^{[\iota]}(k) = u(\iota + k, k) = \bar{u}(\iota + k, k) = \frac{1 + q^2/p}{1 + \iota^2/p} x^{[q]}(k)$$

for all $k, \iota \in \mathbb{Z}$. In other words,

$$\hat{x}^{[\iota]}(k) = (1 + \iota^2/p) x^{[\iota]} = (1 + q^2/p) x^{[q]} = \hat{x}^{[q]}(k)$$

for all $k, \iota \in \mathbb{Z}$, which shows that $\hat{x}^{[\iota]}$ is independent of ι (since q is fixed). □

Thus, property (i) holds, which completes the proof of the theorem. □

We emphasize that Theorem 3 does not require the boundedness of the sequence L_m in (1). On the other hand, if property (1) holds, then the operator R_p defined by (34) has domain $\mathcal{D}(R_p) = E^p$ and it is automatically bounded (since it is closed). For a direct argument, note that if property (1) holds, then

$$|(R_p x)(m+1)| \le |x(m+1)| + \|L_m\| \cdot \|x_m\|.$$

and so

$$\|(R_p x)_{m+1}\| \leq \|x_{m+1}\| + c \max_{s\in I_r} \|x_{m+s}\|$$
$$\leq \|x_{m+1}\| + c \max_{s\in I_r} \max\{1, c\}^{s+r} \|x_{m-r}\|.$$

In particular, $R_p x \in E^p$ when $x \in E^p$, with

$$\|R_p\| \leq 1 + c \max\{1, c\}^r.$$

5. Ulam–Hyers stability

In this section we consider the Ulam–Hyers stability property. We show that it is a consequence of hyperbolicity, and that this property or, more precisely, the stronger property of uniform Ulam–Hyers stability, is equivalent for equations (2) and (16).

Given $p \in [1, +\infty]$, let $\bar{L}\colon E^p \to X^{\mathbb{Z}}$ be the operator defined by

$$(\bar{L}x)(m) = L_{m-1} x_{m-1} \quad \text{for } m \in \mathbb{Z}$$

and let $\bar{\mathfrak{L}}\colon F^p \to X^{\mathbb{Z}^2}$ be the operator defined by

$$(\bar{\mathfrak{L}}u)(m) = \mathfrak{L} u_{m-1} \quad \text{for } m \in \mathbb{Z}$$

Note that

$$(\bar{\mathfrak{L}}u)(m, k) = (\bar{\mathfrak{L}}u)(m)(k) = (\bar{\mathfrak{L}}u_{m-1})(k) = L_{k-1} u^{k-1}_{m-1}$$

for $m, k \in \mathbb{Z}$. Moreover, if the operators $L_m\colon Y \to X$ satisfy (1), then clearly $\bar{L}(E^p) \subset E^p$ and $\bar{\mathfrak{L}}(F^p) \subset F^p$.

For each $p \in [1, +\infty]$, we say that equation (2) is *uniformly Ulam–Hyers stable* with respect to E^p if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and $x, y \in E^p$ satisfying

$$\|x - \bar{L}x - y\|'_p < \varepsilon \tag{41}$$

there exists $z \in E^p$ satisfying

$$z = \bar{L}z + y \quad \text{and} \quad \|x - z\|'_p < \kappa\varepsilon.$$

Analogously, we say that equation (16) is *uniformly Ulam–Hyers stable* with respect to F^p if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and $u, v \in F^p$ satisfying

$$\|u - \bar{\mathfrak{L}}u - v\|'_p < \varepsilon$$

there exists $w \in F^p$ satisfying

$$w = \bar{\mathfrak{L}}w + v \quad \text{and} \quad \|u - w\|'_p < \kappa\varepsilon.$$

Theorem 4. *Let $L_m\colon Y \to X$, for $m \in \mathbb{Z}$, be bounded linear operators satisfying (1). Then for each $p \in [1, +\infty]$ the following properties hold:*

(i) *if the pair* (E^p, E^p) *is admissible, then equation (2) is uniformly Ulam–Hyers stable with respect to* E^p;

(ii) *if the pair* (F^p, F^p) *is admissible, then equation (16) is uniformly Ulam–Hyers stable with respect to* F^p.

Proof. Take $\varepsilon > 0$ and $x, y \in E^p$ satisfying (41). Consider the sequence $\bar{y} = (\bar{y}(m))_{m\in\mathbb{Z}}$ defined by

$$\bar{y}(m) = (x - \bar{L}x - y)(m) = x(m) - L_{m-1}x_{m-1} - y(m)$$

for $m \in \mathbb{Z}$. Clearly, $\bar{y} \in E^p$ because $x, y \in E^p$ and the operators L_m satisfy (1). Since (E^p, E^p) is admissible, there exists a unique sequence $\bar{x} \in E^p$ such that

$$\bar{x}(m+1) - L_m\bar{x}_m = \bar{y}(m+1) = x(m+1) - L_m x_m - y(m+1)$$

for $m \in \mathbb{Z}$. On the other hand, using the linear operator $R_p\colon \mathcal{D}(R_p) \to E^p$ in (34) we obtain

$$\|\bar{x}\|'_p \le \|R_p^{-1}\| \cdot \|\bar{y}\|'_p < \|R_p^{-1}\|\varepsilon.$$

Moreover, setting $z = x - \bar{x}$ we have

$$\begin{aligned} z(m+1) &= x(m+1) - \bar{x}(m+1) \\ &= L_m x_m - L_m\bar{x}_m + y(m+1) = L_m z_m + y(m+1), \end{aligned}$$

and so

$$\|x - z\|'_p = \|\bar{x}\|'_p < \kappa\varepsilon,$$

where $\kappa = \|R_p^{-1}\|$. This concludes the proof of statement 1.

The proof of statement 2 can be obtained in the same manner. Note that since the operators L_m satisfy (1), by construction we have $\bar{\mathcal{L}}u \in F^p$ for $u \in F^p$. This completes the proof of the theorem. □

Finally, we show that equation (2) is uniformly Ulam–Hyers stable if and only if the same happens to equation (16).

Theorem 5. *Let* $L_m\colon Y \to X$, *for* $m \in \mathbb{Z}$, *be bounded linear operators. Then for each* $p \in [1, +\infty]$ *the following properties are equivalent:*

(i) *equation (2) is uniformly Ulam–Hyers stable with respect to* E^p;

(ii) *equation (16) is uniformly Ulam–Hyers stable with respect to* F^p.

Proof. We first assume that property (i) holds and we show that property (ii) holds. Take $p = \infty$. Given $\varepsilon > 0$, let $u, v \in F^p$ be such that

$$\sup_{m\in\mathbb{Z}} \|u(m+1) - \mathcal{L}u_m - v(m+1)\|_\infty < \varepsilon. \tag{42}$$

For each $\iota \in \mathbb{Z}$ we consider the sequences $x^{[\iota]}$ and $y^{[\iota]}$ defined in (28). Proceeding as in the proof of Lemma 1, it follows from (42) that

$$\sup_{m\in\mathbb{Z}} \|x^{[\iota]}(m+1) - L_m x_m^{[\iota]} - y^{[\iota]}(m+1)\| < \varepsilon.$$

By property (i) there exist $\kappa > 0$ (independent of ε and ι) and $z^{[\iota]} \in E^\infty$ satisfying

$$z^{[\iota]}(m+1) = L_m z_m^{[\iota]} + y^{[\iota]}(m+1) \quad \text{for}\, m \in \mathbb{Z} \tag{43}$$

such that

$$\sup_{m\in\mathbb{Z}} \|x^{[\iota]}(m) - z^{[\iota]}(m)\| < \kappa\varepsilon. \tag{44}$$

Define $w(m,n) = z^{[n-m]}(n)$. By (28) we have $v(m,n) = y^{[n-m]}(n)$ and so it follows from (43) and Lemma 1 that

$$w(m+1) = \mathfrak{L} w_m + v(m+1) \quad \text{for}\, m \in \mathbb{Z}. \tag{45}$$

Moreover, by (44) we get

$$\sup_{m\in\mathbb{Z}} \|u(m) - w(m)\|_\infty = \sup_{m\in\mathbb{Z}} \sup_{n\in\mathbb{Z}} \|x^{[n-m]}(n) - z^{[n-m]}(n)\| \le \kappa\varepsilon.$$

This proves property (ii) for $p = \infty$.

Now take $p < +\infty$. Given $\varepsilon > 0$, let $u, v \in F^p$ be such that

$$(\|u - \bar{\mathfrak{L}}u - v\|_p')^p = \sum_{m,k\in\mathbb{Z}} \|u_m^k - (\bar{\mathfrak{L}}u)_m^k - v_m^k\|^p < \varepsilon^p. \tag{46}$$

We continue to consider the sequences $x^{[\iota]}$ and $y^{[\iota]}$ defined in (28). By (46) together with the fact that

$$(\bar{\mathfrak{L}}u)(\iota+k,k) = (\mathfrak{L}u_{\iota+k-1})(k) = L_{k-1}u_{\iota+k-1}^{k-1} = L_{k-1}x_{k-1}^{[\iota]} = (\bar{L}x^{[\iota]})(k),$$

it follows as in (31) that

$$\sum_{\iota\in\mathbb{Z}} \big(\|x^{[\iota]} - \bar{L}x^{[\iota]} - y^{[\iota]}\|_p'\big)^p = \big(\|u - \bar{\mathfrak{L}}u - v\|_p'\big)^p < \varepsilon^p.$$

Take positive numbers α_ι for $\iota \in \mathbb{Z}$ such that

$$\sum_{\iota\in\mathbb{Z}} \alpha_\iota^p < \varepsilon^p \quad \text{and} \quad \|x^{[\iota]} - \bar{L}x^{[\iota]} - y^{[\iota]}\|_p' < \alpha_\iota \;\text{for all}\, \iota \in \mathbb{Z}. \tag{47}$$

By property (i) there exist $\kappa > 0$ (independent of ε and ι) and $z^{[\iota]} \in E^p$ satisfying (43) such that

$$\|x^{[\iota]} - z^{[\iota]}\|_p' < \kappa\alpha_\iota.$$

Define $w(m,n) = z^{[n-m]}(n)$. Then (45) holds. Moreover, proceeding as in (31), it follows from (47) that

$$\begin{aligned}(\|u - v\|_p')^p &= \sum_{m,n\in\mathbb{Z}} \|x_n^{[n-m]} - z_n^{[n-m]}\|^p \\ &= \sum_{\iota\in\mathbb{Z}} \big(\|x^{[\iota]} - z^{[\iota]}\|_p'\big)^p < \kappa^p\varepsilon^p.\end{aligned}$$

This proves property (ii) for $p < +\infty$.

Now we assume that property (ii) holds and we show that property (i) holds. Take $p = \infty$. Given $\varepsilon > 0$, let $x, y \in E^\infty$ be such that

$$\sup_{m\in\mathbb{Z}} \|x(m+1) - L_m x_m - y(m+1)\| < \varepsilon.$$

Define

$$u(m,n) = x(n) \quad \text{and} \quad v(m,n) = y(n).$$

Note that

$$\begin{aligned} u(m)(n) - (\mathfrak{L}u_{m-1})(n) - v(m)(n) &= u(m,n) - L_{n-1}u^{n-1}_{m-1} - v(m,n) \\ &= x(n) - L_{n-1}x_{n-1} - y(n) \end{aligned}$$

since $u^{n-1}_{m-1} = x_{n-1}$. Hence,

$$\begin{aligned} \sup_{m\in\mathbb{Z}} \|u(m) - \mathfrak{L}u_{m-1} - v(m)\|_\infty &= \sup_{m,n\in\mathbb{Z}} \|u(m,n) - (\mathfrak{L}u_{m-1})(n) - v(m)(n)\| \\ &= \sup_{n\in\mathbb{Z}} \|x(n) - L_{n-1}x_{n-1} - y(n)\| < \varepsilon. \end{aligned}$$

By property (ii), there exist $\kappa > 0$ (independent of ε) and $w \in F^\infty$ satisfying

$$w(m+1) = \mathfrak{L}w_m + v(m+1) \quad \text{for } m \in \mathbb{Z} \tag{48}$$

such that

$$\sup_{m\in\mathbb{Z}} \|u(m) - w(m)\|_\infty < \kappa\varepsilon.$$

By Lemma 1, for each $\iota \in \mathbb{Z}$, setting $z^{[\iota]}(k) = w(\iota + k, k)$ we have that

$$z^{[\iota]}(m+1) = L_m z^{[\iota]}_m + y(m+1) \quad \text{for } m \in \mathbb{Z},$$

since $y^{[\iota]}(k) = v(\iota + k, k) = y(k)$. Moreover,

$$\sup_{m\in\mathbb{Z}} \|x(m) - z^{[\iota]}(m)| = \sup_{m\in\mathbb{Z}} \|u(\iota + m, m) - w(\iota + m, m)\| < \kappa\varepsilon.$$

Taking any ι property (i) follows for $p = \infty$.

Now take $p < +\infty$. Given $\varepsilon > 0$, let $x, y \in E^p$ be such that

$$\|x - \bar{L}x - y\|'_p < \varepsilon.$$

Define

$$u(m,n) = \frac{x(n)}{1 + (m-n)^2} \quad \text{and} \quad v(m,n) = \frac{y(n)}{1 + (m-n)^2}.$$

Note that

$$\begin{aligned} u(m)(n) - (\mathfrak{L}u_{m-1})(n) - v(m)(n) &= u(m,n) - L_{n-1}u^{n-1}_{m-1} - v(m,n) \\ &= \frac{x(n) - L_{n-1}x_{n-1} - y(n)}{1 + (m-n)^2}, \end{aligned}$$

since in a similar manner to that in (36), we have

$$u^{n-1}_{m-1} = \frac{x_{n-1}}{1 + (m-n)^2}.$$

Hence, proceeding as in (37) we get

$$(\|u - \bar{\mathfrak{L}}u - v\|'_p)^p = (\|x - \bar{L}x - y\|'_p)^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2)^p} < \varepsilon^p \sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2)^p}.$$

By property (ii), there exists $\kappa > 0$ (independent of ε) and $w \in F^p$ satisfying (48) such that

$$\|u - w\|'_p < \kappa\varepsilon \left(\sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2)^p} \right)^{1/p}. \tag{49}$$

By Lemma 1, for each $\iota \in \mathbb{Z}$, setting $z^{[\iota]}(k) = w(\iota + k, k)$ we have that

$$z^{[\iota]}(m+1) = L_m z^{[\iota]}_m + y^{[\iota]}(m+1) \quad \text{for } m \in \mathbb{Z},$$

with

$$y^{[\iota]}(k) = v(\iota + k, k) = y(k)/(1+\iota^2).$$

Now take $q^{[\iota]}(k) = (1+\iota^2)z^{[\iota]}(k)$. Clearly,

$$q^{[\iota]}(m+1) = L_m q^{[\iota]}_m + y(m+1) \quad \text{for } m \in \mathbb{Z}.$$

Taking $\iota = 0$ and using (49) we obtain

$$\begin{aligned}\|x - q^{[0]}\|'_p = \|x^{[0]} - z^{[0]}\|'_p &\leq \left(\sum_{\iota\in\mathbb{Z}} (\|x^{[\iota]} - z^{[\iota]}\|'_p)^p \right)^{1/p} \\ &= \|u - v\|'_p < \kappa\varepsilon \left(\sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2)^p} \right)^{1/p}.\end{aligned}$$

This completes the proof of the theorem. □

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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