Aequat. Math.

https://doi.org/10.1007/s00010-024-01113-w

Aequationes Mathematicae

Multivariable generalizations of bivariate means via invariance

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Abstract. For a given p-variable mean $M\colon I^p \to I$ (I is a subinterval of $\mathbb{R}$), following (Horwitz in J Math Anal Appl 270(2):499–518, 2002) and (Lawson and Lim in Colloq Math 113(2):191–221, 2008), we can define (under certain assumptions) its $(p+1)$-variable β-invariant extension as the unique solution $K\colon I^{p+1} \to I$ of the functional equation

$$\begin{aligned}&K\big(M(x_2,\dots,x_{p+1}),M(x_1,x_3,\dots,x_{p+1}),\dots,M(x_1,\dots,x_p)\big)\\&\quad=K(x_1,\dots,x_{p+1}),\ \text{for all } x_1,\dots,x_{p+1}\in I\end{aligned}$$

in the family of means. Applying this procedure iteratively we can obtain a mean which is defined for vectors of arbitrary lengths starting from the bivariate one. The aim of this paper is to study the properties of such extensions.

Mathematics Subject Classification. 26E60, 39B12, 39B22.

Keywords. Invariant means, Extended means, Multivariable generalizations, Envelopes.

1. Introduction

The problem of the multi-variable generalization of bivariate means is very natural. Regrettably, such extensions remain unknown for many families of means. For example, in the family of generalized logarithmic means $\mathcal{E}_s\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ ($s \in \mathbb{R}\setminus\{-1,0\}$) defined as

$$\mathcal{E}_s(x,y) := \Big(\frac{x^{s+1}-y^{s+1}}{(s+1)(x-y)}\Big)^{1/s},$$

Stolarsky means, the Heronian mean, several means related to the Pythagorean one, etc. (see for example [8] for their definitions).

The purpose of this note is to provide a broad approach to this problem. Namely, we extend some ideas of Aumann [1,2], Horwitz [21] and Lawson and Lim [26] to generalize bivariate means to the multivariable setting. More precisely, for a given k-variable symmetric, continuous, and strict mean on an

Published online: 09 September 2024
Birkhäuser

interval, we can apply the so-called barycentric operator to generate the $(k+1)$-variable mean on the same interval. Then we use this procedure iteratively in order to get the desired extension.

At the very beginning, let us introduce the family of Gini means, which will be very helpful in illustrating the problem. Namely, in 1938, Gini [17] introduced the generalization of power means. For $r, s \in \mathbb{R}$, the *Gini mean* $\mathcal{G}_{r,s}$ of positive variables $x_1, \ldots, x_n$ ($n \in \mathbb{N}$) is defined as follows:

$$\mathcal{G}_{r,s}(x_1, \ldots, x_n) := \begin{cases} \left(\dfrac{x_1^r + \cdots + x_n^r}{x_1^s + \cdots + x_n^s}\right)^{\frac{1}{r-s}} & \text{if } r \neq s, \\ \exp\left(\dfrac{x_1^r \ln(x_1) + \cdots + x_n^r \ln(x_n)}{x_1^r + \cdots + x_n^r}\right) & \text{if } r = s. \end{cases}$$

Clearly, in the particular case $s = 0$, the mean $\mathcal{G}_{r,0}$ becomes the rth Power mean $\mathcal{P}_r$. It is also obvious that $\mathcal{G}_{s,r} = \mathcal{G}_{r,s}$.

It can be easily shown that for all $r \in \mathbb{R}$ and $x, y \in \mathbb{R}_+$ we have $\mathcal{G}_{r,-r}(x, y) = \sqrt{xy}$. This equality, however, fails to be valid for more than two arguments. Thus (a priori) it could happen that two different means coincide in the bivariate setting. As a consequence, we cannot recover the multi-variable mean based only on its two-variable restriction.

On the other hand, if $f \colon I \to \mathbb{R}$ (I is an interval) is a continuous, strictly monotone function and $M = A^{[f]} \colon \bigcup_{n=1}^{\infty} I^n \to I$ is a quasiarithmetic mean (see Sect. 5.1 a for definition) then it solves the functional equation

$$\begin{aligned} M(x_1, \ldots, x_{k+1}) = M\big(&M(x_2, \ldots, x_{k+1}), M(x_1, x_3, \ldots, x_{k+1}), \ldots, \\ &M(x_1, \ldots, x_k)\big) \text{ for all } k \geq 2 \text{ and } x_1 \ldots, x_{k+1} \in I. \end{aligned} \tag{1.1}$$

Moreover, one can show that $M = A^{[f]}$ is the only mean which solves this equation and such that $M(x_1, x_2) = A^{[f]}(x_1, x_2)$ for all $x_1, x_2 \in I$. As a consequence, we can utilize (1.1) to calculate the value of a quasiarithmetic mean for a vector of arbitrary length based only on its bivariate restriction. The aim of this paper is to generalize the procedure above, to extend other bivariate means to vectors of arbitrary length.

1.1. General framework

Formally, a *mean of order* k, or k*-mean* for short, on a set X is a function $\mu \colon X^k \to X$ satisfying $\mu(x, \ldots, x) = x$ for all $x \in X$. In the twentieth century, the theory of topological means, that is, symmetric means on topological spaces for which the mean operation is continuous, was of great interest. This work was pioneered by Aumann [1], who showed, among other things, that no sphere admits such a mean [2].

Now we proceed to the notion of β-invariant extension introduced by Horwitz [21]. Given a set X and a k-mean $\mu \colon X^k \to X$, the *barycentric operator*

$\beta = \beta_\mu \colon X^{k+1} \to X^{k+1}$ is defined by

$$\beta_\mu(x) := (\mu(x^{\vee 1}), \dots, \mu(x^{\vee (k+1)})),$$

where $x^{\vee j}$ is a vector which is obtained by removing the j-th coordinate in the vector x, that is $x^{\vee j} = (x_i)_{i \neq j}$. For a topological k-mean, we say that the barycentric map β is *power convergent* if for each $x \in X^{k+1}$, we have $\lim_{n\to\infty} \beta^n(x) = (x^*, \dots, x^*)$ for some $x^* \in X$.

A mean $\nu \colon X^{k+1} \to X$ is a β-invariant extension of $\mu \colon X^k \to X$ if $\nu \circ \beta_\mu = \nu$, that is

$$\nu(\mu(x^{\vee 1}), \dots, \mu(x^{\vee (p+1)})) = \nu(x), \quad x \in X^{k+1}.$$

Let us now recall an important result by Lawson and Lim [26].

Proposition 1.1. ([26], Proposition 2.4) *Assume that $\mu \colon X^k \to X$ is a topological k-mean and that the corresponding barycentric operator β_μ is power convergent. Define $\widetilde{\mu} : X^{k+1} \to X$ by $\widetilde{\mu}(x) = x^*$, where* $\lim_{n\to\infty} \beta_\mu^n(x) = (x^*, \dots, x^*)$. *Then*

- (i) *$\tilde{\mu} : X^{k+1} \to X$ is a $(k+1)$-mean on X that is a β-invariant extension of μ.*
- (ii) *Any continuous mean on X^{k+1} that is a β-invariant extension of μ must equal $\tilde{\mu}$.*
- (iii) *If μ is symmetric, so is $\tilde{\mu}$.*

1.2. Properties of means on the interval

For the sake of completeness, let us introduce formally $\mathbb{N} := \{1, \dots\}$, and $\mathbb{N}_p := \{1, \dots, p\}$ (where $p \in \mathbb{N}$).

Throughout this note, I is a subinterval of $\mathbb{R}$. For a given $p \in \mathbb{N}$, a function $M \colon I^p \to I$ is a *p-variable mean on I* if

$$\min(x) \le M(x) \le \max(x) \text{ for every } x \in I^p.$$

We can define some natural properties such as symmetry, continuity, convexity, etc. which refer to properties of M as a p-variable function. A mean M is called *strict* if $\min(x) < M(x) < \max(x)$ for every non-constant vector $x \in I^p$.

Now, define the order $\prec$ on vectors of real numbers of the same lengths by

$$x \prec y :\iff \big(x_i \le y_i \text{ for every } i\big).$$

Then, a mean $M \colon I^p \to I$ is *monotone* if $M(x) \le M(y)$ for all $x, y \in I^p$ with $x \prec y$.

A function $M \colon \bigcup_{p=1}^\infty I^p \to I$ is a *mean* if all its p-variable restrictions $M\restriction_p := M|_{I^p}$ are means for all $p \ge 1$. Such a mean is called *symmetric* (resp. *continuous*, etc.) if all $M\restriction_p$-s admit this property.

1.3. Invariance in a family of means

For $p \in \mathbb{N}$ a selfmapping $\mathbf{M}\colon I^p \to I^p$ is called a *mean-type mapping* if $\mathbf{M} = (M_1, \dots, M_p)$ for some p-variable means $M_1, \dots, M_p$ on I. A mean $K\colon I^p \to I$ is called $\mathbf{M}$-*invariant* if $K \circ \mathbf{M} = K$.

The most classical result by Borwein-Borwein [7] states that if all means M_i are continuous and strict, then there exists exactly one $\mathbf{M}$-invariant mean. This result has several generalisations (see, for example, Matkowski [33], and Matkowski-Pasteczka [35]). For details we refer the reader to the rich literature on the subject, the classical ones being Lagrange [25], Gauss [16], Foster-Philips [15], Lehmer [27], Schoenberg [44] as well as the more recent ones Baják–Páles [3–6], Daróczy–Páles [9,11,12], Deregowska–Pasteczka [14], Głazowska [18,19], Jarczyk–Jarczyk [22], Matkowski [30–33], Matkowski–Páles [36], Matkowski–Pasteczka [34,35] and Pasteczka [38,39,41,42].

In the next section, we will only recall results from [42], as they are the most suitable ones for our purposes.

In this restricted (interval) setting, we know that a β-invariant extension is uniquely defined (in the family of means) if and only if the barycentric operator is power convergent (see [35, Theorem 1]). Moreover, the barycentric operator is a special case of a mean-type mapping consisting of extended means (see [42]). Therefore, in what follows, we deliver a short introduction to these objects.

2. Extended means

Observe that for $d, p \in \mathbb{N}$ and a d-variable mean $M\colon I^d \to I$ we can construct a p-variable mean by choosing d indexes and applying the mean M to the so-obtained vector of length d. Formally, for $d, p \in \mathbb{N}$ and a vector $\alpha \in \mathbb{N}_p^d$ we define a p-variable mean $M^{(p;\alpha)}\colon I^p \to I$ by

$$M^{(p;\alpha)}(x_1, \dots, x_p) := M(x_{\alpha_1}, \dots, x_{\alpha_d}) \text{ for all } (x_1, \dots, x_p) \in I^p. \tag{2.1}$$

Let us emphasize, that for $\alpha = (1, \dots, p) \in \mathbb{N}_p^p$, we have $M^{(p;\alpha)} = M$, thus (purely formally) each mean is an extended mean. However, this approach allowed us to establish several interesting results related to mean-type mappings consisting of such means. We are going to recall them in the following section.

In a simple case, if $\mathcal{A}\colon \mathbb{R}^2 \to \mathbb{R}$ is a bivariate arithmetic mean, $p \geq 3$ and $\alpha = (2, 3)$ then $\mathcal{A}^{(p;\alpha)}\colon I^p \to I$ is given by

$$\mathcal{A}^{(p;\alpha)}(x_1, \dots, x_p) = \mathcal{A}^{(p;2,3)}(x_1, \dots, x_p) = \tfrac{x_2+x_3}{2} \text{ for all } (x_1, \dots, x_p) \in I^p.$$

2.1. Invariance of extended means

Before we proceed to discuss the invariance, we need to build a mean-type mapping. The idea is to use an extended mean at each coordinate. Therefore, in some sense, we need to vectorise the previous approach.

For $p \in \mathbb{N}$ and a vector $\mathbf{d} = (d_1, \dots, d_p) \in \mathbb{N}^p$, let $\mathbb{N}_p^{\mathbf{d}} := \mathbb{N}_p^{d_1} \times \cdots \times \mathbb{N}_p^{d_p}$. Using this notations, a sequence of means $\mathbf{M} = (M_1, \dots, M_p)$ is called a **d**-*averaging* mapping on I if each M_i is a d_i-variable mean on I.

In the next step, for a **d**-averaging mapping $\mathbf{M}$ and a vector of indexes $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}_p^{d_1} \times \cdots \times \mathbb{N}_p^{d_p} = \mathbb{N}_p^{\mathbf{d}}$ let us define a mean-type mapping $\mathbf{M}_\alpha \colon I^p \to I^p$ by

$$\mathbf{M}_\alpha := \Big(M_1^{(p;\alpha_1)}, \dots, M_p^{(p;\alpha_p)}\Big),$$

where $M_i^{(p,\alpha_i)}$-s are defined by (2.1).

Observe that α is a p-tuple of sequences of elements in $\mathbb{N}_p$. Such an object can be represented as a directed graph. Therefore for a given $p \in \mathbb{N}$, $\mathbf{d} = (d_1, \dots, d_p) \in \mathbb{N}^p$, and $\alpha \in \mathbb{N}_p^{\mathbf{d}}$, we define the α-*incidence* graph $G_\alpha = (V_\alpha, E_\alpha)$ as follows:

$$V_\alpha := \mathbb{N}_p \quad \text{and} \quad E_\alpha := \big\{(\alpha_{i,j}, i) \colon i \in \mathbb{N}_p \text{ and } j \in \mathbb{N}_{d_i}\big\}.$$

Since the graph G_α plays a very important role in the invariance of $\mathbf{M}_\alpha$, let us recall some elementary definitions from graph theory.

A sequence $(v_0, \dots, v_n)$ of elements in V such that $(v_{i-1}, v_i) \in E$ for all $i \in \{1, \dots, n\}$ is called a *walk* from v_0 to v_n. The number n is the *length* of the walk. If for all $v, w \in V$ there exists a walk from v to w, then G is called *irreducible*. A *cycle* is a non-empty walk in which only the first and last vertices are equal. A directed graph is said to be *aperiodic* if no integer $k > 1$ divides the length of every cycle of the graph. A graph is called *ergodic* if it is simultaneously irreducible and aperiodic.

Now we are ready to recall the main theorem from the paper [42]. It turns out that the natural assumption to guarantee that an $\mathbf{M}_\alpha$-invariant mean is uniquely determined, is that G_α is ergodic.

Proposition 2.1. ([42], Theorem 2) *Given an interval $I \subset \mathbb{R}$, parameters $p \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^p$, and a* **d**-*averaging mapping $\mathbf{M} = (M_1, \dots, M_p)$ on I such that all M_i-s are strict. For all $\alpha \in \mathbb{N}_p^{\mathbf{d}}$ such that G_α is ergodic:*

(a) *there exists a unique $\mathbf{M}_\alpha$-invariant mean $K_\alpha \colon I^p \to I$;*
(b) *K_α is continuous;*
(c) *K_α is strict;*
(d) *$\mathbf{M}_\alpha^n$ converges, uniformly on compact subsets of I^p, to the mean-type mapping $\mathbf{K}_\alpha \colon I^p \to I^p$, $\mathbf{K}_\alpha = (K_\alpha, \dots, K_\alpha)$;*
(e) *$\mathbf{K}_\alpha \colon I^p \to I^p$ is $\mathbf{M}_\alpha$-invariant, that is $\mathbf{K}_\alpha = \mathbf{K}_\alpha \circ \mathbf{M}_\alpha$;*

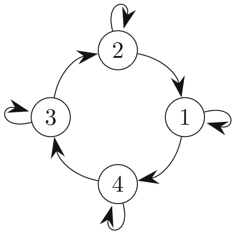

Figure 1. Graph G_α related to Example 1

(f) *if* $M_1, \dots, M_p$ *are nondecreasing with respect to each variable then so is* K_α*;*
(g) *if* $I = (0, +\infty)$ *and* $M_1, \dots, M_p$ *are positively homogeneous, then every iterate of* $\mathbf{M}_\alpha$ *and* K_α *are positively homogeneous.*

At this stage, let us recall a straightforward application of this result.

Example 1. ([42], Example 2) Let $\mathbf{M}\colon \mathbb{R}_+^4 \to \mathbb{R}_+^4$ be given by

$$\mathbf{M}(x, y, z, t) := \left(\frac{2xy}{x+y}, \sqrt{yz}, \frac{z+t}{2}, \sqrt{\frac{t^2+x^2}{2}} \right). \tag{2.2}$$

We show that there exists a unique $\mathbf{M}$-invariant mean $K\colon \mathbb{R}_+^4 \to \mathbb{R}_+$. Additionally, K is continuous and strict (Fig. 1).

Indeed, in the framework of $\mathbf{d}$-averaging mappings, we express $\mathbf{M}$ defined in (2.2) as $\bar{\mathbf{M}}_\alpha$, where $\bar{\mathbf{M}}$ consists of bivariate power means, that is

$$\bar{\mathbf{M}} = (\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2), \text{ and } \alpha = \big((1,2), (2,3), (3,4), (4,1)\big).$$

The vector $\mathbf{d}$ contains the lengths of the elements in α (since $\alpha \in \mathbb{N}_4^{\mathbf{d}}$), thus $\mathbf{d} = (2, 2, 2, 2)$. Obviously all means in $\bar{\mathbf{M}}$, being power means, are continuous and strict. Moreover, the α-incidence graph (see Fig. 1) is aperiodic (since every vertex has a loop) and irreducible (since (4321) is its Hamiltonian cycle). Consequently the α-incidence graph is ergodic.

Thus, in view of Proposition 2.1, there exists exactly one $\mathbf{M}$-invariant mean $K\colon \mathbb{R}_+^4 \to \mathbb{R}_+$. Moreover, by the same theorem, we know that it is continuous and strict.

3. Results

3.1. Auxiliary results

In what follows we show two results. The second one shall be considered a direct application of Proposition 2.1. It is preceded by a simple (and purely technical) lemma, which shows that a key assumption of this proposition is satisfied.

Lemma 3.1. *For every $p \geq 3$ the graph $Q_p := \big(\mathbb{N}_p, \{(i,j) \in \mathbb{N}_p^2 \colon i \neq j\}\big)$ is ergodic.*

Indeed, we can observe that Q_p is a full directed graph (without loops), and hence it is ergodic. Based on this lemma, in view of Proposition 2.1, part (a) the following proposition immediately follows.

Proposition 3.2. *Let $p \in \mathbb{N}$ with $p \geq 2$, and $M \colon I^p \to I$ be a symmetric, continuous, and strict mean. Then there exists a unique β_M-invariant extension. Equivalently, the functional equation*

$$K(M(x^{\vee 1}), \dots, M(x^{\vee (p+1)})) = K(x), \quad x \in I^{p+1} \tag{3.1}$$

has exactly one solution in the family of means $K \colon I^{p+1} \to I$.

Proof. Let $\alpha \in \mathbb{N}_{p+1}^{(p+1)\times p}$ be given by

$$\alpha = \big(\mathbb{N}_{p+1} \setminus \{1\}, \mathbb{N}_{p+1} \setminus \{2\}, \dots, \mathbb{N}_{p+1} \setminus \{p+1\}\big). \tag{3.2}$$

Then $G_\alpha = Q_{p+1}$ and, by Lemma 3.1, it is ergodic. Thus, applying Proposition 2.1 part (a) to the $\mathbf{d}$-averaging mapping $\mathbf{M}$, where

$$\mathbf{d} := (\underbrace{p, \dots, p}_{(p+1)\text{ times}}) \text{ and } \mathbf{M} := (\underbrace{M, \dots, M}_{(p+1)\text{ times}}), \tag{3.3}$$

we get that there exists a unique $\mathbf{M}_\alpha$-invariant mean. In other words, the functional equation $K \circ \mathbf{M}_\alpha = K$ has exactly one solution in the family of means $K \colon I^{p+1} \to I$. However, in this setup,

$$\begin{aligned}\mathbf{M}_\alpha(x) &= \big(M^{(p,\alpha_1)}(x), \dots, M^{(p,\alpha_{p+1})}(x)\big)\\ &= \Big(M\big((x_i)_{i\in\mathbb{N}_{p+1}\setminus\{1\}}\big), \dots, M\big((x_i)_{i\in\mathbb{N}_{p+1}\setminus\{p+1\}}\big)\Big)\\ &= (M(x^{\vee 1}), \dots, M(x^{\vee (p+1)})),\end{aligned}$$

thus the proof is complete. □

Remark 1. Observe that in the case when $p \geq 2$ and $M := \mathcal{A}_p$ is the p-variable arithmetic mean (on $\mathbb{R}$) then equality (3.1) takes the form

$$K\Big(\frac{x_2 + \cdots + x_{p+1}}{p}, \frac{x_1 + x_3 \cdots + x_{p+1}}{p}, \dots, \frac{x_1 + \cdots + x_p}{p}\Big) = K(x).$$

If we define $m := \mathcal{A}_{p+1}(x_1, \dots, x_{p+1}) := \frac{x_1+\cdots+x_{p+1}}{p+1}$ then we can rewrite this equality in a simpler form

$$K\Big(\frac{(p+1)m - x_1}{p}, \frac{(p+1)m - x_2}{p}, \dots, \frac{(p+1)m - x_p}{p}\Big) = K(x).$$

Clearly $K = \mathcal{A}_{p+1}$ is a solution of this equation. Moreover, by Proposition 3.2, this solution is unique.

3.2. Main result

In view of Proposition 3.2, for every symmetric, continuous, and strict mean $M\colon I^p \to I$, we define its *β-invariant extension* $\widetilde{M}\colon I^{p+1} \to I$ (denoted also as $M^{\sim}$) as the unique solution K of equation (3.1) in the family of means.

As we have already checked in Remark 1, $\widetilde{\mathcal{A}_p} = \mathcal{A}_{p+1}$ for all $p \ge 2$. In other words, the β-invariant extension of the p–variable arithmetic mean is simply a $(p+1)$–variable arithmetic mean. It turns out, see Lemma 5.3 below, that this property remains valid also for quasiarithmetic means.

In the following result we prove that a mean shares several properties with its β-invariant extension.

Theorem 1. *Let $p \in \mathbb{N}$ with $p \ge 2$, and $M\colon I^p \to I$ be a symmetric, continuous, and strict mean. Then*

(a) *$\widetilde{M}$ is continuous;*
(b) *$\widetilde{M}$ is symmetric;*
(c) *$\widetilde{M}$ is strict;*
(d) *if M is monotone, then so is $\widetilde{M}$;*
(e) *if M is convex (concave) and monotone, then so is $\widetilde{M}$.*
(f) *if $I = (0,+\infty)$ and M is positively homogeneous, then so is $\widetilde{M}$;*

Proof. Set α, $\mathbf{d}$, and $\mathbf{M}$ by equations (3.2), and (3.3). Then $\widetilde{M} = K_\alpha$ is the unique $\mathbf{M}_\alpha$-invariant mean. By Lemma 3.1, we know that G_α is ergodic.

Therefore parts (a), (c), (d), (f), of this statement are implied by Proposition 2.1 parts (b), (c), (f), (g), respectively. Now, we only need to prove statements (b) and (e).

To show (b), observe that for every vector $x \in I^{p+1}$ and a permutation $\sigma\colon \mathbb{N}_{p+1} \to \mathbb{N}_{p+1}$ we have

$$\begin{aligned}\mathbf{M}_\alpha(x\circ\sigma) &= \big(M((x\circ\sigma)^{\vee 1}),\dots,M((x\circ\sigma)^{\vee(p+1)})\big)\\ &= \big(M(x^{\vee\sigma(1)}),\dots,M(x^{\vee\sigma(p+1)})\big)\\ &= \big([\mathbf{M}_\alpha(x)]_{\sigma(1)},\dots,[\mathbf{M}_\alpha(x)]_{\sigma(p+1)}\big) = \mathbf{M}_\alpha(x)\circ\sigma.\end{aligned}$$

Therefore, by Proposition 2.1 part (d), for every $x \in I^{p+1}$ we have

$$\mathbf{K}_\alpha(x\circ\sigma) = \lim_{n\to\infty}\mathbf{M}_\alpha^n(x\circ\sigma) = \big(\lim_{n\to\infty}\mathbf{M}_\alpha^n(x)\big)\circ\sigma = \mathbf{K}_\alpha(x)\circ\sigma,$$

thus, since K_α is an(y) entry in $\mathbf{K}_\alpha$, we get $K_\alpha(x\circ\sigma) = K_\alpha(x)$, which proves that K_α is symmetric.

Finally, to prove part (e) assume that M is convex and monotone and fix two vectors $x, y \in I^p$ and $t \in (0,1)$. As the intermediate step shows that, for all $n \ge 0$,

$$\big[\mathbf{M}_\alpha^n(tx+(1-t)y)\big]_k \le t\big[\mathbf{M}_\alpha^n(x)\big]_k + (1-t)\big[\mathbf{M}_\alpha^n(y)\big]_k \quad \text{for all } k \in \mathbb{N}_{p+1}. \tag{3.4}$$

For $n = 0$ inequality (3.4) obviously holds (even with equality). Now assume that (3.4) is valid for some $n \ge 0$. Then, for every $k_0 \in \mathbb{N}_{p+1}$, we have

$$\begin{aligned}\big[\mathbf{M}_\alpha^{n+1}(tx+(1-t)y)\big]_{k_0} &= M\Big(\big(\mathbf{M}_\alpha^n(tx+(1-t)y)\big)^{\vee k_0}\Big)\\ &\le M\Big(t\mathbf{M}_\alpha^n(x)^{\vee k_0}+(1-t)\mathbf{M}_\alpha^n(y)^{\vee k_0}\Big)\\ &\le tM\big(\mathbf{M}_\alpha^n(x)^{\vee k_0}\big)+(1-t)M\big(\mathbf{M}_\alpha^n(y)^{\vee k_0}\big)\\ &= t\big[\mathbf{M}_\alpha^{n+1}(x)\big]_{k_0}+(1-t)\big[\mathbf{M}_\alpha^{n+1}(y)\big]_{k_0}.\end{aligned}$$

Since k_0 is an arbitrary element in $\mathbb{N}_{p+1}$ we obtain (3.4) with n replaced by $n+1$. Consequently (3.4) is valid for all $n \ge 0$. In the limit case as $n \to \infty$, in view of Proposition 2.1 part (d), we get

$$K_\alpha(tx+(1-t)y) \le tK_\alpha(x)+(1-t)K_\alpha(y),$$

which shows that $\widetilde{M} = K_\alpha$ is convex and, by the already proved part (d), monotone. The case when M is concave and monotone is analogous. This completes part (e) of the proof. $\square$

In the following proposition, we show that this extension preserves the comparability of means.

Proposition 3.3. *Let $p \in \mathbb{N}$ with $p \ge 2$, and $M, N\colon I^p \to I$ be symmetric, continuous, monotone, and strict means. Then $M \le N$ yields $\widetilde{M} \le \widetilde{N}$.*

Proof. Let α, $\mathbf{d}$, and $\mathbf{M}$ be like in equations (3.2), and (3.3). Additionally, define $\mathbf{N}\colon (I^p)^{p+1} \to I^{p+1}$ by $\mathbf{N} := (N, \dots, N)$. However, $x^{\vee k} \prec y^{\vee k}$ for every $x, y \in I^{p+1}$ with $x \prec y$ and $k \in \mathbb{N}_{p+1}$. Therefore, since M is monotone, we obtain

$$M(x^{\vee k}) \le M(y^{\vee k}) \le N(y^{\vee k}) \text{ for every } k \in \mathbb{N}_{p+1},$$

which can be rewritten in a compact form as $\mathbf{M}_\alpha(x) \prec \mathbf{N}_\alpha(y)$. Thus, by simple induction, $x \prec y$ implies $\mathbf{M}_\alpha^n(x) \prec \mathbf{N}_\alpha^n(y)$ for every $n \in \mathbb{N}$. In the limit case as $n \to \infty$, by Proposition 2.1 part (d), we obtain that $\widetilde{M}(x) \le \widetilde{N}(y)$ for every $x, y \in I^p$ with $x \prec y$.

Finally, for every $x \in I^p$, we have $x \prec x$. Therefore, we obtain the desired inequality $\widetilde{M}(x) \le \widetilde{N}(x)$. $\square$

4. Multivariable generalization of bivariate means

Let $M\colon I^2 \to I$ be a symmetric, continuous, and strict mean on an interval I. Then its *iterative β-invariant extension* is a mean $M^e\colon \bigcup_{p=1}^\infty I^p \to I$ defined by

$$M^e(x_1,\dots,x_p)=\begin{cases} x_1 & \text{for } p=1;\\ M(x_1,x_2) & \text{for } p=2;\\ M^{\sim(p-2)}(x_1,\dots,x_p) & \text{for } p>2, \end{cases} \tag{4.1}$$

where $M^{\sim p}$ is the p-th iteration of the β-invariant extension operator, that is

$$M^{\sim 1}=M^{\sim}=\widetilde{M} \text{ and } M^{\sim p}=(M^{\sim(p-1)})^{\sim} \text{ for } p\ge 2.$$

Now we adapt the result for β-invariant extensions to the iterative setting.

Proposition 4.1. *Let $M\colon I^2\to I$ be a symmetric, continuous, and strict mean. Then*

(a) *M^e is continuous;*
(b) *M^e is symmetric;*
(c) *M^e is strict;*
(d) *if M is monotone, then so is M^e;*
(e) *if M is convex (concave) and monotone then so is $\widetilde{M}$;*
(f) *if $I=(0,+\infty)$ and M is positively homogeneous, then so is $\widetilde{M}$.*

Proof. To show that M^e is continuous recall that, in view of Theorem 1 part (a), the β-invariant extension operator preserves continuity. Therefore, once we know that $M^e\restriction_p$ is continuous for certain $p\ge 2$, then so is $M^e\restriction_{p+1}=(M^e\restriction_p)^{\sim}$. Thus, since $M^e\restriction_2=M$ is continuous, by simple induction we find that $M^e\restriction_p$ is continuous for all $p\ge 2$ which is equivalent to the continuity of M^e and completes part (a) of the proof.

Analogously, applying other parts of Theorem 1, we can show all the remaining parts of this statement. □

Next, we show that this operator is monotone with respect to the ordering of means.

Proposition 4.2. *Let $M,\ N\colon I^2\to I$ be symmetric, continuous, monotone, and strict means. Then $M\le N$ if and only if $M^e\le N^e$.*

Proof. First observe that, by Theorem 1 parts (a)–(d), all iterates $M^{\sim p}$ and $N^{\sim p}$ $(p\in\mathbb{N})$ are symmetric, continuous, monotone, and strict.

If $M\le N$ then, by Proposition 3.3, one can (inductively) show that $M^{\sim p}\le N^{\sim p}$ for all $p\ge 1$, which is equivalent to the inequality $M^e\le N^e$.

Conversely, if $M^e\le N^e$ then $M=M^e\restriction_2\le N^e\restriction_2=N$. □

4.1. Conjugacy of means

For $n \in \mathbb{N}$, a mean $M\colon I^p \to I$ and a continuous, monotone function $\varphi\colon J \to I$ we define a *conjugated mean* $M^{[\varphi]}\colon J^p \to J$ by

$$M^{[\varphi]}(x_1, \dots, x_p) := \varphi^{-1} \circ M\big(\varphi(x_1), \dots, \varphi(x_p)\big).$$

Conjugacy of means is a generalization of the concept of quasiarithmetic means. As a matter of fact, one can easily check that quasiarithmetic means are simply conjugates of the arithmetic mean. In the next statement, we prove that iterative β-invariant extensions commute with conjugacy.

Proposition 4.3. *Let $I, J \subset \mathbb{R}$ be intervals and $M\colon I^2 \to I$ be a symmetric, continuous, monotone, strict mean, and $\varphi\colon J \to I$ be a continuous and strictly monotone function. Then $(M^e)^{[\varphi]} = (M^{[\varphi]})^e$.*

Proof. First, we show that the β-invariant extension commutes with conjugates. More precisely for every symmetric, continuous and monotone mean $N\colon I^p \to I$ we have

$$(\widetilde{N})^{[\varphi]} = (N^{[\varphi]})^{\sim}. \tag{4.2}$$

Indeed, define the mappings

$$\begin{aligned}
\mathbf{M}\colon I^{p+1} \ni x &\mapsto \big(N(x^{\vee 1}), \dots, N(x^{\vee (p+1)})\big) \in I^{p+1};\\
\mathbf{P}\colon J^{p+1} \ni y &\mapsto \big(N^{[\varphi]}(y^{\vee 1}), \dots, N^{[\varphi]}(y^{\vee (p+1)})\big) \in J^{p+1};\\
\Phi\colon I^{p+1} \ni x &\mapsto \big(\varphi(x_1), \dots, \varphi(x_{p+1})\big) \in J^{p+1}.
\end{aligned}$$

Then, for all $y \in J^{p+1}$, we have

$$\begin{aligned}
\Phi \circ \mathbf{P}(y) &= \Phi\big(N^{[\varphi]}(y^{\vee 1}), \dots, N^{[\varphi]}(y^{\vee (p+1)})\big)\\
&= \big(N(\Phi(y)^{\vee 1}), \dots N(\Phi(y)^{\vee (p+1)})\big) = \mathbf{M} \circ \Phi(y).
\end{aligned}$$

Therefore $\mathbf{P} = \Phi^{-1} \circ \mathbf{M} \circ \Phi$, and whence

$$\mathbf{P}^n = \Phi^{-1} \circ \mathbf{M}^n \circ \Phi \text{ for all } n \geq 1. \tag{4.3}$$

By Proposition 2.1 part (d) we know that the sequence of iterates $(\mathbf{M}^n)_{n=1}^{\infty}$ converges to $\widetilde{N}$ in each coordinate while $(\mathbf{P}^n)_{n=1}^{\infty}$ converges to $(N^{[\varphi]})^{\sim}$ in each coordinate. Whence, if we take the limit $n \to \infty$, equality (4.3) implies $(N^{[\varphi]})^{\sim} = \varphi^{-1} \circ \widetilde{N} \circ \Phi = (\widetilde{N})^{[\varphi]}$. Therefore (4.2) holds.

By (4.2) we easily obtain $(M^{\sim p})^{[\varphi]} = (M^{[\varphi]})^{\sim p}$ for all $p \geq 1$, which implies $(M^e)^{[\varphi]} = (M^{[\varphi]})^e$. □

5. Applications to classical families of means

5.1. Quasiarithmetic envelopes

Quasiarithmetic means were introduced as a generalization of Power Means in the 1920s/30 s in a series of nearly simultaneous papers by de Finetti [13], Knopp [23], Kolmogorov [24], and Nagumo [37]. For an interval I and a continuous and strictly monotone function $f: I \to \mathbb{R}$ (from now on $\mathcal{CM}(I)$ is a family of continuous, strictly monotone functions on I) we define the *quasiarithmetic mean* $A^{[f]}: \bigcup_{n=1}^{\infty} I^n \to I$ by

$$A^{[f]}(x_1, \dots, x_n) := f^{-1}\left(\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right),$$

where $n \in \mathbb{N}$ and $x_1, \dots, x_n \in I$. The function f is called a *generator* of the quasiarithmetic mean.

It is well known that for $I = \mathbb{R}_+$, $\pi_r(t) := t^r$ for $r \neq 0$ and $\pi_0(t) := \ln t$, the mean $A^{[\pi_r]}$ coincides with the r-th power mean (this fact had already been noticed by Knopp [23]).

There were a number of results related to quasiarithmetic means. For example (see [20]) $A^{[f]} = A^{[g]}$ if and only if their generators are affine transformations of each other, i.e. there exist α, $\beta \in \mathbb{R}$ such that $g = \alpha f + \beta$.

Generalizing the approach from the paper [40], we are going to introduce quasiarithmetic envelopes. First, for a given mean $M: I^p \to I$ we define sets of quasiarithmetic means which are below and above M. More precisely let

$$\mathcal{F}^-(M) := \{f \in \mathcal{CM}(I) \colon A^{[f]}(x) \le M(x) \text{ for all } x \in I^p\};$$
$$\mathcal{F}^+(M) := \{f \in \mathcal{CM}(I) \colon A^{[f]}(x) \ge M(x) \text{ for all } x \in I^p\}.$$

Now, for a given mean $M: I^p \to I$, we define *local lower* and *upper quasiarithmetic envelopes* $L_M, U_M: I^p \to I$ by

$$L_M(x) := \begin{cases} \sup\{A^{[f]}(x) \colon f \in \mathcal{F}^-(M)\} & \text{if } \mathcal{F}^-(M) \neq \emptyset, \\ \min(x) & \text{otherwise;} \end{cases}$$
$$U_M(x) := \begin{cases} \inf\{A^{[f]}(x) \colon f \in \mathcal{F}^+(M)\} & \text{if } \mathcal{F}^+(M) \neq \emptyset, \\ \max(x) & \text{otherwise.} \end{cases},$$

for all $x \in I^p$.

In the case $M: \bigcup_{p=1}^{\infty} I^p \to I$ we have two approaches to this problem. First, similarly to the previous case, we define two pairs $L_M, U_M: \bigcup_{p=1}^{\infty} I^p \to I$ by

$$L_M(x_1, \dots, x_p) := L_{M\restriction_p}(x_1, \dots, x_p),$$
$$U_M(x_1, \dots, x_p) := U_{M\restriction_p}(x_1, \dots, x_p),$$

where $p \in \mathbb{N}$ is arbitrary and $x_1, \dots, x_p \in I$.

On the other hand, we can repeat the previous setting from the beginning and define

$$\mathcal{G}^-(M) := \{f \in \mathcal{CM}(I)\colon A^{[f]}(x) \le M(x) \text{ for all } x \in \bigcup_{p=1}^{\infty} I^p\};$$

$$\mathcal{G}^+(M) := \{f \in \mathcal{CM}(I)\colon A^{[f]}(x) \ge M(x) \text{ for all } x \in \bigcup_{p=1}^{\infty} I^p\}.$$

Then we set *(global) lower* and *upper quasiarithmetic envelopes* as follows. Let $\mathcal{L}_M, \mathcal{U}_M\colon \bigcup_{p=1}^{\infty} I^p \to I$ be given by

$$\mathcal{L}_M(x) := \begin{cases} \sup\{A^{[f]}(x)\colon f \in \mathcal{G}^-(M)\} & \text{if } \mathcal{G}^-(M) \ne \emptyset, \\ \min(x) & \text{otherwise;} \end{cases}$$

$$\mathcal{U}_M(x) := \begin{cases} \inf\{A^{[f]}(x)\colon f \in \mathcal{G}^+(M)\} & \text{if } \mathcal{G}^+(M) \ne \emptyset, \\ \max(x) & \text{otherwise.} \end{cases},$$

for all $x \in \bigcup_{p=1}^{\infty} I^p$.

Before we proceed, let us illustrate these concepts with a simple example.

Example 2. Define $M\colon \bigcup_{p=1}^{\infty} \mathbb{R}_+^p \to \mathbb{R}_+$ by

$$M(x_1,\dots,x_p) := \left(\frac{x_1^{1/p} + \dots + x_p^{1/p}}{p}\right)^p \qquad p \in \mathbb{N} \text{ and } x_1,\dots,x_p \in \mathbb{R}_+.$$

Then, since the restriction of M to a given (fixed) number of parameters is a power mean (and therefore a quasiarithmetic mean), we get $L_M = U_M = M$. Meanwhile, global quasiarithmetic envelopes bound M for a vector of parameters of arbitrary length. Using the fact that power means are ordered, we get $\mathcal{L}_M = \mathcal{P}_0$ and $\mathcal{U}_M = \mathcal{P}_1$.

Now we are going to show a few basic properties of operators L, U and $\mathcal{L}$, $\mathcal{U}$. We bind them into two propositions. The first one, refers to L and U, while the second to $\mathcal{L}$ and $\mathcal{U}$.

Proposition 5.1. *The following properties are valid:*

(i) *For every mean $M\colon I^p \to I$ we have $L_M \le M \le U_M$.*
(ii) *Let $M, N\colon I^p \to I$ be two means with $M \le N$. Then $L_M \le L_N$ and $U_M \le U_N$.*
(iii) *For every $\varphi \in \mathcal{CM}(I)$ we have $L_{A^{[\varphi]}} = U_{A^{[\varphi]}} = A^{[\varphi]}$.*
(iv) *For every mean $M\colon I^p \to I$ and a monotone, continuous function $\varphi\colon J \to I$ we have $L_M^{[\varphi]} = L_{M^{[\varphi]}}$ and $U_M^{[\varphi]} = U_{M^{[\varphi]}}$.*

Proof. We restrict the proof of this proposition to results concerning the lower envelope. Parts concerning the upper envelope are analogous.

To show (i), fix $x \in I^p$ arbitrarily. If $\mathcal{F}^-(M) = \emptyset$ then

$$L_M(x) = \min(x) \le M(x).$$

Otherwise we have $A^{[f]}(x) \le M(x)$ for all $f \in \mathcal{F}^-(M)$ which yields

$$L_M(x) = \sup\{A^{[f]}(x) \colon f \in \mathcal{F}^-(M)\} \le M(x),$$

which validates the inequality $L_M \le M$ and completes the proof of part (i).

To prove part (ii), it is sufficient to observe that, under the assumption $M \le N$, we have $\mathcal{F}^-(M) \subseteq \mathcal{F}^-(N)$, which easily implies $L_M \le L_N$.

Next, for every $\varphi \in \mathcal{CM}(I)$ we have $\varphi \in \mathcal{F}^-(A^{[\varphi]})$, which shows the inequality $L_{A^{[\varphi]}} \ge A^{[\varphi]}$. Therefore $L_{A^{[\varphi]}} = A^{[\varphi]}$, since the second inequality was already proved in (i). Thus we get (iii).

Finally, in order to prove (iv) assume that φ is strictly increasing and note that

$$\begin{aligned}
\mathcal{F}^-(M^{[\varphi]}) &= \{f \in \mathcal{CM}(J) \colon A^{[f]}(x) \le M^{[\varphi]}(x) \text{ for all } x \in J^p\} \\
&= \{f \in \mathcal{CM}(J) \colon A^{[f \circ \varphi^{-1}]}(x) \le M(x) \text{ for all } x \in I^p\} \\
&= \{f \in \mathcal{CM}(J) \colon f \circ \varphi^{-1} \in \mathcal{F}^-(M)\} \\
&= \{g \circ \varphi \in \mathcal{CM}(J) \colon g \in \mathcal{F}^-(M)\}.
\end{aligned}$$

Therefore either $\mathcal{F}^-(M^{[\varphi]}) = \mathcal{F}^-(M) = \emptyset$ and the equality is trivial or for every $x = (x_1, \dots, x_p) \in I^p$ we have

$$\begin{aligned}
L_{M^{[\varphi]}}(x) &= \sup\{A^{[f]}(x) \colon f \in \mathcal{F}^-(M^{[\varphi]})\} \\
&= \sup\{A^{[g \circ \varphi]}(x) \colon g \in \mathcal{F}^-(M)\} \\
&= \sup\big\{\varphi^{-1}\big(A^{[g]}\big(\varphi(x_1), \dots, \varphi(x_p)\big)\big) \colon g \in \mathcal{F}^-(M)\big\} \\
&= \varphi^{-1}\Big(\sup\big\{A^{[g]}\big(\varphi(x_1), \dots, \varphi(x_p)\big) \colon g \in \mathcal{F}^-(M)\big\}\Big) = L_M^{[\varphi]}(x),
\end{aligned}$$

and we get (iv), which was the last unproved part of this statement. □

In the same spirit, we can establish the analogous result for global envelopes.

Proposition 5.2. *The following properties are valid:*

(i) *For every mean* $M \colon \bigcup_{p=1}^{\infty} I^p \to I$ *we have* $\mathcal{L}_M \le M \le \mathcal{U}_M$.

(ii) *Let* $M, N \colon \bigcup_{p=1}^{\infty} I^p \to I$ *be symmetric, continuous and strict means with* $M \le N$. *Then* $\mathcal{L}_M \le \mathcal{L}_N$ *and* $\mathcal{U}_M \le \mathcal{U}_N$.

(iii) *For every* $\varphi \in \mathcal{CM}(I)$ *we have* $\mathcal{L}_{A^{[\varphi]}} = \mathcal{U}_{A^{[\varphi]}} = A^{[\varphi]}$.

(iv) *For every mean* $M \colon I^p \to I$ *and a monotone, continuous function* $\varphi \colon J \to I$ *we have* $\mathcal{L}_M^{[\varphi]} = \mathcal{L}_{M^{[\varphi]}}$ *and* $\mathcal{U}_M^{[\varphi]} = \mathcal{U}_{M^{[\varphi]}}$.

Its proof follows the lines of that of Proposition 5.1, but we need to replace the set $\mathcal{F}^-(M)$ by $\mathcal{G}^-(M)$.

In the next lemma, we prove that the β-invariant extension of a bivariate quasiaritmetic mean is the quasiarithmetic mean generated by the same function.

Lemma 5.3. *If $\varphi \in \mathcal{CM}(I)$ then $\big(A^{[\varphi]}\restriction_2\big)^e = A^{[\varphi]}$.*

Proof. Let $\varphi \in \mathcal{CM}(I)$ and denote briefly $M := A^{[\varphi]}\restriction_2$. We prove by induction that

$$M^e\restriction_p = A^{[\varphi]}\restriction_p \tag{5.1}$$

for all $p \ge 1$. For $p = 1$ and $p = 2$, this statement is trivial.

Now assume that (5.1) holds for some $p = p_0 \ge 2$. Then

$$M^e\restriction_{p_0+1} = M^{\sim(p_0-1)} = (M^{\sim(p_0-2)})^{\sim} = (M^e\restriction_{p_0})^{\sim} = (A^{[\varphi]}\restriction_{p_0})^{\sim}.$$

Consequently, for all $x \in I^{p_0+1}$, we have

$$\begin{aligned}
&A^{[\varphi]}\big(A^{[\varphi]}(x^{\vee 1}), \dots, A^{[\varphi]}(x^{\vee(p_0+1)})\big) \\
&= A^{[\varphi]}\Bigg(\varphi^{-1}\Big(\frac{\varphi(x_1) + \dots + \varphi(x_{p_0+1}) - \varphi(x_1)}{p_0}\Big), \dots, \\
&\quad \varphi^{-1}\Big(\frac{\varphi(x_1) + \dots + \varphi(x_{p_0+1}) - \varphi(x_{p_0+1})}{p_0}\Big)\Bigg) \\
&= \varphi^{-1}\Bigg(\frac{1}{p_0+1}\sum_{k=1}^{p_0+1}\frac{\varphi(x_1) + \dots + \varphi(x_{p_0+1}) - \varphi(x_k)}{p_0}\Bigg) \\
&= \varphi^{-1}\Bigg(\frac{1}{p_0+1}\sum_{k=1}^{p_0+1}\varphi(x_k)\Bigg) = A^{[\varphi]}(x).
\end{aligned}$$

Thus $M^e\restriction_{p_0+1} = (A^{[\varphi]}\restriction_{p_0})^{\sim} = A^{[\varphi]}\restriction_{p_0+1}$, i.e. (5.1) holds for $p = p_0 + 1$. By induction we obtain that (5.1) holds for all $p \ge 1$, which is precisely the equality $(A^{[\varphi]}\restriction_2)^e = M^e = A^{[\varphi]}$. □

Now we proceed to the discussion of comparability between envelopes. First, we show that local envelopes approximate the mean better than global ones, as expected.

Proposition 5.4. *For every mean $M\colon \bigcup_{p=1}^{\infty} I^p \to I$ we have*

$$\mathcal{L}_M \le L_M \le M \le U_M \le \mathcal{U}_M.$$

Proof. Take $p \in \mathbb{N}$ and $x \in I^p$ arbitrarily. If $\mathcal{G}^-(M) = \emptyset$ then $\mathcal{L}_M = \min \le L_M$. Otherwise, by $\mathcal{G}^-(M) \subseteq \mathcal{F}^-(M)$, we have

$$\mathcal{L}_M(x) = \sup\{A^{[f]}(x) \colon f \in \mathcal{G}^-(M)\} \le \sup\{A^{[f]}(x) \colon f \in \mathcal{F}^-(M)\} = L_M(x),$$

which shows that

$$\mathcal{L}_M(x) \le L_M(x).$$

Similarly we obtain that either $\mathcal{F}^-(M) = \emptyset$ and $L_M = \min$ or $A^{[f]}(x) \le M(x)$ for all $f \in \mathcal{F}^-(M)$, which yields

$$L_M(x) = \sup\{A^{[f]}(x) \colon f \in \mathcal{F}^-(M)\} \le M(x),$$

which can be concluded as $L_M(x) \le M(x)$. Analogously we can prove the dual inequalities $M(x) \le U_M(x) \le \mathcal{U}_M(x)$. □

In the next result, we show how the extension of a mean affects its envelopes.

Theorem 2. *Let $M \colon I^2 \to I$ be a symmetric, continuous, monotone, and strict mean. Then*

(a) $\mathcal{G}^-(M^e) = \mathcal{F}^-(M)$ *and* $\mathcal{G}^+(M^e) = \mathcal{F}^+(M)$*;*
(b) $\mathcal{L}_{M^e} = \sup\{A^{[f]} \colon f \in \mathcal{F}^-(M)\}$ *and* $\mathcal{U}_{M^e} = \inf\{A^{[f]} \colon f \in \mathcal{F}^+(M)\}$*;*
(c) $\mathcal{L}_{M^e} = L_M$ *and* $\mathcal{U}_{M^e} = U_M$ *on* I^2*;*
(d) $\mathcal{L}_{M^e} \le (L_M)^e \le M^e \le (U_M)^e \le \mathcal{U}_{M^e}$*.*

Proof. First, for all $f \in \mathcal{CM}(I)$, we have

$$\begin{aligned} f \in \mathcal{G}^-(M^e) &\iff A^{[f]}(x) \le M^e(x) \text{ for all } x \in \bigcup_{p=1}^{\infty} I^p \\ &\iff (A^{[f]})^e(x) \le M^e(x) \text{ for all } x \in \bigcup_{p=1}^{\infty} I^p \\ &\iff A^{[f]}(x) \le M(x) \text{ for all } x \in I^2 \\ &\iff f \in \mathcal{F}^-(M). \end{aligned}$$

Thus $\mathcal{G}^-(M^e) = \mathcal{F}^-(M)$. The second equality is dual, thus we have proved (a). In view of this, we trivially obtain (b).

To show (c) note that, in the case $\mathcal{F}^-(M) \ne \emptyset$, we have

$$\mathcal{L}_{M^e}(x) = \sup\{A^{[f]}(x) \colon f \in \mathcal{F}^-(M)\} = L_M(x) \text{ for all } x \in I^2.$$

For $\mathcal{F}^-(M) = \emptyset$ we obviously have $\mathcal{L}_{M^e} = \min = L_M$ on I^2. Similarly $\mathcal{U}_{M^e} = U_M$ on I^2.

To prove (d), we focus on inequalities $\mathcal{L}_{M^e} \le (L_M)^e \le M^e$, since the remaining ones are dual. Additionally one can assume that $\mathcal{F}^-(M) \ne \emptyset$, since the second case is trivial.

First, note that

$$\mathcal{L}_{M^e} = \sup\{A^{[f]} \colon f \in \mathcal{F}^-(M)\} = \sup\{(A^{[f]})^e \colon f \in \mathcal{F}^-(M)\}.$$

However, for all $f \in \mathcal{F}^-(M)$ we have

$$(A^{[f]})^e \le \big(\sup\{A^{[g]} \colon g \in \mathcal{F}^-(M)\}\big)^e = (L_M)^e,$$

thus $\mathcal{L}_{M^e} \le (L_M)^e$. Finally, Proposition 4.2 implies $(L_M)^e \le M^e$. □

5.2. Gini means

Let us recall two results characterizing the comparison in the family of Gini means.

Proposition 5.5. ([10]) *Let $p, q, r, s \in \mathbb{R}$. Then the following conditions are equivalent:*

- $\mathcal{G}_{p,q}(x) \le \mathcal{G}_{r,s}(x)$ *for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}^n$*;*
- $\min(p,q) \le \min(r,s)$*, and* $\max(p,q) \le \max(r,s)$*;*
- $(p,q,r,s) \in \Delta_\infty$*, where*

$$\Delta_\infty := \{(p,q,r,s) \in \mathbb{R}^4 \colon \min(p,q) \le \min(r,s) \text{ and } \max(p,q) \le \max(r,s)\}.$$

Proposition 5.6. ([43], Theorem 3) *Let $p, q, r, s \in \mathbb{R}$. Then the following conditions are equivalent:*

- *For all* $x, y > 0$, $\mathcal{G}_{p,q}(x,y) \le \mathcal{G}_{r,s}(x,y)$*;*
- $p + q \le r + s$, $m(p,q) \le m(r,s)$*, and* $\mu(p,q) \le \mu(r,s)$*, where*

$$m(p,q) := \begin{cases} \min(p,q) & \text{if } p,q \ge 0, \\ 0 & \text{if } pq < 0, \\ \max(p,q) & \text{if } p,q \le 0; \end{cases} \qquad \mu(p,q) := \begin{cases} \frac{|p|-|q|}{p-q} & \text{if } p \ne q, \\ \operatorname{sign}(p) & \text{if } p = q; \end{cases}$$

- $(p,q,r,s) \in \Delta_2$*, where*

$$\Delta_2 := \{(p,q,r,s) \in \mathbb{R}^4 \colon p+q \le r+s,\ m(p,q) \le m(r,s),\ \mu(p,q) \le \mu(r,s)\}.$$

By [28,29] we know that $\mathcal{G}_{p,q}$ is monotone if and only if

$$(p,q) \in \mathrm{Mon}_G := \{(p,q) \in \mathbb{R}^2 \colon pq \le 0\} = m^{-1}(0).$$

As a straightforward application of Proposition 4.2 we get.

Corollary 5.7. *Let $p, q, r, s \in \mathbb{R}$ with $(p,q), (r,s) \in \mathrm{Mon}_G$.*

Then $\mathcal{G}^e_{p,q}(x) \le \mathcal{G}^e_{r,s}(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^n_+$ if and only if $(p,q,r,s) \in \Delta_2$.

This corollary shows that the equality $\mathcal{G}_{p,q} = \mathcal{G}^e_{p,q}$ fails to be valid for a large subclass of parameters (p,q). For example if $(p,q,r,s) \in (\mathrm{Mon}_G^2 \cap \Delta_2) \setminus \Delta_\infty$. Then we have $\mathcal{G}^e_{p,q}(x) \le \mathcal{G}^e_{r,s}(x)$ for all $x \in \bigcup_{n=1}^\infty \mathbb{R}^n_+$ while there exists a vector $\bar{x} \in \bigcup_{n=1}^\infty \mathbb{R}^n_+$ such that $\mathcal{G}_{p,q}(\bar{x}) > \mathcal{G}_{r,s}(\bar{x})$. Consequently, at least one of the equalities $\mathcal{G}_{p,q} = \mathcal{G}^e_{p,q}$ or $\mathcal{G}_{r,s} = \mathcal{G}^e_{r,s}$ is not valid.

Furthermore, for all $p \in \mathbb{R}$ and $x, y \in \mathbb{R}_+$ we have $\mathcal{G}_{p,-p}(x,y) = \sqrt{xy} = \mathcal{P}_0(x,y)$ and therefore $G^e_{p,-p} = \mathcal{P}^e_0 = \mathcal{P}_0$. However, in general, the equality $\mathcal{G}_{p,-p}(x_1,\dots,x_n) = \mathcal{P}_0(x_1,\dots,x_n)$ is not valid for $n > 2$.

6. Conclusions and further research

6.1. Conclusions

Let us now review the most important outcomes of this paper.

(C1) β-invariant extension is the tool to construct a $(k+1)$-variable mean based on a k-variable one. Then we applied the iterative approach to generalize bivariate means to multivariable ones—operator $(\cdot)^e$.

(C2) Both of these operations are inner, in the sense that the only tools are either taking the value of the mean or the limit. In particular, there are no additional parameters in this process.

(C3) These operations are monotone and preserve the class of quasiarithmetic means. It was used to show lower and upper bounds for the value of M^e.

(C4) The way of calculating $M^{\sim k}$ at this stage requires recursive application of k infinite processess. Therefore, it has the order-type complexity ω^k.

6.2. Further research

Finally, since this is a new concept, there arise several possible ways of studying this problem.

(R1) At this stage, the explicit value of the iterative β-invariant extension is known only in the family of quasiaritmetic means. It is worth asking if we can obtain an explicit form for some other families of means.

(R2) It is a natural question what the image of the β-invariant extension is. More precisely, to characterize $(p+1)$-variable means $N\colon I^{p+1} \to I$ such that $N = \widetilde{M}$ for some p-variable mean $M\colon I^p \to I$.

(R3) Once we know (C4), we can search for a single iteration process which converges to $M^{\sim 2}$ (or even $M^{\sim k}$) for a given mean M.

(R4) There appears a natural question to characterize those means $M\colon I^2 \to I$ so that M^e is repetition invariant, that is the equality

$$M^e(\underbrace{x_1,\dots,x_1}_{m\text{-times}},\dots,\underbrace{x_n,\dots,x_n}_{m\text{-times}}) = M^e(x_1,\dots,x_n)$$

holds for all $n, m \in \mathbb{N}$ and $(x_1,\dots,x_n) \in I^n$.

(R5) Motivated by Theorem 1 and Proposition 4.1, we can look for other constraints preserved by the extensions (for example differentiability and higher order regularity assumptions).

(R6) One can ask about the stability properties of such an extension, that is, the correspondence between $\|M - N\|$ and $\|M^{\sim} - N^{\sim}\|$ (in a given space).

(R7) The family of continuous, strict means $M\colon I^p \to I$ (I and p are fixed) forms a convex subset of the space $L^\infty(I^n)$. In this sense, the β-invariant extension maps a convex subset of $L^\infty(I^n)$ to $L^\infty(I^{n+1})$. There appear natural questions, for example if this mapping is one-to-one or continuous (in a given sense, uniformly, pointwise, etc.).

(R8) Finally, it could happen that the β-invariant extension is not the most suitable way of extending means. Finding an extension preserving a wider class of means (not only quasiarithmetic ones) could be a great improvement not only in this narrow field, but it might also provide a better understanding of the theory of means.

Author contributions PP is the excusive author.

Data availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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Received: February 26, 2024
Revised: August 21, 2024
Accepted: August 23, 2024