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Set-valued dynamics related to convex-valued m-mappings

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Abstract. In this article, we study the set-valued dynamics related to some Euler-Lagrange type functional equations of convex-valued m-mappings. We deal with perturbations of these equations. In order to do this, we use the Banach contraction principle and the Hausdorff distance. Several outcomes on approximate solutions of a few important classic equations are discussed and some applications are given.

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Keywords. Set-valued dynamics, m-mapping, Banach contraction principle, Approximate solution.

1. Introduction

Let X be a normed space, Y a Banach space and $\varepsilon > 0$. Smajdor [28] and Gajda and Ger [10] observed that if a mapping $f : X \to Y$ satisfies

$$f(x+y) - f(x) - f(y) \in B(0,\varepsilon), \qquad x, y \in X,$$

where $B(0,\varepsilon)$ is the closed ball of radius ε centered at 0, then the set-valued mapping

$$F(x) = f(x) + B(0,\varepsilon), \qquad x \in X,$$

is subadditive (i.e., $F(x+y) \subseteq F(x) + F(y)$ for $x, y \in X$) and the mapping $g : X \to Y$, which satisfies

$$f(x) - g(x) \in B(0,\varepsilon), \qquad x \in X,$$

is an additive selection of F (i.e., $g(x) \in F(x)$ and $g(x+y) = g(x) + g(y)$ for $x, y \in X$).

There arises the natural question under what conditions a subadditive set-valed function admits an additive selection. An answer to this question can be

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found in Gajda and Ger [10]. In fact, they proved that if $(S, +)$ is a commutative semigroup with zero element, Y is a Banach space and $F : S \to \mathcal{P}(Y)$ is a subadditive mapping with bounded diameter and nonempty, convex and closed values, then F admits a unique additive selection.

It is interesting that, once we have obtained a result of Gajda-Ger type, we can prove the stability of the functional equations corresponding to the functional inclusions considered. For more information, see, e.g., [3–6,9,16–18,21–26].

Let f be a mapping between two vector spaces and $a \neq \pm 1$ be a fixed nonzero integer. For $m = 1, 2, 3, 4$, the functional equation

$$\begin{aligned} f(ax+y) + f(ax-y) &= a^{m-2}[f(x+y) + f(x-y)] \\ &\quad + 2(a^2-1)[a^{m-2}f(x) + \frac{(m-2)(1-(m-2)^2)}{6} f(y)], \end{aligned} \tag{1.1}$$

is equivalent to the additive, quadratic, cubic and quartic functional equations, respectively. For convenience, a solution of the functional equation (1.1) is said to be an m-mapping; see, e.g., [11,12,29].

Let $n > 1$ be an integer, $M = \{1, \ldots, n\}$ and $\mathcal{W} = \{I \subseteq M : 1 \in I\}$. Denote $M \setminus I$ by I^c for $I \in \mathcal{W}$. In this paper, we consider the following Euler-Lagrange type functional equations

$$\begin{aligned} \sum_{I \in \mathcal{W}} f\Big(\sum_{i \in I} a_i x_i - \sum_{i \in I^c} a_i x_i\Big) &= 2^{n-2} a_1^{m-2} \sum_{i=2}^{n} a_i^2 \left[f(x_1 + x_i) + f(x_1 - x_i)\right] \\ &\quad + 2^{n-1} a_1^{m-2} \Big(a_1^2 - \sum_{i=2}^{n} a_i^2\Big) f(x_1), \end{aligned} \tag{1.2}$$

$$\begin{aligned} \sum_{I \in \mathcal{W}} f\Big(\sum_{i \in I} a_i x_i - \sum_{i \in I^c} a_i x_i\Big) &= 2^{n-2} \sum_{1 \le i < j \le n} a_i^2 a_j^2 \left[f(x_i + x_j) + f(x_i - x_j)\right] \\ &\quad + 2^{n-1} \sum_{i=1}^{n} a_i^2 \Big(a_i^2 - \sum_{j=1, j \neq i}^{n} a_j^2\Big) f(x_i), \end{aligned} \tag{1.3}$$

where $m \in \{1, 2, 3\}$, f is a mapping between two vector spaces and $a_1, \ldots, a_n$ are fixed nonzero integers with $a_1 \neq \pm 1$ and $a_n = 1$. Using the Banach contraction principle and the Hausdorff distance, we deal with perturbations of set-valued versions of the functional equations (1.2) and (1.3). Some particular cases of our results are discussed. More importantly, the corresponding single-valued functional equations acting as special cases will be included in our results.

Note that if f satisfies (1.3), then f is quartic, see [14, Theorem 2.2] (the single-valued and set-valued versions of (1.3) with $n = 2$, $a_1 = 2$ and $a_2 = 1$ were studied in [20,24]). If f satisfies (1.2) with $m = 3$, then f is cubic, see [13, Theorem 2.2] (the single-valued and set-valued versions of (1.2) with $m = 3$,

$n = 2$, $a_1 = 2$ and $a_2 = 1$ were studied in [15,24,25]). It is easy to verify from [19, Lemma 3.2] that if f satisfies (1.2) with $m = 2$, then f is quadratic (the equation (1.2) with $m = 2$, $n = 2$, $a_1 = 2$ and $a_2 = 1$ was studied in [7]). Also, it is easy to verify from [12, Theorem 2.1] that if f satisfies (1.2) with $m = 1$, then f is additive (the equation (1.2) with $m = 1$, $n = 2$, $a_1 = 2$ and $a_2 = 1$ was studied in [1]).

2. Set-valued dynamics related to functional equations (1.2) and (1.3)

The Banach fixed point theorem [2] (also known as the Banach contraction principle) is an important tool in the theory of metric spaces because it guarantees the existence and uniqueness of fixed points of certain self mappings of metric spaces and provides a constructive method to find those fixed points.

Theorem 2.1. (The Banach contraction principle). *Let (X, d) be a complete metric space, and consider a mapping $\Lambda : X \to X$, which is strictly contractive, i.e., $d(\Lambda x, \Lambda y) \leq L d(x, y)$ for all $x, y \in X$ and some (Lipschitz constant) $L \in (0, 1)$. Then:*

(i) *The mapping Λ has one and only one, fixed point $e = \Lambda(e)$,*
(ii) *The fixed point e is globally attractive, i.e., $\lim_{n\to\infty} \Lambda^n x = e$ for any starting point $x \in X$,*
(iii) *The estimation inequality $d(x, e) \leq \frac{1}{1-L} d(x, \Lambda x)$ holds for all $x \in X$.*

Let Y be a Banach space. We denote the set of all nonempty subsets of Y by $\mathcal{P}_0(Y)$ and the set of all convex closed bounded members of $\mathcal{P}_0(Y)$ by $C_{clb}(Y)$. The number

$$\operatorname{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$$

is said to be the diameter of $A \in \mathcal{P}_0(Y)$.

For $A, B \in \mathcal{P}_0(Y)$ and $\lambda, \eta \in \mathbb{R}$, we write $A + B = \{x + y : \ x \in A, \ y \in B\}$ and $\lambda A = \{\lambda x : \ x \in A\}$; it is well known that $\lambda(A + B) = \lambda A + \lambda B$ and $(\lambda + \eta)A \subseteq \lambda A + \eta A$. Furthermore, when A is convex and $\lambda\eta \geq 0$, we obtain $(\lambda + \eta)A = \lambda A + \eta A$.

For convex closed elements $A_1, \ldots, A_n \in \mathcal{P}_0(Y)$, we define

$$\bigoplus_{i=1}^{n} A_i = A_1 \oplus \cdots \oplus A_n = \overline{A_1 + \cdots + A_n},$$

where $\overline{A_1 + \cdots + A_n}$ denotes the closure of $A_1 + \cdots + A_n$.

For any closed bounded elements $A, A' \in \mathcal{P}_0(Y)$, the Hausdorff distance h between A and A' is defined by

$$h(A, A') = \inf\{\lambda \geq 0 : \ A \subseteq A' + \lambda B_Y, \ A' \subseteq A + B_Y\},$$

where B_Y is the closed unit ball in Y. The following result is proved from the definition of the Hausdorff distance and can be found in [8].

Proposition 2.2. *Let $A, A', B, B', C \in C_{clb}(Y)$ and $\lambda > 0$. Then*

(i) $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$;
(ii) $h(\lambda A, \lambda B) = \lambda h(A, B)$;
(iii) $h(A \oplus C, B \oplus C) = h(A, B)$.

Let $(C_{clb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. In [8, Chapter II], it was proved that $(C_{clb}(Y), \oplus, h)$ is a complete metric semigroup. Rådström proved that $(C_{clb}(Y), \oplus, h)$ is isometrically embedded in a Banach space, see [27, Theorem 2].

From now on, unless otherwise specified, let $n > 1$ be an integer, V be a vector space, Y be a Banach space, $\mathbb{R}_+ = (0, \infty)$, $M = \{1, \ldots, n\}$, $\mathcal{W} = \{I \subseteq M : 1 \in I\}$, $I^c = M \backslash I$ for $I \in \mathcal{W}$, $a_1, \ldots, a_n$ be fixed positive integers with $a_1 \neq 1$ and $a_n = 1$, $m \in \{1, 2, 3\}$ be fixed and $\xi \in \{-1, 1\}$ be fixed.

Let us now consider the following set-valued functional equations related to the Euler-Lagrange type functional equations (1.2) and (1.3).

Let f from V to $C_{clb}(Y)$ be a set-valued mapping. For $m = 1$, 2 and 3, f is called convex-valued additive, quadratic and cubic, respectively, if it satisfies the following set-valued functional equation

$$\begin{aligned} &\bigoplus_{I \in \mathcal{W}} f\Big(\sum_{i \in I} a_i x_i - \sum_{i \in I^c} a_i x_i\Big) \oplus 2^{n-1} a_1^{m-2} \Big(\sum_{i=2}^{n} a_i^2\Big) f(x_1) \\ &\quad = 2^{n-2} a_1^{m-2} \bigoplus_{i=2}^{n} a_i^2 \left[f(x_1 + x_i) \oplus f(x_1 - x_i)\right] \oplus 2^{n-1} a_1^{m} f(x_1) \end{aligned} \tag{2.1}$$

for $x_1, \ldots, x_n \in V$. For convenience, a solution of (2.1) is said to be a convex-valued m-mapping. Also, f is called convex-valued quartic if it satisfies the following set-valued functional equation

$$\begin{aligned} &\bigoplus_{I \in \mathcal{W}} f\Big(\sum_{i \in I} a_i x_i - \sum_{i \in I^c} a_i x_i\Big) \oplus 2^{n-1} \bigoplus_{i=1}^{n} a_i^2 \Big(\sum_{j=1, j \neq i}^{n} a_j^2\Big) f(x_i) \\ &\quad = 2^{n-2} \bigoplus_{1 \leq i < j \leq n} a_i^2 a_j^2 \left[f(x_i + x_j) \oplus f(x_i - x_j)\right] \oplus 2^{n-1} \bigoplus_{i=1}^{n} a_i^4 f(x_i) \end{aligned} \tag{2.2}$$

for $x_1, \ldots, x_n \in V$.

Theorem 2.3. *Let $\omega_m : V^n \to \mathbb{R}_+$ be a function such that*

$$a_1^{\xi m} \omega_m(x_1, \ldots, x_n) \leq L \omega_m\left(a_1^{\xi} x_1, \ldots, a_1^{\xi} x_n\right), \quad (x_1, \ldots, x_n) \in V^n \tag{2.3}$$

for an $L \in (0,1)$. *If* $f : V \to (C_{clb}(Y), \oplus, h)$ *is a set-valued mapping satisfying*

$$\begin{aligned} h\Bigg(& \bigoplus_{I \in \mathcal{W}} f\Big(\sum_{i \in I} a_i x_i - \sum_{i \in I^c} a_i x_i\Big) \oplus 2^{n-1} a_1^{m-2} \Big(\sum_{i=2}^{n} a_i^2\Big) f(x_1), \\ & 2^{n-2} a_1^{m-2} \bigoplus_{i=2}^{n} a_i^2 \left[f(x_1 + x_i) \oplus f(x_1 - x_i)\right] \oplus 2^{n-1} a_1^m f(x_1) \Bigg) \\ & \leq \omega_m(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in V^n, \end{aligned} \tag{2.4}$$

then there exists a unique convex-valued m*-mapping* $c_m^* : V \to (C_{clb}(Y), \oplus, h)$ *such that*

$$h(f(x), c_m^*(x)) \leq \frac{L^{\frac{1+\xi}{2}}}{2^{n-1} a_1^m (1-L)} \omega_m(x, 0, \ldots, 0), \qquad x \in V. \tag{2.5}$$

Moreover, if V *is a normed space and* t, π *are positive real numbers such that* $\xi m < \xi t$ *and*

$$diam\,(f(x)) \leq \pi \|x\|^t, \qquad x \in V, \tag{2.6}$$

then c_m^* *is single-valued.*

Proof. From (2.4) with $(x_1, \ldots, x_n) = (x, 0, \ldots, 0)$, we obtain

$$\begin{aligned} h\Bigg(& \underbrace{f(a_1 x) \oplus \cdots \oplus f(a_1 x)}_{2^{n-1} \text{ times}} \oplus 2^{n-1} a_1^{m-2} \Big(\sum_{i=2}^{n} a_i^2\Big) f(x), \\ & 2^{n-2} a_1^{m-2} \bigoplus_{i=2}^{n} a_i^2 \left[f(x) \oplus f(x)\right] \oplus 2^{n-1} a_1^m f(x) \Bigg) \\ & \leq \omega_m(x, 0, \ldots, 0), \qquad x \in V. \end{aligned}$$

Since the range of f is convex, it follows from the last inequality, (2.3) and Proposition 2.2 that

$$h\left(a_1^{-m} f\left(a_1 x\right), f(x)\right) \leq \frac{1}{2^{n-1} a_1^m} \omega_m(x, 0, \ldots, 0), \qquad x \in V,$$

and

$$h\left(a_1^{m} f\left(a_1^{-1} x\right), f(x)\right) \leq \frac{L}{2^{n-1} a_1^m} \omega_m(x, 0, \ldots, 0), \qquad x \in V.$$

Hence

$$h\left(a_1^{\xi m} f\left(a_1^{-\xi} x\right), f(x)\right) \leq \frac{L^{\frac{1+\xi}{2}}}{2^{n-1} a_1^m} \omega_m(x, 0, \ldots, 0), \qquad x \in V, \tag{2.7}$$

where $\xi \in \{-1, 1\}$ is fixed.

Let us consider a complete generalized metric space (Υ, d), where

$$\Upsilon = \{g \mid g : V \to C_{clb}(Y)\}$$

and

$$d(g,g') = \sup_{x\in X} \frac{h\big(g(x), g'(x)\big)}{\omega_m(x,0,\ldots,0)} < \infty, \qquad g,g' \in \Upsilon.$$

Put also

$$\Gamma g(x) = a_1^{\xi m} g\left(a_1^{-\xi} x\right), \qquad x\in V,\ g\in\Upsilon.$$

We show that $\Gamma : \Upsilon \to \Upsilon$ is a strictly contractive operator with the Lipschitz constant L. To do this, fix $g, g' \in \Upsilon$, $x \in V$ and $C_{g,g'} \in [0,\infty)$ with $d(g,g') \le C_{g,g'}$. Then

$$\begin{aligned}\frac{h\big((\Gamma g)(x), (\Gamma g')(x)\big)}{\omega_m(x,0,\ldots,0)} &= \frac{a_1^{\xi m} h\left(g\left(a_1^{-\xi}x\right), g'\left(a_1^{-\xi}x\right)\right)}{\omega_m(x,0,\ldots,0)} \\ &\le \frac{Lh\left(g\left(a_1^{-\xi}x\right), g'\left(a_1^{-\xi}x\right)\right)}{\omega_m\left(a_1^{-\xi}x,0,\ldots,0\right)} \le LC_{g,g'},\end{aligned}$$

and consequently $d(\Gamma g, \Gamma g') \le L d(g,g')$ for $g,g' \in \Upsilon$, as claimed.

We deduce from (2.7) that

$$d(f,\Gamma f) \le \frac{L^{\frac{1+\xi}{2}}}{2^{n-1} a_1^m}.$$

We can now apply Theorem 2.1 to deduce that the sequence $\{\Gamma^j f\}_{j\mathbb{N}}$ is convergent in (Υ, d) and its limit

$$c_m^*(x) = \lim_{j\to\infty} \Gamma^j f(x), \qquad x\in V, \tag{2.8}$$

is a unique fixed point of Γ. Moreover,

$$d(f, c_m^*) \le \frac{1}{1-L} d(f,\Gamma f) \le \frac{L^{\frac{1+\xi}{2}}}{2^{n-1}a_1^m(1-L)},$$

which proves (2.5).

It follows from (2.3), (2.4) and Proposition 2.2 that

$$\begin{aligned}&h\Bigg(\bigoplus_{I\in\mathcal{W}} a_1^{\xi mn} f\Big(a_1^{-\xi n}\sum_{i\in I} a_i x_i - a_1^{-\xi n}\sum_{i\in I^c} a_i x_i\Big) \oplus 2^{n-1} a_1^{\xi mn+m-2}\Big(\sum_{i=2}^n a_i^2\Big) f\left(a_1^{-\xi n}x_1\right), \\ &\quad 2^{n-2}a_1^{\xi mn+m-2}\bigoplus_{i=2}^n a_i^2\left[f\left(a_1^{-\xi n}x_1 + a_1^{-\xi n}x_i\right) \oplus f\left(a_1^{-\xi n}x_1 - a_1^{-\xi n}x_i\right)\right] \\ &\quad \oplus 2^{n-1}a_1^{\xi mn+m} f\left(a_1^{-\xi n}x_1\right)\Bigg) \\ &\le a_1^{\xi mn}\omega_m\left(a_1^{-\xi n}x_1,\ldots,a_1^{-\xi n}x_n\right) \le L^n \omega_m(x_1,\ldots,x_n), \qquad (x_1,\ldots,x_n)\in V^n.\end{aligned}$$

Taking the limit as $n \to \infty$, we observe that

$$h\left(\bigoplus_{I\in\mathcal{W}} c_m^*\Big(\sum_{i\in I} a_i x_i - \sum_{i\in I^c} a_i x_i\Big) \oplus 2^{n-1} a_1^{m-2}\Big(\sum_{i=2}^{n} a_i^2\Big) c_m^*(x_1),\right.$$
$$\left. 2^{n-2} a_1^{m-2} \bigoplus_{i=2}^{n} a_i^2 \left[c_m^*(x_1+x_i) \oplus c_m^*(x_1-x_i)\right] \oplus 2^{n-1} a_1^m c_m^*(x_1)\right) = 0,$$

and thus c_m^* is a solution of (2.1).

Furthermore, by (2.6), we obtain

$$\operatorname{diam}\left(a_1^{\xi mn} f(a_1^{-\xi n} x)\right) \le a_1^{\xi mn} \pi \|a_1^{-\xi n} x\|^t = \left(a_1^{\xi m - \xi t}\right)^n \pi \|x\|^t, \qquad x \in V,$$

taking the limit as $n \to \infty$ and using (2.8), we observe that c_m^* is a singleton set. This completes the proof of this theorem. □

Remark 2.4. Theorem 2.3 can be applied to $\omega_m(x_1, \dots, x_n) := \zeta \sum_{i=1}^{n} \|x_i\|^p$ for $x_1, \dots, x_n \in V$, where V is a normed space and ζ, p are positive real numbers with $\xi m < \xi p$.

Theorem 2.5. *Let $\varpi : V^n \to \mathbb{R}_+$ be a function such that*

$$a_1^{4\xi} \varpi(x_1, \dots, x_n) \le L \varpi\left(a_1^{\xi} x_1, \dots, a_1^{\xi} x_n\right), \qquad (x_1, \dots, x_n) \in V^n \tag{2.9}$$

for an $L \in (0,1)$. If $f : V \to (C_{clb}(Y), \oplus, h)$ is a set-valued mapping satisfying $f(0) = \{0\}$ and

$$h\left(\bigoplus_{I\in\mathcal{W}} f\Big(\sum_{i\in I} a_i x_i - \sum_{i\in I^c} a_i x_i\Big) \oplus 2^{n-1} \bigoplus_{i=1}^{n} a_i^2 \Big(\sum_{j=1, j\neq i}^{n} a_j^2\Big) f(x_i),\right.$$
$$\left. 2^{n-2} \bigoplus_{1\le i<j\le n} a_i^2 a_j^2 \left[f(x_i+x_j) \oplus f(x_i-x_j)\right] \oplus 2^{n-1} \bigoplus_{i=1}^{n} a_i^4 f(x_i)\right)$$
$$\le \varpi(x_1, \dots, x_n), \qquad (x_1, \dots, x_n) \in V^n, \tag{2.10}$$

then there exists a unique convex-valued quartic mapping $c^* : V \to (C_{clb}(Y), \oplus, h)$ such that

$$h(f(x), c^*(x)) \le \frac{L^{\frac{1+\xi}{2}}}{2^{n-1} a_1^4 (1-L)} \varpi(x, 0, \dots, 0), \qquad x \in V.$$

Moreover, if V is a normed space and t, π are positive real numbers such that $4\xi < \xi t$ and $\operatorname{diam}(f(x)) \le \pi \|x\|^t$ for $x \in V$, then c^* is single-valued.

Proof. Setting $(x_1,\ldots,x_n)=(x,0,\ldots,0)$ in (2.10) and using (2.9), the convexity of the range of f and Proposition 2.2, we have

$$h\left(a_1^{4\xi}f\left(a_1^{-\xi}x\right),f(x)\right)\le\frac{L^{\frac{1+\xi}{2}}}{2^{n-1}a_1^4}\varpi(x,0,\ldots,0),\qquad x\in V. \tag{2.11}$$

Let us consider a complete generalized metric space (Υ,d), where

$$\Upsilon=\{g\mid g:V\to C_{clb}(Y),\ f(0)=\{0\}\}$$

and

$$d(g,g')=\sup_{x\in X}\frac{h\big(g(x),g'(x)\big)}{\varpi(x,0,\ldots,0)}<\infty,\qquad g,g'\in\Upsilon.$$

Put also

$$\Lambda g(x)=a_1^{4\xi}g\left(a_1^{-\xi}x\right),\qquad x\in V,\ g\in\Upsilon.$$

We deduce from (2.11) that

$$d(f,\Gamma f)\le\frac{L^{\frac{1+\xi}{2}}}{2^{n-1}a_1^4}.$$

The rest of the proof is similar to the proof of Theorem 2.3. □

Remark 2.6. Theorem 2.5 can be applied to $\varpi(x_1,\ldots,x_n):=\zeta\sum_{i=1}^n\|x_i\|^p$ for $x_1,\ldots,x_n\in V$, where V is a normed space and ζ,p are positive real numbers with $4\xi<\xi p$.

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