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Spherical and hyperbolic bicentric polygons

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Abstract. Relations of circumradius, inradius and the distance between the circumcenter and incenter of Euclidean bicentric polygons are generalized into spherical geometry and hyperbolic geometry. The asymptotic behavior of these generalized formulas with small circumradius are studied. Relations for hyperbolic hyper-ideal bicentric polygons are derived.

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1. Introduction

1.1. Euclidean bicentric polygons

Given a Euclidean triangle, its circumradius R, inradius r, and the distance d between its circumcenter and incenter satisfy the relation

$$d^2 = R^2 - 2rR.$$

This formula was discovered by William Chapple in 1746 [1] and by Leonhard Euler in 1765 [2,3].

A *bicentric polygon* is a polygon with all vertices on a circle, called the *circumcircle*, and all sides tangent to another circle, called the *incircle*.

Given a Euclidean bicentric quadrilateral, its circumradius R, inradius r, and the distance d between its circumcenter and incenter satisfy the relation

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2).$$

This formula was discovered by Nicolaus Fuss in 1797 [4].

Given a Euclidean bicentric convex pentagon, its circumradius R, inradius r, and the distance d between its circumcenter and incenter satisfy the relation [5,6]

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$$8d^2Rr^3 + 4R^2r^2(R^2 - d^2) - 2Rr(R^2 - d^2)^2 - (R^2 - d^2)^3 = 0.$$

Equivalent forms of this formula were discovered by Nicolaus Fuss in 1802 [7] and Jakob Steiner in 1827 [8].

For a self-intersecting bicentric pentagon, the relation ((3.15) in [9]) is

$$8d^2Rr^3 - 4R^2r^2(R^2 - d^2) - 2Rr(R^2 - d^2)^2 + (R^2 - d^2)^3 = 0.$$

More about the history of these formulas and their relation with Poncelet's closure theorem [10] can be found in [11–16].

1.2. Hyperbolic bicentric polygons

Alabdullatif [17] generalized Chapple's formula and Fuss' formula into hyperbolic geometry. Given a hyperbolic triangle, its circumradius R, inradius r, and the distance d between its circumcenter and incenter satisfy the relation

$$\tanh r = \frac{\tanh(R+d)(\cosh^2 R \sinh^2 r - \cosh^2 R + \cosh^2(r+d))}{\cosh^2(r+d) - \cosh^2 R \cosh^2 r}.$$

Given a hyperbolic bicentric quadrilateral, its circumradius R, inradius r, and the distance d between its circumcenter and incenter satisfy the relation

$$\tanh^4 r = \frac{(s^2(R-d) - s^2(r))^2(s^2(R+d) - s^2(r))^2}{c^4(r)((c^4(R) + s^4(d))(s^2(R-d) - s^2(r))(s^2(R+d) - s^2(r)) - s^4(r)c^4(R)s^4(d))}$$

where $s(x) = \sinh x$ and $c(x) = \cosh x$.

The two formulas are derived by using Poncelet's closure theorem in hyperbolic geometry which is established in [17]. Poncelet's closure theorem for the hyperbolic plane (and general hyperbolic space) was first established in [18].

For a hyperbolic bicentric n-gon, the relation between its circumradius R, inradius r, and the distance d between its circumcenter and incenter is also derived in [17] using hyperbolic trigonometry.

After checking that, when R approaches 0, Chapple's formula (or Fuss' formula) appears as a factor of the lowest order terms of its hyperbolic analogue, Alabdullatif [17] conjectured that this holds for any bicentric n-gon.

Conjecture 1. (Alabdullatif, 2016) Let C and D be two disjoint circles in the hyperbolic plane, with D inside C. Let R denote the radius of C, r denote the radius of D and d denote the distance between the two circles' centres. Assume that there is a bicentric embedded n-gon between them, then, we can write the hyperbolic general formula as $f_n(R, r, d) = 0$, by using the expressions

$$\cosh(x) \simeq 1 + \frac{x^2}{2}$$
$$\sinh(x) \simeq x$$

for R small, $x = R, r, d, R+d, ..., R+d+r$, such that

$$f_n(R,r,d) = \Sigma_{i=1}^{k} g_i(R,r,d)$$

where $g_i(R,r,d)$ is homogeneous of order i, then the lowest order non zero g_i has the Euclidean version as a factor.

1.3. Spherical and hyperbolic triangles

Cho and Naranjo [19] generalized Chapple's formula into spherical geometry and hyperbolic geometry and derived a unified formula for the three geometries.

$$\begin{aligned} d^2 &= (R-r)^2 - r^2, \text{ in Euclidean geometry,} \\ \sin^2 d &= \sin^2(R-r) - \sin^2 r \cos^2 R, \text{ in spherical geometry,} \\ \sinh^2 d &= \sinh^2(R-r) - \sinh^2 r \cosh^2 R, \text{ in hyperbolic geometry.} \end{aligned}$$

The three equations can be unified as

$$s(2d)^2 = s(2R-2r)^2 - s(2r)^2(1 - Ks(2R)^2)$$

where

$$s(x) = \begin{cases} \frac{x}{2} & \text{for Euclidean geometry} \\ \sin\frac{x}{2} & \text{for Spherical geometry} \\ \sinh\frac{x}{2} & \text{for hyperbolic geometry.} \end{cases}$$

as defined in [20].

And K is the curvature,

$$K = \begin{cases} 0 & \text{for Euclidean geometry} \\ 1 & \text{for Spherical geometry} \\ -1 & \text{for hyperbolic geometry.} \end{cases}$$

1.4. Main results

In this paper, given a spherical or hyperbolic bicentric polygon, a relation between its circumradius R, inradius r, and the distance d between its circumcenter and incenter is derived. This relation is derived by using a different method and expressed in a different form from [17].

Define

$$x_* = \begin{cases} x & \text{for Euclidean geometry} \\ \tan x & \text{for Spherical geometry} \\ \tanh x & \text{for hyperbolic geometry.} \end{cases}$$

Theorem 2. *In Euclidean, hyperbolic or spherical geometries, given two circles with radius R and r, let d be the distance between the centers of the two circles. Suppose $R > r + d$. There exists an n-gon inscribed in the circle with radius R and circumscribed about the circle with radius r if and only if d_*, r_* and R_* satisfy the polynomial equation $U_n^{(K)}(d_*, r_*, R_*) = 0$.*

Inspired by Alabdullatif's conjecture, we study the behavior of $U_n^{(K)}$ when R is small.

Theorem 3. *In hyperbolic or spherical geometry, when R approaches 0, using $x_* \simeq x$ for $x = d, r, R$, $U_n^{(\pm 1)}(d_*, r_*, R_*)$ is written in terms of d, r, R. Then its homogeneous part with the lowest degree equals $U_n^{(0)}(d, r, R)$.*

Let $l_1, l_2, ..., l_n$ be n mutually disjoint geodesics in the hyperbolic plane. For each pair $\{l_i, l_{i+1}\}$ (where $n + 1 \equiv 1$), there is a unique geodesic segment P_i orthogonal to both l_i and l_{i+1}, and the length of the segment realizes the distance between l_i and l_{i+1}. The region with finite area bounded by the geodesics $l_1, l_2, ..., l_n$ and $P_1, P_2, ..., P_n$ is a *hyperbolic hyperideal polygon*.

A hyperbolic hyperideal polygon is bicentric if there exists a hyperbolic circle tangent to $P_1, P_2, ..., P_n$ and a hyperbolic circle tangent to $l_1, l_2, ..., l_n$.

Theorem 4. *Given two hyperbolic circles with radius R and r respectively, there exists a bicentric hyperbolic hyperideal n-gon whose sides are tangent to the two circles if and only if $U_n^{(-1)}(d_*, r_*, R_*^{-1}) = 0$. And the polynomial $U_n^{(-1)}(d_*, r_*, R_*^{-1})$ is symmetric in r and R.*

1.5. Organization of the paper

In Sect. 2, the main tool, Cayley's conditions, is recalled. In Sect. 3, Cayley's conditions are used to derive the unified formulas. In Sect. 4, the asymptotic behavior of the formulas in hyperbolic and spherical geometries are studied when R is small. In Sect. 5, formulas for bicentric hyperbolic hyperideal polygons are derived.

2. Cayley's conditions

Arthur Cayley first found in 1853 [21,22] explicit conditions determining, for two given conics, the existence of an n-gon inscribed in one and circumscribed about the second conic. These conditions were republished in 1861 [23,24] in a more complete form. More about the history and modern proofs of Cayley's conditions can be found in [12–15,25].

Theorem 5. *[Cayley, 1861] Let $C[x,y,z]=0$ and $D[x,y,z]=0$ be the homogeneous quadratic equations in projective coordinates that define suitable conics C and D.*

Let $Q(t)$be the 3×3 symmetric matrix of the quadratic form $tC[x,y,z]+D[x,y,z]$ and let A_i be defined by

$$\sqrt{\det Q(t)} = A_0 + A_1 t + A_2 t^2 + \dots$$

Let $n \geq 3$. There exists an n-gon inscribed in C and circumscribed about D if and only if

$$\det\begin{pmatrix} A_2 & \dots & A_{m+1} \\ & \dots & \\ A_{m+1} & \dots & A_{2m} \end{pmatrix} = 0 \textit{ for } n = 2m+1;$$
$$\det\begin{pmatrix} A_3 & \dots & A_{m+1} \\ & \dots & \\ A_{m+1} & \dots & A_{2m-1} \end{pmatrix} = 0 \textit{ for } n = 2m.$$

2.1. Examples [5,6]

Let

$$x^2+y^2 = R^2,$$
$$x^2+(y-d)^2 = r^2$$

be two circles in the Euclidean plane.

The corresponding homogeneous polynomials are

$$x^2+y^2-R^2z^2=0,$$
$$x^2+y^2-2dyz+(d^2-r^2)z^2=0.$$

Then

$$Q(t) = t\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -R^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -d \\ 0 & -d & d^2-r^2 \end{pmatrix}$$
$$= \begin{pmatrix} t+1 & 0 & 0 \\ 0 & t+1 & -d \\ 0 & -d & -tR^2+d^2-r^2 \end{pmatrix}.$$

Now let $E_n, n \geq 0$, denote the coefficients in the case of Euclidean geometry, i.e.,

$$\begin{aligned} & E_0 + E_1 t + E_2 t^2 + E_3 t^3 + E_4 t^4 + \dots \\ =& \sqrt{\det Q(t)} \\ =& \sqrt{-(1+t)(r^2 + (r^2 + R^2 - d^2)t + R^2 t^2)} \\ =& ir\sqrt{(1+t)\left(1 + \frac{r^2 + R^2 - d^2}{r^2} t + \frac{R^2}{r^2} t^2\right)}. \end{aligned}$$

Then $E_n, n \geq 0$, can be determined. For example,

$$\begin{aligned} E_0 &= ir \\ E_1 &= ir \cdot \frac{2r^2 + R^2 - d^2}{2r^2} \\ E_2 &= ir \cdot \frac{4r^2R^2 - (R^2 - d^2)^2}{8r^4} \\ E_3 &= ir \cdot \frac{-2r^2(R^2 - d^2)(R^2 + d^2) + (R^2 - d^2)^3}{16r^6} \\ E_4 &= ir \cdot \frac{-16d^4r^4 + 8r^2(R^2 - d^2)^2(R^2 + 2d^2) - 5(R^2 - d^2)^4}{128r^8} \\ E_5 &= ir \cdot \frac{32d^4r^6 + 48r^4d^4(R^2 - d^2) - 10r^2(R^2 - d^2)^3(R^2 + 3d^2) + 7(R^2 - d^2)^5}{256r^{10}} \end{aligned}$$

In general, each

$$E_n(d, r, R) = ir \cdot \frac{T_n(d, r, R)}{t_n r^{2n}} \tag{1}$$

where t_n is a power of 2 and $T_n(d, r, R)$ is a homogeneous polynomial of degree $2n$ in terms of d, r and R.

Cayley's condition for the existence of a triangle inscribed in the first circle and circumscribed about the second circle is $E_2 = 0$ which is equivalent to Chapple's formula $R^2 - d^2 = 2rR$.

Cayley's condition for the existence of a bicentric quadrilateral inscribed in the first circle and circumscribed about the second circle is $E_3 = 0$ which is equivalent to Fuss' formula $(R^2 - d^2)^2 = 2r^2(R^2 + d^2)$.

Cayley's condition for the existence of a bicentric pentagon inscribed in the first circle and circumscribed about the second circle is $E_2E_4 = E_3^2$ which is equivalent to

$$\begin{aligned} &(8d^2Rr^3 + 4R^2r^2(R^2 - d^2) - 2Rr(R^2 - d^2)^2 - (R^2 - d^2)^3) \cdot \\ &(8d^2Rr^3 - 4R^2r^2(R^2 - d^2) - 2Rr(R^2 - d^2)^2 + (R^2 - d^2)^3) = 0. \end{aligned}$$

which contains Fuss and Steiner's formula for a convex bicentric pentagon and the formula for a self-intersecting bicentric pentagon.

3. Unified formulas

Cayley's type conditions corresponding to Poncelet's closure theorem for the hyperbolic plane (and general hyperbolic space) were derived in [18]. The following proof investigates some concrete examples of the general theory in [18].

Proof of Theorem 2. **Step 1.** In the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$, consider the first circle in $\mathbb{S}^2$ with center $(0, 0, 1)$ and radius R (measured by the spherical metric). It is the intersection of $\mathbb{S}^2$ with the cone in $\mathbb{R}^3$ defined by

$$x^2 + y^2 = z^2 \tan^2 R.$$

Consider the second circle in $\mathbb{S}^2$ with center $(0, \sin d, \cos d)$ and radius r. It is the intersection of $\mathbb{S}^2$ with the cone defined as follows.

Let (x, y, z) be a point on the cone. It also denotes the vector from $(0, 0, 0)$ to this point. The angle between the vector (x, y, z) and the vector $(0, \sin d, \cos d)$ is r. Then

$$\begin{aligned}
& (x, y, z) \cdot (0, \sin d, \cos d) = |(x, y, z)|\ |(0, \sin d, \cos d)| \cos r. \\
\iff \quad & y \sin d + z \cos d = \sqrt{x^2 + y^2 + z^2} \cos r \\
\iff \quad & (y \sin d + z \cos d)^2 = \frac{x^2 + y^2 + z^2}{1 + \tan^2 r} \\
\iff \quad & (1 + \tan^2 d)x^2 + (y - z \tan d)^2 = \tan^2 r(y \tan d + z)^2.
\end{aligned}$$

Step 2. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two vectors in $\mathbb{R}^3$. The Lorentzian inner product is defined as

$$(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = x_1x_2 + y_1y_2 - z_1z_2.$$

The Lorentzian norm of (x, y, z) is

$$||(x, y, z)|| = \sqrt{x^2 + y^2 - z^2}.$$

The hyperboloid model of the hyperbolic plane is

$$\mathbb{H}^2 = \{(x, y, z) \in R^3 : ||(x, y, z)||^2 = -1,\ z > 0\}.$$

Consider the first circle in $\mathbb{H}^2$ with center $(0, 0, 1)$ and radius R (measured by the hyperbolic metric). It is the intersection of $\mathbb{H}^2$ with the cone defined as follows [26].

$$\begin{aligned}
& (x, y, z) \circ (0, 0, 1) = ||(x, y, z)||\ ||(0, 0, 1)|| \cosh R \\
\iff \quad & -z = \sqrt{x^2 + y^2 - z^2}\sqrt{-1} \cosh R \\
\iff \quad & x^2 + y^2 = z^2 \tanh^2 R.
\end{aligned}$$

Consider the second circle in $\mathbb{H}^2$ with center $(0, \sinh d, \cosh d)$ and radius r. It is the intersection of $\mathbb{H}^2$ with the cone defined as follows.

$$\begin{aligned}
& (x,y,z)\circ(0,\sinh d,\cosh d) = ||(x,y,z)||\ ||(0,\sinh d,\cosh d)||\cosh r \\
\iff\quad & y\sinh d - z\cosh d = \sqrt{x^2+y^2-z^2}\sqrt{-1}\cosh r \\
\iff\quad & (y\sinh d - z\cosh d)^2 = \frac{z^2-x^2-y^2}{1-\tanh^2 r} \\
\iff\quad & (1-\tanh^2 d)x^2 + (y - z\tanh d)^2 = \tanh^2 r(z - y\tanh d)^2
\end{aligned}$$

Step 3. A unified formula is derived in terms of the function x_* and the curvature K.

In spherical, Euclidean or hyperbolic geometries, consider the two circles with radius R and r and the distance between their centers is d. The corresponding cones are

$$x^2+y^2 = z^2R_*^2$$
$$(1+Kd_*^2)x^2 + (y-zd_*)^2 = r_*^2(z+yKd_*)^2.$$

The second one is equivalent to

$$(1+Kd_*^2)x^2 + (1-K^2r_*^2d_*^2)y^2 - 2d_*(1+Kr_*^2)yz + (d_*^2-r_*^2)z^2 = 0.$$

Then by Cayley's conditions,

$$\begin{aligned}
Q(t) &= t\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -R_*^2\end{pmatrix} + \begin{pmatrix}1+Kd_*^2 & 0 & 0\\ 0 & 1-K^2r_*^2d_*^2 & -d_*(1+Kr_*^2)\\ 0 & -d_*(1+Kr_*^2) & d_*^2-r_*^2\end{pmatrix}\\
&= \begin{pmatrix}t+1+Kd_*^2 & 0 & 0\\ 0 & t+1-K^2r_*^2d_*^2 & -d_*(1+Kr_*^2)\\ 0 & -d_*(1+Kr_*^2) & -tR_*^2+d_*^2-r_*^2\end{pmatrix}.
\end{aligned}$$

Now

$$\begin{aligned}
&(A_0 + A_1t + A_2t^2 + A_3t^3 + A_4t^4 + \dots)^2\\
=&\det Q(t)\\
=&-(t+1+Kd_*^2)(R_*^2t^2 + (r_*^2+R_*^2-d_*^2-K^2d_*^2r_*^2R_*^2)t + r_*^2(1+Kd_*^2)^2)\\
=&-R_*^2t^3 + (d_*^2-r_*^2-2R_*^2-Kd_*^2R_*^2+K^2d_*^2r_*^2R_*^2)t^2\\
&+(1+Kd_*^2)(d_*^2-2r_*^2-R_*^2-Kd_*^2r_*^2+K^2d_*^2r_*^2R_*^2)t - r_*^2(1+Kd_*^2)^3
\end{aligned}$$

where

$$A_n = \begin{cases} E_n & \text{for Euclidean geometry}\\ S_n & \text{for Spherical geometry}\\ H_n & \text{for hyperbolic geometry.}\end{cases}$$

After finding $A_n, n \geq 0$, Cayley's conditions can produce unified formulas in terms of d_*, r_*, R_* and K. Corresponding to $K=0$, the relations are exactly the ones in Euclidean geometry. $\square$

For example, comparing the coefficients of $t^n, n = 0, 1, 2, 3,$ we have

$$\begin{aligned}
A_0^2 &= -r_*^2(1+Kd_*^2)^3 \\
2A_0A_1 &= (1+Kd_*^2)(d_*^2 - 2r_*^2 - R_*^2 - Kd_*^2r_*^2 + K^2d_*^2r_*^2R_*^2) \\
A_1^2 + 2A_0A_2 &= d_*^2 - r_*^2 - 2R_*^2 - Kd_*^2R_*^2 + K^2d_*^2r_*^2R_*^2 \\
2A_0A_3 + 2A_1A_2 &= -R_*^2.
\end{aligned}$$

Cayley's condition for the existence of a triangle inscribed in the first circle and circumscribed about the second circle is

$$\begin{aligned}
&A_2 = 0 \\
&\Longleftrightarrow (1+Kd^2)^2(d_*^2(1-Kr_*R_*)^2 + Kr_*^2) - R_*^2 - 2r_*R_*) \\
&\quad (d_*^2(1+Kr_*R_*)^2 + Kr_*^2) - R_*^2 + 2r_*R_*) = 0 \\
&\Longleftrightarrow d_*^2 = \frac{R_*^2 - 2r_*R_*}{(1+Kr_*R_*)^2 + Kr_*^2} \\
&\Longleftrightarrow \begin{cases} d^2 = (R-r)^2 - r^2 & \text{in Euclidean geometry} \\ \sin^2 d = \sin^2(R-r) - \sin^2\cos^2 R & \text{in spherical geometry} \\ \sinh^2 d = \sinh^2(R-r) - \sinh^2\cosh^2 R & \text{in hyperbolic geometry.} \end{cases}
\end{aligned}$$

Cayley's condition for the existence of a bicentric quadrilateral inscribed in the first circle and circumscribed about the second circle is

$$\begin{aligned}
&A_3 = 0 \\
&\quad \Longleftrightarrow \\
&d_*^4((1-K^2r_*^2R_*^2)^2 - K^2r_*^4) \\
&\quad - 2d_*^2((1+K^2r_*^2R_*^2)(R_*^2 + r_*^2) + Kr_*^2(r_*^2 + 4R_*^2)) + R_*^4 - 2r_*^2R_*^2 = 0.
\end{aligned}$$

In Euclidean geometry, it becomes

$$\begin{aligned}
&d^4 - 2d^2(R^2 + r^2) + R^4 - 2r^2R^2 = 0 \\
&\qquad \Longleftrightarrow (R^2 - d^2)^2 = 2r^2(R^2 + d^2).
\end{aligned}$$

4. Asymptotic behavior when R is small

Proof of Theorem 3. In the case of hyperbolic geometry, where $K = -1$,

$$
\begin{aligned}
&H_0 + H_1 t + H_2 t^2 + H_3 t^3 + H_4 t^4 + \dots \\
&\quad = \sqrt{\det Q(t)} \\
&\quad = i r_*(1+Kd_*^2)^{\frac{3}{2}} \sqrt{1 + \frac{1}{1+Kd_*^2} t} \\
&\qquad \sqrt{1 + \frac{r_*^2 + R_*^2 - d_*^2 - K^2 d_*^2 r_*^2 R_*^2}{r_*^2(1+Kd_*^2)^2} t + \frac{R_*^2}{r_*^2(1+Kd_*^2)^2} t^2}.
\end{aligned}
$$

Comparing with the expansion in the case of Euclidean geometry, we have

$$
\begin{aligned}
H_0 &= i r_*(1+Kd_*^2)^{\frac{3}{2}} \\
H_1 &= i r_*(1+Kd_*^2)^{\frac{3}{2}} \cdot \frac{T_1(d_*, r_*, R_*) + \lambda_1}{r_*^2(1+Kd_*^2)^3} \\
H_2 &= i r_*(1+Kd_*^2)^{\frac{3}{2}} \cdot \frac{T_2(d_*, r_*, R_*) + \lambda_2}{8 r_*^4(1+Kd_*^2)^6} \\
H_3 &= i r_*(1+Kd_*^2)^{\frac{3}{2}} \cdot \frac{T_3(d_*, r_*, R_*) + \lambda_3}{16 r_*^6(1+Kd_*^2)^9} \\
&\dots \\
H_n &= i r_*(1+Kd_*^2)^{\frac{3}{2}} \cdot \frac{T_n(d_*, r_*, R_*) + \lambda_n}{t_n r_*^{2n}(1+Kd_*^2)^{3n}}
\end{aligned}
$$

where T_n is the homogeneous polynomial in the formula (1) of E_n the coefficients in Euclidean geometry and λ_n is a polynomial in terms of d_*, r_* and R_* with the lowest degree $2n+2, n \geq 1$.

When R is small, then r, d are small, therefore d_*, r_*, R_* are small and approximately equal to d, r, R respectively. By ignoring the term with degree greater than 1, we have

$$
\begin{aligned}
H_n(d_*, r_*, R_*) &= i r_*(1+Kd_*^2)^{\frac{3}{2}} \cdot \left(\frac{T_n(d_*, r_*, R_*)}{t_n r_*^{2n}} + \frac{\lambda_n}{t_n r_*^{2n}} \right) \cdot (1+Kd_*^2)^{-3n} \\
&\simeq i r_* \cdot \frac{T_n(d_*, r_*, R_*)}{t_n r_*^{2n}} \\
&= E_n(d_*, r_*, R_*) \\
&\simeq E_n(d, r, R).
\end{aligned}
$$

Therefore the lowest term in a relation of H_i's determined by Cayley's conditions is a relation of E_i's determined by the same conditions.

The same arguments work for spherical bicentric polygons. □

5. Hyperideal

Proof of Theorem 4. Let (x, y, z) be a space-like vector in $\mathbb{R}^3$, i.e., satisfying $||(x, y, z)|| > 0$. Consider the set of all vectors in $\mathbb{R}^3$ Lorentz orthogonal to (x, y, z):

$$\{(a, b, c) \in \mathbb{R}^3 : (a, b, c) \circ (x, y, z) = 0\}$$

which is a plane passing through $(0, 0, 0)$. Its intersection with the hyperboloid model $\mathbb{H}^2$ is a geodesic P in $\mathbb{H}^2$.

A cone is obtained by rotating the ray in the direction (x, y, z) about the z-axis. The equation of the cone is

$$\begin{aligned} & (x, y, z) \circ (0, 0, 1) = ||(x, y, z)|| \ \ | \, ||(0, 0, 1)|| \, | \ \ \sinh R \\ \Longleftrightarrow \quad & -z = \sqrt{x^2 + y^2 - z^2} \ |\sqrt{-1}| \ \sinh R \\ \Longleftrightarrow \quad & x^2 + y^2 = \frac{z^2}{\tanh^2 R} \end{aligned}$$

where R is the distance form the point $(0, 0, 1)$ to the geodesic P (Theorem 3.2.12 in [26]).

As in the case of hyperbolic polygons, the circle in $\mathbb{H}^2$ with center $(0, \sinh d, \cosh d)$ and radius r is the intersection of $\mathbb{H}^2$ with the cone defined by

$$(1 - \tanh^2 d)x^2 + (y - z \tanh d)^2 = \tanh^2 r(z - y \tanh d)^2.$$

The intersections of the two cones with the plane defined by $z = 1$ are a circle and an ellipse. For a fixed integer $n \geq 3$, Cayley's conditions are satisfied if and only if there exists a Euclidean n-gon inscribed in the circle and circumscribed about the ellipse. The i-th side of the Euclidean polygon determines a plane L_i passing through $(0, 0, 0)$ and this side. The intersection of L_i with $\mathbb{H}^2$ is a geodesic l_i tangent to the hyperbolic circle with center $(0, \sinh d, \cosh d)$ and radius r.

Let (x_i, y_i, z_i) be the vector from $(0, 0, 0)$ to the vertex of the Euclidean polygon incident with the i-th side and $(i+1)$-th side (where $n+1 \equiv 1$). Then (x_i, y_i, z_i) is a space-like vector and it determines a geodesic P_i in $\mathbb{H}^2$ whose distance to the point $(0, 0, 1)$ is R, and is therefore tangent to the hyperbolic circle with center $(0, 0, 1)$ and radius R. And P_i is orthogonal to the two geodesics l_i and l_{i+1}. Then $\{l_1, \ldots, l_n\}$ together with $\{P_1, \cdots, P_n\}$ bound a hyperbolic hyperideal polygon.

Since each P_i is tangent to a hyperbolic circle and each l_i is tangent to another hyperbolic circle, this hyperideal polygon is bicentric.

Comparing the two cones in this case with the two cones in the case of hyperbolic polygons, we can see that once a relation of d, R and r of a hyperbolic bicentric polygon is obtained, a relation of d, R and r of a hyperbolic hyperideal bicentric polygon can be derived by replacing $\tanh R$, $\sinh R$ and $\cosh R$ by $\tanh^{-1} R$, $i \cosh R$ and $i \sinh R$ respectively.

And the formula for a hyperbolic hyperideal polygon is symmetric in R and r, because the difference between a circumcircle and an incircle does not exist in this case. □

For example, from the relation for a hyperbolic triangle, we get the relation for a hyperbolic hyperideal triangle:

$$\begin{aligned} & d_*^2 = \frac{1 - 2r_* R_*}{(R_* - r_*)^2 - r_*^2 R_*^2}, \quad \text{where } x_* = \tanh x \\ \iff \quad & \sinh^2 d = -\cosh^2(R - r) + \sinh^2 r \sinh^2 R. \end{aligned}$$

The relation for a hyperbolic hyperideal bicentric quadrilateral is

$$\begin{aligned} & d_*^4((R_*^2 - r_*^2)^2 - r_*^4 R_*^4) \\ & \quad - 2d_*^2((R_*^2 + r_*^2)(1 + r_*^2 R_*^2) - r_*^2 R_*^2(r_*^2 R_*^2 + 4)) + 1 - 2r_*^2 R_*^2 = 0 \end{aligned}$$

where $x_* = \tanh x$.

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Declarations
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