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On the variable-order fractional derivatives with respect to another function

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Abstract. In this paper, we present various concepts concerning generalized fractional calculus, wherein the fractional order of operators is not constant, and the integral kernel depends on a function. We observe that in the case of variable order, the concepts are distinct, and we present relations between them. Formulas for approximating fractional derivatives are provided, involving only integer-order derivatives. Finally, we conclude the work with some simulations to exemplify the method.

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1. Introduction

Fractional calculus arises as a dynamic and interdisciplinary field, expanding the boundaries of classical calculus by incorporating concepts of fractional-order derivatives and integrals. Its origins trace back to the 17th century, with the pioneering works of Leibniz and L'Hôpital, but it has gained prominence in recent decades due to its applications in a variety of fields, including physics, engineering, biology, economics, and computer science. With this, it began to attract the attention of a large community of researchers, not only in the field of mathematics, as they realized that fractional derivatives can naturally incorporate into their models not only the imperfections that always occur in nature but also the noise observed in the data, among other factors [4,7,9,10].

Unlike ordinary derivatives, which are local concepts limited to a specific instant, fractional derivatives are typically defined through integrals, encompassing a range from the starting point of the process to the current time. This characteristic endows fractional operators with a memory element, capturing past information. Conversely, the derivative's order remains constant

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throughout the process. Hence, it becomes plausible to conceive that the order of the fractional derivative evolves over time, much like a function defined within the temporal scope of the dynamic process. This perspective highlights the dynamic nature of fractional calculus, enabling a nuanced understanding of complex systems' behavior. Since Samko and Ross's pioneering works on variable order fractional calculus [11,12], numerous studies have continued to emerge in this field. For example, in [5], new variable-order operators are obtained by generalizing constant-order operators within the Laplace transform domain. In [15], a Leibniz-type rule and a generalized chain rule of the composite function are proven. A survey of recent relevant literature and findings regarding primary definitions, models, numerical methods, and their applications is presented in [13]. Additionally, [8] provides a comprehensive overview of the advancements achieved in the development of variable-order fractional calculus, particularly applied to simulating complex physical systems. Concerning fixed-order fractional operators, it is widely acknowledged that multiple definitions have arisen over the decades. To mitigate this abundance of definitions, one approach is to explore derivatives and integrals with respect to another arbitrary function [1,6]. Through specific instances of this function, we can rediscover some of the classical definitions. Hence, the natural generalization is to combine the two aforementioned concepts: variable-order fractional operators with respect to another function. This work has already initiated in [2,3], and our objective here is to further its advancement.

The paper is structured as follows. In Sect. 2, we introduce new concepts regarding fractional operators of variable order and with dependence on an arbitrary function g. In the following Sect. 3, we establish relationships among the different fractional derivatives and demonstrate their distinct nature in the case of variable order. Furthermore, we will prove that they are all bounded operators. Finally, in Sect. 4, we will present expansion formulas involving only integer-order derivatives for the various fractional derivatives.

2. Preliminaries

We will begin our work by presenting some concepts of fractional calculus, presenting definitions of fractional derivatives and integrals of variable order. As we shall see, different definitions can be given when the order is variable with time.

Throughout this work, γ is a differentiable function defined on $[a, b]$ and taking values in $(0, 1)$. We will also consider two real differentiable functions u and g, with domain $[a, b]$, such that $g'(t) > 0$ for all t.

The left fractional integral of the function u, of order γ and with respect to the kernel g, is defined as follows (see [2,12])

$$\mathbb{I}_{a+}^{\gamma(t)}u(t) = \frac{1}{\Gamma(\gamma(t))}\int_a^t g'(s)(g(t)-g(s))^{\gamma(t)-1}u(s)\,ds,$$

whereas the right fractional integral is defined by

$$\mathbb{I}_{b-}^{\gamma(t)}u(t) = \frac{1}{\Gamma(\gamma(t))}\int_t^b g'(s)(g(s)-g(t))^{\gamma(t)-1}u(s)\,ds.$$

With respect to Riemann–Liouville fractional derivatives, we present two distinct ones, which we refer to as type 1 and type 2. The left Riemann–Liouville fractional derivative type 1 of the function u (order γ, kernel g), is defined as

$${}_1\mathbb{D}_{a+}^{\gamma(t)}u(t) = \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}u(s)\,ds,$$

and the right one is defined as

$${}_1\mathbb{D}_{b-}^{\gamma(t)}u(t) = \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\int_t^b g'(s)(g(s)-g(t))^{-\gamma(t)}u(s)\,ds.$$

For the Riemann–Liouville fractional derivative type 2, the left and right operators are defined as

$${}_2\mathbb{D}_{a+}^{\gamma(t)}u(t) = \left(\frac{1}{g'(t)}\frac{d}{dt}\right)\left[\frac{1}{\Gamma(1-\gamma(t))}\int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}u(s)\,ds\right]$$

and

$${}_2\mathbb{D}_{b-}^{\gamma(t)}u(t) = \left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\left[\frac{1}{\Gamma(1-\gamma(t))}\int_t^b g'(s)(g(s)-g(t))^{-\gamma(t)}u(s)\,ds\right],$$

respectively.

With respect to Caputo fractional derivatives, we present three types. The left Caputo fractional derivative type 1 of the function u (order γ, kernel g), is defined as

$$\begin{aligned}{}^C_1\mathbb{D}_{a+}^{\gamma(t)}u(t) = {}_1\mathbb{D}_{a+}^{\gamma(t)}(u(t)-u(a)) &= \frac{1}{\Gamma(1-\gamma(t))}\\ &\times\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}(u(s)-u(a))\,ds,\end{aligned}$$

and the right one is defined as

$$\begin{aligned}{}^C_1\mathbb{D}_{b-}^{\gamma(t)}u(t) = {}_1\mathbb{D}_{b-}^{\gamma(t)}(u(t)-u(b)) &= \frac{1}{\Gamma(1-\gamma(t))}\\ &\times\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\int_t^b g'(s)(g(s)-g(t))^{-\gamma(t)}(u(s)-u(b))\,ds.\end{aligned}$$

For the Caputo fractional derivative type 2, the left and right operators are defined as

$$
\begin{aligned}
{}^{C}_{2}\mathbb{D}^{\gamma(t)}_{a+}u(t) &= {}_{2}\mathbb{D}^{\gamma(t)}_{a+}(u(t)-u(a))\\
&= \left(\frac{1}{g'(t)}\frac{d}{dt}\right)\left[\frac{1}{\Gamma(1-\gamma(t))}\int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}(u(s)-u(a))\,ds\right]
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{C}_{2}\mathbb{D}^{\gamma(t)}_{b-}u(t) &= {}_{2}\mathbb{D}^{\gamma(t)}_{b-}(u(t)-u(b))\\
&= \left(\frac{-1}{g'(t)}\frac{d}{dt}\right)\left[\frac{1}{\Gamma(1-\gamma(t))}\int_t^b g'(s)(g(s)-g(t))^{-\gamma(t)}(u(s)-u(b))\,ds\right],
\end{aligned}
$$

respectively. Finally, the left and right operators type 3 are defined as

$$
{}^{C}_{3}\mathbb{D}^{\gamma(t)}_{a+}u(t) = \frac{1}{\Gamma(1-\gamma(t))}\int_a^t (g(t)-g(s))^{-\gamma(t)}u'(s)\,ds
$$

and

$$
{}^{C}_{3}\mathbb{D}^{\gamma(t)}_{b-}u(t) = \frac{-1}{\Gamma(1-\gamma(t))}\int_t^b (g(s)-g(t))^{-\gamma(t)}u'(s)\,ds,
$$

respectively.

We note that the Caputo fractional derivatives of type 3 were previously introduced in the study [3].

3. Relations between the different fractional derivatives

Next, we present certain connections between these operators and some results are proven, generalizing the results presented in [14]. As we will observe, with additional assumptions, it can be demonstrated that they are equivalent. In all cases, we will only establish the proof for the first relation (for the left fractional operators), as for the right fractional operators is comparable and will require appropriate modifications.

Theorem 1. *The subsequent two relations are valid:*

$$
\begin{aligned}
{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t) = {}^{C}_{3}\mathbb{D}^{\gamma(t)}_{a+}u(t) &+ \frac{\gamma'(t)}{g'(t)\Gamma(2-\gamma(t))}\int_a^t (g(t)-g(s))^{1-\gamma(t)}u'(s)\\
&\times\left[\frac{1}{1-\gamma(t)}-\ln(g(t)-g(s))\right]ds
\end{aligned}
$$

and

$$
{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{b-}u(t) = {}^{C}_{3}\mathbb{D}^{\gamma(t)}_{b-}u(t) + \frac{\gamma'(t)}{g'(t)\Gamma(2-\gamma(t))}\int_t^b (g(s)-g(t))^{1-\gamma(t)}u'(s)
$$

$$\times \left[\frac{1}{1-\gamma(t)} - \ln(g(s)-g(t)) \right] ds.$$

Proof. Starting with the definition and applying the integration by parts formula, we derive the following:

$${}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t) = \frac{1}{\Gamma(1-\gamma(t))} \left(\frac{1}{g'(t)} \frac{d}{dt} \right) \int_a^t \frac{(g(t)-g(s))^{1-\gamma(t)}}{1-\gamma(t)} u'(s)\, ds.$$

Upon differentiation of the integral with respect to the variable t, we obtain:

$$\begin{aligned}
{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t) &= \frac{1}{\Gamma(1-\gamma(t))g'(t)} \left[\frac{\gamma'(t)}{(1-\gamma(t))^2} \int_a^t (g(t)-g(s))^{1-\gamma(t)} u'(s)\, ds + \frac{1}{1-\gamma(t)} \right. \\
&\quad \left. \times \int_a^t (g(t)-g(s))^{1-\gamma(t)} \left[-\gamma'(t) \ln(g(t)-g(s)) + (1-\gamma(t)) \frac{g'(t)}{g(t)-g(s)} \right] u'(s)\, ds \right] \\
&= {}^{C}_{3}\mathbb{D}^{\gamma(t)}_{a+}u(t) + \frac{\gamma'(t)}{g'(t)\Gamma(2-\gamma(t))} \int_a^t (g(t)-g(s))^{1-\gamma(t)} u'(s) \\
&\quad \times \left[\frac{1}{1-\gamma(t)} - \ln(g(t)-g(s)) \right] ds.
\end{aligned}$$

□

Theorem 2. *The following two relations hold true:*

$${}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t) = {}^{C}_{2}\mathbb{D}^{\gamma(t)}_{a+}u(t) - \frac{\gamma'(t)\Gamma'(1-\gamma(t))}{g'(t)\Gamma^2(1-\gamma(t))} \int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}(u(s)-u(a))\, ds$$

and

$${}^{C}_{1}\mathbb{D}^{\gamma(t)}_{b-}u(t) = {}^{C}_{2}\mathbb{D}^{\gamma(t)}_{b-}u(t) + \frac{\gamma'(t)\Gamma'(1-\gamma(t))}{g'(t)\Gamma^2(1-\gamma(t))} \int_t^b g'(s)(g(s)-g(t))^{-\gamma(t)}(u(s)-u(b))\, ds.$$

Proof. It is a direct consequence of applying the chain rule. □

Based on the two preceding theorems, the conclusion can be drawn that if γ is a constant function, then

$${}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t) = {}^{C}_{2}\mathbb{D}^{\gamma(t)}_{a+}u(t) = {}^{C}_{3}\mathbb{D}^{\gamma(t)}_{a+}u(t) \quad \text{and} \quad {}^{C}_{1}\mathbb{D}^{\gamma(t)}_{b-}u(t) = {}^{C}_{2}\mathbb{D}^{\gamma(t)}_{b-}u(t) = {}^{C}_{3}\mathbb{D}^{\gamma(t)}_{b-}u(t).$$

However, in general, they are distinct. As we shall demonstrate next, by computing the fractional derivative of a power function, we will illustrate the differences between these fractional operators. Firstly, we need to recall two essential functions. The first is the Beta function, defined as

$$\mathrm{B}(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\, ds, \quad \text{for } x,y>0.$$

A fundamental property of the Beta function is its close relationship with the Gamma function:

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The second function is the Digamma function, defined as the logarithmic derivative of the Gamma function:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Theorem 3. *Assuming $\beta > 0$, consider the two functions $u(t) = (g(t) - g(a))^\beta$ and $v(t) = (g(b) - g(t))^\beta$. Then,*

$$\begin{aligned}{}^C_1\mathbb{D}^{\gamma(t)}_{a+}u(t) = & \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+1)}(g(t)-g(a))^{\beta-\gamma(t)} \\ & + \frac{\gamma'(t)\Gamma(\beta+1)}{g'(t)\Gamma(\beta-\gamma(t)+2)}(g(t)-g(a))^{\beta-\gamma(t)+1} \\ & \times \Big[-\ln(g(t)-g(a)) + \psi(\beta-\gamma(t)+2) - \psi(1-\gamma(t))\Big]\end{aligned}$$

and

$$\begin{aligned}{}^C_1\mathbb{D}^{\gamma(t)}_{b-}u(t) = & \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+1)}(g(b)-g(t))^{\beta-\gamma(t)} \\ & - \frac{\gamma'(t)\Gamma(\beta+1)}{g'(t)\Gamma(\beta-\gamma(t)+2)}(g(b)-g(t))^{\beta-\gamma(t)+1} \\ & \times \Big[-\ln(g(b)-g(t)) + \psi(\beta-\gamma(t)+2) - \psi(1-\gamma(t))\Big].\end{aligned}$$

Proof. According to the definition,

$$\begin{aligned}{}^C_1\mathbb{D}^{\gamma(t)}_{a+}u(t) &= \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}(g(s)-g(a))^\beta\,ds \\ &= \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\Bigg[(g(t)-g(a))^{-\gamma(t)} \\ &\quad \times \int_a^t g'(s)\left(1-\frac{g(s)-g(a)}{g(t)-g(a)}\right)^{-\gamma(t)}(g(s)-g(a))^\beta\,ds\Bigg].\end{aligned}$$

By substituting $\chi = \frac{g(s)-g(a)}{g(t)-g(a)}$ in the integral through a change of variable, we arrive at the following:

$$\begin{aligned}{}^C_1\mathbb{D}^{\gamma(t)}_{a+}u(t) &= \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\left[(g(t)-g(a))^{\beta-\gamma(t)+1}\int_0^1(1-\chi)^{-\gamma(t)}\chi^\beta\,d\chi\right] \\ &= \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\left[(g(t)-g(a))^{\beta-\gamma(t)+1}\mathrm{B}(1-\gamma(t),\beta+1)\right] \\ &= \frac{1}{\Gamma(1-\gamma(t))}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)\left[(g(t)-g(a))^{\beta-\gamma(t)+1}\frac{\Gamma(1-\gamma(t))\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+2)}\right].\end{aligned}$$

Differentiating the expression within the brackets with respect to the variable t and subsequently simplifying, we arrive at the desired expression. □

Theorem 4. *Given $\beta > 0$, consider the two functions $u(t) = (g(t) - g(a))^\beta$ and $v(t) = (g(b) - g(t))^\beta$. Then,*

$$\begin{aligned} {}_{2}^{C}\mathbb{D}_{a+}^{\gamma(t)} u(t) = & \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+1)}(g(t)-g(a))^{\beta-\gamma(t)} \\ & + \frac{\gamma'(t)\Gamma(\beta+1)}{g'(t)\Gamma(\beta-\gamma(t)+2)}(g(t)-g(a))^{\beta-\gamma(t)+1} \\ & \times \Big[-\ln(g(t)-g(a)) + \psi(\beta-\gamma(t)+2) \Big] \end{aligned}$$

and

$$\begin{aligned} {}_{2}^{C}\mathbb{D}_{b-}^{\gamma(t)} u(t) = & \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+1)}(g(b)-g(t))^{\beta-\gamma(t)} \\ & - \frac{\gamma'(t)\Gamma(\beta+1)}{g'(t)\Gamma(\beta-\gamma(t)+2)}(g(b)-g(t))^{\beta-\gamma(t)+1} \\ & \times \Big[-\ln(g(b)-g(t)) + \psi(\beta-\gamma(t)+2) \Big]. \end{aligned}$$

Proof. The proof follows a similar structure to the one presented in Theorem 3, with the necessary adjustments made. □

Lastly, we provide the formula for the Caputo fractional derivative type 3.

Theorem 5. [3] *Given $\beta > 0$, consider the two functions $u(t) = (g(t) - g(a))^\beta$ and $v(t) = (g(b) - g(t))^\beta$. Then,*

$$ {}_{3}^{C}\mathbb{D}_{a+}^{\gamma(t)} u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+1)}(g(t)-g(a))^{\beta-\gamma(t)} $$

and

$$ {}_{3}^{C}\mathbb{D}_{b-}^{\gamma(t)} u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(t)+1)}(g(b)-g(t))^{\beta-\gamma(t)}. $$

Assuming higher smoothness conditions in the function space, we are able to eliminate the singularity present in the integrand function in ${}_{3}^{C}\mathbb{D}_{a+}^{\gamma(t)} u$ and ${}_{3}^{C}\mathbb{D}_{b-}^{\gamma(t)} u$, as exemplified in the following result. The proof is omitted, as it is an immediate consequence of the integration by parts formula.

Theorem 6. *If $u \in C^2[a, b]$, then*

$$\begin{aligned} {}_{3}^{C}\mathbb{D}_{a+}^{\gamma(t)} u(t) = & \frac{(g(t)-g(a))^{1-\gamma(t)}}{\Gamma(2-\gamma(t))} \frac{u'(a)}{g'(a)} \\ & + \frac{1}{\Gamma(2-\gamma(t))} \int_a^t (g(t)-g(s))^{1-\gamma(t)} \times \frac{d}{ds}\left(\frac{u'(s)}{g'(s)}\right) ds \end{aligned}$$

and

$$
{}^C_3\mathbb{D}^{\gamma(t)}_{b-}u(t) = -\frac{(g(b)-g(t))^{1-\gamma(t)}}{\Gamma(2-\gamma(t))}\frac{u'(b)}{g'(b)}
+ \frac{1}{\Gamma(2-\gamma(t))}\int_t^b (g(s)-g(t))^{1-\gamma(t)} \times \frac{d}{ds}\left(\frac{u'(s)}{g'(s)}\right) ds.
$$

Following this, we will establish an upper bound for the norm of the fractional operators, consequently concluding that the fractional derivative is zero at the extremes of their domains. To start, given $u \in C^1[a,b]$, define

$$
L_u = \max_{t\in[a,b]} \left|\frac{u'(t)}{g'(t)}\right|.
$$

Theorem 7. *The following two formulas hold:*

$$
\left|{}^C_3\mathbb{D}^{\gamma(t)}_{a+}u(t)\right| \leq \frac{L_u}{\Gamma(2-\gamma(t))}(g(t)-g(a))^{1-\gamma(t)}
$$

and

$$
\left|{}^C_3\mathbb{D}^{\gamma(t)}_{b-}u(t)\right| \leq \frac{L_u}{\Gamma(2-\gamma(t))}(g(b)-g(t))^{1-\gamma(t)},
$$

for all $t \in [a,b]$.

Proof. It directly follows from the definition of the fractional derivative. □

Thus, we conclude that ${}^C_3\mathbb{D}^{\gamma(t)}_{a+}u(a) = 0 = {}^C_3\mathbb{D}^{\gamma(t)}_{b-}u(b)$.

Theorem 8. *The following two formulas hold:*

$$
\left|{}^C_1\mathbb{D}^{\gamma(t)}_{a+}u(t)\right| \leq \frac{L_u}{\Gamma(2-\gamma(t))}M_1(t)(g(t)-g(a))^{1-\gamma(t)}
$$

and

$$
\left|{}^C_1\mathbb{D}^{\gamma(t)}_{b-}u(t)\right| \leq \frac{L_u}{\Gamma(2-\gamma(t))}M_2(t)(g(b)-g(t))^{1-\gamma(t)},
$$

for all $t \in [a,b]$, where

$$
M_1(t) = 1 + \frac{|\gamma'(t)|}{g'(t)(2-\gamma(t))}(g(t)-g(a))\left[\frac{1}{1-\gamma(t)} + \frac{1}{2-\gamma(t)} + \ln(g(t)-g(a))\right] \tag{1}
$$

and

$$
M_2(t) = 1 + \frac{|\gamma'(t)|}{g'(t)(2-\gamma(t))}(g(b)-g(t))\left[\frac{1}{1-\gamma(t)} + \frac{1}{2-\gamma(t)} + \ln(g(b)-g(t))\right]. \tag{2}
$$

Proof. By Theorem 1, we conclude that

$$\left|{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t)\right| \leq \left|{}^{C}_{3}\mathbb{D}^{\gamma(t)}_{a+}u(t)\right| + \left|\frac{\gamma'(t)}{g'(t)(1-\gamma(t))\Gamma(2-\gamma(t))}\int_a^t g'(s)(g(t)-g(s))^{1-\gamma(t)}\frac{u'(s)}{g'(s)}\,ds\right|$$
$$+\left|\frac{\gamma'(t)}{g'(t)\Gamma(2-\gamma(t))}\int_a^t g'(s)(g(t)-g(s))^{1-\gamma(t)}\frac{u'(s)}{g'(s)}\ln(g(t)-g(s))\,ds\right|.$$

Obviously,

$$\left|\int_a^t g'(s)(g(t)-g(s))^{1-\gamma(t)}\frac{u'(s)}{g'(s)}\,ds\right| \leq \frac{L_u}{2-\gamma(t)}(g(t)-g(a))^{2-\gamma(t)}.$$

Regarding the last integral, applying the integration by parts formula yields

$$\left|\int_a^t g'(s)(g(t)-g(s))^{1-\gamma(t)}\ln(g(t)-g(s))\,ds\right|$$
$$\leq \frac{1}{2-\gamma(t)}(g(t)-g(a))^{2-\gamma(t)}\left(\frac{1}{2-\gamma(t)}+\ln(g(t)-g(a))\right).$$

By combining Theorem 7 with the two preceding inequalities, we arrive at the desired conclusion. ☐

Therefore, we deduce that ${}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(a) = 0 = {}^{C}_{1}\mathbb{D}^{\gamma(t)}_{b-}u(b)$.

Theorem 9. *The following two formulas are valid:*

$$\left|{}^{C}_{2}\mathbb{D}^{\gamma(t)}_{a+}u(t)\right| \leq \frac{L_u}{\Gamma(2-\gamma(t))}M_3(t)(g(t)-g(a))^{1-\gamma(t)}$$

and

$$\left|{}^{C}_{2}\mathbb{D}^{\gamma(t)}_{b-}u(t)\right| \leq \frac{L_u}{\Gamma(2-\gamma(t))}M_4(t)(g(b)-g(t))^{1-\gamma(t)},$$

for all $t\in[a,b]$*, where*

$$M_3(t) = M_1(t) + \frac{|\gamma'(t)\Gamma'(1-\gamma(t))|}{g'(t)(2-\gamma(t))\Gamma(1-\gamma(t))}(g(t)-g(a))$$

and

$$M_4(t) = M_2(t) + \frac{|\gamma'(t)\Gamma'(1-\gamma(t))|}{g'(t)(2-\gamma(t))\Gamma(1-\gamma(t))}(g(b)-g(t)),$$

with the functions M_1 *and* M_2 *given by* (1) *and* (2)*, respectively.*

Proof. By Theorem 2,

$$\left|{}^{C}_{2}\mathbb{D}^{\gamma(t)}_{a+}u(t)\right| \leq \left|{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t)\right|$$
$$+\left|\frac{\gamma'(t)\Gamma'(1-\gamma(t))}{g'(t)\Gamma^2(1-\gamma(t))}\int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}(u(s)-u(a))\,ds\right|,$$

integrating the last integral by parts and applying Theorem 8, we achieve the desired result. ☐

Consequently, we derive that ${}^{C}_{2}\mathbb{D}^{\gamma(t)}_{a+}u(a) = 0 = {}^{C}_{2}\mathbb{D}^{\gamma(t)}_{b-}u(b)$.

4. Integer-order expansions of the fractional derivative

We now present formulas for series expansions of the given fractional derivatives, which involve only integer-order derivatives. By considering finite sums and substituting fractional derivatives with these numerical decompositions, we can convert the fractional problem into a classical problem (of integer-order derivatives) and thus apply already known methods for its resolution. To begin, we recall the result presented in [3], where the previously mentioned formula is presented for the Caputo derivative of type 3. Afterward, we prove the respective formulas for the remaining two types of derivatives.

To start, let us introduce some auxiliary functions. We will consider the sequences of functions $(a_k)_{k\in\mathbb{N}_0}$, $(b_k)_{k\in\mathbb{N}}$, $(V_k)_{k\in\mathbb{N}}$, and $(W_k)_{k\in\mathbb{N}}$, all defined on the interval $[a,b]$, and defined as follows:

$$\begin{aligned}
a_k(t) &= 1+\sum_{p=m-k+1}^{\infty}\frac{\Gamma(p-1+\gamma(t)-m)}{\Gamma(\gamma(t)-1-k)(p-m+k)!},\\
b_k(t) &= \frac{\Gamma(k-1+\gamma(t)-m)}{\Gamma(1-\gamma(t))\Gamma(\gamma(t))(k-m-1)!},\\
V_k(t) &= \int_a^t (g(s)-g(a))^{k-1}u'(s)ds,\\
W_k(t) &= \int_t^b (g(b)-g(s))^{k-1}u'(s)ds,
\end{aligned}$$

where $t\in[a,b]$ and $m\in\mathbb{N}_0$ is a given integer. The result is the following:

Theorem 10. [3] *Consider a function $u\in C^{m+2}[a,b]$. Then,*

$$\begin{aligned}
{}_{3}^{C}\mathbb{D}_{a+}^{\gamma(t)}u(t) &= \sum_{k=0}^{m}\frac{a_k(t)(g(t)-g(a))^{1-\gamma(t)+k}}{\Gamma(2-\gamma(t)+k)}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{k+1}u(t)\\
&\quad+\sum_{k=m+1}^{\infty} b_k(t)(g(t)-g(a))^{1-\gamma(t)+m-k}V_{k-m}(t)
\end{aligned}$$

and

$$\begin{aligned}
{}_{3}^{C}\mathbb{D}_{b-}^{\gamma(t)}u(t) &= \sum_{k=0}^{m}\frac{a_k(t)(g(b)-g(t))^{1-\gamma(t)+k}}{\Gamma(2-\gamma(t)+k)}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^{k+1}u(t)\\
&\quad-\sum_{k=m+1}^{\infty} b_k(t)(g(b)-g(t))^{1-\gamma(t)+m-k}W_{k-m}(t).
\end{aligned}$$

Using the previous result, together with Theorems 1 and 2, we obtain the remaining formulas.

Theorem 11. *Consider a function $u\in C^{m+2}[a,b]$. Then,*

$$
{}_{1}^{C}\mathbb{D}_{a+}^{\gamma(t)}u(t) = \sum_{k=0}^{m}\frac{a_k(t)(g(t)-g(a))^{1-\gamma(t)+k}}{\Gamma(2-\gamma(t)+k)}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{k+1}u(t)
$$

$$+\sum_{k=m+1}^{\infty} b_k(t)(g(t)-g(a))^{1-\gamma(t)+m-k}V_{k-m}(t)$$
$$+\frac{\gamma'(t)(g(t)-g(a))^{1-\gamma(t)}}{g'(t)\Gamma(2-\gamma(t))}\Big[\Big(\frac{1}{1-\gamma(t)}-\ln(g(t)-g(a))\Big)$$
$$\times\sum_{l=0}^{\infty}\frac{\Gamma(l+\gamma(t)-1)V_{l+1}(t)}{\Gamma(\gamma(t)-1)\,l!\,(g(t)-g(a))^l}$$
$$+\sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\frac{\Gamma(l+\gamma(t)-1)V_{l+n+1}(t)}{\Gamma(\gamma(t)-1)\,l!\,n\,(g(t)-g(a))^{l+n}}\Big]$$

and

$$\begin{aligned}{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{b-}u(t) &= \sum_{k=0}^{m}\frac{a_k(t)(g(b)-g(t))^{1-\gamma(t)+k}}{\Gamma(2-\gamma(t)+k)}\left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^{k+1}u(t)\\ &-\sum_{k=m+1}^{\infty} b_k(t)(g(b)-g(t))^{1-\gamma(t)+m-k}W_{k-m}(t)\\ &+\frac{\gamma'(t)(g(b)-g(t))^{1-\gamma(t)}}{g'(t)\Gamma(2-\gamma(t))}\Big[\Big(\frac{1}{1-\gamma(t)}-\ln(g(b)-g(t))\Big)\\ &\times\sum_{l=0}^{\infty}\frac{\Gamma(l+\gamma(t)-1)W_{l+1}(t)}{\Gamma(\gamma(t)-1)\,l!\,(g(b)-g(t))^l}+\sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\frac{\Gamma(l+\gamma(t)-1)W_{l+n+1}(t)}{\Gamma(\gamma(t)-1)\,l!\,n\,(g(b)-g(t))^{l+n}}\Big].\end{aligned}$$

Proof. According to Theorem 1,

$$\begin{aligned}{}^{C}_{1}\mathbb{D}^{\gamma(t)}_{a+}u(t) &= {}^{C}_{3}\mathbb{D}^{\gamma(t)}_{a+}u(t)+\tfrac{\gamma'(t)}{g'(t)\Gamma(2-\gamma(t))}\int_a^t(g(t)-g(s))^{1-\gamma(t)}u'(s)\\ &\qquad\times\left[\tfrac{1}{1-\gamma(t)}-\ln(g(t)-g(s))\right]ds\end{aligned}$$

and leveraging Theorem 10, our task is simplified to expanding the second term on the right-hand side of the last equality. Applying the binomial formula

$$\begin{aligned}(g(t)-g(s))^{1-\gamma(t)} &= (g(t)-g(a))^{1-\gamma(t)}\left[1-\frac{g(s)-g(a)}{g(t)-g(a)}\right]^{1-\gamma(t)}\\ &= (g(t)-g(a))^{1-\gamma(t)}\sum_{l=0}^{\infty}\binom{1-\gamma(t)}{l}(-1)^l\left(\frac{g(s)-g(a)}{g(t)-g(a)}\right)^l\\ &= (g(t)-g(a))^{1-\gamma(t)}\sum_{l=0}^{\infty}\frac{\Gamma(l+\gamma(t)-1)}{\Gamma(\gamma(t)-1)\,l!}\left(\frac{g(s)-g(a)}{g(t)-g(a)}\right)^l,\end{aligned}$$

and the Taylor series formula

$$\begin{aligned}\ln(g(t)-g(s)) &= \ln(g(t)-g(a))+\ln\left(1-\frac{g(s)-g(a)}{g(t)-g(a)}\right)\\ &= \ln(g(t)-g(a))-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{g(s)-g(a)}{g(t)-g(a)}\right)^n,\end{aligned}$$

and substituting them back into the initial formula, we obtain the desired result. □

Theorem 12. *Consider a function $u \in C^{m+2}[a,b]$. Then,*

$$
\begin{aligned}
{}^C_2\mathbb{D}^{\gamma(t)}_{a+}u(t) =& \sum_{k=0}^{m} \frac{a_k(t)(g(t)-g(a))^{1-\gamma(t)+k}}{\Gamma(2-\gamma(t)+k)} \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{k+1} u(t) \\
&+ \sum_{k=m+1}^{\infty} b_k(t)(g(t)-g(a))^{1-\gamma(t)+m-k} V_{k-m}(t) + \frac{\gamma'(t)(g(t)-g(a))^{1-\gamma(t)}}{g'(t)\Gamma(2-\gamma(t))} \\
&\times \Big[\Big(\psi(2-\gamma(t)) - \ln(g(t)-g(a))\Big) \\
&\times \sum_{l=0}^{\infty} \frac{\Gamma(l+\gamma(t)-1)V_{l+1}(t)}{\Gamma(\gamma(t)-1)\, l!\, (g(t)-g(a))^l} + \sum_{l=0}^{\infty}\sum_{n=1}^{\infty} \frac{\Gamma(l+\gamma(t)-1)V_{l+n+1}(t)}{\Gamma(\gamma(t)-1)\, l!\, n\, (g(t)-g(a))^{l+n}}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
{}^C_2\mathbb{D}^{\gamma(t)}_{b-}u(t) =& \sum_{k=0}^{m} \frac{a_k(t)(g(b)-g(t))^{1-\gamma(t)+k}}{\Gamma(2-\gamma(t)+k)} \left(\frac{-1}{g'(t)}\frac{d}{dt}\right)^{k+1} u(t) \\
&- \sum_{k=m+1}^{\infty} b_k(t)(g(b)-g(t))^{1-\gamma(t)+m-k} W_{k-m}(t) + \frac{\gamma'(t)(g(b)-g(t))^{1-\gamma(t)}}{g'(t)\Gamma(2-\gamma(t))} \\
&\times \Big[\Big(\psi(2-\gamma(t)) - \ln(g(b)-g(t))\Big) \\
&\times \sum_{l=0}^{\infty} \frac{\Gamma(l+\gamma(t)-1)W_{l+1}(t)}{\Gamma(\gamma(t)-1)\, l!\, (g(b)-g(t))^l} + \sum_{l=0}^{\infty}\sum_{n=1}^{\infty} \frac{\Gamma(l+\gamma(t)-1)W_{l+n+1}(t)}{\Gamma(\gamma(t)-1)\, l!\, n\, (g(b)-g(t))^{l+n}}\Big].
\end{aligned}
$$

Proof. Starting with the formula presented in Theorem 2,

$$
\begin{aligned}
{}^C_2\mathbb{D}^{\gamma(t)}_{a+}u(t) =& {}^C_1\mathbb{D}^{\gamma(t)}_{a+}u(t) + \frac{\gamma'(t)\Gamma'(1-\gamma(t))}{g'(t)\Gamma^2(1-\gamma(t))} \\
&\times \int_a^t g'(s)(g(t)-g(s))^{-\gamma(t)}(u(s)-u(a))\,ds
\end{aligned}
$$

and integrating by parts, we derive

$$
{}^C_2\mathbb{D}^{\gamma(t)}_{a+}u(t) = {}^C_1\mathbb{D}^{\gamma(t)}_{a+}u(t) + \frac{\gamma'(t)\Gamma'(1-\gamma(t))}{g'(t)\Gamma^2(1-\gamma(t))} \int_a^t \frac{(g(t)-g(s))^{1-\gamma(t)}}{1-\gamma(t)} u'(s)\,ds.
$$

Similarly to what was done in the proof of Theorem 11, by employing the binomial formula on the term $(g(t)-g(s))^{1-\gamma(t)}$, then simplifying the expressions and considering the formula

$$
\psi(z+1) = \psi(z) + \frac{1}{z},
$$

we arrive at the desired conclusion. □

We illustrate the previous results with some numerical simulations. In all cases, the test function is given by $u(t) = (g(t)-g(0))^4$, with $t \in [0,1]$. For the fractional order and the kernel, two cases are considered: $\gamma(t) = (t+1)/4$ and

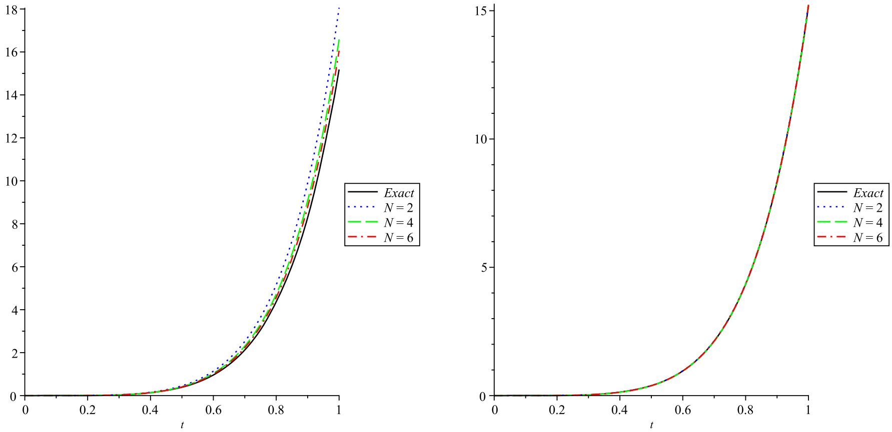

Figure 1. Illustration of Theorem 11: $\gamma(t) = (t+1)/4$ and $g(t) = \exp(t)$; $m = 0$ (left) and $m = 3$ (right)

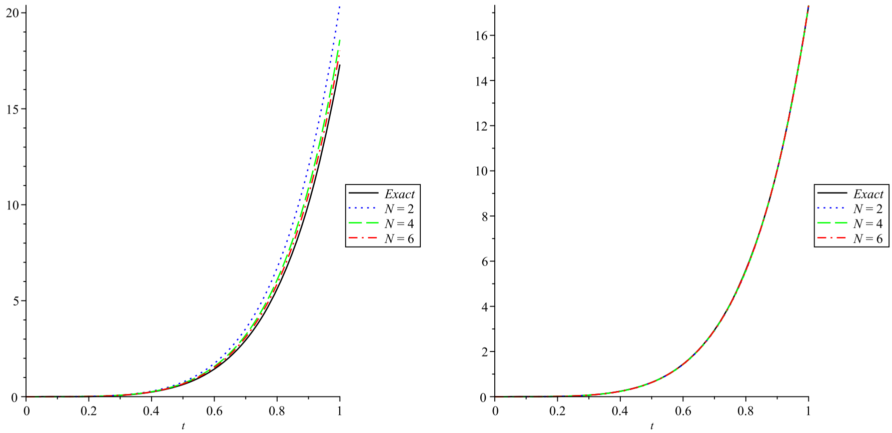

Figure 2. Illustration of Theorem 11: $\gamma(t) = \cos(t)/2$ and $g(t) = t^2 + t$; $m = 0$ (left) and $m = 3$ (right)

$g(t) = \exp(t)$ (Figs. 1 and 3), and in the other situation, $\gamma(t) = \cos(t)/2$ and $g(t) = t^2 + t$ (Figs. 2 and 4). Furthermore, to understand the importance of the parameter $m \in \mathbb{N}_0$ in the approximation, two cases are studied: $m \in \{0, 3\}$. The approximation consists of considering the formulas presented in Theorems 11 and 12, but replacing the given series by finite sums, ending at $N \in \mathbb{N}$ with $N \geq m+1$. The plots illustrated in Figs. 1 and 2 correspond to the Caputo

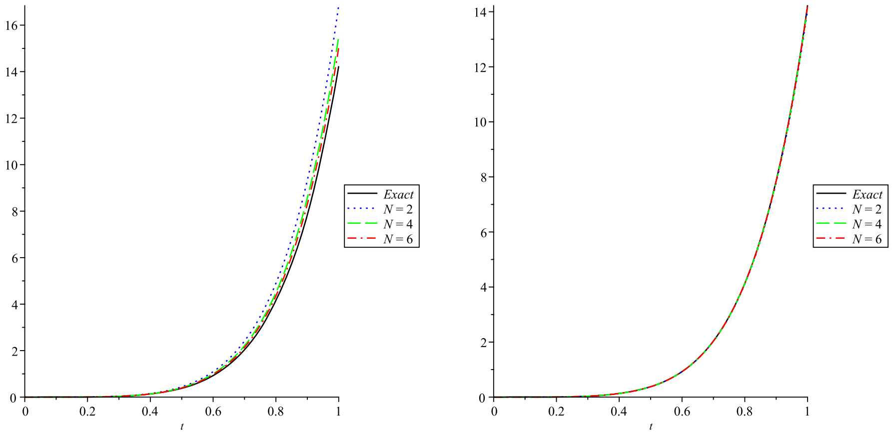

Figure 3. Illustration of Theorem 12: $\gamma(t) = (t+1)/4$ and $g(t) = \exp(t)$; $m = 0$ (left) and $m = 3$ (right)

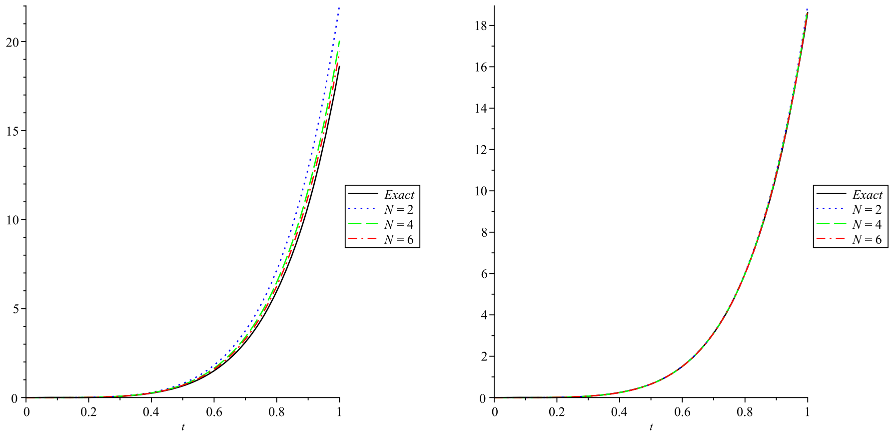

Figure 4. Illustration of Theorem 12: $\gamma(t) = \cos(t)/2$ and $g(t) = t^2 + t$; $m = 0$ (left) and $m = 3$ (right)

derivative of type 1 (Theorem 11), and in Figs. 3 and 4 they correspond to the Caputo derivative of type 2 (Theorem 12).

Table 1 displays the absolute errors for the previous simulations, with $N = 6$, encompassing both of the preceding theorems. In columns 2 and 3, we utilize the expansion provided in Theorem 11, with $\gamma(t) = (t+1)/4$ and $g(t) = \exp(t)$, while the remaining two columns employ the expansion outlined in Theorem 12, with $\gamma(t) = \cos(t)/2$ and $g(t) = t^2 + t$.

TABLE 1. Absolute errors, for Theorem 11 (columns 2 and 3) and Theorem 12 (columns 4 and 5)

t_1	$m = 0$	$m = 3$	$m = 0$	$m = 3$
0.1	1.2674552×10^{-5}	1.417886×10^{-7}	5.4399209×10^{-5}	2.951876×10^{-7}
0.2	2.3845188×10^{-4}	3.279880×10^{-6}	8.0316036×10^{-4}	1.290876×10^{-5}
0.3	1.4864830×10^{-3}	1.835490×10^{-5}	4.1723867×10^{-3}	1.183140×10^{-4}
0.4	5.8489679×10^{-3}	5.101329×10^{-5}	1.3926913×10^{-2}	5.577790×10^{-4}
0.5	1.7794236×10^{-2}	5.805311×10^{-5}	3.6171115×10^{-2}	1.779061×10^{-3}
0.6	4.5839782×10^{-2}	1.762463×10^{-4}	7.9704050×10^{-2}	4.285358×10^{-3}
0.7	1.0498700×10^{-1}	1.314852×10^{-3}	1.5611504×10^{-1}	8.066582×10^{-3}
0.8	2.2010913×10^{-1}	5.010451×10^{-3}	2.7944617×10^{-1}	1.134126×10^{-2}
0.9	4.3046332×10^{-1}	1.489584×10^{-2}	4.6514647×10^{-1}	8.335220×10^{-3}
1	7.9542868×10^{-1}	3.828839×10^{-2}	7.2788984×10^{-1}	1.410165×10^{-2}

Table 2. Absolute errors, for Theorem 11 and $m = 0$

t_1	$N = 100$	$N = 200$	$N = 300$
0.1	1.0686744×10^{-6}	3.8111130×10^{-7}	2.0790952×10^{-7}
0.2	1.5309673×10^{-5}	5.4159480×10^{-6}	2.9405975×10^{-6}
0.3	7.5751698×10^{-5}	2.6447826×10^{-5}	1.4248636×10^{-5}
0.4	2.3665248×10^{-4}	8.1143310×10^{-5}	4.3247610×10^{-5}
0.5	5.6605850×10^{-4}	1.8970445×10^{-4}	9.9743273×10^{-5}
0.6	1.1325255×10^{-3}	3.6932290×10^{-4}	1.9102071×10^{-4}
0.7	1.9909946×10^{-3}	6.2931920×10^{-4}	3.1942530×10^{-4}
0.8	3.1737201×10^{-3}	9.6942012×10^{-4}	4.8193230×10^{-4}
0.9	4.6903420×10^{-3}	1.3822280×10^{-3}	6.7222895×10^{-4}
1	6.5374600×10^{-3}	1.8594200×10^{-3}	8.8492400×10^{-4}

Table 3. Absolute errors, for Theorem 11 and $m = 3$

t_1	$N = 100$	$N = 200$	$N = 300$
0.1	9.5855040×10^{-9}	3.8274400×10^{-9}	2.2181192×10^{-9}
0.2	4.4834970×10^{-7}	1.8050900×10^{-7}	1.0534860×10^{-7}
0.3	4.3026300×10^{-6}	1.7413900×10^{-6}	1.0157580×10^{-6}
0.4	2.1027284×10^{-5}	8.5259000×10^{-6}	4.9735625×10^{-6}
0.5	6.9414456×10^{-5}	2.8118180×10^{-5}	1.6365660×10^{-5}
0.6	1.7507300×10^{-4}	7.0859173×10^{-5}	4.1140094×10^{-5}
0.7	3.5806510×10^{-4}	1.4512195×10^{-4}	8.4074942×10^{-5}
0.8	6.0827846×10^{-4}	2.4836570×10^{-4}	1.4387710×10^{-4}
0.9	8.5204340×10^{-4}	3.5651976×10^{-4}	2.0772133×10^{-4}
1	9.1487000×10^{-4}	4.1358316×10^{-4}	2.4654668×10^{-4}

An important observation to make is that as we increase the value of the parameter m, we observe a notable acceleration in the convergence of our numerical approximation towards the exact function. This phenomenon underscores the significance of the parameter m in refining our approximation and achieving results that closely align with the true behavior of the function. Furthermore, as we augment m, our expansion encompasses higher-order derivatives up to the $(m+1)$-th order. This implies that our approximation becomes more comprehensive, incorporating additional detail from the function. Differing from Taylor expansion-type approximations, which involve consideration of higher-order derivatives, these approaches do not necessitate such stringent conditions. Indeed, for $m = 0$, the function must belong to class C^2, and for $m = 3$, the function must belong to class C^5. In order to reduce the approximation error, it suffices to augment the value of N, without imposing more stringent differentiability conditions. For example, let us reconsider the result given by Theorem 11, where $\gamma = \cos(t)/2$ and $g(t) = t^2 + t$, for three different N values: $N \in \{100, 200, 300\}$, and for $m \in \{0, 3\}$. We recall that for $m = 0$, the expansion only depends on the first derivative of the test function, whereas for $m = 3$, it depends on the third derivative of the function. As expected, increasing the value of N leads to a decrease in error. The results are illustrated in the Tables 2 (for $m = 0$) and 3 (for $m = 3$).

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Data Availibility No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no Conflict of interest.

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