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Aequationes Mathematicae

A functional equation related to Wigner's theorem

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Abstract. An open problem posed by G. Maksa and Z. Páles is to find the general solution of the functional equation

$$\{\|f(x) - \beta f(y)\| : \beta \in \mathbb{T}_n\} = \{\|x - \beta y\| : \beta \in \mathbb{T}_n\} \quad (x, y \in H)$$

where $f : H \to K$ is between two complex normed spaces and $\mathbb{T}_n := \{e^{i\frac{2k\pi}{n}} : k = 1, \cdots, n\}$ is the set of the nth roots of unity. With the aid of the celebrated Wigner's unitary-antiunitary theorem, we show that if $n \geq 3$ and H and K are complex inner product spaces, then f satisfies the above equation if and only if there exists a phase function $\sigma : H \to \mathbb{T}_n$ such that $\sigma \cdot f$ is a linear or anti-linear isometry. Moreover, if the solution f is continuous, then f is a linear or anti-linear isometry.

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1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers, and $\mathbb{T}$ be the unit circle in $\mathbb{C}$. Let n be a fixed positive integer and $\mathbb{T}_n$ denote the finite cyclic subgroup of $\mathbb{T}$ consisting of the nth roots of unity, i.e., $\mathbb{T}_n := \{e^{i\frac{2k\pi}{n}} : k = 1, \cdots, n\}$.

The celebrated Wigner's theorem plays a fundamental role in the foundations of quantum mechanics and in representation theory in physics. It states that any quantum mechanical symmetry transformation can be represented by a unitary or an anti-unitary operator on a complex inner product space. In

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mathematical language, the result can be reformulated in the following way. For complex inner product spaces H and K, a mapping $f: H \to K$ satisfies

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \quad (x, y \in H) \tag{1.1}$$

if and only if there exist a function (called *phase function*) $\sigma : H \to \mathbb{T}$ and a linear or anti-linear (i.e. conjugate linear) isometry $U : H \to K$ such that

$$f(x) = \sigma(x)U(x) \quad (x \in H). \tag{1.2}$$

We then say that f is phase equivalent to a linear or anti-linear isometry. There are several proofs of this result, see [1,4,6,7,9,18,22] to list just some of them. For further generalizations of this fundamental result, we mention the papers [3,5,8,16,17,19].

The real version of Wigner's theorem was also obtained by Rätz [18, Corollary 8 (a)] and by Turnšek [22, Theorem 2.4 (i)] as follows. A mapping $f : H \to K$ between real inner product spaces satisfies (1.1) if and only if there exist a function $\sigma : H \to \{-1, 1\}$ and a real linear isometry $U : H \to K$ such that (1.2) holds. We then say that f is phase equivalent to a real linear isometry. By an easy argument, Maksa and Páles [15, Theorem 2] obtained that if H and K are real inner product spaces then $f : H \to K$ satisfies the functional equation

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\} \quad (x, y \in H), \tag{1.3}$$

if and only if f satisfies (1.1). Combining the two results above, Maksa and Páles [15, Theorem 2] then proved that the general solution of the functional Eq. (1.3) is phase equivalent to a real linear isometry.

Maksa and Páles [15] also at last formulated two open problems as follows.

Problem 1.1. [15] *Under what conditions, when H and K are real normed but not necessarily inner product spaces, the solutions of (1.3) are phase equivalent to a real linear isometry?*

Problem 1.2. [15] *Let H and K be complex normed spaces and n be a fixed positive integer. Under what conditions the solutions $f : H \to K$ of the following generalization of (1.3):*

$$\{\|f(x) - \beta f(y)\| : \beta \in \mathbb{T}_n\} = \{\|x - \beta y\| : \beta \in \mathbb{T}_n\} \quad (x, y \in H) \tag{1.4}$$

have the form (1.2) for some phase function $\sigma : H \to \mathbb{T}_n$ and some linear or anti-linear isometry $U : H \to K$?

There are several recent papers dealing with Problem 1.1, see [10–14,20, 21,23]. Recently, Ilišević, Turnšek, etc. gave a positive answer to Problem 1.1. Namely, they proved that the solutions of Eq. (1.3) for a mapping $f : H \to K$, where H and K are real normed spaces, are phase equivalent to a real linear isometry under the condition that either f is surjective [12] or K is strictly convex [13]. Huang and Tan in [10] introduced a weaker version of Eq. (1.3) as follows:

$$\min\{\|f(x)+f(y)\|,\|f(x)-f(y)\|\}=\min\{\|x+y\|,\|x-y\|\}\quad (x,y\in H).$$

They also proved that every surjective mapping satisfying this equation between two real normed spaces is phase equivalent to a real linear isometry.

In this paper, we give a positive answer to Problem 1.2 for complex inner product spaces. Let H and K be two complex inner product spaces and f be a mapping $f: H \to K$. We will show in Theorem 2.5 that for $n \geq 3$, a mapping f satisfies (1.4) if and only if there is a phase function $\sigma: H \to \mathbb{T}_n$ such that $\sigma \cdot f$ is a linear or anti-linear isometry. Moreover, we will show in Corollary 2.7 that for $n \geq 3$, a continuous mapping f satisfies (1.4) if and only if f is a linear or anti-linear isometry.

2. Results

Throughout this section, let H and K be complex inner product spaces, and n be a fixed positive integer.

First we present a simple and useful lemma. For convenience, we introduce two equations

$$\{\|f(x)+\beta f(y)\|:\beta\in\mathbb{T}_n\}=\{\|x+\beta y\|:\beta\in\mathbb{T}_n\}\quad (x,y\in H)\tag{2.1}$$

and

$$\{\mathrm{Re}\beta\langle f(x),f(y)\rangle:\beta\in\mathbb{T}_n\}=\{\mathrm{Re}\beta\langle x,y\rangle:\beta\in\mathbb{T}_n\}\quad (x,y\in H).\tag{2.2}$$

Lemma 2.1. *For $n \geq 2$ and a mapping $f: H \to K$, the following statements are equivalent:*

(i) *f satisfies Eq. (1.4);*
(ii) *f satisfies Eq. (2.1);*
(iii) *f satisfies Eq. (2.2).*

Moreover, if this is the case, then f is norm preserving, i.e., $\|f(x)\| = \|x\|$ for every $x \in H$.

Proof. Putting $y = x$, we deduce from each of Eqs. (1.4), (2.1) and (2.2) that f is norm preserving. For $x, y \in H$ and $\beta \in \mathbb{T}_n$, we obtain

$$\|x\pm\beta y\|^2=\|x\|^2+\|y\|^2\pm 2\mathrm{Re}\langle x,\beta y\rangle=\|x\|^2+\|y\|^2\pm 2\mathrm{Re}\overline{\beta}\langle x,y\rangle,$$

and

$$\|f(x)\pm\beta f(y)\|^2=\|f(x)\|^2+\|f(y)\|^2\pm 2\mathrm{Re}\overline{\beta}\langle f(x),f(y)\rangle.$$

By the norm-preserving property and the fact that $\beta \in \mathbb{T}_n$ if and only if $\overline{\beta} \in \mathbb{T}_n$, each of Eqs. (1.4) and (2.1) is equivalent to Eq. (2.2). The proof is complete. □

Remark 2.2. In the case $n = 1$, Eqs. (1.4), (2.1) and (2.2) turn out to be

$$\|f(x) - f(y)\| = \|x - y\| \quad (x, y \in H), \tag{2.3}$$
$$\|f(x) + f(y)\| = \|x + y\| \quad (x, y \in H), \tag{2.4}$$

and

$$\mathrm{Re}\langle f(x), f(y)\rangle = \mathrm{Re}\langle x, y\rangle \quad (x, y \in H), \tag{2.5}$$

respectively. By definition, f satisfies Eq. (2.3) if and only if f is an isometry. Since an inner product space is strictly convex, Baker [2] showed that $f - f(0)$ is a real linear isometry if f is an isometry. However, an easy argument [15, Theorem 1] shows that Eqs. (2.4) and (2.5) are equivalent to each other and each of them is equivalent to f being a real linear isometry. Hence, in general, Lemma 2.1 does not hold for $n = 1$.

In the case $n = 2$, each of Eqs. (1.4) and (2.1) is the same as Eq. (1.3), and Eq. (2.2) becomes

$$|\mathrm{Re}\langle f(x), f(y)\rangle| = |\mathrm{Re}\langle x, y\rangle| \quad (x, y \in H). \tag{2.6}$$

Note that every complex linear space is obviously a real linear space and if $\langle \cdot, \cdot \rangle$ is a complex inner product on H, then $\mathrm{Re}\langle \cdot, \cdot \rangle$ is a real inner product on H which induces the same norm. Therefore, from the real version of Wigner's theorem, the following theorem follows immediately. See [15, Theorem 2] or [22, Theorem 2.4 (i)] for its proof.

Theorem 2.3. ([15,22]) *For a mapping $f : H \to K$, the following statements are equivalent:*

(i) *f satisfies Eq. (1.3);*
(ii) *f satisfies Eq. (2.6);*
(iii) *there exist a phase function $\sigma : H \to \{-1, 1\}$ and a real linear isometry $U : H \to K$ such that $f = \sigma \cdot U$.*

It may be worth noting that in (iii) of Theorem 2.3, the phrase "a real linear isometry" cannot be replaced by "a linear or anti-linear isometry". To show this, we construct an example as follows.

Example. Let $H = \mathbb{C}$ and $K = \mathbb{C}^2$, both equipped with the usual inner products, i.e.,

$$\langle x, y\rangle = x\overline{y} \quad (x, y \in \mathbb{C})$$

and

$$\langle (x_1, x_2), (y_1, y_2)\rangle = x_1\overline{y_1} + x_2\overline{y_2} \quad ((x_1, x_2), (y_1, y_2) \in \mathbb{C}^2).$$

Define $f : \mathbb{C} \to \mathbb{C}^2$ by

$$f(x) = \frac{1}{\sqrt{2}}(x, \overline{x}), \quad x \in \mathbb{C}.$$

An easy calculation shows that f satisfies Eq. (2.6), but for any phase function $\sigma : H \to \{-1, 1\}$, $\sigma \cdot f$ is neither linear nor anti-linear.

However, in the case $n \geq 3$, in sharp contrast with the cases $n = 1, 2$, each of (1.4), (2.1) and (2.2) is equivalent to there being a phase function $\sigma : H \to \mathbb{T}_n$ such that $\sigma \cdot f$ is a linear or anti-linear isometry. This is the content of Theorem 2.5. To obtain it, we need an elementary lemma as follows.

Lemma 2.4. *For $n \geq 3$ and $s, t \in \mathbb{C}$, the equation*

$$\{Re\beta s : \beta \in \mathbb{T}_n\} = \{Re\beta t : \beta \in \mathbb{T}_n\} \tag{2.7}$$

holds if and only if $s = \beta_0 t$ or $s = \beta_0 \bar{t}$ for some $\beta_0 \in \mathbb{T}_n$.

Proof. Sufficiency. If $s = \beta_0 t$ for some $\beta_0 \in \mathbb{T}_n$, then

$$\{\mathrm{Re}\beta s : \beta \in \mathbb{T}_n\} = \{\mathrm{Re}\beta_0\beta t : \beta \in \mathbb{T}_n\} = \{\mathrm{Re}\beta t : \beta \in \mathbb{T}_n\},$$

where the second equality follows from the fact that $\beta_0 \mathbb{T}_n = \mathbb{T}_n$. If $s = \beta_0 \bar{t}$, then

$$\begin{aligned}\{\mathrm{Re}\beta s : \beta \in \mathbb{T}_n\} &= \{\mathrm{Re}\beta_0\beta \bar{t} : \beta \in \mathbb{T}_n\} = \{\mathrm{Re}\beta \bar{t} : \beta \in \mathbb{T}_n\} \\ &= \{\mathrm{Re}\overline{\beta} t : \beta \in \mathbb{T}_n\} = \{\mathrm{Re}\beta t : \beta \in \mathbb{T}_n\},\end{aligned}$$

where the fourth equality follows from the fact that $\beta \in \mathbb{T}_n$ if and only if $\overline{\beta} \in \mathbb{T}_n$.

Necessity. Suppose that Eq. (2.7) holds. We first claim that $|s| = |t|$. Indeed, taking $t = |t|e^{i\theta}$ and $\beta_k = e^{i\frac{2k\pi}{n}} \in \mathbb{T}_n$ for $k = 1, \cdots, n$, we see that

$$\begin{aligned}(\mathrm{Re}\beta_k t)^2 = |t|^2 \cos^2\left(\theta + \frac{2k\pi}{n}\right) &= \frac{|t|^2}{2}\left(1 + \cos\left(2\theta + \frac{4k\pi}{n}\right)\right) \\ &= \frac{|t|^2}{2}\left(1 + \cos 2\theta \cos\frac{4k\pi}{n} - \sin 2\theta \sin\frac{4k\pi}{n}\right).\end{aligned}$$

Let us recall the formula of partial sums of trigonometric series

$$\sum_{k=1}^{n} \cos ku = \frac{\sin \frac{n}{2}u \cdot \cos \frac{n+1}{2}u}{\sin \frac{u}{2}} \quad \text{and} \quad \sum_{k=1}^{n} \sin ku = \frac{\sin \frac{n}{2}u \cdot \sin \frac{n+1}{2}u}{\sin \frac{u}{2}}$$

for $u \in (0, 2\pi)$. It follows that

$$\sum_{k=1}^{n}(\mathrm{Re}\beta_k t)^2 = \frac{|t|^2}{2}\left(n + \cos 2\theta \cdot \sum_{k=1}^{n}\cos k\frac{4\pi}{n} - \sin 2\theta \cdot \sum_{k=1}^{n}\sin k\frac{4\pi}{n}\right) = \frac{|t|^2 n}{2}.$$

Similarly,

$$\sum_{k=1}^{n}(\mathrm{Re}\beta_k s)^2 = \frac{|s|^2 n}{2}.$$

Therefore, Eq. (2.7) implies that $|s| = |t|$ and the claim is proved. Furthermore, let us take $s = |t|e^{i\theta_1}$ and $t = |t|e^{i\theta}$. It follows from Eq. (2.7) that

$$\left\{\cos\left(\theta_1 + \frac{2k\pi}{n}\right) : k = 1, \cdots, n\right\} = \left\{\cos\left(\theta + \frac{2k\pi}{n}\right) : k = 1, \cdots, n\right\}.$$

Choose $k_0 \in \{1, \cdots, n\}$ such that $\cos\theta_1 = \cos(\theta + \frac{2k_0\pi}{n})$, or equivalently,

$$\pm\theta_1 = \theta + \frac{2k_0\pi}{n} + 2m\pi$$

for some integer m. Therefore, we have $s = \beta_{k_0} t$ or $s = \overline{\beta_{k_0}}\,\overline{t}$. □

Now we present our main theorem. For convenience, we introduce the equation

$$\langle f(x), f(y)\rangle = \beta(x,y)\langle x, y\rangle \text{ or } \langle f(x), f(y)\rangle = \beta(x,y)\langle y, x\rangle \quad \text{for some } \beta(x,y) \in \mathbb{T}_n \quad (x, y \in H). \tag{2.8}$$

Theorem 2.5. *For $n \geq 3$ and a mapping $f : H \to K$, the following statements are equivalent:*

(i) *f satisfies Eq. (1.4);*
(ii) *f satisfies Eq. (2.1);*
(iii) *f satisfies Eq. (2.2);*
(iv) *f satisfies Eq. (2.8);*
(v) *there exist a phase function $\sigma : H \to \mathbb{T}_n$ and a linear or anti-linear isometry $U : H \to K$ such that $f = \sigma \cdot U$.*

Proof. The equivalences (i) $\iff$ (ii) $\iff$ (iii) and (iii) $\iff$ (iv) are contained in Lemmas 2.1 and 2.4, respectively. We will complete the proof by establishing the implications (v) $\Longrightarrow$ (i) and (iv) $\Longrightarrow$ (v).

(v) $\Longrightarrow$ (i). Suppose that $f = \sigma \cdot U$ for some phase function $\sigma : H \to \mathbb{T}_n$ and some linear isometry U (respectively, some anti-linear isometry U). For $x, y \in H$ and $\beta \in \mathbb{T}_n$,

$$\begin{aligned}\|f(x) - \beta f(y)\| &= \|\sigma(x)U(x) - \sigma(y)\beta U(y)\| = \|U(\sigma(x)x - \sigma(y)\beta y)\| \\ &= \|\sigma(x)x - \sigma(y)\beta y\| = \|x - \overline{\sigma(x)}\sigma(y)\beta y\|\end{aligned}$$

(respectively,

$$\begin{aligned}\|f(x) - \beta f(y)\| &= \|\sigma(x)U(x) - \sigma(y)\beta U(y)\| = \|U(\overline{\sigma(x)}x - \overline{\sigma(y)\beta}y)\| \\ &= \|\overline{\sigma(x)}x - \overline{\sigma(y)\beta}y\| = \|x - \sigma(x)\overline{\sigma(y)\beta}y\|).\end{aligned}$$

Note that $\beta\mathbb{T}_n = \mathbb{T}_n$ and $\overline{\beta} \in \mathbb{T}_n$ for every $\beta \in \mathbb{T}_n$. It follows that Eq. (1.4) holds.

(iv) $\Longrightarrow$ (v). Suppose that Eq. (2.8) holds. Therefore Eq. (1.1) holds. By Wigner's theorem, there exist a phase function $\sigma_1 : H \to \mathbb{T}$ and a linear or anti-linear isometry $U_1 : H \to K$ such that $f = \sigma_1 \cdot U_1$. Next, we will divide our proof into two cases: $\dim H = 1$ and $\dim H \geq 2$.

(I) The case that $\dim H = 1$. In this case, we have that $f(H) \subset U_1(H)$ and $\dim U_1(H) = 1$. After a suitable identification, the mapping f can be regarded as a mapping $f : \mathbb{C} \to \mathbb{C}$. Set

$$A := \left\{x \in \mathbb{C}\backslash\{0\} : \frac{x}{\overline{x}} \in \mathbb{T}_n\right\},$$
$$B := \left\{x \in \mathbb{C}\backslash\{0\} : \frac{x}{\overline{x}} \notin \mathbb{T}_n \text{ and } f(x) = \beta(x,1)f(1)x\right\},$$

and

$$C := \left\{x \in \mathbb{C}\backslash\{0\} : \frac{x}{\overline{x}} \notin \mathbb{T}_n \text{ and } f(x) = \beta(x,1)f(1)\overline{x}\right\}.$$

Obviously, the sets A, B and C are disjoint. By Eq. (2.8), we have that $A \cup B \cup C = \mathbb{C}\backslash\{0\}$. We claim that $B = \emptyset$ or $C = \emptyset$. Suppose, on contrary that $x \in B$ and $y \in C$. Then

$$f(x) = \beta(x,1)f(1)x \quad \text{and} \quad f(y) = \beta(y,1)f(1)\overline{y}.$$

Substituting these into Eq. (2.8), we obtain that

$$\beta(x,1)\overline{\beta(y,1)}xy = \beta(x,y)x\overline{y} \quad \text{or} \quad \beta(x,1)\overline{\beta(y,1)}xy = \beta(x,y)\overline{x}y,$$

which means that

$$\frac{y}{\overline{y}} = \frac{\beta(x,y)}{\beta(x,1)\overline{\beta(y,1)}} \in \mathbb{T}_n \quad \text{or} \quad \frac{x}{\overline{x}} = \frac{\beta(x,y)}{\beta(x,1)\overline{\beta(y,1)}} \in \mathbb{T}_n.$$

However, the fact $x \in B$ and $y \in C$ implies that

$$\frac{x}{\overline{x}} \notin \mathbb{T}_n \quad \text{and} \quad \frac{y}{\overline{y}} \notin \mathbb{T}_n.$$

This contradiction proves the claim. If $C = \emptyset$, then we set

$$A_1 := \left\{x \in \mathbb{C}\backslash\mathbb{R} : \frac{x}{\overline{x}} \in \mathbb{T}_n \text{ and } f(x) = \beta(x,1)f(1)\overline{x}\right\}.$$

If $x \in A_1$, then

$$\frac{\overline{x}}{x} \in \mathbb{T}_n \quad \text{and} \quad f(x) = \beta(x,1)f(1)\overline{x} = \beta(x,1)\frac{\overline{x}}{x}f(1)x.$$

If $x \in A\backslash A_1$ or $x \in B$, then Eq. (2.8) implies that

$$f(x) = \beta(x,1)f(1)x.$$

Define

$$\sigma(x) = \begin{cases} \beta(x,1)\frac{\overline{x}}{x}, & x \in A_1 \\ \beta(x,1), & x \in (A\backslash A_1) \cup B \\ 1, & x = 0 \end{cases}$$

and $U(x) = f(1)x,\ x \in \mathbb{C}$. Then $\sigma(\mathbb{C}) \subset \mathbb{T}_n$, U is a linear isometry and $f = \sigma \cdot U$. If $B = \emptyset$, a similar argument shows that there exist a phase function $\sigma : \mathbb{C} \to \mathbb{T}_n$ and an anti-linear isometry $U : \mathbb{C} \to \mathbb{C}$ such that $f = \sigma \cdot U$. This completes the proof of the case that $\dim H = 1$.

(II) The case that $\dim H \geq 2$. In this case, fix a unit vector $e \in H$. Define

$$\sigma(x) = \begin{cases} \overline{\sigma_1(e)}\sigma_1(x), & x \in H\backslash\{0\} \\ 1, & x = 0 \end{cases}$$

and $U = \sigma_1(e)U_1$. Then $\sigma : H \to \mathbb{T}$ is a phase function, $U : H \to K$ is a linear or anti-linear isometry and $f = \sigma \cdot U$. It suffices to show that $\sigma(H\backslash\{0\}) \subset \mathbb{T}_n$. From Eq. (2.8) it follows that if $\langle x, y\rangle \in \mathbb{R}\backslash\{0\}$ then

$$\beta(x, y)\langle x, y\rangle = \langle f(x), f(y)\rangle = \sigma(x)\overline{\sigma(y)}\langle x, y\rangle,$$

which implies that $\sigma(x)\overline{\sigma(y)} = \beta(x, y) \in \mathbb{T}_n$. This is equivalent to saying that $\sigma(x) \in \mathbb{T}_n$ if and only if $\sigma(y) \in \mathbb{T}_n$ whenever $\langle x, y\rangle \in \mathbb{R}\backslash\{0\}$. Let $Z = (\mathbb{C}e)^{\perp} = \{z \in H : \langle z, e\rangle = 0\}$ and choose arbitrarily $z \in Z\backslash\{0\}$. Since $\sigma(e) = 1$, the equations $\langle e, e + z\rangle = 1$ and $\langle z, e + z\rangle = \|z\|^2$ imply that $\sigma(e + z) \in \mathbb{T}_n$ and hence $\sigma(z) \in \mathbb{T}_n$. Moreover, for every $t \in \mathbb{C}\backslash\{0\}$, the equation $\langle z, te + z\rangle = \|z\|^2$ implies that $\sigma(te + z) \in \mathbb{T}_n$, and then the equation $\langle te, te + z\rangle = |t|^2$ implies that $\sigma(te) \in \mathbb{T}_n$. This completes the proof of the case that $\dim H \geq 2$.

The proof is complete. □

Remark 2.6. Note that in (v) of Theorem 2.5, the phrase "a linear or anti-linear isometry" cannot be replaced either by "a linear isometry" or by "an anti-linear isometry". For example, the mappings $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ defined by $f_1(x) = \overline{x}$ and $f_2(x) = x$ both satisfy Eq. (2.8). However, it is easy to check that for any phase function $\sigma : H \to \mathbb{T}_n$ where $n \geq 3$, $\sigma \cdot f_1$ cannot be linear and $\sigma \cdot f_2$ cannot be anti-linear.

The following corollary describes the continuous solutions of (1.4).

Corollary 2.7. *For $n \geq 3$ and a continuous mapping $f : H \to K$, each of Eqs. (1.4), (2.1), (2.2) and (2.8) holds if and only if f is a linear or anti-linear isometry.*

Proof. By Theorem 2.5, we need only to prove that Eq. (1.4) implies that f is a linear or anti-linear isometry. Assume that f is a continuous mapping satisfying Eq. (1.4). Then there exists a phase function $\sigma : H \to \mathbb{T}_n$ such that $\sigma \cdot f$ is a linear or anti-linear isometry. Thus,

$$\sigma(x)\overline{\sigma(y)}\langle f(x), f(y)\rangle = \langle x, y\rangle \quad (x, y \in H)$$

or

$$\sigma(x)\overline{\sigma(y)}\langle f(x), f(y)\rangle = \langle y, x\rangle \quad (x, y \in H).$$

If $y \neq 0$, then there exists an open ball V_y centered at y such that

$$\sigma(x) = \sigma(y)\frac{\langle x, y\rangle}{\langle f(x), f(y)\rangle} \quad (x \in V_y)$$

or

$$\sigma(x) = \sigma(y)\frac{\langle y, x\rangle}{\langle f(x), f(y)\rangle} \quad (x \in V_y).$$

This, by the continuity of f, shows that σ is continuous on V_y. Since the phase function $\sigma : H \to \mathbb{T}_n$ takes only finite values, we have that σ is constant on V_y. By the fact that $H\backslash\{0\}$ is connected, σ is constant on $H\backslash\{0\}$, f must be a linear or anti-linear isometry. □

Remark 2.8. The proof of Corollary 2.7 is similar to that of [15, Corollary 3], where the case $n = 2$ was considered as follows. For a continuous mapping $f : H \to K$, each of Eqs. (1.3) and (2.6) holds if and only if f is a real linear isometry.

Author contributions All the authors contributed equally and significantly in writing this paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

[1] Almeida, D.F., Sharma, C.S.: The first mathematical proof of Wigner's theorem. J. Nat. Geom. **2**, 113–123 (1992)

[2] Baker, J.A.: Isometries in normed spaces. Am. Math. Monthly **78**, 655–658 (1971)

[3] Bakić, D., Guljaš, B.: Wigner's theorem in Hilbert C^*-modules over C^*-algebras of compact operators. Proc. Am. Math. Soc. **130**(8), 2343–2349 (2002)

[4] Bargmann, V.: Note on Wigner's theorem on symmetry operations. J. Math. Phys. **5**, 862–868 (1964)

[5] Chevalier, G.: Handbook of Quantum Logic and Quantum Structures. Wigner's theorem and its generalizations, pp. 429–475. Elsevier Science BV, Amsterdam (2007)

[6] Freed, D.S.: On Wigner's theorem., vol. 18, pp. 83–89. Geom. Topol. Publ., Coventry (2012)

[7] Gy, P.: Gehér, An elementary proof for the non-bijective version of Wigner's theorem. Phys. Lett. A **378**(30–31), 2054–2057 (2014)

[8] Gy, P.: Gehér, Wigner's theorem on Grassmann spaces. J. Funct. Anal. **273**(9), 2994–3001 (2017)

[9] Győry, M.: A new proof of Wigner's theorem. Rep. Math. Phys. **54**, 159–167 (2004)

[10] Huang, X., Tan, D.: Min-phase-isometries and Wigner's theorem on real normed spaces. Results Math. **77**(4) (2022). Paper No. 152, pp. 15
[11] Huang, X., Tan, D.: Wigner's theorem in atomic L_p-spaces ($p>0$). Publ. Math. Debrecen. **92**(3–4), 411–418 (2018)
[12] Ilišević, D., Omladič, M., Turnšek, A.: Phase-isometries between normed spaces. Linear Algebra Appl. **612**, 99–111 (2021)
[13] Ilišević, D., Turnšek, A.: On Wigner's theorem in strictly convex normed spaces. Publ. Math. Debrecen **97**(3–4), 393–401 (2020)
[14] Li, Y., Tan, D.: Wigner's theorem on the Tsirelson space T. Ann. Funct. Anal. **10**(4), 515–524 (2019)
[15] Maksa, G., Páles, Z.: Wigner's theorem revisited. Publ. Math. Debrecen **81**(1–2), 243–249 (2012)
[16] Molnár, L.: Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem. J. Funct. Anal. **194**(2), 248–262 (2002)
[17] Qian, W., Wang, L., Wu, W., Yuan, W.: Wigner-type theorem on transition probability preserving maps in semifinite factors. J. Funct. Anal. **276**(6), 1773–1787 (2019)
[18] Rätz, J.: On Wigner's theorem: remarks, complements, comments, and corollaries. Aequationes Math. **52**(1–2), 1–9 (1996)
[19] Šemrl, P.: Generalized symmetry transformations on quaternionic indefinite inner product spaces: an extension of quaternionic version of Wigner's theorem. Comm. Math. Phys. **242**(3), 579–584 (2003)
[20] Tan, D., Huang, X.: Phase-isometries on real normed spaces. J. Math. Anal. Appl. **488**(1), 124058 (2020)
[21] Tan, D., Huang, X.: The Wigner property for CL-spaces and finite-dimensional polyhedral Banach spaces. Proc. Edinb. Math. Soc. (2) **64**(2), 183–199 (2021)
[22] Turnšek, A.: A variant of Wigner's functional equation. Aequationes Math. **89**(4), 949–956 (2015)
[23] Wang, R., Bugajewski, D.: On normed spaces with the Wigner property. Ann. Funct. Anal. **11**(3), 523–539 (2020)

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