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Aequationes Mathematicae

Some characterizations of the disc by properties of isoptic triangles

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Abstract. The main result in this article is the following: Let $K \subset \mathbb{R}^2$ be a regular convex body and let α, β, θ, be three angles such that K has α-chords, β-chords, and θ-chords of constant length and $\alpha+\beta+\theta=\pi$, then K is a disc. We also prove another characterization of the disc with respect to properties of its (α,β,θ)-circumscribed triangles.

Mathematics Subject Classification. 52A10.

1. Introduction

Let K be a strictly convex body in the plane, i.e., a compact and convex set with non-empty interior and without segments in its boundary. Denote by $\ell(t)$ the support line of K with outward normal vector $u(t)=(\cos t, \sin t)$, for every real number t. Consider now a triangle $\Delta = \triangle ABC$ with given angles α, β, and θ. For every $t \in [0,2\pi]$ there exists exactly one triangle similar to Δ circumscribed to K, with its side $A(t)C(t)$ over the line $\ell(t)$, and with the angles α, β, and θ in the counter clockwise sense as shown in Fig. 1. We denote such a triangle by $\Delta(t)=A(t)B(t)C(t)$ and name it (α,β,θ)-triangle. Let $D(t)$, $E(t)$, and $F(t)$ be the contact points between K and the sides of $\Delta(t)$. When K is a disc, the following conditions hold:

(1) $\overline{A(t)D(t)}=\overline{A(t)F(t)}$, $\overline{B(t)D(t)}=\overline{B(t)E(t)}$, $\overline{C(t)E(t)}=\overline{C(t)F(t)}$,
(2) $\frac{\overline{A(t)D(t)}}{\overline{D(t)B(t)}}=\lambda_1$, $\frac{\overline{B(t)E(t)}}{\overline{E(t)C(t)}}=\lambda_2$, $\frac{\overline{C(t)F(t)}}{\overline{F(t)A(t)}}=\lambda_3$, for three fixed numbers λ_1, λ_2, λ_3,
(3) $\overline{D(t)F(t)}=\lambda_\alpha$, $\overline{D(t)E(t)}=\lambda_\beta$, $\overline{E(t)F(t)}=\lambda_\theta$, for some fixed numbers λ_α, λ_β, λ_θ,

for every $t \in [0,2\pi]$.

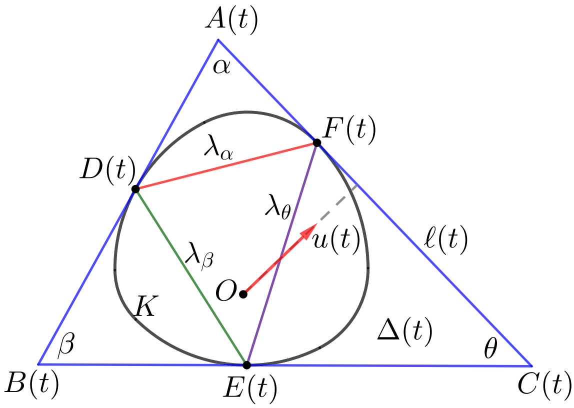

FIGURE 1. A triangle similar to Δ circumscribed to K

However, what happens if for a convex body K, any one of conditions (1), (2) or (3) holds for every $t \in [0, 2\pi]$? Is K a disc?

As we will see in the following sections, the answer is positive and it relies on results about isoptic curves. We may think that if the size of $\Delta(t)$ is independent of t, i.e., it always has the same size, then it is sufficient to ensure that K must be a disc. However, this is not true: Let K be a convex body in the plane and let $\mathcal{P}$ be a convex polygon. It is said that K is a rotor in $\mathcal{P}$ if for every rotation ρ, there is a translate of $\mathcal{P}$ that contains $\rho(K)$ and all sides of $\mathcal{P}$ are tangent to K. In the case where the polygon $\mathcal{P}$ is a triangle with angles α, β, θ, it is known that there exist rotors different from discs if $\frac{\alpha}{\pi}$, $\frac{\beta}{\pi}$, and $\frac{\theta}{\pi}$ are all rational numbers, see for instance [3], and for the particular case of rotors in equilateral triangles see [11].

Similar problems were recently studied: a convex body K is a disc if for some angles $\alpha \in (0, \pi)$ and $\beta = \alpha$, it holds that $\overline{A(t)D(t)} = \overline{D(t)B(t)}$, for every $t \in [0, 2\pi]$ (see [9]). If for some $\alpha \in (0, \pi)$ it holds that $\overline{B(t)D(t)}$ has a constant value for every $t \in [0, 2\pi]$, K is a disc (see [4]). If $D(t)F(t) = \lambda_\alpha$, for every $t \in [0, 2\pi]$ and for a constant number λ_α, and K has constant width or has rotational symmetry of angle $\pi - \alpha$, then K is a disc (see [7]). We can see that condition (1) implies that K must be a disc: just notice that the points $D(t)$, $E(t)$, $F(t)$, are points of contact between the incircle of $\triangle A(t)B(t)C(t)$ and its sides. Now we use Lemma 3.3 in [6] and conclude that K is a disc. However, there are convex bodies different from discs, for which $\overline{A(t)D(t)} = \overline{A(t)F(t)}$ for every $t \in [0, 2\pi]$ (see [8]). Indeed, there are convex bodies, different from discs, for which this condition holds for three (or more) different angles $\alpha, \beta, \theta \in (0, \pi)$, but in this case the condition $\alpha + \beta + \theta = \pi$ does not hold.

The main purpose of this paper is to prove that a convex body for which condition (3) holds, must be a disc.

2. Basic concepts of isoptic curves

The function $p : \mathbb{R} \longrightarrow \mathbb{R}$, defined as $p(t) = \max_{x\in K}\langle u(t), x\rangle$, is known by the name of support function of K. When the origin O is contained in K, $p(t)$ is nothing else but the distance from O to the support line $\ell(t)$. The distance between the support lines $\ell(t)$ and $\ell(t+\pi)$ is called the width of K in direction $u(t)$ and it is denoted by $w(t)$, in other words, $w(t) = p(t)+p(t+\pi)$. If $w(t)$ is constant, independently of t, we say that K is a body of constant width. For any $\alpha \in (0, \pi)$, the α-isoptic K_α of K is defined as the locus of points at which two tangent lines to K intersect at an angle α. Using the support function, ∂K is parameterized (see for instance [10]) by

$$\gamma(t) = p(t)u(t) + p'(t)u'(t), \text{ for } t \in [0, 2\pi].$$

The isoptic curve K_α can be parameterized by the same angle by the formula (see [2] or [8])

$$\gamma_\alpha(t) = p(t)u(t) + \left[p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\pi-\alpha)\right]u'(t).$$

By Cauchy's formula, the perimeter of K can be obtained by (see [10])

$$L(K) = \int_0^{2\pi} p(t)dt. \tag{1}$$

For any $t \in \mathbb{R}$ we define (see Fig. 2)

$$\begin{aligned} a_\alpha(t) &= |\gamma_\alpha(t) - \gamma(t)|,\\ b_\alpha(t) &= |\gamma_\alpha(t) - \gamma(t+\pi-\alpha)|,\\ q_\alpha(t) &= |\gamma(t) - \gamma(t+\pi-\alpha)|. \end{aligned}$$

By some simple calculations we can express the lengths $a(t)$ and $b(t)$ in terms of the support function of K:

$$a_\alpha(t) = \frac{1}{\sin\alpha}[p(t+\pi-\alpha) + p(t)\cos\alpha - p'(t)\sin\alpha], \tag{2}$$

$$b_\alpha(t) = \frac{1}{\sin\alpha}[p(t+\pi-\alpha)\cos\alpha + p'(t+\pi-\alpha)\sin\alpha + p(t)]. \tag{3}$$

3. Some useful lemmas about isoptic curves

Lemma 1. *Let K be a strictly convex body in the plane and let $\alpha, \theta \in (0, \pi)$ be angles such that $a_\alpha(t) = a_\alpha(t+\theta)$, for every $t \in [0, 2\pi]$. Then, K has rotational symmetry of angle θ.*

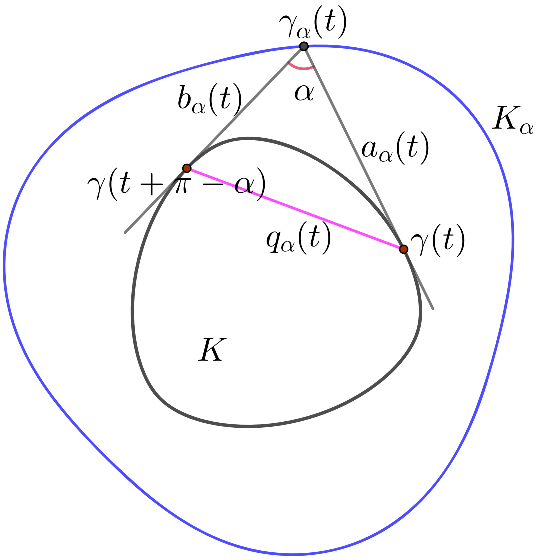

FIGURE 2. Parameters of the isoptic curve

Proof. Recall that $a_\alpha(t) = \frac{1}{\sin\alpha}[p(t+\pi-\alpha) + p(t)\cos\alpha - p'(t)\sin\alpha]$. The hypothesis $a_\alpha(t) = a_\alpha(t+\theta)$ implies that

$$\begin{aligned} p(t+\pi-\alpha) + p(t)\cos\alpha - p'(t)\sin\alpha = p(t+\theta+\pi-\alpha) \\ + p(t+\theta)\cos\alpha - p'(t+\theta)\sin\alpha, \end{aligned}$$

which is equivalent to

$$[p'(t+\theta) - p'(t)]\sin\alpha = [p(t+\theta) - p(t)]\cos\alpha + p(t+\theta+\pi-\alpha) - p(t+\pi-\alpha).$$

Let $y(t) = p(t+\theta) - p(t)$. Using the previous equality we obtain the following differential equation

$$y'(t)\sin\alpha = y(t)\cos\alpha + y(t+\pi-\alpha). \tag{4}$$

We express y and y' in terms of their Fourier Series:

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

$$y'(t) = \sum_{n=-\infty}^{\infty} nic_n e^{int}.$$

Then,

$$\sum_{n=-\infty}^{\infty} ni\sin\alpha c_n e^{int} = \sum_{n=-\infty}^{\infty} \cos\alpha c_n e^{int} + \sum_{n=-\infty}^{\infty} (-1)^n e^{-in\alpha} c_n e^{int}$$

and we have

$$ni \sin \alpha c_n = \cos \alpha + (-1)^n e^{-in\alpha} c_n.$$

We claim that $c_n = 0$ for every n not equal to 1 or -1. Suppose this is not the case. Then,

$$ni \sin \alpha = \cos \alpha + (-1)^n \cos n\alpha - (-1)^n i \sin(n\alpha).$$

Equivalently

$$n \sin \alpha = -(-1)^n \sin(n\alpha) \text{ and } \cos \alpha + (-1)^n \cos n\alpha = 0.$$

Notice that the first equation is not satisfied when $n \neq 0, 1, -1$ while the second is false for $n = 0$. The claim now follows and we conclude that

$$p(t + \theta) - p(t) = y(t) = c_{-1} e^{-it} + c_1 e^{it} = a_1 \cos t + b_1 \sin t,$$

where $c_1 = a_1 - b_1 i = \overline{c_{-1}}$. Using vector notation we obtain

$$p(t + \theta) = p(t) + \langle (a_1, b_1), u(t) \rangle.$$

Let t_0 be such that $u(t_0 + \theta)$ is parallel to (a_1, b_1). Notice that

$$\begin{aligned}
p(t_0 + 2\theta) &= p(t_0 + \theta) + \langle (a_1, b_1), u(t_0 + \theta) \rangle = p(t_0 + \theta) + ||(a_1, b_1)||, \\
p(t_0 + 3\theta) &= p(t_0 + 2\theta) + ||(a_1, b_1)|| = p(t_0 + \theta) + 2||(a_1, b_1)||, \\
&\vdots \\
p(t_0 + n\theta) &= p(t_0 + \theta) + (n - 1)||(a_1, b_1)||.
\end{aligned}$$

Since the support function is bounded, we must have $(a_1, b_1) = (0, 0)$. We conclude that $p(t + \theta) = p(t)$ for every t, which implies that K has rotational symmetry of angle θ. □

As an application of Lemma 1, we have the following characterization of the disc, which we will prove for every kind of triangles in the following section.

Proposition 1. *Let K be a convex body and $\alpha, \beta \in (0, \pi)$ such that $2\alpha + \beta = \pi$. Suppose that $q_\alpha(t) = \lambda_\alpha$ and $q_\beta(t) = \lambda_\beta$, for every $t \in [0, 2\pi]$ and for some positive numbers λ_α and λ_β. Then K is a disc.*

Proof. Let $\measuredangle \gamma(t)\gamma(t+\pi-\alpha)\gamma(t-2\alpha) = \phi$ and $\measuredangle \gamma(t)\gamma(t+\pi-\alpha)\gamma_\alpha(t) = \alpha_1(t)$, for every $t \in [0, 2\pi]$, as shown in Fig. 3. Notice that $\alpha_1(t + \pi - \alpha) = \alpha_1(t) + \phi - \alpha$, for every t. Let $t_1, t_2 \in [0, 2\pi]$ be such that $\alpha(t_1) \leq \alpha(t) \leq \alpha(t_2)$ for every $t \in [0, 2\pi]$. Since $\alpha_1(t_1 + \pi - \alpha) = \alpha_1(t_1) + \phi - \alpha$, we must have that $\phi - \alpha \geq 0$. Similarly, $\alpha(t_2 + \pi - \alpha) = \alpha_1(t_2) + \phi - \alpha$ implies that $\phi - \alpha \leq 0$. We conclude that $\alpha = \phi$ and $\alpha_1(t + \pi - \alpha) = \alpha_1(t)$ for every $t \in [0, 2\pi]$. Using the previous equality and the Law of sines for the triangles $\Delta\gamma_\alpha(t+\pi-\alpha)\gamma(t-2\alpha)\gamma(t+\pi-\alpha)$

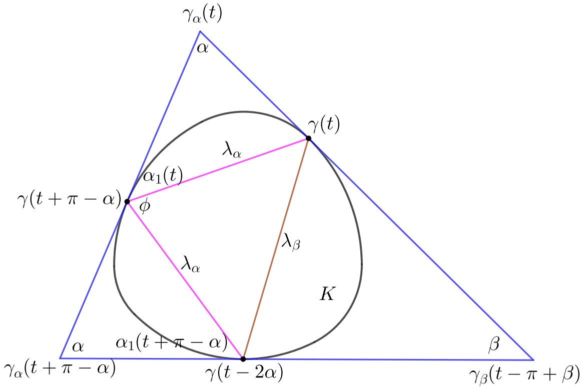

Figure 3. An isosceles circumscribed triangle

and $\Delta\gamma_\alpha(t)\gamma(t+\pi-\alpha)\gamma(t)$ we conclude that for every t the following equalities hold

$$\frac{a_\alpha(t)}{\sin\alpha_1(t)} = \frac{\lambda_\alpha}{\sin\alpha} = \frac{a_\alpha(t+\pi-\alpha)}{\sin\alpha_1(t)}.$$

It follows that $a_\alpha(t) = a_\alpha(t+\pi-\alpha)$ for every t. By Lemma 1, K has rotational symmetry of angle α. The result now follows from Theorem 2 in [7]. □

Lemma 2. *Let K be a strictly convex body in the plane and let $\alpha \in (0,\pi)$ be a given angle. Then there exist two real numbers $t_0, t_1 \in [0, 2\pi]$ such that $a_\alpha(t_0) = b_\alpha(t_0)$ and $a_\alpha(t_1+\pi-\alpha) = b_\alpha(t_1)$.*

Proof. From (2) and (3) we have that

$$\sin\alpha[a_\alpha(t) - b_\alpha(t)]$$
$$= p(t+\pi-\alpha)(1-\cos\alpha) - p(t)(1-\cos\alpha) - p'(t)\sin\alpha - p'(t+\pi-\alpha)\sin\alpha.$$

By Cauchy's formula for the perimeter and since p is a periodic function with period equal to 2π, we have that

$$\int_0^{2\pi} \sin\alpha[a_\alpha(t) - b_\alpha(t)]dt$$
$$= L(K)(1-\cos\alpha) - L(K)(1-\cos\alpha) - p(t)\sin\alpha|_0^{2\pi} - p(t+\pi-\alpha)\sin\alpha|_0^{2\pi},$$

hence

$$\int_0^{2\pi} \sin\alpha[a_\alpha(t) - b_\alpha(t)]dt = 0.$$

Since a and b are continuous functions, we have that there exists a number t_0 such that $a_\alpha(t_0) = b_\alpha(t_0)$.

The proof of the existence of t_1 such that $a_\alpha(t_1 + \pi - \alpha) = b_\alpha(t_1)$ is completely analogous. □

The following lemma gives another characterization of the disc.

Lemma 3. *Let K be a strictly convex body in the plane and let $\alpha \in (0, \pi/2)$ be a given angle. Suppose $a_\alpha(t) = \lambda a_{\pi-\alpha}(t)$, for every $t \in [0, 2\pi]$ and for $\lambda > 1$. Then K is a disc.*

Proof. We know that

$$a_\alpha(t) = \frac{1}{\sin\alpha}[p(t + \pi - \alpha) + p(t)\cos\alpha - p'(t)\sin\alpha]$$

and

$$\lambda a_{\pi-\alpha}(t) = \frac{\lambda}{\sin\alpha}[p(t + \alpha) - p(t)\cos\alpha - p'(t)\sin\alpha].$$

Then,

$$p(t + \pi - \alpha) + p(t)\cos\alpha - p'(t)\sin\alpha = \lambda p(t + \alpha) - \lambda p(t)\cos\alpha - \lambda p'(t)\sin\alpha,$$

or

$$(\lambda - 1)p'(t)\sin\alpha + p(t + \pi - \alpha) - \lambda p(t + \alpha) + (\lambda + 1)p(t)\cos\alpha = 0. \quad (5)$$

Let the Fourier series of p be given by

$$p(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

By equation (5) we have

$$\sum_{n=-\infty}^{\infty} i(\lambda - 1)n\sin\alpha c_n e^{int} + \sum_{-\infty}^{\infty}(-1)^n c_n e^{-in\alpha}e^{int} - \sum_{n=-\infty}^{\infty} \lambda c_n e^{in\alpha}e^{int}$$
$$+ \sum_{n=-\infty}^{\infty} i(\lambda + 1)\cos\alpha c_n e^{int} = 0.$$

We conclude that

$$[i(\lambda - 1)n\sin\alpha + (-1)^n e^{-in\alpha} - \lambda e^{in\alpha} + (\lambda + 1)\cos\alpha]c_n = 0,$$

i.e.,

$$[i(\lambda - 1)n\sin\alpha + (-1)^n\cos n\alpha - (-1)^n i\sin n\alpha - \lambda\cos n\alpha - i\lambda\sin n\alpha$$
$$+ (\lambda + 1)\cos\alpha]c_n = 0.$$

For $n = 0$ we obtain

$$1 - \lambda + (\lambda + 1)\cos\alpha = 0,$$

which implies

$$\lambda = \frac{1+\cos\alpha}{1-\cos\alpha}. \tag{6}$$

Notice that if $c_n \neq 0$ we must have

$(\lambda-1)n\sin\alpha - (\lambda+(-1)^n)\sin n\alpha = 0$ and $((-1)^n-\lambda)\cos n\alpha + (\lambda+1)\cos\alpha = 0.$

Now, if $n \neq 1$ is an odd natural number, the first equation simplifies to

$$n\sin\alpha + \sin n\alpha = 0,$$

which is never satisfied. It follows that $c_n = 0$ for every odd natural number $n \neq 1$. On the other hand, for every natural even number $n \neq 0$, we have

$$(1-\lambda)\cos n\alpha + (\lambda+1)\cos\alpha = 0.$$

Using equation (6) we conclude that

$$\cos n\alpha = -1,$$

which is impossible, since $0 < \alpha < \pi/2$. It follows that $c_n = 0$ for every even number $n > 0$ and that $p(t) = c_0 + ce^{it}$. Thus, K is a disc. □

4. Main results

The first result we prove here is concerns property (2) mentioned in the introduction.

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a strictly convex body and let $\alpha, \beta, \theta \in (0,\pi)$ be three angles such that $\alpha+\beta+\theta = \pi$. Suppose that for every $t \in [0, 2\pi]$*

$$\frac{b_\alpha(t)}{a_\beta(t+\pi-\alpha)} = \lambda_1, \quad \frac{b_\beta(t+\pi-\alpha)}{a_\theta(t-\pi+\theta)} = \lambda_2, \quad \frac{b_\theta(t-\pi+\theta)}{a_\alpha(t)} = \lambda_3,$$

for some constants λ_1, λ_2, and λ_3. Then K is a disc.

Proof. Since all (α,β,θ)-triangles are similar and the points $\gamma(t+\pi-\alpha)$, $\gamma(t-\pi+\theta)$, $\gamma(t)$, divide the corresponding sides in the given ratios λ_1, λ_2, λ_3, we have that

$$\frac{a_\alpha(t)}{b_\alpha(t)}, \quad \frac{a_\beta(t+\pi-\alpha)}{b_\beta(t+\pi-\alpha)}, \quad \text{and} \quad \frac{a_\theta(t-\pi+\theta)}{b_\theta(t-\pi+\theta)}$$

are also constant. Now, by Lemma 2 we have that $a_\alpha(t) = b_\alpha(t)$, $a_\beta(t) = b_\beta(t)$, $a_\theta(t) = b_\theta(t)$ for every $t \in [0, 2\pi]$. It follows that the points $\gamma(t+\pi-\alpha)$, $\gamma(t-\pi+\theta)$, $\gamma(t)$ are the contact points between the incircle of the triangle $\triangle\gamma_\alpha(t)\gamma_\beta(t+\pi-\alpha)\gamma_\theta(t-\pi+\theta)$ and its sides, for every $t \in [0, 2\pi]$. Now, the hypothesis of Lemma 3.3 in [6] holds, and so we conclude that K is a disc. □

Now we present the main result of this work.

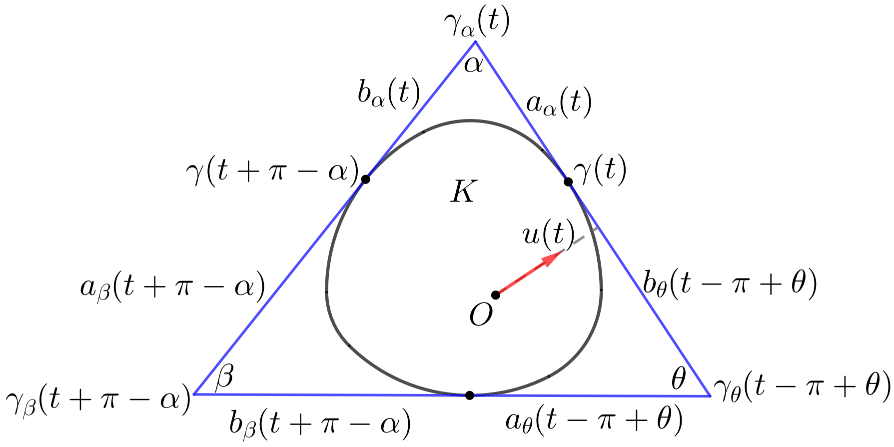

FIGURE 4. $\gamma(t+\pi-\alpha)$, $\gamma(t-\pi+\theta)$, $\gamma(t)$, divide the corresponding sides in the given ratios

Theorem 2. *Let $K \subset \mathbb{R}^2$ be a regular convex body and let $\alpha, \beta, \theta \in (0, \pi)$ be three angles such that $\alpha + \beta + \theta = \pi$. Suppose that for every $t \in [0, 2\pi]$, $q_\alpha(t) = \lambda_\alpha$, $q_\beta(t) = \lambda_\beta$, $q_\theta(t) = \lambda_\theta$, for some constants λ_α, λ_β, and λ_θ. Then K is a disc.*

In the proof of Theorem 2 we will use the following lemma (see [11]). For the sake of completeness we give a proof here.

Lemma 4. *Let C be one of the points of intersection between two circles Γ_1 and Γ_2 with centres O_1 and O_2, respectively. The unique chord AB, with $A \in \Gamma_1$, $B \in \Gamma_2$, through C and with maximum length is obtained when AB is orthogonal to the common chord between Γ_1 and Γ_2, i.e., when AB is parallel to O_1O_2.*

Proof. Let AB be any chord through C, as shown in Fig. 5. Let M_1 and M_2 be the orthogonal projections of O_1 and O_2 onto AB. We know that the length of M_1M_2 is half the length of AB. Suppose the orthogonal projection, T, of O_1 onto the line O_2M_2 lies in the segment O_2M_2. Since $O_1M_1M_2T$ is a rectangle, we have that the lengths of O_1T and M_1M_2 are equal. From here we see that the maximum length of M_1M_2 and hence of AB is when AB is parallel to O_1O_2. The case when the orthogonal projection of O_2 onto the line O_1M_1 lies in the segment O_1M_1, is completely analogous. □

Proof of Theorem 2. Let $t \in [0, 2\pi]$ be any angle and let $\Delta(t) = \triangle\gamma_\alpha(t)\gamma_\beta(t+\pi-\alpha)\gamma_\theta(t-\pi+\theta)$ be the corresponding circumscribed (α, β, θ)-triangle. The contact points between the sides of $\Delta(t)$ and the boundary of K are $\gamma(t)$,

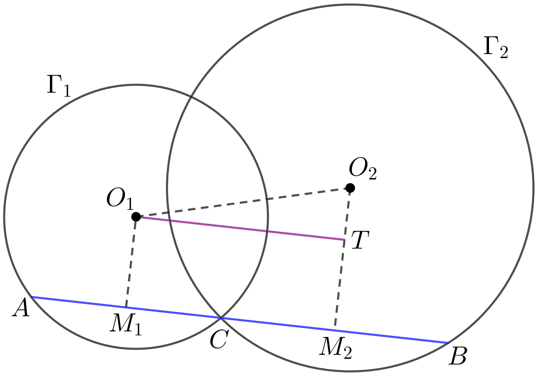

FIGURE 5. Maximum chord AB is when AB is parallel to O_1O_2

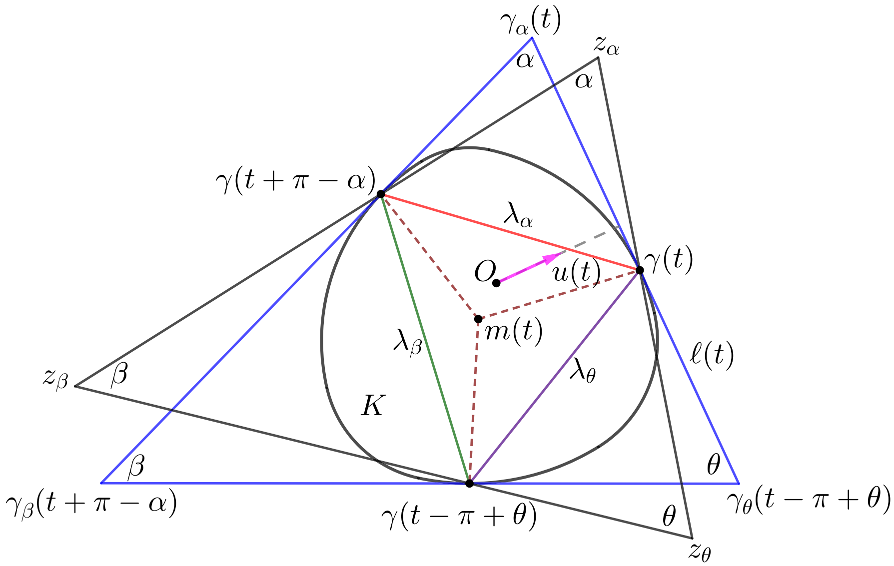

FIGURE 6. All circumscribed (α,β,θ)-triangles are maximal

$\gamma(t+\pi-\alpha)$, and $\gamma(t-\pi+\theta)$, respectively. By Miquel's theorem we know that there exists a point $m(t)$ in common to the circumcircles of triangles $\triangle\gamma_\alpha(t)\gamma(t+\pi-\alpha)\gamma(t)$, $\triangle\gamma_\beta(t+\pi-\alpha)\gamma(t-\pi+\theta)\gamma(t+\pi-\alpha)$, and $\triangle\gamma_\theta(t-\pi+\theta)\gamma(t)\gamma(t-\pi+\theta)$. By Lemma 4 we obtain that the maximum (α,β,θ)-triangle circumscribed to $\triangle\gamma(t)\gamma(t+\pi-\alpha)\gamma(t-\pi+\theta)$ is obtained when $[m(t),\gamma(t)]$, $[m(t),\gamma(t+\pi-\alpha)]$, $[m(t),\gamma(t-\pi+\theta)]$ are orthogonal to the corresponding sides of $\Delta(t)$. Suppose that this is not the case and let $\triangle z_\alpha z_\beta z_\theta$ be the maximum $((\alpha, \beta, \theta)$-triangle circumscribed to $\triangle\gamma(t)\gamma(t+\pi-\alpha)\gamma(t-\pi+\theta)$, as shown in Fig. 6. Since the boundary of K is regular, the sides of $\triangle z_\alpha z_\beta z_\theta$ intersect the interior of K. If we consider the corresponding support lines of K, parallel to the sides of $\triangle z_\alpha z_\beta z_\theta$, we obtain an (α, β, θ)-triangle circumscribed to K with size bigger than the size of $\triangle z_\alpha z_\beta z_\theta$. This is a contradiction since such a triangle must touch the boundary of K in three points which are vertices of a triangle congruent to $\Delta(t)$. It follows that the triangle $\Delta(t)$, for every $t \in [0, 2\pi]$, is maximal. In particular, we have that the length of $a_\alpha(t)$ is constant for every $t \in [0, 2\pi]$. We apply Lemma 2 in [4] and conclude that K is a disc. □

Remark 1. Indeed, we have that the points $\gamma(t)$, $\gamma(t+\pi-\alpha)$, and $\gamma(t-\pi+\theta)$ are the contact points between the incircle of $\Delta(t)$ and the sides of $\Delta(t)$. We can also conclude the proof of the theorem using Lemma 3.3 in [6].

Author contributions JJ and FJ wrote the manuscript and JJ produce the figures. All authors reviewed the manuscript

Declarations

Conflict of interest The authors declare no competing interests.

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