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Aequationes Mathematicae

Some sufficient conditions for path-factor uniform graphs

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Abstract. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph H of G is called an $\mathcal{H}$-factor of G if each component of H is isomorphic to an element of $\mathcal{H}$. A graph G is called an $\mathcal{H}$-factor uniform graph if for any two edges e_1 and e_2 of G, G has an $\mathcal{H}$-factor covering e_1 and excluding e_2. Let each component in $\mathcal{H}$ be a path with at least d vertices, where $d \geq 2$ is an integer. Then an $\mathcal{H}$-factor and an $\mathcal{H}$-factor uniform graph are called a $P_{\geq d}$-factor and a $P_{\geq d}$-factor uniform graph, respectively. In this article, we verify that (i) a 2-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if $\delta(G) > \frac{\alpha(G)+4}{2}$; (ii) a $(k+2)$-connected graph G of order n with $n \geq 5k+3-\frac{3}{5\gamma-1}$ is a $P_{\geq 3}$-factor uniform graph if $|N_G(A)| > \gamma(n-3k-2)+k+2$ for any independent set A of G with $|A| = \lfloor \gamma(2k+1) \rfloor$, where k is a positive integer and γ is a real number with $\frac{1}{3} \leq \gamma \leq 1$.

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1. Introduction

The graphs considered here are finite, undirected and simple. Let G be a graph with edge set $E(G)$ and vertex set $V(G)$. We use $i(G)$, $\omega(G)$, $\alpha(G)$ and $\delta(G)$ to denote the number of isolated vertices, the number of connected components, the independence number and the minimum degree of G, respectively. Let $N_G(x)$ denote the set of neighbours of a vertex x in G. By $d_G(x)$ we mean $|N_G(x)|$ and we call it the degree of a vertex x in G. For any $X \subseteq V(G)$ or $X \subseteq E(G)$ the symbol $G[X]$ denotes the subgraph of G induced by X. We write $N_G(X) = \bigcup_{x \in X} N_G(x)$ and $G - X = G[V(G) \setminus X]$ for $X \subseteq V(G)$, and denote by $G - X$ the subgraph derived from G by deleting edges of X for $X \subseteq E(G)$. The edge joining vertices x and y is denoted by xy. A vertex subset X of G is called an independent set if $X \cap N_G(X) = \emptyset$. Let P_n and K_n denote the path and the complete graph with n vertices, respectively. We denote by $K_{m,n}$ the complete bipartite graph with the bipartition (X, Y), where $|X| = m$ and $|Y| = n$. Let G_1 and G_2 be two graphs. By $G_1 \cup G_2$ we mean a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. By $G_1 \vee G_2$ we mean a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{e = xy : x \in V(G_1), y \in V(G_2)\}$. Recall that $\lfloor r \rfloor$ is the greatest integer with $\lfloor r \rfloor \leq r$, where r is a real number.

A subgraph of G is spanning if the subgraph includes all the vertices of G. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph H of G is called an $\mathcal{H}$-factor of G if each component of H is isomorphic to an element of $\mathcal{H}$. A graph G is called an $\mathcal{H}$-factor covered graph if G admits an $\mathcal{H}$-factor covering e for any $e \in E(G)$. A graph G is called an $\mathcal{H}$-factor uniform graph if $G - e$ is an $\mathcal{H}$-factor covered graph for any $e \in E(G)$. Let each component in $\mathcal{H}$ be a path with at least d vertices, where $d \geq 2$ is an integer. Then an $\mathcal{H}$-factor, an $\mathcal{H}$-factor covered graph and an $\mathcal{H}$-factor uniform graph are called a $P_{\geq d}$-factor, a $P_{\geq d}$-factor covered graph and a $P_{\geq d}$-factor uniform graph, respectively.

Amahashi and Kano [1] derived a characterization for a graph with a $\{K_{1,l} : 1 \leq l \leq m\}$-factor. Kano and Saito [11] posed a sufficient condition for the existence of $\{K_{1,l} : m \leq l \leq 2m\}$-factors in graphs. Kano, Lu and Yu [10] investigated the existence of $\{K_{1,2}, K_{1,3}, K_5\}$-factors and $P_{\geq 3}$-factors in graphs depending on the number of isolated vertices. Bazgan et al. [2] put forward a toughness condition for a graph to have a $P_{\geq 3}$-factor. Zhou, Bian and Pan [22], Zhou, Wu and Bian [28], Zhou, Wu and Xu [30], Wang and Zhang [13], Zhou [20] obtained some results on $P_{\geq 3}$-factors in graphs with given properties. Johansson [7] presented a sufficient condition for a graph to have a path-factor. Gao, Chen and Wang [4] showed an isolated toughness condition for the existence of $P_{\geq 3}$-factors in graphs with given properties. Kano, Lee and Suzuki [9] verified that each connected cubic bipartite graph with at least eight vertices admits a $P_{\geq 8}$-factor. Wang and Zhang [14], Zhou and Liu [23] presented some degree conditions for the existence of graph factors. Zhou, Wu and Liu [29], Zhou [21], Yuan and Hao [16] established some relationships between independence numbers and graph factors. Enomoto, Plummer and Saito [3], Zhou, Liu and Xu [25], Zhou [18,19], Zhou and Sun [26] derived some neighborhood conditions for the existence of graph factors. some other results on graph factors can be found in Wang and Zhang [15], Zhou and Liu [24].

A graph H is factor-critical if $H - x$ has a perfect matching for each $x \in V(H)$. To characterize a graph with a $P_{\geq 3}$-factor, Kaneko [8] introduced the concept of a sun. A sun is a graph formed from a factor-critical graph H by adding n new vertices $x_1, x_2, \dots, x_n$ and n new edges $y_1x_1, y_2x_2, \dots, y_nx_n$, where $V(H) = \{y_1, y_2, \dots, y_n\}$. According to Kaneko, K_1 and K_2 are also suns. A sun with at least six vertices is called a big sun. A component of G is called a sun component if it is isomorphic to a sun. Let $sun(G)$ denote the number of sun components of G. Kaneko [8] put forward a criterion for a graph with a $P_{\geq 3}$-factor.

Theorem 1.1. [8]. *A graph G admits a $P_{\geq 3}$-factor if and only if*

$$sun(G - X) \leq 2|X|$$

for all $X \subseteq V(G)$.

Later, Zhou and Zhang [17] *improved Theorem* 1.1 *and acquired a criterion for a $P_{\geq 3}$-factor covered graph.*

Theorem 1.2. [17]. *Let G be a connected graph. Then G is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G - X) \leq 2|X| - \varepsilon(X)$$

for any vertex subset X of G, where $\varepsilon(X)$ is defined by

$$\varepsilon(X) = \begin{cases} 2, \text{ if } X \text{ is not an independent set}; \\ 1, \text{ if } X \text{ is a nonempty independent set and } G - X \text{ has} \\ \quad \text{a non} - \text{sun component}; \\ 0, \text{ otherwise}. \end{cases}$$

Zhou and Sun [27] *got a binding number condition for the existence of $P_{\geq 3}$-factor uniform graphs. Gao and Wang* [5]*, Liu* [12] *improved the above result on $P_{\geq 3}$-factor uniform graphs. Hua* [6] *investigated the relationship between isolated toughness and $P_{\geq 3}$-factor uniform graphs. It is natural and interesting to put forward some new sufficient conditions to guarantee that a graph is a $P_{\geq 3}$-factor uniform graph. In this article, we proceed to study $P_{\geq 3}$-factor uniform graphs and pose some new graphic parameter conditions for the existence of $P_{\geq 3}$-factor uniform graphs, which are shown in the following.*

Theorem 1.3. *Let G be a 2-edge-connected graph. If G satisfies*

$$\delta(G) > \frac{\alpha(G) + 4}{2},$$

then G is a $P_{\geq 3}$-factor uniform graph.

Theorem 1.4. *Let k be a positive integer and γ be a real number with $\frac{1}{3} \leq \gamma \leq 1$, and let G be a $(k+2)$-connected graph of order n with $n \geq 5k + 3 - \frac{3}{5\gamma - 1}$. If*

$$|N_G(A)| > \gamma(n - 3k - 2) + k + 2$$

for any independent set A of G with $|A| = \lfloor \gamma(2k+1) \rfloor$, then G is a $P_{\geq 3}$-factor uniform graph.

The proofs of Theorems 1.3 *and* 1.4 *will be given in Sections 2 and 3.*

2. The proof of Theorem 1.3

Proof of Theorem 1.3. For any $e = xy \in E(G)$, let $G' = G - e$. To verify Theorem 1.3, we only need to prove that G' is a $P_{\geq 3}$-factor covered graph. Suppose, to the contrary, that G' is not a $P_{\geq 3}$-factor covered graph. Then it follows from Theorem 1.2 that

$$sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \tag{2.1}$$

for some vertex subset X of G'.
Claim 1. $X \neq \emptyset$.

Proof. Assume that $X = \emptyset$. Then from (2.1) and $\varepsilon(X) = 0$ we have $sun(G') \geq 1$. On the other hand, since G is 2-edge-connected, G' is connected, which implies that $\omega(G') = 1$. Thus, we derive that $1 \leq sun(G') \leq \omega(G') = 1$, that is, $sun(G') = 1$. Note that $|V(G')| = |V(G)| \geq 3$ by G being a 2-edge-connected graph. Hence, G' is a big sun, which implies that there exist at least three vertices x_1, x_2, x_3 with $d_{G'}(x_i) = 1$, $i = 1, 2, 3$. Thus, there exists at least one vertex with degree 1 in G, which contradicts that G is 2-edge-connected. Claim 1 is proved. □

Claim 2. $|X| \geq 2$.

Proof. Let $|X| \leq 1$. Combining this with Claim 1, we get $|X| = 1$.

If $G' - X$ admits a non-sun component, then $\varepsilon(X) = 1$ by the definition of $\varepsilon(X)$. According to (2.1) and $\varepsilon(X) = 1$, we obtain

$$sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 = 2|X| = 2. \tag{2.2}$$

Note that $G' - X$ includes a non-sun component. Combining this with (2.2), we get

$$\alpha(G') \geq sun(G' - X) + 1. \tag{2.3}$$

Since $G' = G - e$, we deduce $\alpha(G) \geq \alpha(G') - 2$. Then using (2.2) and (2.3), we infer

$$\alpha(G) \geq \alpha(G') - 2 \geq sun(G' - X) - 1 \geq 2 - 1 = 1. \tag{2.4}$$

By virtue of (2.2), $G' - X$ has at least two sun components, which implies that $G - X$ admits one vertex v with $d_{G-X}(v) = 1$. Thus, we derive

$$\delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| = |X| + 1 = 2. \tag{2.5}$$

It follows from (2.4), (2.5) and $\delta(G) > \frac{\alpha(G)+4}{2}$ that

$$2 \geq \delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{5}{2},$$

which is a contradiction.

If $G' - X$ does not admit a non-sun component, then $\varepsilon(X) = 0$ by the definition of $\varepsilon(X)$. By means of (2.1), $|X| = 1$ and $\varepsilon(X) = 0$, we get

$$\alpha(G') \geq sun(G' - X) \geq 2|X| + 1 = 3. \tag{2.6}$$

From (2.6), we have

$$\alpha(G) \geq \alpha(G') - 2 \geq 3 - 2 = 1. \tag{2.7}$$

Note that $sun(G - X) \geq sun(G' - X) - 2 \geq 3 - 2 = 1$ by (2.6), which implies that $G - X$ has at least one vertex v with $d_{G-X}(v) \leq 1$. Thus, we infer

$$\delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| \leq |X| + 1 = 2. \tag{2.8}$$

In terms of (2.7), (2.8) and $\delta(G) > \frac{\alpha(G)+4}{2}$, we derive

$$2 \geq \delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{5}{2},$$

which is a contradiction. This completes the proof of Claim 2. □

Suppose that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \ldots, H_c$, where $|V(H_i)| \geq 6$, in $G' - X$, and so

$$sun(G' - X) = a + b + c. \tag{2.9}$$

It follows from (2.1), (2.9), $\varepsilon(X) \leq 2$ and Claim 2 that

$$a + b + c = sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \geq 3. \tag{2.10}$$

Claim 3. $\delta(G) \leq |X| + 1$.

Proof. If $a \neq 0$, then $d_{G'-X}(v) = 0$ for any $v \in V(aK_1)$. Note that $G' = G - e$. Thus, we infer $d_{G-X}(v) \leq 1$ for any $v \in V(aK_1)$, and so

$$\delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| \leq |X| + 1.$$

If $a = 0$, then $b + c \neq 0$, which implies that $G' - X$ admits at least two vertices with degree 1, and so $G - X$ has at least one vertex v with $d_{G-X}(v) = 1$. Thus, we obtain

$$\delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| = |X| + 1.$$

This completes the proof of Claim 3. □

Next, we consider two cases by the value of $a + c$.

Case 1. $a + c = 0$.

In this case, $b \geq 3$ by (2.10).

Claim 4. $\alpha(G) \geq b$.

Proof. If $x \notin V(bK_2)$ or $y \notin V(bK_2)$, then we easily see that $\alpha(G) \geq b$. If $x \in V(bK_2)$ and $y \in V(bK_2)$, then $G - X$ has $(b-2)$ K_2's and a P_4 component, and so we easily see that $\alpha(G) \geq (b - 2) + 2 = b$. We have finished the proof of Claim 4. □

According to (2.10), $a+c=0$, Claim 4 and $\delta(G) > \frac{\alpha(G)+4}{2}$, we deduce

$$\begin{aligned}\delta(G) &> \frac{\alpha(G)+4}{2} \geq \frac{b+4}{2} = \frac{a+b+c+4}{2}\\ &\geq \frac{2|X|-1+4}{2} = \frac{2|X|+3}{2} > |X|+1,\end{aligned}$$

which contradicts Claim 3.

Case 2. $a+c \neq 0$.

Subcase 2.1. $a \neq 0$.

If $x \notin V(aK_1)$ and $y \notin V(aK_1)$, then $d_{G-X}(v) = 0$ for any $v \in V(aK_1)$. Thus, we derive

$$\delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| = |X|. \tag{2.11}$$

It follows from (2.10), (2.11), $\delta(G) > \frac{\alpha(G)+4}{2}$ and $\alpha(G) \geq sun(G-X) \geq sun(G'-X) - 2$ that

$$\begin{aligned}|X| \geq \delta(G) &> \frac{\alpha(G)+4}{2} \geq \frac{sun(G'-X)-2+4}{2} = \frac{sun(G'-X)+2}{2}\\ &\geq \frac{2|X|-1+2}{2} = |X| + \frac{1}{2},\end{aligned}$$

which is a contradiction. In what follows, we discuss the case with $x \in V(aK_1)$ or $y \in V(aK_1)$. Without loss of generality, let $x \in V(aK_1)$. We write $Y = V(H_1) \cup \cdots \cup V(H_c)$.

Subcase 2.1.1. $y \in V(bK_2) \cup Y$.

In this subcase, we deduce $\alpha(G) \geq a+b+c$. Combining this with (2.10) and $\delta(G) > \frac{\alpha(G)+4}{2}$, we infer

$$\delta(G) > \frac{\alpha(G)+4}{2} \geq \frac{a+b+c+4}{2} \geq \frac{2|X|-1+4}{2} = \frac{2|X|+3}{2} > |X|+1,$$

which contradicts Claim 3.

Subcase 2.1.2. $y \in V(G) \setminus (V(bK_2) \cup Y)$.

In this subcase, we have $sun(G-X) \geq sun(G'-X) - 1$. Combining this with (2.10), $\alpha(G) \geq sun(G-X)$ and $\delta(G) > \frac{\alpha(G)+4}{2}$, we derive

$$\begin{aligned}\delta(G) &> \frac{\alpha(G)+4}{2} \geq \frac{sun(G-X)+4}{2} \geq \frac{sun(G'-X)-1+4}{2}\\ &= \frac{sun(G'-X)+3}{2} \geq \frac{2|X|-1+3}{2} = |X|+1,\end{aligned}$$

which contradicts Claim 3.

Subcase 2.2. $c \neq 0$.

Obviously, $\alpha(G') \geq a+b+\sum_{i=1}^{c} \frac{|V(H_i)|}{2} \geq a+b+3c$ by $|V(H_i)| \geq 6$. Combining this with (2.10), $c \neq 0$ and $\alpha(G) \geq \alpha(G') - 2$, we obtain

$$\alpha(G) \geq \alpha(G') - 2 \geq a+b+3c-2 \geq a+b+c \geq 2|X|-1. \tag{2.12}$$

By virtue of (2.12), Claim 3 and $\delta(G) > \frac{\alpha(G)+4}{2}$, we deduce

$$|X| + 1 \geq \delta(G) > \frac{\alpha(G)+4}{2} \geq \frac{2|X|-1+4}{2} = |X| + \frac{3}{2},$$

which is a contradiction. This completes the proof of Theorem 1.3. □

3. The proof of Theorem 1.4

Proof of Theorem 1.4. For any $e \in E(G)$, we write $G' = G - e$. To prove Theorem 1.4, we only need to justify that G' is a $P_{\geq 3}$-factor covered graph. Suppose, to the contrary, that G' is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 1.2, we have

$$sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \tag{3.1}$$

for some vertex subset X of G'. We write $a = i(G - X)$ and $b = \lfloor \gamma(2k+1) \rfloor$.

Claim 1. $b \geq a + 1$.

Proof. Let $b \leq a$. We may choose b isolated vertices $x_1, x_2, \ldots, x_b$ in $G - X$. Write $A = \{x_1, x_2, \ldots, x_b\}$. Then A is an independent set of G. Thus, we infer

$$\gamma(n - 3k - 2) + k + 2 < |N_G(A)| \leq |X|. \tag{3.2}$$

It follows from (3.1), (3.2) and $\varepsilon(X) \leq 2$, $\frac{1}{3} \leq \gamma \leq 1$ and $n \geq 5k + 3 - \frac{3}{5\gamma - 1}$ that

$$\begin{aligned}
0 &\geq |X| + sun(G' - X) - n \geq |X| + 2|X| - \varepsilon(X) + 1 - n \\
&\geq 3|X| - n - 1 > 3(\gamma(n - 3k - 2) + k + 2) - n - 1 \\
&= (3\gamma - 1)n - 3\gamma(3k + 2) + 3k + 5 \\
&\geq (3\gamma - 1)\left(5k + 3 - \frac{3}{5\gamma - 1}\right) - 3\gamma(3k + 2) + 3k + 5 \\
&= (3\gamma - 1)(2k + 1) - \frac{3(3\gamma - 1)}{5\gamma - 1} + 3 \\
&\geq 3 - \frac{3(3\gamma - 1)}{5\gamma - 1} > 3 - 3 = 0,
\end{aligned}$$

which is a contradiction. We have finished the proof of Claim 1. □

In what follows, we consider four cases by the value of $|X|$ and derive a contradiction in each case.

Case 1. $|X| = 0$.

Note that $G' = G - e$ and G is $(k+2)$-connected. Hence, G' is $(k+1)$-connected and $\omega(G') = 1$. Combining this with (3.1) and $\varepsilon(X) = 0$, we obtain $1 = \omega(G') \geq sun(G') \geq 1$. Thus, we have $sun(G') = \omega(G') = 1$. Then using $n \geq 5k + 3 - \frac{3}{5\gamma - 1} \geq 8 - \frac{3}{5 \times \frac{1}{3} - 1} = \frac{7}{2} > 3$, we see that G' is a big sun, and so G' has at least three vertices with degree 1, which contradicts that G' is a $(k+1)$-connected graph.

Case 2. $1 \le |X| \le k$.

Note that $1 \le |X| \le k$ and G' is $(k+1)$-connected. We derive $\omega(G'-X) = 1$. According to (3.1) and $\varepsilon(X) \le |X|$, we get

$$1 = \omega(G' - X) \ge sun(G' - X) \ge 2|X| - \varepsilon(X) + 1 \ge |X| + 1 \ge 2,$$

which is a contradiction.

Case 3. $|X| = k + 1$.

Since G is $(k+2)$-connected, $G - X$ is connected, and so $\omega(G - X) = 1$. Note that $G' = G - e$. Thus, we deduce

$$\omega(G' - X) \le \omega(G - X) + 1 = 2. \tag{3.3}$$

By virtue of (3.1), (3.3), $k \ge 1$ and $\varepsilon(X) \le 2$, we infer

$$\begin{aligned} 2 &\ge \omega(G' - X) \ge sun(G' - X) \ge 2|X| - \varepsilon(X) + 1 \ge 2|X| - 1 \\ &= 2(k + 1) - 1 = 2k + 1 \ge 3, \end{aligned}$$

which is a contradiction.

Case 4. $|X| \ge k + 2$.

In light of (3.1), $\varepsilon(X) \le 2$ and $\frac{1}{3} \le \gamma \le 1$, we derive

$$\begin{aligned} sun(G - X) &\ge sun(G' - X) - 2 \ge 2|X| - \varepsilon(X) + 1 - 2 \ge 2|X| - 3 \\ &\ge 2(k + 2) - 3 = 2k + 1 \ge \gamma(2k + 1) \ge \lfloor \gamma(2k + 1) \rfloor = b, \end{aligned}$$

which implies that $G - X$ admits an independent set of order at least b. Then using Claim 1, we may choose a isolated vertices $x_1, x_2, \ldots, x_a$ and $(b - a)$ nonadjacent vertices $x_{a+1}, \ldots, x_b$ with $d_{G-X}(x_i) = 1$ for $a + 1 \le i \le b$, in $G - X$. Set $A = \{x_1, x_2, \ldots, x_a, x_{a+1}, \ldots, x_b\}$. Then A is an independent set of G. Thus, we deduce

$$\gamma(n - 3k - 2) + k + 2 < |N_G(A)| \le |X| + b - a,$$

that is,

$$|X| > \gamma(n - 3k - 2) + k + 2 - b + a. \tag{3.4}$$

It follows from (3.1), (3.4), $\varepsilon(X) \le 2$ and $n \ge 5k + 3 - \frac{3}{5\gamma - 1}$ that

$$\begin{aligned} 0 &\ge |X| + 2sun(G' - X) - i(G' - X) - n \\ &\ge |X| + 2(2|X| - \varepsilon(X) + 1) - (i(G - X) + 2) - n \\ &\ge |X| + 2(2|X| - 1) - (a + 2) - n \\ &= 5|X| - a - 4 - n \\ &> 5(\gamma(n - 3k - 2) + k + 2 - b + a) - a - 4 - n \end{aligned}$$

$$\begin{aligned}
&= (5\gamma - 1)n - 5\gamma(3k+2) + 5k + 10 - 5b + 4a - 4\\
&\geq (5\gamma - 1)\left(5k + 3 - \frac{3}{5\gamma - 1}\right) - 5\gamma(3k+2) + 5k + 6 - 5b\\
&= 5\gamma(2k+1) - 5b\\
&= 5\gamma(2k+1) - 5\lfloor\gamma(2k+1)\rfloor\\
&\geq 0,
\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 1.4. □

4. Remarks

Remark 1. Next, we show that the condition $\delta(G) > \frac{\alpha(G)+4}{2}$ in Theorem 1.3 cannot be replaced by $\delta(G) \geq \frac{\alpha(G)+4}{2}$. We construct a graph $G = K_{3+t} \vee (4+2t)K_2$, where t is a nonnegative integer. Then G is $(3+t)$-connected, $\delta(G) = 4 + t$ and $\alpha(G) = 4 + 2t$. Thus, we have $\delta(G) = \frac{\alpha(G)+4}{2}$. For any $e \in E((4+2t)K_2)$, let $G' = G - e = K_{3+t} \vee ((3+2t)K_2 \cup (2K_1))$. Select $X = V(K_{3+t}) \subseteq V(G')$. Then $|X| = 3 + t$ and $\varepsilon(X) = 2$. Thus, we derive

$$sun(G' - X) = 5 + 2t > 4 + 2t = 2(3+t) - 2 = 2|X| - \varepsilon(X).$$

By Theorem 1.2, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $P_{\geq 3}$-factor uniform graph.

Remark 2. The conditions with a $(k+2)$-connected graph and $|N_G(A)| > \gamma(n - 3k - 2) + k + 2$ in Theorem 1.4 cannot be replaced by a $(k+1)$-connected graph and $|N_G(A)| \geq \gamma(n - 3k - 2) + k + 1$. Let γ be a rational number such that $\frac{1}{3} \leq \gamma \leq 1$. Then we can write $\gamma = \frac{b}{2k+1}$ for nonnegative integers b and k. Let $G = K_{k+1} \vee ((2k+1)K_2)$, where k is a positive integer. Then G is $(k+1)$-connected and $n = |V(G)| = 5k + 3 > 5k + 3 - \frac{3}{5\gamma-1}$. If A is an independent set of order $b = \gamma(2k+1)$, then

$$\gamma(n - 3k - 2) + k + 2 > |N_G(A)| = \gamma(2k+1) + k + 1 = \gamma(n - 3k - 2) + k + 1.$$

For any $e \in E((2k+1)K_2)$, let $G' = G - e = K_{k+1} \vee ((2k)K_2 \cup (2K_1))$. Select $X = V(K_{k+1}) \subseteq V(G')$. Then $|X| = k + 1$ and $\varepsilon(X) = 2$. Thus, we infer

$$sun(G' - X) = 2k + 2 > 2k = 2(k+1) - 2 = 2|X| - \varepsilon(X).$$

According to Theorem 1.2, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $P_{\geq 3}$-factor uniform graph.

5. Conclusion

The concept of path-factor uniform graph was first presented by Zhou and Sun [27], and they showed a binding number condition for the existence of

$P_{\geq 3}$-factor uniform graphs. Gao and Wang [5], Liu [12] improved Zhou and Sun's above result. Hua [6] gave toughness and isolated toughness conditions for graphs to be $P_{\geq 3}$-factor uniform graphs. In our article, we study the relationships between some graphic parameters (for instance, minimum degree, independence number and neighborhood, and so on) and the existence of $P_{\geq 3}$-factor uniform graphs. The theorems derived in this article belong to existence theorems, that is, under what kind of conditions the path-factor uniform graph exists. However, in a specific computer network, it needs to use a certain algorithm to determine the values of some graphic parameters of the fix network graph and show the eligible path-factor uniform graph from the algorithm point of view. The problems of such algorithms are worthy of consideration in future research.

So far, results on the existence of path-factor uniform graphs are very few. There are many problems on graphs which can be considered for path-factor uniform graphs. For example, we can consider the structures and properties of path-factor uniform graphs. In what follows, we put forward open problems as the end of our article.

Problem 1. Find the necessary and sufficient conditions for a graph to be a path-factor uniform graph.

Problem 2. Find relationships between other graphic parameters and path-factor uniform graphs.

Problem 3. What are the structures and properties in path-factor uniform graphs?

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Declaration

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