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Characterizations of derivations on spaces of smooth functions

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Dedicated to Professor Wojciech Kryszewski on the occasion of his 65-th birthday.

Abstract. We provide a list of equivalent conditions under which an additive operator acting on a space of smooth functions on a compact real interval is a multiple of the derivation.

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1. Introduction

By $\mathbb{R}$ we denote the set of reals, $\mathbb{Q}$ are rationals, $\mathbb{Z}$ are integers, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $I \subseteq \mathbb{R}$ is an interval and $k \in \mathbb{N}_0$, then $C^k(I)$ is the space of real-valued functions on I that are k-times continuously differentiable on the interior of I. If $k = 0$, then we write simply $C(I)$. The space $C^k(I)$ is furnished with the standard pointwise algebraic operations and hence it is a real commutative algebra.

Definition. (*e.g. Kuczma* [12, page 391]) Assume that Q is a commutative ring and P is a subring of Q. A function $f\colon P \to Q$ is called *derivation* if it is *additive*:

$$f(x+y) = f(x) + f(y), \quad x, y \in P \tag{1}$$

and it satisfies the *Leibniz rule*:

$$f(xy) = xf(y) + yf(x), \quad x, y \in P. \tag{2}$$

The following theorem describes derivations over fields of characteristic zero.

Theorem 1. [12, Theorem 14.2.1] *Let K be a field of characteristic zero, F be a subfield of K, S be an algebraic base of K over F if it exists, and let $S = \varnothing$ otherwise. If $f\colon F \to K$ is a derivation, then, for every function $u\colon S \to K$ there exists a unique derivation $g\colon K \to K$ such that $g = f$ on F and $g = u$ on S.*

From this theorem it follows in particular that nonzero derivations $f\colon \mathbb{R} \to \mathbb{R}$ exist. It is well known they are discontinuous and very irregular mappings. For an exhaustive discussion of the notion of derivation and related functional equations the reader is referred to Gselmann [5,6], Gselmann, Kiss, Vincze [7] and the references therein. Recently Ebanks [2,3] studied derivations and derivations of higher order on rings.

The "model" example of a derivation is the operator of derivative on the space $C^k(I)$ for $k > 0$. Indeed, if we define $T\colon C^k(I) \to C(I)$ as $T(f) = f'$ for $f \in C^k(I)$, then clearly $C^k(I)$ is a subring of $C(I)$, T is additive and it satisfies the Leibniz rule:

$$T(f \cdot g) = f \cdot T(g) + g \cdot T(f). \tag{3}$$

Crucial results about equation (3) on the space $C^k(I)$ are due to H. König and V. Milman. We refer the reader to their recent monograph [11]. They studied several operator equations and inequalities that are related to derivatives on the spaces of smooth functions. Later on, we will utilize their elegant result [11, Theorem 3.1] regarding (3). Briefly, if I is an open set, then the general solution of (3) for all $f, g \in C^k(I)$ is of the form

$$T(f) = c \cdot f \cdot \ln|f| + d \cdot f', \quad f \in C^k(I) \tag{4}$$

for some continuous functions $c, d \in C(I)$, if $k > 0$, and

$$T(f) = c \cdot f \cdot \ln|f|, \quad f \in C^k(I) \tag{5}$$

if $k = 0$ (in formulas (4) and (5) the convention that $0 \cdot \ln 0 = 0$ is adopted). Note that no additivity is assumed.

It is a natural question to characterize real-to-real derivations among additive functions with the aid of a relation which is weaker than (2). In particular, the very first article published in the first volume of *Aequationes Mathematicae* by Nishiyama and Horinouchi [14] addresses this question. The authors studied the following relations, each of which is a direct consequence of (2) alone and together with (1) implies (2):

$$f(x^2) = 2xf(x), \quad x \in \mathbb{R}, \tag{6}$$

$$f(x^{-1}) = -x^{-2}f(x), \quad x \in \mathbb{R},\, x \neq 0, \tag{7}$$

and

$$f(x^n) = ax^{n-m}f(x^m), \quad x \in \mathbb{R},\, x \neq 0, \tag{8}$$

where $a \neq 1$ and n, m are integers such that $am = n \neq 0$. Further similar results, as well as some generalizations, are due to Jurkat [8], Kannappan and Kurepa [9,10], Kurepa [13], among others. Ebanks [4] generalized and extended these results to arbitrary fields. A recent paper by Amou [1] provides some n-dimensional generalizations of the results of [8–10,13].

This paper provides versions of the above-mentioned results for operators $T\colon C^k(I) \to C(I)$. Therefore, we seek conditions which are equivalent to (3).

2. Main results

Throughout this section let us fix $k \in \mathbb{N}_0$ and an interval $I \subseteq \mathbb{R}$. We will study conditions upon an additive operator $T\colon C^k(I) \to C(I)$ which yield analogues to Eqs. (6), (14) and (8). Therefore, we will focus on the following operator relations:

$$T(f^2) = 2f \cdot T(f), \tag{9}$$

$$T(f) = -f^2 \cdot T\left(\frac{1}{f}\right), \tag{10}$$

$$T(f^n) = nf^{n-1} \cdot T(f). \tag{11}$$

Our first theorem is a simple observation that some reasonings concerning derivations from the real-to-real case can be extended to arbitrary commutative rings without substantial changes. We adopted parts of the proof of [12, Theorem 14.3.1].

Theorem 2. *Assume that Q is a commutative ring, P is a subring of Q and $T\colon P \to Q$ is an additive operator. Then, the following conditions are pairwise equivalent:*

(i) *T satisfies $T(f^2) = 2f \cdot T(f)$ for all $f \in P$,*
(ii) *T satisfies $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ for all $f, g \in P$,*
(iii) *T satisfies $T(f^n) = nf^{n-1} \cdot T(f)$ for all $f \in P$ and $n \in \mathbb{N}$.*

Proof. (i) ⇒ (ii). Fix arbitrarily $f, g \in P$. By (9) we get

$$T((f+g)^2) = 2(f+g) \cdot T(f+g).$$

Since T is additive,

$$T(f^2) + 2T(f \cdot g) + T(g^2) = 2f \cdot T(f) + 2g \cdot T(f) + 2f \cdot T(g) + 2g \cdot T(g).$$

Using (9) again, after reductions we obtain (3).

(ii) ⇒ (iii). If $n = 1$, then (11) reduces to an identity. Assume that (11) holds for some $n \in \mathbb{N}$ and all $f \in P$. Then, by (3) and the induction hypothesis we have

$$\begin{aligned} T(f^{n+1}) &= T(f^n \cdot f) = f^n \cdot T(f) + f \cdot T(f^n) \\ &= f^n \cdot T(f) + nf^{n-1+1} \cdot T(f) = (n+1)f^n \cdot T(f). \end{aligned}$$

(iii) $\Rightarrow$ (i). Take $n = 2$. □

The next corollary will be utilized later on.

Corollary 1. *Assume that* $T: C^k(I) \to C(I)$ *is an additive operator. Then, the following conditions are pairwise equivalent:*

(i) T *satisfies* $T(f^2) = 2f \cdot T(f)$ *for all* $f \in C^k(I)$,
(ii) T *satisfies* $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ *for all* $f, g \in C^k(I)$,
(iii) T *satisfies* $T(f^n) = nf^{n-1} \cdot T(f)$ *for all* $f \in C^k(I)$ *and* $n \in \mathbb{N}$.

Our next result characterizes the Leibniz rule (3) on a domain restricted to functions separated from zero. Thus, we can consider conditions (10) and (11) for negative n, which involve the function $1/f$. The situation is a bit more complicated, but Theorem 3 below has a mainly technical role.

Theorem 3. *Assume that* $T: C^k(I) \to C(I)$ *is an additive operator and* $\varepsilon_1 \in (0,1)$, $\varepsilon_2 \in (0,1)$ *and* $c \in (1, +\infty]$ *are constants. Consider the following conditions:*

(i) T *satisfies* $T(f) = -f^2 \cdot T\left(\frac{1}{f}\right)$ *for all* $f \in C^k(I)$, $c > f > \varepsilon_1$,
(ii) T *satisfies* $T(f^2) = 2f \cdot T(f)$ *for all* $f \in C^k(I)$, $f > \varepsilon_2$,
(iii) T *satisfies* $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ *for all* $f, g \in C^k(I)$, $f > \varepsilon_2$, $g > \varepsilon_2$,
(iv) T *satisfies* $T(f^n) = nf^{n-1} \cdot T(f)$ *for all* $n \in \mathbb{Z}$ *and all* $f \in C^k(I)$ *such that* $\varepsilon_2 < f < 1/\varepsilon_2$, *and* $f^{n-1} > \varepsilon_2$ *if* $n > 0$ *and* $f^{n+1} > \varepsilon_2$ *if* $n < 0$.

Then: (i) *with* $c = +\infty$ *implies* (ii) *with* $\varepsilon_2 > \sqrt{\varepsilon_1}$, (ii) *and* (iii) *are equivalent,* (iii) *implies* (iv), (iv) *implies* (i) *with* $\varepsilon_1 = \varepsilon_2$ *and* $c = 1/\varepsilon_2$.

Proof. (i) $\Rightarrow$ (ii). First, note that by applying (10) for $f = 1$ and using the rational homogeneity of T we get that T vanishes on each constant function equal to a rational number. Observe that for an arbitrary rational $\delta > 0$ (which will be chosen later) the identity

$$\frac{1}{f^2 - \delta^2} = \frac{1}{2\delta}\left(\frac{1}{f - \delta} - \frac{1}{f + \delta}\right) \tag{12}$$

holds for $f \in C^k(I)$ such that $f > \delta$. Next, if $\varepsilon_1 > 0$ is given and $\varepsilon_2 > \sqrt{\varepsilon_1}$, then we will find some rational $\delta > 0$ such that $\varepsilon_2 > \varepsilon_1 + \delta$ and $\varepsilon_2^2 > \varepsilon_1 + \delta^2$. Consequently, if $f \in C^k(I)$ and $f > \varepsilon_2$, then $f \pm \delta > \varepsilon_1$ and $f^2 - \delta^2 > \varepsilon_1$.

Using (i) three times together with (12) and the additivity of T we obtain

$$\begin{aligned} T(f^2) &= T(f^2-\delta^2) = -(f^2-\delta^2)^2 T\left(\frac{1}{f^2-\delta^2}\right) \\ &= -\frac{1}{2\delta}(f^2-\delta^2)^2 T\left(\frac{1}{f-\delta} - \frac{1}{f+\delta}\right) \\ &= -\frac{1}{2\delta}(f+\delta)^2(f-\delta)^2\left[T\left(\frac{1}{f-\delta}\right) - T\left(\frac{1}{f+\delta}\right)\right] \\ &= \frac{1}{2\delta}\left[(f+\delta)^2 T(f-\delta) - (f-\delta)^2 T(f+\delta)\right] = 2fT(f). \end{aligned}$$

(ii) $\Leftrightarrow$ (iii). Analogously as in Theorem 2 for $f > \varepsilon_2$ and $g > \varepsilon_2$. (iii) $\Rightarrow$ (iv). If $n = 1$, then (11) is trivially satisfied. Assume that f, n and ε_2 satisfy the assumptions of (iv). For $n > 1$ we proceed like in Theorem 2. If $n = 0$, then (iv) reduces to $T(1) = 0$, which follows from (iii). If $n = -1$, then for $1/\varepsilon_2 > f > \varepsilon_2$ we have

$$0 = T(1) = T\left(f \cdot \frac{1}{f}\right) = \frac{1}{f} \cdot T(f) + f \cdot T\left(\frac{1}{f}\right).$$

Assume that $n < -1$. By downward induction, one can check that for $f^{n+1} > \varepsilon_2$ we have from (3)

$$\begin{aligned} T(f^n) &= T\left(f^{n+1} \cdot \frac{1}{f}\right) = f^{n+1} \cdot T\left(\frac{1}{f}\right) + \frac{1}{f} \cdot T\left(f^{n+1}\right) \\ &= -f^{n+1} \cdot f^{-2} T(f) + \frac{n+1}{f} \cdot f^n \cdot T(f) = nf^{n-1}T(f). \end{aligned}$$

$(iv) \Rightarrow (i)$. Take $n = -1$. □

If we assume additionally that interval I is compact, then the situation clarifies considerably.

Theorem 4. *Assume that I is compact and $T\colon C^k(I) \to C(I)$ is an additive operator. Then, the following conditions are pairwise equivalent:*

(i) *T satisfies $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ for all $f, g \in C^k(I)$,*
(ii) *T satisfies $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ for all $f, g \in C^k(I), f > 0$, $g > 0$,*
(iii) *T satisfies $T(f^2) = 2f \cdot T(f)$ for all $f \in C^k(I)$,*
(iv) *T satisfies $T(f^2) = 2f \cdot T(f)$ for all $f \in C^k(I)$, $f > 0$,*
(v) *T satisfies $T(f) = -f^2 \cdot T\left(\frac{1}{f}\right)$ for all $f \in C^k(I)$, $f > 0$,*
(vi) *T satisfies $T(f^n) = nf^{n-1} \cdot T(f)$ for all $f \in C^k(I)$ and $n \in \mathbb{N}$,*
(vii) *T satisfies $T(f^n) = nf^{n-1} \cdot T(f)$ for all $f \in C^k(I)$, $f > 0$ and $n \in \mathbb{N}$.*

Proof. This statement is a consequence of Corollary 1 and Theorem 3. Since I is compact, f attains its global extrema. Thus, we will find some rational $r, q \in \mathbb{Q}$ such that $1/2 < rf + q < 2$. Moreover, as it was already observed in the proof of Theorem 3, each of the conditions of Theorem 4 implies that

$T(1) = 0$ and then T vanishes on constant functions equal to a rational number. Consequently, we have $T(rf + q) = rT(f) + T(q) = rT(f)$ and therefore Theorem 3 applies to the conditions (ii), (iv), (v) and (vii) with appropriately chosen ε_1 and ε_2. The remaining conditions are equivalent by Corollary 1. Therefore, we are done if we prove for example the implication (iv) $\Rightarrow$ (iii).

Fix $f \in C^k(I)$ arbitrarily and choose $r, q \in \mathbb{Q}$ such that $1/2 < rf + q < 2$. By (iv) we get

$$T((rf+q)^2) = 2(rf+q)T(rf+q).$$

Then using additivity we obtain

$$r^2T(f^2) + 2rqT(f) + T(q^2) = 2r^2fT(f) + 2rqT(f)$$

and after reduction

$$T(f^2) + 0 = 2fT(f)$$

i.e. condition (iii). □

One can join Corollary 1 and Theorem 4 with the mentioned result of H. König and V. Milman to obtain a corollary.

Corollary 2. *Under the assumptions of Corollary 1 or Theorem 4, if $k > 0$, then each of the conditions listed there is equivalent to the following one:*

(x) there exists some $d \in C(I)$ such that $T(f) = d \cdot f'$ for all $f \in C^k(I)$

and if $k = 0$, then $T = 0$ is the only additive operator that fulfils any of the equivalent conditions.

Proof. Consider $f(x) = x$ on I and denote $\tilde{d} := T(f) \in C(I)$. Next, note that by [11, Theorem 3.1] the formulas (4) and (5), respectively hold on the interior of I with some $c, d \in C(\mathrm{int}I)$. The additivity of T implies that $c = 0$. Therefore $\tilde{d}$ is a continuous extension of d to the whole interval I. □

3. Final remarks

Remark. The inequalities between f, g and constants ε_1 and ε_2 in Theorem 3 are not optimal. This however was not our goal since the role of this result is auxiliary only. Similarly, the inequality $f > 0$ in some of the conditions of Theorem 4 can be equivalently replaced by an estimate from above or from below by any other fixed constant.

Moreover, in the proof of Theorem 4 we showed more than is stated. Namely, it is equivalently enough to assume, instead of $f > 0$, that f is bilaterally bounded by two rational numbers, like $1/2$ and 2. However, since this generalization is only apparent and easy, we do not include it in the formulation of the theorem.

Example 1. Assume that $\varphi\colon (1,\infty) \to \mathbb{R}$ is a smooth mapping that satisfies the equation

$$\varphi(2x) = 2\varphi(x), \quad x \in (1,\infty). \tag{13}$$

Such mappings exist in abundance. In fact, every map φ_0 defined on $(1,2]$ can be uniquely extended to a solution of (13). Next, let $d\colon (e,\infty) \to \mathbb{R}$ be defined as

$$d(x) = x \cdot \varphi(\ln x), \quad x \in (e,\infty).$$

It is easy to see that

$$d(x^2) = 2xd(x), \quad x \in (e,\infty)$$

and

$$d(xy) \neq xd(y) + yd(x)$$

in general, unless φ is additive. Define $T\colon C^1((e,\infty)) \to C((e,\infty))$ as follows:

$$T(f) = d \circ f, \quad f \in C^1((e,\infty)).$$

One can see that T satisfies (9) for all $f, g \in C((e,\infty))$, but fails to satisfy the Leibniz rule (3). Thus, the assumption of additivity in all our results is essential. Observe also that T has the property that it vanishes on constant functions equal to a rational. This fact, as a consequence of additivity, was frequently used in the proofs of our Theorems 3 and 4. Therefore, the additivity assumption cannot be relaxed to this property.

Example 2. Assume that I is an interval and T is given by the formula

$$T(f) = f'' - \frac{(f')^2}{f}, \quad f \in C^2(I),\ f > 0.$$

Then T satisfies (3) for all $f, g \in C^2(I)$ such that $f > 0$ and $g > 0$. This observation is a particular case of the second part of [11, Corollary 3.4]. Clearly, T is not additive. Moreover, T cannot be extended in such a way that it satisfies (3) on the whole space $C^2(I)$.

The following examples show that if the domain of operator T is changed, then the conditions discussed in our results are no longer equivalent and various situations are possible.

Example 3. Let $\mathcal{S}$ be the space of all functions $f \in C^1((0,\infty))$ which satisfy the functional equation

$$f(x+1) = 2f(x), \quad x \in (0,\infty). \tag{14}$$

Note that $\mathcal{S}$ is not closed under multiplication. Moreover, each function $f_0\colon (0,1] \to \mathbb{R}$ can be uniquely extended to a solution of (14). Therefore, $\mathcal{S}$

is an infinite-dimensional subspace of $C^1((0,\infty))$. Define $T\colon C^1((0,\infty)) \to C^1((0,\infty))$ by the formula

$$T(f)(x) = f(x+1), \quad f \in C^1((0,\infty)),\, x \in (0,\infty).$$

It is easy to check that T is additive and satisfies (3) for $f, g \in \mathcal{S}$. Thus, there are more solutions of (3) if the domain of T is restricted to a particular subspace of $C^k(I)$.

Example 4. Let $P[x]$ be the space of all real polynomials of variable x. By $\deg(f)$ we denote the degree of a polynomial $f \in P[x]$. Define $T\colon P[x] \to P[x]$ by

$$T(f) = \deg(f) \cdot f, \quad f \in P[x].$$

Then T is not additive, it satisfies (3) and there exists no extension of T to the whole space $C^k(\mathbb{R})$ which is a solution of (3).

Example 5. Let

$$\mathcal{S} := \{f\colon (0,\infty) \to \mathbb{R} : f(x) = x^k \text{ for some } k \in \mathbb{Z} \text{ and } x \in (0,\infty)\}.$$

Note that $\mathcal{S}$ is closed under multiplication but it is not a linear space. Next, let a double sequence φ on $\mathbb{Z}$ of natural numbers be defined as follows: $\varphi(0) = 0$, $\varphi(k)$ is arbitrary but $\neq k$ if k is odd, and if $k = 2^n \cdot m$ with some $n \in \mathbb{N}$ and odd $m \in \mathbb{Z}$, then

$$\varphi(k) := 2^{\frac{n^2-n}{2}} \cdot m^n \cdot \varphi(m).$$

Note that we have

$$\begin{aligned} \varphi(2k) &= \varphi(2^{n+1} \cdot m) = 2^{\frac{n^2+n}{2}} \cdot m^{n+1} \cdot \varphi(m) \\ &= 2^n \cdot m \cdot 2^{\frac{n^2-n}{2}} \cdot m^n \cdot \varphi(m) = k \cdot \varphi(k), \quad k \in \mathbb{Z}. \end{aligned} \tag{15}$$

Define $T\colon \mathcal{S} \to C((0,\infty))$ by

$$T(f)(x) := k \cdot x^{\varphi(k)}, \quad x \in (0,\infty) \tag{16}$$

if $f(x) = x^k$ for $x \in (0,\infty)$. One can see that if f is of this form, then by (15)

$$T(f^2)(x) = 2k \cdot x^{\varphi(2k)} = 2k \cdot x^{k \cdot \varphi(k)} = 2f(x)T(f)(x)$$

for all $x \in (0,\infty)$, i.e. T satisfies (9).

Moreover, one can see that (10) is equivalent to the equality

$$\varphi(k) - \varphi(-k) = 2k, \quad k \in \mathbb{Z},\, k \neq 0.$$

Therefore, we can construct a sequence φ such that T defined by (16) satisfies (10) as well as another sequence φ' for which T does not satisfy (10). Finally, (3) is not true on $\mathcal{S}$. Indeed, note that if (3) is satisfied by T given by (16),then:

$$\varphi(k+l) = \varphi(k) + l = \varphi(l) + k, \quad k, l \in \mathbb{Z}, k \neq 0, l \neq 0,$$

which does not hold.

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