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On path-factor critical deleted (or covered) graphs

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Abstract. Let $k \geq 2$ be an integer. A $P_{\geq k}$-factor of a graph G is its spanning subgraph each of whose components is a path of order at least k. A graph G is $P_{\geq k}$-factor covered if for any edge e of G, G has a $P_{\geq k}$-factor containing e. A graph G is $P_{\geq k}$-factor deleted if for any edge e of G, G has a $P_{\geq k}$-factor excluding e. A graph G is $(P_{\geq k}, n)$-factor critical covered if for any $Q \subseteq V(G)$ with $|Q| = n$, the graph $G - Q$ is $P_{\geq k}$-factor covered. A graph G is $(P_{\geq k}, n)$-factor critical deleted if for any $Q \subseteq V(G)$ with $|Q| = n$, the graph $G - Q$ is $P_{\geq k}$-factor deleted. Zhou et al. (Contribut Dis Math 14(1): 167–174, 2019) introduced the sun toughness of a graph G, which is denoted by $s(G)$ and defined by

$$s(G) = \min\left\{\frac{|X|}{sun(G - X)} : X \subseteq V(G), sun(G - X) \geq 2\right\}$$

if G is not a complete graph; otherwise, $s(G) = +\infty$. In this article, we prove that (i) an $(n + r + 2)$-connected graph G is $(P_{\geq 3}, n)$-factor critical deleted if $s(G) > \frac{n+r+3}{2(r+2)}$, where $n \geq 0$ and $r \geq 0$ are two integers; (ii) an $(n + r + 1)$-connected graph G is $(P_{\geq 3}, n)$-factor critical covered if $s(G) > \frac{n+r+1}{2r+1}$, where $n \geq 0$ and $r \geq 1$ are two integers.

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1. Introduction

We consider only undirected finite graphs without loops or multiple edges, unless explicitly stated otherwise. Let G be a graph, and let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex x in G, denoted by $d_G(x)$, is the number of vertices adjacent to x in G. We use $i(G)$ and $\omega(G)$ to denote the number of isolated vertices and connected

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components of G, respectively. Let X be a vertex set of G and E' be an edge set of G. Then $G-X$ denotes the resulting graph after removing the vertices of X from G, and $G-E'$ denotes the subgraph derived from G by removing E'. For notational simplicity, we write $G-x=G-\{x\}$ for $x\in V(G)$ and $G-e=G-\{e\}$ for $e\in E(G)$. A vertex set X of G is independent if no two elements in X are adjacent. Let G_1 and G_2 be two graphs. We denote by $G_1\cup G_2$ and $G_1\vee G_2$ the union and the join of G_1 and G_2, respectively. Let P_n, C_n and K_n denote the path, the cycle and the complete graph with n vertices, respectively.

A path-factor is a spanning subgraph F of G such that every component of F is a path. Let $k\geq 2$ be an integer. A $P_{\geq k}$-factor of a graph G is its spanning subgraph each of whose components is a path of order at least k. A graph G is $P_{\geq k}$-factor covered if for any edge e of G, G has a $P_{\geq k}$-factor containing e. A graph G is $P_{\geq k}$-factor deleted if for any edge e of G, G has a $P_{\geq k}$-factor excluding e. A graph G is said to be $(P_{\geq k},n)$-factor critical covered if for any $Q\subseteq V(G)$ with $|Q|=n$, the graph $G-Q$ is $P_{\geq k}$-factor covered. A graph G is said to be $(P_{\geq k},n)$-factor critical deleted if for any $Q\subseteq V(G)$ with $|Q|=n$, the graph $G-Q$ is $P_{\geq k}$-factor deleted.

Bazgan, Benhamdine, Li and Woźniak [1] verified that a 1-tough graph G of order at least 3 admits a $P_{\geq 3}$-factor. Kano, Lu and Yu [5] justified that a graph G contains a $P_{\geq 3}$-factor if $i(G-X)\leq \frac{2}{3}|X|$ for all $X\subseteq V(G)$. Wang [7] claimed that a bipartite graph G admits a $P_{\geq 3}$-factor if and only if $i(G-X-M)\leq 2|X|+|M|$ for any $X\subseteq V(G)$ and independent $M\subseteq E(G)$. Kelmans [6] showed some sufficient conditions for graphs to have path-factors. Zhou [10], Zhou, Bian and Sun [13] derived some sufficient conditions for graphs to be $P_{\geq 3}$-factor covered graphs. Gao, Wang and Chen [2] got some results on the existence of $P_{\geq 3}$-factor deleted graphs. Zhou [9], Zhou, Sun and Liu [15] obtained some results on the $P_{\geq 3}$-factor with given properties. Zhou [11], Zhou, Bian and Pan [12] presented some sufficient conditions for graphs to be $(P_{\geq 3},n)$-factor critical deleted graphs.

To characterize graphs admitting $P_{\geq 3}$-factors, Kaneko [3] introduced the concept of a sun. If $R-x$ admits a perfect matching for any $x\in V(R)$, then R is called a factor-critical graph. Let R be a factor-critical graph with vertex set $V(R)=\{x_1,x_2,\cdots,x_n\}$. By adding new vertices $y_1,y_2,\cdots,y_n$ together with new edges $x_1y_1,x_2y_2,\cdots,x_ny_n$ to R, we derive a new graph H, which is called a *sun*. By Kaneko, K_1 and K_2 are also suns. Especially, a sun with at least six vertices is called a big sun. We denote by $sun(G)$ the number of sun components of G.

Kaneko [3] posed a criterion for a graph to have a $P_{\geq 3}$-factor. Kano, Katona and Király [4] came up with a simple proof.

Theorem 1. *([3, 4]). A graph G contains a $P_{\geq 3}$-factor if and only if*

$$sun(G-X)\leq 2|X|$$

for all $X \subseteq V(G)$.

Later, Zhang and Zhou [8] extended Theorem 1, and derived a characterization for $P_{\geq 3}$-factor covered graphs.

Theorem 2. *([8]). A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G - X) \leq 2|X| - \varepsilon(X)$$

for all $X \subseteq V(G)$, where $\varepsilon(X)$ is defined by

$$\varepsilon(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set\ and\ G - X\ has\ a\ non-sun\ component; \\ 0, & otherwise. \end{cases}$$

Zhou, Sun and Liu [16] introduced the sun toughness of a graph G, which is denoted by $s(G)$ and defined by

$$s(G) = \min \left\{ \frac{|X|}{sun(G - X)} : X \subseteq V(G), sun(G - X) \geq 2 \right\}$$

if G is not a complete graph; otherwise, $s(G) = +\infty$. Then they showed two sun toughness conditions for graphs to be $P_{\geq 3}$-factor deleted graphs or $P_{\geq 3}$-factor covered graphs.

Theorem 3. *([16]). A 2-edge-connected graph G is a $P_{\geq 3}$-factor deleted graph if its sun toughness $s(G) \geq 1$.*

Theorem 4. *([16]). A connected graph G of order at least 3 is a $P_{\geq 3}$-factor covered graph if its sun toughness $s(G) \geq 1$.*

Zhou, Bian and Pan [12] presented a binding number condition for graphs to be $(P_{\geq 3}, n)$-factor critical deleted graphs.

Theorem 5. *([12]). Let n be a nonnegative integer, and let G be an $(n + 2)$-connected graph. If $bind(G) > \frac{3+n}{2}$, then G is $(P_{\geq 3}, n)$-critical deleted.*

Zhou and Sun [14] gave a binding number condition for graphs to be $(P_{\geq 3}, n)$-factor critical covered graphs.

Theorem 6. *[14]. Let n be an integer with $n \geq 1$, and let G be an $(n + 1)$-connected graph with $|V(G)| \geq n + 3$. If $bind(G) \geq \frac{4+n}{3}$, then G is $(P_{\geq 3}, n)$-factor-critical covered.*

In this article, we pose sun toughness conditions for graphs to be $(P_{\geq 3}, n)$-factor critical deleted graphs or $(P_{\geq 3}, n)$-factor critical covered graphs, respectively. Our main results are two generalizations of Theorems 3 and 4.

Theorem 7. *An $(n + r + 2)$-connected graph G is a $(P_{\geq 3}, n)$-factor critical deleted graph if its sun toughness $s(G) > \frac{n+r+3}{2(r+2)}$, where n and r are two nonnegative integers.*

Theorem 8. *An $(n+r+1)$-connected graph G is a $(P_{\geq 3}, n)$-factor critical covered graph if its sun toughness $s(G) > \frac{n+r+1}{2r+1}$, where $n \geq 0$ and $r \geq 1$ are integers.*

2. Proof of Theorem 7

We first show the following lemma, which will be used in the proof of Theorem 7.

Lemma 1. *Let n and r be two nonnegative integers, let G be an $(n+r+2)$-connected graph, and let $H = G - Q - e$ for any $Q \subseteq V(G)$ with $|Q| = n$ and any $e \in E(G-Q)$. If $sun(H-X) \geq 2|X|+1$ for $X \subseteq V(H)$, then $|X| \geq r+2$.*

Proof. Assume that $sun(H-X) \geq 2|X|+1$. Then $X \neq \emptyset$ since otherwise $G-Q$ is 2-connected and so $H = G-Q-e$ is a sun having at most two end-vertices, but every big sun has at least 3 end-vertices. Thus $\omega(G-Q-X) \geq \omega(G-Q-X-e)-1 \geq sun(H-X)-1 \geq 2|X| \geq 2$. Thus $|X| \geq r+2$ since G is $(n+r+2)$-connected. □

Proof of Theorem 7. Theorem 7 obviously holds for a complete graph. Next, we assume that G is not a complete graph. Let $Q \subseteq V(G)$ with $|Q| = n$ and $e = uv \in E(G-Q)$, and let $H = G-Q-e$. It suffices to justify that H contains a $P_{\geq 3}$-factor. For a contradiction, suppose that H has no $P_{\geq 3}$-factor. Then by Theorem 1, there exists a vertex set X of H that satisfies

$$sun(H-X) \geq 2|X|+1. \tag{1}$$

□

Claim 1. $s(G) \leq \frac{n+|X|+1}{2|X|}$.

Proof. It is obvious that

$$sun(G-Q-X-e) \leq sun(G-Q-X)+2.$$

Thus we prove Claim 1 by considering the following two cases.

Case 1. $sun(G-Q-X-e) \leq sun(G-Q-X)+1$.

Because of (1) and Lemma 1, we obtain

$$\begin{aligned} sun(G-Q-X) &\geq sun(G-Q-X-e)-1 = sun(H-X)-1 \\ &\geq 2|X| \geq 2(r+2). \end{aligned}$$

Thus, we derive

$$s(G) \leq \frac{|Q \cup X|}{sun(G-Q-X)} \leq \frac{n+|X|}{2|X|} < \frac{n+|X|+1}{2|X|}.$$

Case 2. $sun(G-Q-X-e) = sun(G-Q-X)+2$.

In this case, we may assume that $e = uv$ joins two sun components D_1 and D_2 of $G - Q - X - e$, where $u \in V(D_1)$ and $v \in V(D_2)$. Then it follows from (1) that $sun(G-Q-X-v) \geq sun(G-Q-X-e)-1 = sun(H-X)-1 \geq 2|X| \geq 2(r+2)$, which implies

$$s(G) \leq \frac{|Q \cup X \cup \{v\}|}{sun(G-Q-X-v)} \leq \frac{n+|X|+1}{2|X|}.$$

This completes the proof of Claim 1. □

Using Claim 1 and Lemma 1, we deduce

$$s(G) \leq \frac{n+|X|+1}{2|X|} = \frac{1}{2} + \frac{n+1}{2|X|} \leq \frac{1}{2} + \frac{n+1}{2(r+2)} = \frac{n+r+3}{2(r+2)},$$

which contradicts $s(G) > \frac{n+r+3}{2(r+2)}$. We have verified Theorem 7. □

Remark 1. Now, we explain that the condition on $s(G)$ in Theorem 7 is sharp.

Let n and r be two nonnegative integers with $n \geq r+1$, let H_1 and H_2 be two big suns, and let $G = K_{n+r+2} \vee ((2r+3)K_1 \cup H_1 \cup H_2 \cup \{e\})$, where $e = uv$, $u \in V(H_1)$ and $v \in V(H_2)$. We know that G is $(n+r+2)$-connected and $s(G) = \frac{|V(K_{n+r+2}) \cup \{v\}|}{sun(G-(V(K_{n+r+2}) \cup \{v\}))} = \frac{n+r+3}{2(r+2)}$. Let $Q \subseteq V(K_{n+r+2}) \subseteq V(G)$ with $|Q| = n$. Then $G - Q - e = K_{r+2} \vee ((2r+3)K_1 \cup H_1 \cup H_2)$. Let $X = V(K_{r+2}) \subseteq V(G-Q-e)$. Then we obtain

$$sun(G-Q-e-X) = 2r+5 > 2(r+2) = 2|X|.$$

From Theorem 1, $G-Q-e$ has no $P_{\geq 3}$-factor, and so G is not $(P_{\geq 3}, n)$-factor critical deleted.

Remark 2. Now, we claim that the condition of $(n+r+2)$-connectedness in Theorem 7 is the best possible.

Let $G = K_{n+r+1} \vee ((2r+1)K_1 \cup P_3)$, where n and r are two integers with $n-2 \geq r \geq 0$. Let $P_3 = x_1x_2x_3$. We know that G is $(n+r+1)$-connected and $s(G) = \frac{|V(K_{n+r+1}) \cup \{x_2\}|}{sun(G-(V(K_{n+r+1}) \cup \{x_2\}))} = \frac{n+r+2}{2r+3} > \frac{n+r+3}{2(r+2)}$. Let $Q \subseteq V(K_{n+r+1}) \subseteq V(G)$ with $|Q| = n$ and $e \in E(P_3)$. Then $G-Q-e = K_{r+1} \vee ((2r+2)K_1 \cup K_2)$. Let $X = V(K_{r+1}) \subseteq V(G-Q-e)$. Then we have

$$sun(G-Q-e-X) = 2r+3 > 2(r+1) = 2|X|.$$

Using Theorem 1, $G-Q-e$ has no $P_{\geq 3}$-factor, and so G is not $(P_{\geq 3}, n)$-factor critical deleted.

3. The proof of Theorem 8

Proof of Theorem 8. Theorem 8 obviously holds for a complete graph. Next, we always assume that G is not complete. Let $Q \subseteq V(G)$ with $|Q| = n$, and let $H = G - Q$. It suffices to prove that H is $P_{\geq 3}$-factor covered. For a contradiction, suppose that H is not $P_{\geq 3}$-factor covered. Then by Theorem 2, we have

$$sun(H - X) \geq 2|X| - \varepsilon(X) + 1 \tag{2}$$

for some vertex set X of H.

In the following, we discuss four cases by the value of $|X|$.

Case 1. $|X| = 0$.

Obviously, $\varepsilon(X) = 0$. According to (1), $sun(H) \geq 1$. Note that G is $(n + r + 1)$-connected, and so $|V(G)| \geq n + r + 2$. Hence, H is $(r + 1)$-connected, which implies $\omega(H) = 1$. Thus, we get

$$1 \leq sun(H) \leq \omega(H) = 1,$$

namely,

$$sun(H) = \omega(H) = 1.$$

Combining this with $|V(H)| = |V(G)| - |Q| \geq (n + r + 2) - n = r + 2 \geq 3$, we know that H is a big sun. Then there exists a vertex x in H such that $d_H(x) = 1$, and so

$$d_G(x) \leq d_{G-Q}(x) + |Q| = d_H(x) + n = n + 1,$$

which contradicts that G is $(n + r + 1)$-connected.

Case 2. $|X| = 1$.

In this case, $\varepsilon(X) \leq 1$. In light of (1), we get

$$\begin{aligned} \omega(G - Q - X) = \omega(H - X) \geq sun(H - X) \geq 2|X| \\ -\varepsilon(X) + 1 \geq 2|X| = 2 > 1 = \omega(G), \end{aligned}$$

which implies that G is at most $(n + 1)$-connected, which contradicts that G is $(n + r + 1)$-connected.

Case 3. $2 \leq |X| \leq r$.

From (1), $\varepsilon(X) \leq 2$ and $\omega(G) = 1$, we derive

$$\begin{aligned} \omega(G - Q - X) = \omega(H - X) \geq sun(H - X) \geq 2|X| \\ -\varepsilon(X) + 1 \geq 2|X| - 1 \geq 4 - 1 = 3 > 1 = \omega(G), \end{aligned}$$

which implies that G is at most $(n + r)$-connected, which contradicts that G is $(n + r + 1)$-connected.

Case 4. $|X| \geq r + 1$.

It follows from (1), $\varepsilon(X) \leq 2$ and $r \geq 1$ that

$$\begin{aligned} sun(G - Q - X) = sun(H - X) \geq 2|X| - \varepsilon(X) \\ +1 \geq 2|X| - 1 \geq 2(r + 1) - 1 = 2r + 1 \geq 3. \end{aligned}$$

Combining this with the definition of $s(G)$, we infer

$$s(G) \le \frac{|Q \cup X|}{sun(G - Q - X)} \le \frac{n + |X|}{2|X| - 1} = \frac{1}{2} + \frac{n + \frac{1}{2}}{2|X| - 1}$$
$$\le \frac{1}{2} + \frac{n + \frac{1}{2}}{2(r + 1) - 1} = \frac{n + r + 1}{2r + 1},$$

which contradicts $s(G) > \frac{n+r+1}{2r+1}$. This completes the proof of Theorem 8. □

Remark 3. We now show that the condition $s(G) > \frac{n+r+1}{2r+1}$ in Theorem 8 cannot be replaced by $s(G) \ge \frac{n+r+1}{2r+1}$.

Let $G = K_{n+r+1} \vee ((2r + 1)K_1)$, where $n \ge 0$ and $r \ge 1$ are two integers. We easily see that G is $(n + r + 1)$-connected and $s(G) = \frac{|V(K_{n+r+1})|}{sun(G-V(K_{n+r+1}))} = \frac{n+r+1}{2r+1}$. Let $Q \subseteq V(K_{n+r+1}) \subseteq V(G)$ with $|Q| = n$. Then $G - Q = K_{r+1} \vee ((2r + 1)K_1)$. Select $X = V(K_{r+1})$ in $G - Q$. Then $\varepsilon(X) = 2$ by the definition of $\varepsilon(X)$. Thus, we derive

$$sun(G - Q - X) = 2r + 1 > 2(r + 1) - 2 = 2|X| - \varepsilon(X).$$

In light of Theorem 2, $G - Q$ is not $P_{\ge 3}$-factor covered. So G is not $(P_{\ge 3}, n)$-factor critical covered.

Remark 4. We now claim that the condition of $(n + r + 1)$-connectedness in Theorem 8 is sharp.

Let $G = K_{n+r} \vee ((2rK_1) \cup M)$, where $n > r \ge 1$ are two integers, and M is a connected graph not being a sun. Clearly, G is $(n + r)$-connected and $s(G) = \frac{|V(K_{n+r})|}{sun(G-V(K_{n+r}))} = \frac{n+r}{2r} > \frac{n+r+1}{2r+1}$. Let $Q \subseteq V(K_{n+r}) \subseteq V(G)$ with $|Q| = n$. Then $G - Q = K_r \vee ((2rK_1) \cup M)$. Choose $X = V(K_r)$ in $G - Q$. Then $1 \le \varepsilon(X) \le 2$ by the definition of $\varepsilon(X)$. Thus, we acquire

$$sun(G - Q - X) = 2r > 2r - 1 \ge 2|X| - \varepsilon(X).$$

It follows from Theorem 2 that $G - Q$ is not $P_{\ge 3}$-factor covered. Hence, G is not $(P_{\ge 3}, n)$-factor critical covered.

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