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A note on the Choquet type operators

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Abstract. In this note Choquet type operators are introduced in connection with Choquet's theory of integrability with respect to a not necessarily additive set function. Based on their properties, a quantitative estimate for the nonlinear Korovkin type approximation theorem associated to Bernstein–Kantorovich–Choquet operators is proved. The paper also includes a large generalization of Hölder's inequality within the framework of monotone and sublinear operators acting on spaces of continuous functions.

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1. Introduction

Choquet's theory of integrability (as described by Denneberg [8], Grabisch [12] and Wang and Klir [16]) emphasizes the importance of a new class of nonlinear operators that verify a mix of conditions characteristic of Choquet's integral. Its technical definition is detailed as follows.

Given a Hausdorff topological space X, we will denote by $\mathcal{F}(X)$ the vector lattice of all real-valued functions defined on X endowed with the pointwise ordering. Two important vector sublattices of it are

$$C(X) = \{f \in \mathcal{F}(X) : \ f \text{ continuous}\}$$

and

$$C_b(X) = \{f \in \mathcal{F}(X) : \ f \text{ continuous and bounded}\}.$$

With respect to the sup norm, $C_b(X)$ becomes a Banach lattice. See [15] for the theory of these spaces.

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As is well known, all norms on the N-dimensional real vector space $\mathbb{R}^N$ are equivalent. See Bhatia [2], Theorem 13, p. 16. When endowed with the sup norm and the coordinate-wise ordering, $\mathbb{R}^N$ can be identified (algebraically, isometrically and in order) with the space $C(\{1,\ldots,N\})$, where $\{1,\ldots,N\}$ carries the discrete topology.

Suppose that X and Y are two Hausdorff topological spaces and E and F are respectively ordered vector subspaces of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. An operator $T: E \to F$ is said to be a *Choquet type operator* (respectively a *Choquet type functional when* $F = \mathbb{R}$) if it satisfies the following three conditions:

(Ch1) (*Sublinearity*) T is subadditive and positively homogeneous, that is,

$$T(f+g) \leq T(f) + T(g) \quad \text{and} \quad T(af) = aT(f)$$

for all f, g in E and $a \geq 0$;

(Ch2) (*Comonotone additivity*) $T(f+g) = T(f)+T(g)$ whenever the functions $f, g \in E$ are comonotone in the sense that

$$(f(s) - f(t)) \cdot (g(s) - g(t)) \geq 0 \text{ for all } s, t \in X;$$

(Ch3) (*Monotonicity*) $f \leq g$ in E implies $T(f) \leq T(g)$.

All the aforementioned conditions are independent of each other.

If a nonlinear operator T is monotone and positively homogeneous then necessarily

$$T(0) = 0 \text{ and } f \geq 0 \text{ implies } T(f) \geq 0;$$

the converse works for linear operators but not in the general case.

The Choquet integral associated to a vector capacity with values in $\mathbb{R}^N$ is a natural source of Choquet type operators. See Remark 4. For more examples (important in approximation theory) see [10], where the following extension of Korovkin's approximation theorem to the framework of Choquet type operators was proved.

Theorem 1. (The nonlinear extension of Korovkin's theorem: the several variables case) *Suppose that X is a locally compact subset of the Euclidean space $\mathbb{R}^N$ and E is a vector sublattice of $\mathcal{F}(X)$ that contains the $2N+2$ test functions* $1, \ \pm\mathrm{pr}_1, \ldots, \ \pm\mathrm{pr}_N$ *and* $\sum_{k=1}^N \mathrm{pr}_k^2$. (Here $\mathrm{pr}_k : (x_1,\ldots,x_N) \to x_k$ $(k = 1,\ldots,N)$ denote the canonical projections on $\mathbb{R}^N$).

(i) *If $(T_n)_n$ is a sequence of monotone and sublinear operators from E into E such that*

$$\lim_{n\to\infty} T_n(f) = f \qquad \text{uniformly on the compact subsets of } X$$

for each of the $2N+2$ aforementioned test functions, then the above limit property also holds for all nonnegative functions f in $E \cap C_b(X)$.

(ii) *If, in addition, each operator T_n is comonotone additive, then $(T_n(f))_n$ converges to f uniformly on the compact subsets of X, for every $f \in E \cap C_b(X)$.*

Notice that in both cases (i) and (ii) the family of testing functions can be reduced to $1,\ -\mathrm{pr}_1,\ldots,\ -\mathrm{pr}_N$ *and* $\sum_{k=1}^{N}\mathrm{pr}_k^2$ *when* K *is included in the positive cone of* $\mathbb{R}^N$*. Also, the convergence of* $(T_n(f))_n$ *to* f *is uniform on* X *when* $f\in E$ *is uniformly continuous and bounded on* X.

In this paper we prove a quantitative estimate concerning the above Korovkin-type theorem in the case of Bernstein-Kantorovich-Choquet operators but our argument works also for the Szász-Mirakjan-Kantorovich-Choquet operators, the Baskakov-Kantorovich-Choquet operators etc. See Theorem 4, which is based on a generalization of the Cauchy-Bunyakovsky-Schwarz inequality for Choquet type operators (stated as Lemma 1).

A large generalization of Hölder's inequality within the framework of monotone and sublinear operators acting on spaces of continuous functions makes the objective of Theorem 3.

For the convenience of the reader, we devoted Sect. 2 to an overview of basic facts about monotone capacities and the Choquet integral.

2. Preliminaries on Choquet's integral

Given a nonempty set X, by a *lattice* of subsets of X we mean any collection Σ of subsets that contains $\emptyset$ and X and is closed under finite intersections and unions. A lattice Σ is an *algebra* if in addition it is closed under complementation. An algebra closed under countable unions and intersections is called a σ-algebra.

Of special interest is the case where X is a compact Hausdorff space and Σ is either the lattice $\Sigma_{up}^{+}(X)$ of all upper contour closed sets $S=\{x\in X: f(x)\geq t\}$, or the lattice $\Sigma_{up}^{-}(X)$ of all upper contour open sets $S=\{x\in X: f(x)>t\}$ associated to pairs $f\in C(X)$ and $t\in\mathbb{R}$.

When X is a compact metrizable space, $\Sigma_{up}^{+}(X)$ coincides with the lattice of all closed subsets of X (and $\Sigma_{up}^{-}(X)$ coincides with the lattice of all open subsets of X).

In what follows Σ denotes a lattice of subsets of an abstract set X.

Definition 1. A set function $\mu:\Sigma\to[0,\infty)$ is called a capacity if it verifies the following two conditions:

$(C1)$ $\mu(\emptyset)=0$; and

$(C2)$ $\mu(A)\leq\mu(B)$ for all $A,B\in\Sigma$, with $A\subset B$ (monotonicity).

The capacity μ is called normalized if $\mu(X)=1$.

If Σ is an algebra of subsets of X, then to every capacity μ defined on Σ, one can attach a new capacity $\overline{\mu}$, the *dual* of μ, which is defined by the formula

$$\overline{\mu}(A)=\mu(X)-\mu(X\setminus A).$$

Notice that $\overline{(\bar{\mu})} = \mu$.

The capacities provide a non additive generalization of *probability measures*, that is, of capacities μ having the property of σ-additivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for every sequence $A_1, A_2, A_3, \ldots$ of disjoint sets belonging to Σ such that $\cup_{n=1}^{\infty} A_n \in \Sigma$.

Some other classes of capacities exhibiting extensions of the properties of additivity or σ-additivity are listed below.

A capacity μ is called *submodular* (or strongly subadditive) if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \quad \text{for all } A, B \in \Sigma. \tag{2.1}$$

Every additive measure is also submodular, but the converse fails. A normalized submodular capacity μ defined on an algebra Σ of sets has the property

$$\mu(A) = 0 \text{ implies } \mu(\complement A) = 1. \tag{2.2}$$

A capacity μ is called *lower continuous* (or continuous by ascending sequences) if

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$$

for every nondecreasing sequence $(A_n)_n$ of sets in Σ such that $\cup_{n=1}^{\infty} A_n \in \Sigma$; μ is called *upper continuous* (or continuous by descending sequences) if $\lim_{n\to\infty} \mu(A_n) = \mu\left(\cap_{n=1}^{\infty} A_n\right)$ for every nonincreasing sequence $(A_n)_n$ of sets in Σ such that $\cap_{n=1}^{\infty} A_n \in \Sigma$. If μ is an additive capacity defined on a σ-algebra, then its upper/lower continuity is equivalent to the property of σ-additivity.

If Σ is a σ-algebra, then a capacity $\mu : \Sigma \to [0,1]$ is lower (upper continuous) if and only if its dual $\bar{\mu}$ is upper (lower) continuous.

There are several standard procedures to attach to a probability measure certain not necessarily additive capacities. So is the case of *distorted probabilities,* $\mu(A) = u(P(A))$, obtained from a given probability measure $P : \Sigma \to [0,1]$ and applying to it a distortion $u : [0,1] \to [0,1]$, that is, a nondecreasing and continuous function such that $u(0) = 0$ and $u(1) = 1$. For example, one may chose $u(t) = t^a$ with $\alpha > 0$. When the distortion u is concave (for example, when $u(t) = t^a$ with $0 < \alpha < 1$ or when $u(t) = \frac{2t}{t+1}$), then μ is an example of lower continuous submodular capacity.

The following concept of integrability with respect to a capacity $\mu : \Sigma \to [0,\infty)$ was introduced by Choquet [5,6]. It concerns the class of *upper measurable* functions, that is, the functions $f : X \to \mathbb{R}$ such that all upper contour sets $\{x \in X : f(x) \geq t\}$ belong to Σ.

Definition 2. The Choquet integral of an upper measurable function f on a set $A \in \Sigma$ is defined as the sum of two Riemann improper integrals,

$$(C)\int_A f \mathrm{d}\mu$$

$$= \int_0^{+\infty} \mu\left(\{x \in A : f(x) \geq t\}\right) \mathrm{d}t + \int_{-\infty}^{0} \left[\mu\left(\{x \in A : f(x) \geq t\}\right) - \mu(A)\right] \mathrm{d}t.$$

Accordingly, f is said to be Choquet integrable if both integrals above are finite.

Every upper measurable and bounded function is Choquet integrable. If $f \geq 0$, then the last integral in the formula appearing in Definition 2 is 0.

When Σ is a σ-algebra, the upper measurability and the Borel measurability are equivalent and the Choquet integral coincides with the Lebesgue integral for σ-additive measures besides, the inequality sign $\geq$ in the above two integrands can be replaced by $>$; see [16], Theorem 11.1, p. 226.

The next remarks summarize the basic properties of the Choquet integral:

Remark 1. (a) If f and g are two upper measurable functions which are Choquet integrable, then

$$f \geq 0 \text{ implies } (\mathrm{C})\int_X f \mathrm{d}\mu \geq 0 \quad \text{(positivity)}$$

$$f \leq g \text{ implies } (\mathrm{C})\int_X f \mathrm{d}\mu \leq (\mathrm{C})\int_X g \mathrm{d}\mu \quad \text{(monotonicity)}$$

$$(\mathrm{C})\int_X af \mathrm{d}\mu = a \cdot (\mathrm{C})\int_X f \mathrm{d}\mu \text{ for all } a \geq 0 \quad \text{(positive homogeneity)}$$

$$(\mathrm{C})\int_X 1 \cdot \mathrm{d}\mu(t) = \mu(X) \quad \text{(calibration).}$$

(b) In general, the Choquet integral is not additive but (as was noticed by Dellacherie [7]), if f and g are comonotonic (that is, $(f(\omega) - f(\omega')) \cdot (g(\omega) - g(\omega')) \geq 0$, for all $\omega, \omega' \in X$), then

$$(\mathrm{C})\int_X (f+g)\mathrm{d}\mu = (\mathrm{C})\int_X f \mathrm{d}\mu + (\mathrm{C})\int_X g \mathrm{d}\mu.$$

An immediate consequence is the property of translation invariance,

$$(\mathrm{C})\int_X (f+c)\mathrm{d}\mu = (\mathrm{C})\int_X f \mathrm{d}\mu + c \cdot \mu(X)$$

for all $c \in \mathbb{R}$ and all Choquet integrable functions f.

(c) If μ is a lower continuous capacity, then the Choquet integral is lower continuous in the sense that

$$\lim_{n\to\infty} \left((C)\int_X f_n \mathrm{d}\mu \right) = (C)\int_X f \mathrm{d}\mu,$$

whenever $(f_n)_n$ is a nondecreasing sequence of bounded random variables that converges pointwise to the bounded variable f.

For (a) and (b), see Denneberg [8], Proposition *5.1,* p. *64; (c)* follows in a straightforward way from the definition of the Choquet integral.

(d) If $\mu \leq \nu$ are two capacities, then $(C)\int_X f\mathrm{d}\mu \leq (C)\int_X f\mathrm{d}\nu$, for all non-negative measurable functions f.

(e) $(C)\int_A -f\mathrm{d}\mu = -(C)\int_A f\mathrm{d}\overline{\mu}$. See [16], Theorem *11.7*, p. *233*.

Remark 2. (The Subadditivity Theorem) If μ is a submodular capacity, then the associated Choquet integral is subadditive, that is,

$$(\mathrm{C})\int_X (f+g)\mathrm{d}\mu \leq (\mathrm{C})\int_X f\mathrm{d}\mu + (\mathrm{C})\int_X g\mathrm{d}\mu$$

for all f and g integrable on X. See [8], Theorem *6.3*, p. *75*. In addition, the following two integral analogs of the modulus inequality hold true,

$$|(\mathrm{C})\int_X f\mathrm{d}\mu| \leq (\mathrm{C})\int_X |f|\mathrm{d}\mu$$

and

$$|(\mathrm{C})\int_X f\mathrm{d}\mu - (\mathrm{C})\int_X g\mathrm{d}\mu| \leq (\mathrm{C})\int_X |f-g|\mathrm{d}\mu.$$

The last assertion is covered by Corollary *6.6*, p. *82*, in [8].

Remark 3. If μ is a submodular capacity, then the associated Choquet integral is a submodular functional in the sense that

$$(C)\int_A \sup\{f,g\}\,\mathrm{d}\mu + (C)\int_A \inf\{f,g\}\mathrm{d}\mu \leq (C)\int_A f\mathrm{d}\mu + (C)\int_A g\mathrm{d}\mu$$

for all f and g integrable on X. For this, integrate term by term the inequality

$$\begin{aligned}&\mu\left(\{x:\sup\{f,g\}(x)\geq t\}\right)+\mu\left(\{x:\inf\{f,g\}(x)\geq t\}\right)\\&\quad\leq \mu\left(\{x:f(x)\geq t\}\right)+\mu\left(\{x:g(x)\geq t\}\right).\end{aligned}$$

The Choquet integral associated to any lower continuous capacity is a comonotonically additive, monotone and lower continuous functional. The converse also holds.

Theorem 2. *Suppose that X is a compact Hausdorff space and $I : C(X) \to \mathbb{R}$ is a comonotonically additive and monotone functional such that $I(1) = 1$. Then I is also lower continuous and there exists a unique lower continuous normalized capacity $\mu : \Sigma_{up}^{-}(X) \to [0,1]$ such that*

$$I(f) = \int_0^{+\infty} \mu\left(\{x\in X : f(x) > t\}\right)\mathrm{d}t + \int_{-\infty}^0 \left[\mu\left(\{x\in X : f(x) > t\}\right) - 1\right]\mathrm{d}t$$

for all $f \in C(X)$. Moreover, if I is submodular in the sense that

$$I(\sup\{f,g\}) + I(\inf\{f,g\}) \leq I(f) + I(g) \quad \textit{for all } f,g \in C(X),$$

then μ is submodular too.

Proof. Let $(f_n)_n$ and f in $C(X)$, with (f_n) nondecreasing and $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in X$. Since I is monotone, it is immediate that

$$\lim_{n\to\infty} I(f_n) \leq I(f).$$

On the other hand, choose any arbitrary $\varepsilon > 0$ and take $g = f - \varepsilon 1$, that is $f = g + \varepsilon 1$. Then, $\lim_{n\to\infty} f_n(x) = f(x) > g(x)$, for all $x \in X$. Since X is compact and (f_n) is a nondecreasing sequence of continuous functions, by Dini's theorem, there is an integer N, such that $f_n(x) > g(x) = f(x) - \varepsilon 1$, for all $x \in X$ and $n \geq N$. Taking into account the comonotonic additivity and monotonicity of I, we infer that

$$I(f_n) \geq I(f - \varepsilon 1) = I(f) - \varepsilon I(1)$$

for all $n \geq N$. Passing to the limit, first as $n \to \infty$ and next as $\varepsilon \to 0$, we obtain $\lim_{n\to\infty} I(f_n) \geq I(f)$. Since the other inequality was already noticed, we conclude that $\lim_{n\to\infty} I(f_n) = I(f)$.

The integral representation of I is part of a more general result due to Cerreia-Vioglio et al. See [4], Proposition 17, p. 907. As concerns the correspondence between the property of submodularity of I and μ, this follows by adapting the argument in [4], Theorem 13 (c), p. 901. □

A result similar to Theorem 2, but for the comonotonically additive, monotone and upper continuous functionals, was shown by Zhou [17].

Remark 4. (Vector capacities) The aforementioned theory of integration with respect to a capacity can be easily extended by considering vector capacities. A simple example is offered by the set functions $\boldsymbol{\mu}$ defined on the lattice $\Sigma_{up}^{+}(X)$ (associated to a compact Hausdorff space X) and taking values in the positive cone of $\mathbb{R}^N$ in such a way that

$$\boldsymbol{\mu}(\emptyset) = 0 \text{ and } \boldsymbol{\mu}(A) \leq \boldsymbol{\mu}(B) \text{ if } A \subset B.$$

The concepts of upper/lower continuity and submodularity extend verbatim to the case of vector capacities. Moreover, a vector capacity $\boldsymbol{\mu}$ is upper continuous (lower continuous, submodular etc.) if and only if all its components $\mu_k = \mathrm{pr}_k \circ \boldsymbol{\mu}$ are scalar capacities in the sense of Definition 2, with the respective property. Therefore, the integral with respect to a submodular vector capacity $\boldsymbol{\mu}$,

$$(C)\int_X f \mathrm{d}\boldsymbol{\mu} = \left((C)\int_X f \mathrm{d}\mu_1, \ldots, (C)\int_X f \mathrm{d}\mu_N\right),$$

defines a Choquet type operator from $C(X)$ to $\mathbb{R}^N$.

According to Theorem 2, this construction generates all Choquet type operators from $C(X)$ to $\mathbb{R}^N$. More general results concerning the theory of Choquet type operators taking values in an arbitrary ordered Banach space are available in [11].

3. The extension of Hölder's inequality

The extension of Hölder's inequality to the framework of Choquet integral was treated by numerous authors, see for example [1,3,13]. By adapting the standard argument based on Young's inequality (see, [14], section 1.2, pp. 11-13), Hölder's inequality for the range of parameters $p \in (1, \infty)$ and $1/p+1/q = 1$ can be further extended to the general framework of sublinear and monotone operators. Recall that Young's inequality for this choice of parameters asserts that for all nonnegative numbers u, v we have

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q} \quad \text{for all } u, v \geq 0 \tag{3.1}$$

and the equality occurs if and only if $u^p = v^q$.

Theorem 3. (Hölder's inequality for $p \in (1, \infty)$ *and* $1/p + 1/q = 1$) *Suppose that* X *and* Y *are two Hausdorff topological spaces and* E *and* F *are respectively vector sublattices of* $C_b(X)$ *and* $C_b(Y)$ *which contain the unit (the function identically* 1*). Then every sublinear and monotone operator* $T : E \to F$ *for which* $T(1) = 1$ *verifies the inequality*

$$T(|fg|) \leq [T(|f|^p)]^{1/p} \cdot [T(|g|^q)]^{1/q} \tag{3.2}$$

for all $f, g \in E$ *such that* $fg \in E$.

Proof. For $y \in Y$ arbitrarily fixed, consider the sublinear and monotone functional $A_y : E \to \mathbb{R}$ defined by the formula

$$A_y(f) = (T(f))\,(y).$$

Clearly, $A_y(1) = 1$.

Assuming $A_y(|f|^p) > 0$ and $A_y(|g|^q) > 0$, we apply inequality (3.1) for $u = |f|/A_y(|f|^p)^{1/p}$ and $v = |g|/A_y(|g|^q)^{1/q}$ to infer that

$$\frac{|f|}{A_y(|f|^p)^{1/p}} \frac{|g|}{A_y(|g|^q)^{1/q}} \leq \frac{1}{p} \cdot \frac{|f|^p}{A_y(|f|^p)} + \frac{1}{q} \cdot \frac{|g|^q}{A_y(|g|^q)}. \tag{3.3}$$

Since the functional A_y is monotone and sublinear, the last inequality implies

$$\frac{A_y(|fg|)}{A_y(|f|^p)^{1/p} \cdot A_y(|g|^q)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

that is, $T(|f \cdot g|)(y) \leq [T(|f|^p)(y)]^{1/p} \cdot [T(|g|^q)(y)]^{1/q}$, which is inequality (3.2) in the statement.

If $A_y(|f|^p) = 0$ and/or $A_y(|g|^q) = 0$, then one repeats the above reasoning by replacing in (3.3) the vanishing number(s) by an $\varepsilon > 0$ arbitrarily small and then passing to the limit as $\varepsilon \to 0$ to conclude that $A_y(|f| \cdot |g|) = 0$. The proof is done. $\square$

Remark 5. (*Conditions for equality in Theorem* 3) We assume that X is a compact Hausdorff space and $T : C(X) \to C(X)$ is a Choquet type operator such that $T(1) = 1$ and

$$T(\sup\{f,g\}) + T(\inf\{f,g\}) \leq T(f) + T(g) \text{ for all } f,g \in C(X);$$

the last condition is nothing but the property of submodularity.

For $x \in X$ arbitrarily fixed, let us consider the comonotone additive and monotone functional

$$A_x : C(X) \to \mathbb{R}, \quad A_x(f) = (T(f))(x).$$

Clearly, $A_x(1) = 1$ and A_x is a submodular functional. According to Theorem 2 there exists a unique normalized, lower-continuous and submodular capacity μ_x on $\Sigma_{up}^{-}(X)$, such that $A_x(f) = (C)\int_X f d\mu_x$. In this case,

$$(C)\int_X |h| \mathrm{d}\mu_x = 0 \text{ is equivalent to } \mu_x(\{t \in X : |h(t)| > 0\}) = 0$$

whenever $h \in C(X)$. See [16], Theorem *11.3*, p. *228*.

We have equality in (3.2) at the point x every time when $A_x(|f|^p) = 0$ and/or $A_x(|g|^q) = 0$, equivalently,

$$\mu_x(\{t \in X : |f(t)| > 0\}) = 0 \quad \text{and/or} \quad \mu_x(\{t \in X : |g(t)| > 0\}) = 0.$$

According to (2.2), this means that equality occurs when

$$|f(t)| = 0 \text{ except for a } \mu_x\text{-null set} \quad \text{and/or} \quad |g(t)| = 0 \text{ except for a } \mu_x\text{-null set.}$$

Suppose now that $A_x(|f|^p) > 0$ and $A_x(|g|^q) > 0$. In this case an inspection of the proof of Theorem 3 shows that equality occurs in (3.2) at the point x if

$$\mu_x\left\{t \in X : \frac{1}{p}\cdot\frac{|f(t)|^p}{A_x(|f|^p)} + \frac{1}{q}\cdot\frac{|g(t)|^q}{A_x(|g|^q)} > \frac{|f(t)|}{A_x(|f|^p)^{1/p}}\frac{|g(t)|}{A_x(|g|^q)^{1/q}}\right\} = 0,$$

equivalently,

$$\frac{1}{p}\cdot\frac{|f(t)|^p}{A_x(|f|^p)} + \frac{1}{q}\cdot\frac{|g(t)|^q}{A_x(|g|^q)} = \frac{|f(t)|}{A_x(|f|^p)^{1/p}}\frac{|g(t)|}{A_x(|g|^q)^{1/q}}, \tag{3.4}$$

except possibly a μ_x-null set. According to the equality case in Young's inequality, this implies the existence of two positive constants α and β such that

$$\alpha|f(t)|^p = \beta|g(t)|^q \tag{3.5}$$

except possibly a μ_x-null set.

If an operator $T : E \to F$ is monotone and subadditive, then it verifies the inequality

$$|T(f) - T(g)| \leq T(|f-g|) \qquad \text{for all } f,g. \tag{3.6}$$

Indeed, $f \leq g + |f - g|$ yields $T(f) \leq T(g) + T(|f - g|)$, that is, $T(f) - T(g) \leq T(|f - g|)$, and interchanging the role of f and g we infer that $-(T(f) - T(g)) \leq T(|f - g|)$.

If in addition $T(0) = 0$ (for example, this happens when T is monotone and sublinear), then (3.6) yields the following inequality that complements (3.2):

$$|T(f)| \leq T(|f|) \text{ for all } f \in E. \tag{3.7}$$

This leads us to *Holder's inequality for* $p = 1$ *and* $q = \infty$:

$$|T(fg)| \leq T(|fg|) \leq T(|f|) \sup_{x \in X} |g(x)| \tag{3.8}$$

for all $f, g \in E$ such that $fg \in E$.

If X is a locally compact Hausdorff space and $T : C_b(X) \to \mathbb{R}$ is a positive linear functional for which $T(1) = 1$, then T admits the integral representation $T(f) = \int_X f d\mu$ for a suitable Borel probability measure μ and the difference

$$T(f^2) - T(f)^2 = \int_X f^2 \mathrm{d}\mu - \left(\int_X f \mathrm{d}\mu\right)^2$$

is just the *variance* of f. The fact that the variance is nonnegative follows from the Cauchy-Bunyakovsky-Schwarz inequality (the particular case of Hölder's inequality for $p = q = 2$). Thus, in the general context of sublinear and monotone operators $T : C_b(X) \to C_b(X)$, the quantity

$$D_T^2(f) = T(1) \cdot T(f^2) - T(f)^2$$

can be interpreted as the T-*variance* of f. The T-*covariance* of a pair of functions f and g in $C_b(X)$ can be introduced via the formula

$$\mathrm{Cov}_T(f, g) = T(1) \cdot T(fg) - T(f)T(g).$$

Problem 1. *Under what conditions on* T *is the following nonlinear version of the Cauchy-Bunyakovsky-Schwarz inequality,*

$$|\mathrm{Cov}_T(f, g)| \leq \sqrt{D_T^2(f)}\sqrt{D_T^2(g)},$$

true?

Some results related to this problem are presented in what follows.

Lemma 1. *If* T *is a monotone and sublinear operator that maps* $C_b(X)$ *into itself, then*

$$D_T^2(-|f|)) = T(1) \cdot T(|f|^2) - |T(-|f|)|^2 \geq 0,$$

for all $f \in C_b(X)$.

Proof. Since T is monotone and subadditive, the fact that $0 \leq (\lambda - |f(x)|)^2$ for all $\lambda > 0$ and $x \in X$ yields

$$0 \leq T[(\lambda - |f|)^2](x) \leq \lambda^2 T(1)(x) + 2\lambda T(-|f|)(x) + T(|f|^2)(x). \tag{3.9}$$

Suppose by reductio ad absurdum that there exists $x_0 \in X$ such that

$$|T(-|f|)(x_0)| > \sqrt{T(1)(x_0) \cdot T(f^2)(x_0)}. \tag{3.10}$$

Then the second degree polynomial in λ,

$$\lambda^2 T(1)(x_0) + 2\lambda T(-|f|)(x_0) + T(|f|^2)(x_0) = 0,$$

will have two positive distinct solutions $\lambda_1 < \lambda_2$. As a consequence, for any $\lambda \in (\lambda_1, \lambda_2)$,

$$\lambda^2 T(1)(x_0) + 2\lambda T(-|f| \cdot |g|)(x_0) + T(f^2 g^2)(x_0) < 0,$$

which contradicts condition (3.9). Therefore (3.10) does not hold and the proof of Lemma 1 is done. □

The next lemma provides a partial answer to Problem 1.

Lemma 2. *Suppose that $T : C_b(X) \to C_b(X)$ is a Choquet type operator. Then for all pairs of functions $f, g \in C_b(X)$ such that $|f|$ and $|g|$ are comonotone we have the inequality*

$$|\mathrm{Cov}_T(-|f|, -|g|)| \leq \sqrt{D_T^2(-|f|)}\sqrt{D_T^2(-|g|)}.$$

Proof. Let $\lambda > 0$ arbitrarily fixed. According to Lemma 1,

$$|T(-|f| - \lambda |g|)|^2 \leq T(1) \cdot T(|f|^2 + 2\lambda |fg| + \lambda^2 |g|^2)$$

while the fact that T is comonotonic additive yields

$$|T(-|f| - \lambda |g|)|^2 = (T(-|f|) + \lambda T(-|g|))^2.$$

Therefore

$$\lambda^2 D^2(-|g|) + 2\lambda (T(1) \cdot T(|fg|) - T(-|f|)T(-|g|)) + D^2(-|f|) \geq 0$$

and taking into account that $\lambda > 0$ was arbitrarily fixed one can conclude (repeating the argument used in the proof of Lemma 1) that

$$|T(1) \cdot T(|fg|) - T(-|f|)T(-|g|)|^2 \leq D_T^2(|f|) D_T^2(|g|).$$

□

4. An application to Korovkin theory

The following examples of Choquet type operators, borrowed from [9] , illustrate both our nonlinear extension of Korovkin's theorem stated in Theorem 1 and the nonlinear Cauchy–Bunyakovsky–Schwarz inequalities stated in Lemmas 1 and 2 :

- the *Bernstein-Kantorovich-Choquet* operators $K_{n,\mu} : C([0,1]) \to C([0,1])$, defined by the formula

$$K_{n,\mu}(f)(x) = \sum_{k=0}^{n} \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\mathrm{d}\mu}{\mu([k/(n+1),(k+1)/(n+1)])} \cdot \binom{n}{k} x^k (1-x)^{n-k};$$

- the *Szász-Mirakjan-Kantorovich-Choquet* operators $S_{n,\mu} : C([0,\infty)) \to C([0,\infty))$, defined by the formula

$$S_{n,\mu}(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(C)\int_{k/n}^{(k+1)/n} f(t)\mathrm{d}\mu}{\mu([k/n,(k+1)/n])} \cdot \frac{(nx)^k}{k!};$$

- the *Baskakov-Kantorovich-Choquet* operators $V_{n,\mu} : C([0,\infty)) \to C([0,\infty))$ defined by the formula

$$V_{n,\mu}(f)(x) = \sum_{k=0}^{\infty} \frac{(C)\int_{k/n}^{(k+1)/n} f(t)\mathrm{d}\mu}{\mu([k/n,(k+1)/n])} \cdot \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

In the above examples μ is a submodular capacity whose restrictions to suitable intervals are normalized by dividing the respective integrals by the length of the interval of integration.

The aim of this section is to prove a quantitative estimate for the Korovkin type result stated in Theorem 1. A basic ingredient is Lemma 1.

Theorem 4. *Let us consider the sequence of monotone, sublinear and comonotone additive Bernstein-Kantorovich-Choquet operators $(K_{n,\nu})_n$ defined as above, but with ν a submodular normalized capacity satisfying an inequality of the form $\nu \leq c\cdot\overline{\nu}$, with $c \geq 1$. Then, for all nonnegative functions $f \in C([0,1])$, all points $x \in [0,1]$ and all indices $n \in \mathbb{N}$, the following quantitative estimate holds:*

$$|K_{n,\nu}(f)(x) - f(x)| \leq (c+1)\omega_1(f; \sqrt{x^2 + 2xK_{n,\nu}(-t)(x) + K_{n,\nu}(t^2)(x)}), \tag{4.1}$$

where $\omega_1(f;\delta) = \sup\{|f(t) - f(x)| : t, x \in [0,1],\ |t-x| \leq \delta)$ denotes the modulus of continuity.

Proof. For x arbitrarily fixed, we have

$$|K_{n,\nu}(f)(x) - f(x)| = |K_{n,\nu}(f)(x) - K_{n,\nu}(f(x))(x) + K_{n,\nu}(f(x)\cdot 1)(x) - f(x)|$$

$$\leq |K_{n,\nu}(f(t)-f(x))(x)| + |f(x)| \cdot |K_{n,\nu}(1)(x)-1|$$
$$\leq K_{n,\nu}(|f(t)-f(x)|)(x) + |f(x)| \cdot |K_{n,\nu}(1)(x)-1|, \tag{4.2}$$

where the last inequality follows from the relation (3.7).

On the other hand, from the properties of the modulus of continuity, for all $t \in [0,1]$ and $\delta > 0$, we have

$$|f(t)-f(x)| \leq \omega_1(f;|t-x|) = \omega_1\left(f;\delta \cdot \frac{|t-x|}{\delta}\right) \leq \left(\frac{|t-x|}{\delta}+1\right) \cdot \omega_1(f;\delta).$$

Choosing $\delta = |K_{n,\nu}(-|t-x|)(x)| = -K_{n,\nu}(-|t-x|)(x)$ (since $K_{n,\nu}(-|t-x|)(x) \leq 0$), we obtain

$$|f(t)-f(x)| \leq \left(\frac{|t-x|}{|K_{n,\nu}(-|t-x|)(x)|}+1\right) \cdot \omega_1(f;|K_{n,\nu}(-|t-x|)(x)|).$$

Applying to the last inequality the monotone and sublinear operator $K_{n,\nu}$, we infer that

$$K_{n,\nu}(|f(t)-f(x)|)(x)$$
$$\leq \left(\frac{K_{n,\nu}(|t-x|)(x)}{|K_{n,\nu}(-|t-x|)(x)|}+K_{n,\nu}(1)(x)\right) \cdot \omega_1(f;|K_{n,\nu}(-|t-x|)(x)|).$$

Combining this fact with the inequality (4.2) we arrive at

$$|K_{n,\nu}(f(t)-f(x))(x)|$$
$$\leq \left(\frac{K_{n,\nu}(|t-x|)(x)}{|K_{n,\nu}(-|t-x|)(x)|}+K_{n,\nu}(1)(x)\right) \cdot \omega_1(f;|K_{n,\nu}(-|t-x|)(x)|)$$
$$+ |f(x)| \cdot |K_{n,\nu}(1)(x)-1|. \tag{4.3}$$

Denote $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, to simplify the appearance of formulas. Taking into account that $\nu \leq c \cdot \overline{\nu}$ we infer from Remark 1 (d) and (e) that

$$K_{n,\nu}(|t-x|)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} |t-x|d\nu(t)}{\nu([k/(n+1),(k+1)/(n+1)])}$$
$$\leq c\sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} |t-x|d\overline{\nu}(t)}{\nu([k/(n+1),(k+1)/(n+1)])}$$
$$= c\sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{|(C)\int_{k/(n+1)}^{(k+1)/(n+1)} -|t-x|d\nu(t)|}{\nu([k/(n+1),(k+1)/(n+1)])} = c \cdot |K_{n,\nu}(-|t-x|)(x)|,$$

which implies $K_{n,\nu}(|t-x|)(x)/|K_{n,\nu}(-|t-x|)(x)| \leq c$.

Now, since $K_{n,\nu}(1) = 1$, the inequality stated by Lemma 1, gives us

$$K_{n,\nu}(-|t-x|)(x) \leq \sqrt{K_{n,\nu}((t-x)^2)(x)} \leq \sqrt{K_{n,\nu}(t^2)(x)+2xK_{n,\nu}(-t)(x)+x^2}.$$

Replacing all these in (4.3), we immediately obtain the inequality (4.1). $\square$

Remark 6. (a) A concrete example of submodular normalized capacity satisfying Theorem 4 is $\nu(A) = u(\mathcal{L}(A))$, where $\mathcal{L}$ denotes the Lebesgue measure, u is the distortion defined by $u(t) = \frac{2t}{t+1}$ and $c = 2$. Indeed, $\nu([0,1]) = 1$ and $\nu(A) = \frac{2\mathcal{L}(A)}{\mathcal{L}(A)+1}$. Denoting $\mathcal{L}(A) = x$, we get $\nu(A) = \frac{2x}{x+1}$ and

$$\overline{\nu}(A) = 1 - \nu([0,1] \setminus A) = 1 - \frac{2\mathcal{L}([0,1] \setminus A)}{\mathcal{L}([0,1] \setminus A) + 1} = 1 - \frac{2(1-x)}{2-x} = \frac{x}{2-x}.$$

Then, a simple computation shows that $\frac{2x}{x+1} \leq 2 \cdot \frac{2}{2-x}$ for all $x \in [0,1]$. Therefore Theorem 4 holds for ν when $c = 2$.

(b) Theorem 4 remains valid for submodular and normalized capacities of the form $\nu(A) = u(\mathcal{L}(A))$, with u a nondecreasing, concave function with $u(0) = 0$, $u(1) = 1$ and a constant $c \geq 1$ such that $u(x) \leq c[1 - u(1-x)]$ for all $x \in [0,1]$.

(c) Theorem 4 can be easily adapted to the case of Szász-Mirakjan-Kantorovich-Choquet operators and Baskakov-Kantorovich-Choquet operators.

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