Aequat. Math. 95 (2021), 281–299

0001-9054/21/020281-19
published online August 20, 2020
https://doi.org/10.1007/s00010-020-00746-x

Aequationes Mathematicae

The small dimension lemma and d'Alembert's equation on semigroups

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Abstract. We derive an algebraic version over an algebraically closed field k with characteristic $\neq 2$ of Yang's Small Dimension Lemma. With its help we describe the k-valued solutions of d'Alembert's functional equation on semigroups S in terms of multiplicative functions and irreducible, 2-dimensional representations of S.

Mathematics Subject Classification. Primary 39B52; Secondary 20M30.

Keywords. Functional equation, d'Alembert, involution, representation, semigroup.

1. Set up

Throughout the paper we enforce the set up below.

k is an algebraically closed field of characteristic $\neq 2$ with identity element 1, and k^* denotes the multiplicative group $k^* := (k \setminus \{0\}, \cdot)$.

$k[X] :=$ the vector space over k of k-valued functions on the set X.

S is a non-empty semigroup equipped with an involution $x \mapsto x^*$ (an anti-automorphism of S such that $(x^*)^* = x$ for all $x, y \in S$) and a homomorphism $\mu : S \to k^*$ satisfying $\mu(xx^*) = 1$ for all $x \in S$. If S is a monoid or a group, we let e denote its identity element.

We shall study *d'Alembert's functional equation*

$$g(xy) + \mu(y)g(xy^*) = 2g(x)g(y) \text{ for all } x, y \in S, \tag{1.1}$$

where $g \in k[S]$ is the unknown function. A *d'Alembert function* is a non-zero solution g of (1.1).

We encounter *Wilson's functional equation* that here means

$$f(xy) + \mu(y)f(xy^*) = 2f(x)g(y) \text{ for all } x, y \in S, \tag{1.2}$$

in which $f \in k[S]$ is the unknown function for fixed $g \in k[S]$. Note that $\mathrm{Wil}(g) := \{f \in k[S] \mid f \text{ satisfies (1.2)}\}$ is a subspace of $k[S]$. The elements f of $\mathrm{Wil}(g)$ are called *Wilson functions* (corresponding to g).

We also encounter *the symmetrized sine addition law*, which is

$$w(xy) + w(yx) = 2w(x)g(y) + 2w(y)g(x),\ x, y \in S, \tag{1.3}$$

in which $w \in k[S]$ is the unknown function for fixed $g \in k[S]$. Note that $\mathrm{W}(g) := \{w \in k[S] \mid w \text{ satisfies (1.3)}\}$ is a subspace of $k[S]$.

2. Introduction

Yang [13,14] proved the Small Dimension Lemma (Theorem 5.1 below) and used it to solve d'Alembert's and Wilson's classic functional equations

$$g(xy) + g(xy^{-1}) = 2g(x)g(y),\ x, y \in G,\ \text{and} \tag{2.1}$$

$$f(xy) + f(xy^{-1}) = 2f(x)g(y),\ x, y \in G, \tag{2.2}$$

on compact groups G. It has long puzzled me, that the ideas behind her ingenious lemma have not been extended to other groups or semigroups. The present paper does so: We derive an algebraic version of Yang's Small Dimension Lemma (Theorem 5.2) and use it to d'Alembert's functional equation (1.1) on semigroups. Our version enables us to address two shortcomings of the existing theory of d'Alembert's functional equation.

1. The domains of definition S of the solutions of (1.1) have progressively been extended from $\mathbb{R}$ via abelian groups to groups and monoids (see Aczél and Dhombres [1], Davison [4] and Stetkær [12] for details and references). The present paper takes the step farther to semigroups S, which is the natural domain in view of the form of (1.1).
2. Our solutions of (1.1) may assume values in any quadratically closed field k of characteristic $\neq 2$. Although this generality of the range space has been accomplished for d'Alembert's classic functional equation (2.1) on abelian groups and P_3-groups (see for example Aczél and Dhombres [1, Chapter 13, Theorem 16] and Corovei [3, Theorem 6]), the rest of the literature deals almost exclusively with $k = \mathbb{C}$.

The function μ in (1.1) is $\mu \equiv 1$ in much of the literature, like it is in the classic functional equations (2.1) and (2.2). The reason we consider also $\mu \neq 1$, is, that this occurs in the literature, for example in Parnami, Singh and Vasudeva [9], Davison [4], Stetkær [11,12] and Elqorachi and Redouani [5]. Parnami et al. [9] calls a special version of (1.1) the exponential-cosine functional equation.

Our way of solving d'Alembert's functional equation is to combine the Algebraic Small Dimension Lemma (Theorem 5.2) with methods by Davison, who in [4] found its complex valued solutions on monoids for $\mu = 1$, and by Stetkær [12]. It turns out, that the k-valued d'Alembert functions come about, when $\mathbb{C}$ is replaced by k in the formulas of the literature. This may not sound surprising, but it is not the case for Wilson functions. The k-valued solutions can on semigroups be described in terms of multiplicative functions and 2-dimensional, irreducible representations, like they have been in previous studies of solutions on groups and monoids.

A number of mathematicians have studied other versions of d'Alembert's and Wilson's functional equations on semigroups and monoids S. Examples are Elqorachi and Redouani [5, Lemma 3.2] and Fadli, Zeglami and Kabbaj [6]. It is crucial for our proof of the Small Dimension Lemma that $x \mapsto x^*$ is an anti-automorphism of S. [5] and [6] work with an automorphism of S, so our considerations can not be transferred to [5] and [6].

This paper is structured as follows. Sections 1 and 3 introduce the set up, notation and terminology. Section 4 contains miscellaneous auxiliary facts about representations. The principal part of the paper starts with Sect. 5 that discusses and proves the Algebraic Small Dimension Lemma. Section 6 derives some general properties of d'Alembert functions. Section 7 treats abelian d'Alembert functions, while Sect. 8 produces the non-abelian ones by combining Sects. 5 and 6. Section 9 notes that the multiplicative functions and 2-dimensional, irreducible representations that describe a continuous d'Alembert function on a topological semigroup are also continuous.

3. Notation and terminology

Throughout the paper we use the set up, notation and terminology described in Sects. 1 and 3.

Definition 3.1. Let Σ be a semigroup, and let $g \in k[\Sigma]$ be fixed.

(a) $f \in k[\Sigma]$ is *multiplicative* if $f(xy) = f(x)f(y)$ for all $x, y \in \Sigma$.
(b) Let $x \in \Sigma$. For any $f \in k[\Sigma]$ we define the *right translate* $R(x)f \in k[\Sigma]$ by $[R(x)f](y) := f(yx)$ for $y \in \Sigma$, and the *left translate* $L'(x)f \in k[\Sigma]$ by $[L'(x)f](y) := f(xy)$ for $y \in \Sigma$. R is called the *right regular representation* of Σ. We define $f_x := L'(x)f - f(x)g \in k[\Sigma]$.
(c) $\mathcal{T}(g) := \operatorname{span}\{g, L'(x)g \mid x \in \Sigma\} = \operatorname{span}\{g, g_x \mid x \in \Sigma\}$. It is a subspace of $k[\Sigma]$.

Definition 3.2. Let F be a function on a semigroup Σ. It is said to be *abelian* if $F(x_1x_2\cdots x_n) = F(x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)})$ for all $x_1, x_2 \ldots, x_n \in \Sigma$, all permutations π of n elements and all $n = 2, 3, \ldots$. It is *non-abelian* if it is not abelian. It is *central* if $F(xy) = F(yx)$ for all $x, y \in \Sigma$.

Definition 3.3. By the help of the given involution $x \mapsto x^*$ of S we associate to any $F \in k[S]$ the function $F^* \in k[S]$ defined by $F^*(x) := \mu(x)F(x^*)$, $x \in S$. The map $F \mapsto F^*$ is an involution of $k[S]$, i.e., an automorphism of the vector space $k[S]$ such that $(F^*)^* = F$. The even part of F is $F_e := (F + F^*)/2$, and its odd part is $F_o := (F - F^*)/2$. We say that F is *even* if $F^* = F$, and that it is *odd* if $F^* = -F$.

Definition 3.4. For any vector space V we let $\mathcal{L}(V)$ denote the algebra of linear operators of V into V. The dual vector space of V is denoted V^*, and we write $\langle \varphi, v\rangle$ for the value of $\varphi \in V^*$ at $v \in V$. If $W \subseteq V$, we define $W^\perp := \{\varphi \in V^* \mid \langle \varphi, w\rangle = 0 \text{ for all } w \in W\}$. The transpose of $A \in \mathcal{L}(V)$ is denoted A^t.

Definition 3.5. Let V be a 2-dimensional vector space. For $A \in \mathcal{L}(V)$ we let $\operatorname{adj}(A) \in \mathcal{L}(V)$ denote the *adjugate operator* of A from linear algebra. Properties of adjugation can easily be derived from its matrix form:

$$\operatorname{adj}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

$\operatorname{adj} : \mathcal{L}(V) \to \mathcal{L}(V)$ is linear, and we have for any $A, B \in \mathcal{L}(V)$ that

$$A + \operatorname{adj}(A) = (\operatorname{tr} A)I,\ A\operatorname{adj}(A) = \operatorname{adj}(A)A = (\det A)I,\ \operatorname{adj}(\operatorname{adj}(A)) = A,$$
$$\operatorname{adj}(AB) = \operatorname{adj}(B)\operatorname{adj}(A),\ \operatorname{adj}(BAB^{-1}) = B\operatorname{adj}(A)B^{-1}.$$

In the last identity B is assumed to be invertible.

Definition 3.6. Let π be a *representation* of a semigroup Σ on a vector space $V \neq \{0\}$ over k, i.e., a map $\pi : \Sigma \to \mathcal{L}(V)$ such that $\pi(xy) = \pi(x)\pi(y)$ for all $x, y \in \Sigma$.

(i) A subspace $W \subseteq V$ of V is π-*invariant* if $\pi(x)W \subseteq W$ for all $x \in \Sigma$.
(ii) π is *irreducible* if $\{0\}$ and V are the only π-invariant subspaces.
(iii) For $\varphi \in V^*$ and $v \in V$ we let $c_{\varphi,v} \in k[\Sigma]$ denote the function $c_{\varphi,v}(x) := \langle \varphi, \pi(x)v\rangle$, $x \in \Sigma$. The *space of matrix coefficients* is the subspace $\operatorname{span}\{c_{\varphi,v} \mid \varphi \in V^*, v \in V\}$ of $k[\Sigma]$. Its elements are called *matrix coefficients* of π.
(iv) C_φ denotes for $\varphi \in V^*$ the subspace $C_\varphi := \{c_{\varphi,v} \mid v \in V\}$ of $k[\Sigma]$.
(v) Let $\dim V < \infty$. The matrix coefficient $\chi_\pi := \operatorname{tr}\pi \in k[\Sigma]$ is called the *character* of π. The space of matrix coefficients of π consists of the functions $x \mapsto \operatorname{tr}(A\pi(x))$, $x \in \Sigma$, where $A \in \mathcal{L}(V)$.

4. Auxiliary results about representations

If the semigroup Σ in Definition 3.6 is a monoid or a group, then the condition $\pi(e) = I$ is usually added to the definition of a representation. We do not do so in the present paper, because it can be proved that $\pi(e) = I$ in the situations that are relevant for us: Proposition 4.1 takes care of Theorem 5.2(c), while Proposition 6.1(h) has π as (a restriction of) the right regular representation R that trivially satisfies $R(e) = I$.

Proposition 4.1. *If π is an irreducible representation of a monoid on a vector space of dimension ≥ 2, then $\pi(e) = I$.*

Proof. Let Σ denote the monoid and V the vector space. $\pi(e) = \pi(e)^2$, so $\pi(e)$ is a projection. It suffices to prove that its range is all of V. $\pi(e)V$ is a π-invariant subspace of V, so $\pi(e)$ has range $\{0\}$ or V by the irreducibility of π. If $\pi(e)V = \{0\}$, then $\pi(e) = 0$, so $\pi(x) = \pi(ex) = \pi(e)\pi(x) = 0$ for all $x \in \Sigma$, so any subspace of V is π-invariant. This contradicts the irreducibility, since $\dim V \geq 2$. Hence $\pi(e)$ has range V. $\square$

Burnside's theorem is an important tool in the study of representations and their matrix coefficients, which are essential for our proof of Theorem 5.2. Burnside's theorem holds for algebraically closed fields, which is one reason we assume that k is algebraically closed. Theorem 4.2 is cited from Lomonosov and Rosenthal [8], which contains a short, elementary and self-contained proof of it.

Theorem 4.2. (Burnside) *The only irreducible algebra of linear transformations on a vector space of finite dimension greater than 1 over an algebraically closed field is the algebra of all linear transformations on the vector space.*

We shall use the following corollary of Burnside's theorem that works for representations of semigroups. It is immediate from Theorem 4.2.

Corollary 4.3. *Let π be an irreducible representation of S on a vector space V over k with $2 \leq \dim V < \infty$. Then*

(a) any $A \in \mathcal{L}(V)$ can be written in the form $A = \sum_{i=1}^{n} a_i \pi(x_i)$ for some $n \in \{1, 2, \ldots\}$, $a_i \in k$ and $x_i \in S$ for $i = 1, 2, \ldots, n$.

(b) Any element of $\mathcal{L}(V^*)$ can be written in the form $\sum_{i=1}^{n} a_i \pi(x_i)^t$ for some $n \in \{1, 2, \ldots\}$, $a_i \in k$ and $x_i \in S$ for $i = 1, 2, \ldots, n$.

By the help of Corollary 4.3 we get Proposition 4.4 that contains some results about matrix coefficients. Proposition 4.4(c) is related to Theorem 8.3(a), while point (d) is a partial converse of Theorem 8.2, which is one of our main results.

Proposition 4.4. *Let ρ be a representation of a semigroup Σ on a 2-dimensional vector space V over k. Define $g \in k[\Sigma]$ and $f_A \in k[\Sigma]$ for $A \in \mathcal{L}(V)$ by $g(x) := \frac{1}{2} \operatorname{tr} \rho(x)$ and $f_A(x) := \frac{1}{2} \operatorname{tr}(A\rho(x))$ for $x \in \Sigma$. Then*

(a) g is central. It is non-abelian, if and only if ρ is irreducible.

(b) $\mathcal{T}(g) \subseteq \{f_A \mid A \in \mathcal{L}(V)\}$ with equality if ρ is irreducible.

(c) $\{f_A \mid A \in \mathcal{L}(V) \text{ has } \operatorname{tr} A = 0\} \subseteq W(g)$.

(d) Let $\Sigma = S$, and assume that $\mu(x)\rho(x^*) = adj(\rho(x))$ for all $x \in S$. Then f_A satisfies Wilson's functional Eq. (1.2). In particular $g = f_I$ satisfies d'Alembert's functional Eq. (1.1).

Proof. (a) Characters of representations are central, as is well known.

Let ρ be irreducible. We prove that g is non-abelian by contradiction, so we assume g is abelian. Then $g(xyz) = g(xzy)$ for all $x, y, z \in \Sigma$, so that

$$0 = \operatorname{tr}(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(z)\rho(y)) = \operatorname{tr}(\rho(x)[\rho(y)\rho(z) - \rho(z)\rho(y)]).$$

Corollary 4.3(a) gives us that $0 = \operatorname{tr}(A[\rho(y)\rho(z) - \rho(z)\rho(y)])$ for all $A \in \mathcal{L}(V)$, which implies that $\rho(y)\rho(z) = \rho(z)\rho(y)$ for all $y, z \in \Sigma$. Applying Corollary 4.3(a) once more we obtain that $BC = CB$ for all $B, C \in \mathcal{L}(V)$. But this contradicts that $\dim V = 2$.

Assume conversely that ρ is not irreducible. Then V possesses a 1-dimensional ρ-invariant subspace W. Choose a basis $\{w, v\}$ of V such that $w \in W$. With respect to this basis ρ has the matrix form

$$\rho(x) = \begin{pmatrix} \rho_{11}(x) & \rho_{12}(x) \\ 0 & \rho_{22}(x) \end{pmatrix} \text{ for } x \in \Sigma.$$

That ρ is a homomorphism implies that the functions ρ_{11} and ρ_{22} are multiplicative and hence abelian. Therefore $g = \frac{1}{2}\operatorname{tr}\rho = \frac{1}{2}(\rho_{11} + \rho_{22})$ is abelian.

(b) The inclusion $\subseteq$. Clearly $g = \frac{1}{2}\operatorname{tr}\rho$ is the element of the right hand side with $A = I$. If $x, y \in \Sigma$, then $[L'(x)g](y) = g(xy) = \frac{1}{2}\operatorname{tr}(\rho(xy)) = \frac{1}{2}\operatorname{tr}(\rho(x)\rho(y))$, so $L'(x)g = f_{\rho(x)}$. This proves the inclusion $\subseteq$, because $\{f_A \mid A \in \mathcal{L}(V)\}$ is a vector space over k.

The inclusion $\supseteq$. According to Corollary 4.3(a) we may write any $A \in \mathcal{L}(V)$ in the form $A = \sum_{i=1}^n a_i\rho(x_i)$, where $n \in \mathbb{N}$, $a_i \in k$ and $x_i \in \Sigma$ for $i = 1, 2, \ldots, n$. For any $x \in \Sigma$ we get

$$\begin{aligned} f_A(x) &= \frac{1}{2}\operatorname{tr}\left(\sum_{i=1}^n a_i\rho(x_i)\rho(x)\right) = \frac{1}{2}\sum_{i=1}^n a_i \operatorname{tr}(\rho(x_i x)) \\ &= \sum_{i=1}^n a_i g(x_i x) = \sum_{i=1}^n a_i [L'(x_i)g](x), \end{aligned}$$

so $f_A = \sum_{i=1}^n a_i L'(x_i)g \in \mathcal{T}(g)$.

(c) Let $A \in \mathcal{L}(V)$ have $\operatorname{tr} A = 0$. For any $x, y \in \Sigma$ we compute that

$$\begin{aligned} &2[2f_A(x)g(y) + 2f_A(y)g(x) - (f_A(xy) + f_A(yx))] \\ &\quad = \operatorname{tr}(A\rho(x))\operatorname{tr}\rho(y) + \operatorname{tr}(A\rho(y))\operatorname{tr}\rho(x) - \operatorname{tr}(A\rho(x)\rho(y)) - \operatorname{tr}(A\rho(y)\rho(x)) \\ &\quad = \operatorname{tr}(A\rho(x)[\operatorname{tr}(\rho(y))I - \rho(y)]) + \operatorname{tr}(A\rho(y)[\operatorname{tr}(\rho(x))I - \rho(x)]). \end{aligned}$$

Due to the identity $C + \operatorname{adj}(C) = \operatorname{tr}(C)I$ for $C \in \mathcal{L}(V)$ this expression equals $\operatorname{tr}(AB)$, where $B := \rho(x)\operatorname{adj}(\rho(y)) + \rho(y)\operatorname{adj}(\rho(x))$. From $\operatorname{adj}(B) = B$ (from the definition of B), we get that $B = cI$ for some $c \in k$. Thus the end result of the computation is that $\operatorname{tr}(AB) = c\operatorname{tr} A = 0$ as desired.

(d) results from the following calculation, in which $x, y \in S$ are arbitrary.

$$2f_A(xy) + 2\mu(y)f_A(xy^*) = \operatorname{tr}(A\rho(xy)) + \mu(y)\operatorname{tr}(A\rho(xy^*))$$

$$= \operatorname{tr}(A\rho(x)[\rho(y) + \mu(y)\rho(y^*)]) = \operatorname{tr}(A\rho(x)[\rho(y) + \operatorname{adj}(\rho(y))])$$
$$= \operatorname{tr}(A\rho(x)\operatorname{tr}(\rho(y))I) = \operatorname{tr}(A\rho(x))2g(y) = 4f_A(x)g(y).$$

□

5. The algebraic small dimension lemma

5.1. Statement and discussion

The Small Dimension Lemma was stated and proved by Yang [13,14]. Theorem 5.1 reproduces its main points as found in [14].

Theorem 5.1. (The Small Dimension Lemma) *Let $U(n)$ be the group of unitary $n \times n$ matrices. Let $\pi : G \to U(n)$, where $2 \leq n < \infty$, be a continuous, irreducible representation of a compact group G. Suppose that there exists a non-zero vector $v_0 \in \mathbb{C}^n$ such that*

$$\left[\pi(x) + \pi(x^{-1})\right] v_0 \in \mathbb{C}v_0 \text{ for all } x \in G. \tag{5.1}$$

Then $n = 2$, and $\det \pi(x) = 1$ for all $x \in G$.

Theorem 5.2 is our extension of Yang's Small Dimension Lemma.

Theorem 5.2. (The Algebraic Small Dimension Lemma) *Let π be a finite dimensional, irreducible representation of S on a vector space V over k such that $\dim V \geq 2$. Suppose that there exist a non-zero vector $v_0 \in V$ and a function $g \in k[S]$ such that*

$$[\pi(x) + \mu(x)\pi(x^*)]v_0 = 2g(x)v_0 \text{ for all } x \in S. \tag{5.2}$$

Then the following statements hold.

- *(a)* $\dim V = 2$.
- *(b)* $\mu(x)\pi(x^*) = adj(\pi(x))$ *for all* $x \in S$.
- *(c) If S is a monoid, then $\pi(e) = I$. If S is a group, and the involution $x \mapsto x^*$ of S is the group inversion, then $\det \pi = \mu$.*
- *(d) The property (5.2) extends from $v_0 \in V$ to all of V, in the sense that $\pi(x) + \mu(x)\pi(x^*) = 2g(x)I$ for all $x \in S$.*
- *(e) Define $f_A \in k[S]$ for $A \in \mathcal{L}(V)$ by $f_A(x) := \frac{1}{2}\operatorname{tr}(A\pi(x))$, $x \in S$. Then f_A satisfies Wilson's functional equation (1.2).*
- *(f) $g = \chi_\pi/2$, and g is a central, non-abelian d'Alembert function.*

Theorem 5.1 is a result in analysis, while Theorem 5.2 is an algebraic result that requires neither compactness, continuity, unitarity, groups nor the field of complex numbers.

(a), (b) and (c) of Theorem 5.2 generalize Theorem 5.1, while (d), (e) and (f) register some connections between (5.2) and d'Alembert's and Wilson's functional equations that Theorem 5.1 does not touch on.

According to Theorem 5.2(d) the identity $\pi(x) + \mu(x)\pi(x^*) = 2g(x)I$ is equivalent to the formally weaker assumption (5.2). The identity of (d) occurs as a hypothesis in [13, Condition (2.1)] instead of (5.1).

A representation π satisfying (5.1) or (5.2) can arise in various ways, given a d'Alembert function g. Yang [13,14] found a π via the operator valued Fourier transformation. In the present paper π is a natural object derived from g: It is the right regular representation of S on $\mathcal{T}(g)$ (the vector space spanned by g and its left translates), or more precisely its restriction to a subspace of $\mathcal{T}(g)$. See Proposition 6.1(h) and the proof of Theorem 8.2.

5.2. Proof of the algebraic small dimension lemma

Note $\dim V \geq 2$, so that Burnside's theorem (Corollary 4.3) applies in the proof. Let ρ denote the representation $\rho(x) := \mu(x^*)\pi(x)$, $x \in S$.

(a) We denote the matrix coefficients of π as in Definition 3.6, while those of ρ are marked with a superscript ρ. Thus $c_{\varphi,v}(x) := \langle \varphi, \pi(x)v\rangle$ and $c^{\rho}_{\varphi,v}(x) := \langle \varphi, \rho(x)v\rangle$ for $\varphi \in V^*,\ x \in S,\ v \in V$.

Let $V_0 := \operatorname{span}(v_0)$. The proof of (a) is based on three claims about the matrix coefficients of π. The key observation is the first claim, which is the only one that uses (5.2).

Claim. If $\varphi_1, \varphi_2 \in V_0^{\perp} \setminus \{0\}$, then $C_{\varphi_1} = C_{\varphi_2}$.

Proof of the claim. Let $\tau(x) := x^*$ for all $x \in S$. It suffices to prove that $C_{\varphi} = \{c^{\rho}_{\psi,v_0} \circ \tau \mid \psi \in V^*\}$ for any $\varphi \in V_0^{\perp} \setminus \{0\}$, because the right hand side does not depend on φ. So let $\varphi \in V_0^{\perp} \setminus \{0\}$. By Corollary 4.3 we have $C_{\varphi} = \{c_{\varphi,v} \mid v \in V\} = \operatorname{span}\{c_{\varphi,\pi(x)v_0} \mid x \in S\}$, which leads us to the computation (using that $\varphi \in V_0^{\perp}$) that

$$\begin{aligned} c_{\varphi,\pi(x)v_0}(y^*) &= \langle \varphi, \pi(y^*)\pi(x)v_0\rangle = \langle \varphi, \pi(y^*x)v_0\rangle \\ &= \langle \varphi, 2g(y^*x)v_0 - \rho((y^*x)^*)v_0\rangle = -\langle \varphi, \rho((y^*x)^*)v_0\rangle \\ &= -\langle \varphi, \rho(x^*)\rho(y)v_0\rangle = -\langle \rho(x^*)^t\varphi, \rho(y)v_0\rangle = -c^{\rho}_{\rho(x^*)^t\varphi,v_0}(y). \end{aligned}$$

Now,

$$\begin{aligned} C_{\varphi} &= \operatorname{span}\{c_{\varphi,\pi(x)v_0} \mid x \in S\} = \operatorname{span}\{c^{\rho}_{\rho(x^*)^t\varphi,v_0} \circ \tau \mid x \in S\} \\ &= \operatorname{span}\{c^{\rho}_{\rho(x)^t\varphi,v_0} \circ \tau \mid x \in S\}. \end{aligned}$$

Since $\varphi \neq 0$, we get from Corollary 4.3(b) that $C_{\varphi} = \operatorname{span}\{c^{\rho}_{\psi,v_0} \circ \tau \mid \psi \in V^*\}$ as desired. $\square$

Claim. $C_{\varphi} \neq \{0\}$, when $\varphi \in V^* \setminus \{0\}$.

Proof of the claim. Let $\varphi \neq 0$. It suffices to prove that $c_{\varphi,v} \neq 0$, when $v \neq 0$. Assume for contradiction that $c_{\varphi,v} = 0$. Then $\langle \varphi, \pi(x)v \rangle = 0$ for all $x \in S$. Corollary 4.3 implies that $\langle \varphi, v' \rangle = 0$ for all $v' \in V$, which means that $\varphi = 0$. But this contradicts that $\varphi \neq 0$. □

Claim. If $\varphi_1, \varphi_2 \in V^*$ are linearly independent, then C_{φ_1} and C_{φ_2} form a direct sum in $k[S]$.

Proof of the claim. We shall for any fixed $v_1, v_2 \in V$ prove that

$$c_{\varphi_1,v_1} + c_{\varphi_2,v_2} = 0 \tag{5.3}$$

implies that $c_{\varphi_1,v_1} = c_{\varphi_2,v_2} = 0$. It suffices to prove that $v_1 = 0$, because this entails that $c_{\varphi_1,v_1} = 0$, and then (5.3) gives $c_{\varphi_2,v_2} = 0$.

(5.3) means that $\varphi_1(\pi(x)v_1) + \varphi_2(\pi(x)v_2) = 0$ for all $x \in G$, or equivalently that $(\pi(x)^t\varphi_1)(v_1) + (\pi(x)^t\varphi_2)(v_2) = 0$ for all $x \in G$. Corollary 4.3(b) applied to V^* gives us that

$$(T\varphi_1)(v_1) + (T\varphi_2)(v_2) = 0 \text{ for all } T \in \mathcal{L}(V^*). \tag{5.4}$$

φ_1 and φ_2 are linearly independent, so to any $\varphi \in V^*$ there exists a $T \in \mathcal{L}(V^*)$ such that $T\varphi_1 = \varphi$ and $T\varphi_2 = 0$. By (5.4) we then get that $\varphi(v_1) = 0$. However, φ is arbitrary in V^*, so $v_1 = 0$ as desired. □

To finish (a) assume for contradiction that $\dim V \geq 3 = \dim V_0 + 2$. Then there exist two linearly independent elements $\varphi_1, \varphi_2 \in V_0^\perp$. Combining the first and third claims we get that $C_{\varphi_1} = C_{\varphi_2} = \{0\}$, which contradicts the second claim. This finishes the proof of (a).

(b) Let us for brevity momentarily write σ for the representation $x \mapsto \sigma(x) := \operatorname{adj}(\rho(x^*))$. Note that the subspace $V_1 := \{v \in V \mid \pi(x)v = \sigma(x)v \text{ for all } x \in S\}$ of V is π-invariant. Indeed, if $v_1 \in V_1$, we get for any $x, y \in S$ that

$$\pi(x)[\pi(y)v_1] = \pi(xy)v_1 = \sigma(xy)v_1 = \sigma(x)\sigma(y)v_1 = \sigma(x)[\pi(y)v_1],$$

so $\pi(y)v_1 \in V_1$. It follows from the irreducibility of π that $V = V_1$ if $V_1 \neq \{0\}$. The content of (b) is that π and σ agree. To prove (b) it suffices to show that $v_0 \in V_1$, so that $V_1 \neq \{0\}$, i.e., to show that $\pi(x)v_0 = \operatorname{adj}(\rho(x^*))v_0$ for all $x \in S$. This we proceed to do.

$\dim V = 2$ by (a), so we may assume that $V = k^2$, and that the non-zero vector v_0 from (5.2) is $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We write π in matrix form as

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}, \text{ where } \pi_{ij} \in k[S] \text{ for } i, j = 1, 2.$$

Similarly for ρ. Note for use below that there exists an $x_0 \in S$ such that $\pi_{21}(x_0) \neq 0$. If not, then the line $\{\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \alpha \in k\}$ is a π-invariant, proper subspace of V, contradicting the irreducibility of π. The second component of (5.2) says that

$$\pi_{21}(x) + \rho_{21}(x^*) = 0 \text{ for all } x \in S. \tag{5.5}$$

Note π_{21} is odd: For any $x \in S$ we find by the definition of ρ that

$$\pi_{21}^*(x) = \mu(x)\pi_{21}(x^*) = -\mu(x)\rho_{21}(x) = -\mu(x)\mu(x^*)\pi_{21}(x) = -\pi_{21}(x).$$

We shall show that $\pi(x)v_0 = \operatorname{adj}(\rho(x^*))v_0$ for all $x \in S$, which in coordinates boils down to the two requirements $\pi_{11}(x) = \rho_{22}(x^*)$ and $\pi_{21}(x) = -\rho_{21}(x^*)$. The latter equality is (5.5), so to get (b) it remains to show that

$$\delta(x) := \pi_{11}(x) - \rho_{22}(x^*) = 0 \text{ for all } x \in S.$$

The homomorphism property $\pi(xy) = \pi(x)\pi(y)$ of π gives us the formula $\pi_{21}(xy) = \pi_{21}(x)\pi_{11}(y) + \pi_{22}(x)\pi_{21}(y)$ for all $x, y \in S$. Combining this with (5.5) we obtain for any $x, y \in S$ that

$$\begin{aligned}\pi_{21}(x)\pi_{11}(y) + \pi_{22}(x)\pi_{21}(y) &= \pi_{21}(xy) = -\rho_{21}((xy)^*) = -\rho_{21}(y^*x^*)\\ &= -\rho_{21}(y^*)\rho_{11}(x^*) - \rho_{22}(y^*)\rho_{21}(x^*) = \pi_{21}(y)\rho_{11}(x^*) + \rho_{22}(y^*)\pi_{21}(x).\end{aligned}$$

Comparing the left and right hand sides we get that

$$\pi_{21}(x)[\pi_{11}(y) - \rho_{22}(y^*)] = \pi_{21}(y)[\rho_{11}(x^*) - \pi_{22}(x)].$$

On the right $\rho_{11}(x^*) - \pi_{22}(x) = \mu(x)\pi_{11}(x^*) - \pi_{22}(x) = \mu(x)[\pi_{11}(x^*) - \mu(x^*)\pi_{22}(x)] = \mu(x)\delta(x^*) = \delta^*(x)$, so we get that $\pi_{21}(x)\delta(y) = \pi_{21}(y)\delta^*(x)$. Since π_{21} is odd and $\pi_{21}(x_0) \neq 0$, we infer that δ is odd, which reduces the formula to $\pi_{21}(x)\delta(y) = -\pi_{21}(y)\delta(x)$. It is elementary for an identity of this form that either $\pi_{21} = 0$ or $\delta = 0$, when $\operatorname{char}(k) \neq 2$ (see [12, Exercise 1.1(b)]). But $\pi_{21}(x_0) \neq 0$, so $\delta = 0$.

(c) If S is a monoid, then $\pi(e) = I$ by Proposition 4.1. If S is a group, and the involution is the group inversion, we get the rest of (c) from

$$\begin{aligned}I = \pi(e) = \pi(xx^{-1}) = \pi(xx^*) = \pi(x)\pi(x^*) &= \pi(x)\mu(x)^{-1}\operatorname{adj}(\pi(x))\\ = \mu(x)^{-1}\pi(x)\operatorname{adj}(\pi(x)) &= \mu(x)^{-1}\det(\pi(x))I.\end{aligned}$$

(d) The identity $\pi(x) + \operatorname{adj}(\pi(x)) = \operatorname{tr}(\pi(x))I$ and (b) give that

$$\pi(x) + \mu(x)\pi(x^*) = \pi(x) + \operatorname{adj}(\pi(x)) = \operatorname{tr}(\pi(x))I. \tag{5.6}$$

Comparing this with (5.2) we see that $\operatorname{tr}\pi = 2g$, so (5.6) is (d).

(e) is immediate from Proposition 4.4(d).

(f) We saw during the proof of (d) that $\operatorname{tr}\pi = 2g$, so $g = \frac{1}{2}\chi_\pi$. Proposition 4.4(a) and Proposition 4.4(d) give that g is a central, non-abelian solution of d'Alembert's functional equation. Furthermore $g \neq 0$, g being non-abelian, so g is a d'Alembert function.

6. Some properties of d'Alembert functions

In this section we derive some properties of k-valued d'Alembert functions on semigroups. For complex valued functions and monoids they are in the literature, and many of the arguments in, say, Davison [4] and Stetkær [12] carry over word for word when we extend from $\mathbb{C}$ to k and from monoids to semigroups.

The notations F^*, f_x, $\mathrm{Wil}(g)$, $W(g)$, R etc. were introduced in Sects. 1 and 3.

Proposition 6.1. *Let $g \in k[S]$ be a d'Alembert function. The following statements hold:*

(a) g is central and $g^* = g$.
(b) g satisfies for all $x, y, z \in S$ the pre-d'Alembert functional equation

$$g(xyz) + g(xzy) = 2g(x)g(yz) + 2g(y)g(xz) + 2g(z)g(xy) - 4g(x)g(y)g(z).$$

(c) If $x \in S$, then $(g_x)^* = -g_x$, and $g_x \in W(g)$.
(d) $d_g(x) := 2g(x)^2 - g(x^2)$, $x \in S$, is a multiplicative function.
(e) For all $x, y \in S$ the following formula holds:

$$\Delta_g(x,y) := \frac{g\left(x^2y^2\right) - g\left((xy)^2\right)}{2} = g_x(x)g_y(y) - g_x(y)^2.$$

(f) g is abelian, if and only if $\Delta_g = 0$.
(g) $\mathcal{T}(g) \subseteq kg + W(g)$.
(h) $\mathcal{T}(g)$ is invariant under the right regular representation R. Let ρ denote the restriction of R to $\mathcal{T}(g)$. Then the map $x \mapsto \rho(x) \in \mathcal{L}(\mathcal{T}(g))$ is a representation of S on $\mathcal{T}(g)$ satisfying

$$\rho(x) + \mu(x)\rho(x^*) = 2g(x)I \text{ for all } x \in S. \tag{6.1}$$

If S is a monoid, then $\rho(e) = I$.

Proof. (a) can be proved like [12, Proposition 9.17(a) and (b)].

(b) Replace $\mathbb{C}$ by k in [4, Proposition 5.2] or [12, Proposition 9.17(c)].

(c) For any $y \in S$ we find, using (1.1) and $g^* = g$, that

$$\begin{aligned}(g_x)^*(y) &= \mu(y)g_x(y^*) = \mu(y)g(xy^*) - \mu(y)g(x)g(y^*)\\ &= [2g(x)g(y) - g(xy)] - g(x)g^*(y) = 2g(x)g(y) - g(xy) - g(x)g(y)\\ &= -[g(xy) - g(x)g(y)] = -g_x(y).\end{aligned}$$

For the last statement note (b) and do the computations in the proof of [12, Lemma 8.8] with k instead of $\mathbb{C}$.

(d) Replacing y by yy^* in d'Alembert's functional equation (1.1) we find that $g(xyy^*) = g(x)g(yy^*)$. Using that g is central we obtain that

$$d_g(xy) = \mu(xy)g(xy(xy)^*) = \mu(x)\mu(y)g(xyy^*x^*) = \mu(x)\mu(y)g(x^*xyy^*)$$

$$= \mu(x)\mu(y)g(x^*x)g(yy^*) = \mu(x)g(xx^*)\mu(y)g(yy^*) = d_g(x)d_g(y).$$

(e) Put $w := g_y \in W(g)$ in [12, Formula (6.6)] and use that g is central, so that $g_x(y) = g_y(x)$ for all $x, y \in S$.

(f) $\Delta_g = 0$ by the definition of Δ_g, when g is abelian. To get the converse we assume that $\Delta_g = 0$ and prove that g is abelian. Since g is central by (a), it suffices to prove that g satisfies Kannappan's condition $g(xyz) - g(xzy) = 0$ for all $x, y, z \in S$. The rest of the proof proceeds like the proof of [4, Proposition 2.7], which is reproduced in [12, Lemma 8.10].

(g) $\mathcal{T}(g)$ is by definition spanned by g, which is in kg, and elements of the form $L'(x)g = g(x)g + g_x$. The elements on the right are in $kg + W(g)$, because $g_x \in W(g)$ according to (c).

(h) The invariance is easy, because g is central. The homomorphism property of ρ is trivial, because R is a representation. Also (6.1) is easy to derive by simple computations from the definition of $\mathcal{T}(g)$. □

We recognize (6.1) as a version of the hypothesis (5.2) of Theorem 5.2. Another central hypothesis of Theorem 5.2 is that the representation is finite dimensional, which here means that $\dim \mathcal{T}(g) < \infty$. By applying methods from Davison [4] we shall extend the existing proof of the finite dimensionality of $\mathcal{T}(g)$ from $\mathbb{C}$ to k and monoids to semigroups (Lemma 8.1(c)). However, ρ may not be irreducible, so we shall work with an irreducible sub-representation π of ρ.

The exposition splits into the cases of g abelian (Sect. 7) and g non-abelian (Sect. 8). We finish this section by presenting examples of solutions of d'Alembert's functional equation, both abelian and non-abelian. They are meant as illustrations of the theory in the next sections.

Examples 6.2. (1) Let $\lambda \in \mathbb{C}$. Some abelian solutions of the equation $g(x + y) + e^{\lambda y} g(x - y) = 2g(x)g(y),\ x, y \in \mathbb{R}$, are $g_\alpha \in C(\mathbb{R})$, where $\alpha \in \mathbb{C}$, defined by

$$g_\alpha(x) = \frac{e^{\alpha x} + e^{(\lambda - \alpha)x}}{2},\ x \in \mathbb{R}.$$

For $\lambda = 0$ and $\alpha = i$ we obtain the well known solution $x \mapsto \cos x$ of d'Alembert's classic functional equation on $\mathbb{R}$.

(2) A non-abelian solution of the equation $g(xy) + (\det y)g(xy^{-1}) = 2g(x)g(y)$, $x, y \in GL(2, k)$, is $g(x) := \frac{1}{2} \operatorname{tr} x$, $x \in GL(2, k)$. Writing

$$a := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } c := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we compute that $g(abc) = 1$ and $g(acb) = -1$, so $g(abc) \neq g(acb)$.

7. Abelian d'Alembert functions

The abelian case does not involve the Small Dimension Lemma.

Theorem 7.1. *The abelian d'Alembert functions in $k[S]$ are the functions $(\chi + \chi^*)/2$, where $\chi \in k[S]$ ranges over the non-zero multiplicative functions on S.*

The decomposition of g is essentially unique: If $g = (\chi_1 + \chi_2)/2$, where $\chi_1, \chi_2 \in k[S]$ are multiplicative, then either $\chi_1 = \chi$ and $\chi_2 = \chi^*$, or $\chi_1 = \chi^*$ and $\chi_2 = \chi$.

Proof. We first prove that any abelian d'Alembert function $g \in k[S]$ has the form described. Note that $g = g^*$ by Proposition 6.1(a). By Proposition 6.1(c) $g_a \in W(g)$ for any $a \in S$, and this reduces to the sine addition law $g_a(xy) = g_a(x)g(y) + g_a(y)g(x)$, since g is abelian. There are two cases.

Case 1. $g_a = 0$ for all $a \in S$. This means that $g(ax) = g(a)g(x)$ for all $a, x \in S$, so g is a multiplicative function. As χ we may choose $\chi := g$, because $g = g^*$.

Case 2. $g_a \neq 0$ for some $a \in S$. Using that k is algebraically closed we get, like in the proof of [12, Theorem 4.1], that there exist multiplicative functions $\chi_1, \chi_2 \in k[S]$, such that $g = (\chi_1 + \chi_2)/2$. Substituting this into d'Alembert's functional equation (1.1) we find that

$$\chi_1(x)[\chi_1^*(y) - \chi_2(y)] + \chi_2(x)[\chi_2^*(y) - \chi_1(y)] = 0 \text{ for all } x, y \in S. \quad (7.1)$$

Now $g = g^*$, so $\chi_1 + \chi_2 = \chi_1^* + \chi_2^*$ or equivalently $\chi_2^* - \chi_1 = \chi_2 - \chi_1^*$, which substituted into (7.1) gives us that $[\chi_1(x) - \chi_2(x)][\chi_1^*(y) - \chi_2(y)] = 0$, so that $\chi_1 = \chi_2$ or $\chi_2 = \chi_1^*$. In the first possibility we may take $\chi := \chi_1$, because $g = g^*$. In the second the formula for g obviously holds.

Conversely let $g = (\chi + \chi^*)/2$, where $\chi \in k[S]$ is a non-zero multiplicative function. It is elementary to verify that g is an abelian solution of d'Alembert's functional equation (1.1). It is left to show that $g \neq 0$, which we do by contradiction. If $\chi + \chi^* = 0$, we get for any $x \in S$ that $\chi(x)^2 = \chi(x^2) = -\chi^*(x^2) = -\chi^*(x)^2 = -(-\chi(x))^2 = -\chi(x)^2$, so that $\chi(x)^2 = 0$. But then $\chi = 0$, contradicting that χ is non-zero.

The essential uniqueness is a general fact about multiplicative functions (modify for instance [12, Corollary 3.19] from $\mathbb{C}$ to k). □

The k-valued solutions of d'Alembert's classic functional equation (2.1) are described in Aczél and Dhombres [1, p. 220-222] for abelian groups, and in Corovei [3, Theorem 6] for P_3-groups. Theorem 7.1 occurs as Stetkær [12, Proposition 9.31], except that $k = \mathbb{C}$ in [12].

8. Non-abelian d'Alembert functions

Throughout Sect. 8 we let $g \in k[S]$ denote a fixed non-abelian d'Alembert function. Our main result, Theorem 8.2, derives the form of g.

We start by a study of the vector spaces $W(g)$ and $\mathcal{T}(g)$ that play an important role in the proof of Theorem 8.2. They are related (Lemma 8.1(c)). Theorem 8.3 provides more information about them.

Lemma 8.1. *(a) Choose $a, b \in S$ such that $\Delta_g(a,b) \neq 0$ (this can be done by Proposition 6.1(f)).*
If $w \in W(g)$ and $w(a) = w(b) = w(ab) = 0$, then $w = 0$.
(b) $\dim W(g) = 3$, *and* $W(g) = span\{g_x \mid x \in S\}$.
(c) $\mathcal{T}(g) = kg + W(g)$, *where the sum is direct.*
(d) $\dim \mathcal{T}(g) = 4$.

Proof. (a) An inspection shows that the computations in the proof of [12, Corollary 6.2] work for any field k with $\mathrm{char}(k) \neq 2$ and not just for $\mathbb{C}$, as is formally the case in [12, Chapter 6]. Note that (in the notation of [12]) $\widetilde{\Delta} = 4\Delta$ due to [12, Lemma B.7(c)] and our Proposition 6.1(a) and (d).

(b) Note that $g_x \in W(g)$ for any $x \in S$ by Proposition 6.1(c). We view elements of k^3 as column vectors. The linear map $L : W(g) \to k^3$, defined by $Lw := (w(a)\ w(b)\ w(ab))^t$ for $w \in W(g)$, is injective according to (a), so it suffices to show that Lg_a, Lg_b and $L(g_{ab} - g_{ba})$ are linearly independent vectors in k^3. Using that g is central (Proposition 6.1(a)) we deduce that

$$g_b(a) = g_a(b), \quad g_{ab}(a) - g_{ba}(a) = g_{ab}(b) - g_{ba}(b) = 0,$$
$$g_{ab}(ab) - g_{ba}(ab) = -2\Delta_g(a,b) \text{ (by the definition of } \Delta_g),$$

which reduces the determinant $\left| Lg_a\ Lg_b\ L(g_{ab} - g_{ba}) \right|$ to

$$\begin{vmatrix} g_a(a) & g_b(a) & g_{ab}(a) - g_{ba}(a) \\ g_a(b) & g_b(b) & g_{ab}(b) - g_{ba}(b) \\ g_a(ab) & g_b(ab) & g_{ab}(ab) - g_{ba}(ab) \end{vmatrix} = \begin{vmatrix} g_a(a) & g_a(b) & 0 \\ g_a(b) & g_b(b) & 0 \\ g_a(ab) & g_b(ab) & -2\Delta_g(a,b) \end{vmatrix}$$
$$= \left(g_a(a)g_b(b) - g_a(b)^2\right)(-2\Delta_g(a,b)) = -2\Delta_g(a,b)^2 \neq 0.$$

(c) We prove that the sum is direct by contradiction. If it is not, then $g \in W(g)$, so $g(xy) + g(yx) = 4g(x)g(y)$ for all $x, y \in S$. Thus $\gamma := 2g$ satisfies $\gamma(xy) + \gamma(yx) = 2\gamma(x)\gamma(y)$, so γ is multiplicative (by Stetkær [10]). But then $g = \gamma/2$ is abelian, contradicting that g is non-abelian.

$\mathcal{T}(g) \subseteq kg + W(g)$ by Proposition 6.1(g). Conversely, $g \in \mathcal{T}(g)$ by the definition of $\mathcal{T}(g)$, and so $g_x = L'(x)g - g(x)g \in \mathcal{T}(g)$. Hence $kg + W(g) \subseteq \mathcal{T}(g)$.

(d) follows from (b) and (c). □

We next show that the given non-abelian d'Alembert function g has the form $g = \frac{1}{2}\chi_\pi$ (Theorem 8.2), and we express the associated vector spaces $W(g)$

and $\mathcal{T}(g)$ in terms of matrix elements of π (Theorem 8.3). Proposition 4.4(d) is a converse of Theorem 8.2.

Theorem 8.2. *There exists an irreducible, 2-dimensional representation π of S with the property that $\mu(x)\pi(x^*) = adj(\pi(x))$ for all $x \in S$, such that $g = \frac{1}{2}\chi_\pi$. If S is a monoid, then $\pi(e) = I$.*

If π' is any irreducible representation of S on a finite dimensional vector space over k, such that $g = \frac{1}{2}\chi_{\pi'}$, then π' and π are equivalent.

Proof. The proof combines the Algebraic Small Dimension Lemma with the fact that $\dim \mathcal{T}(g) = 4$ (Lemma 8.1(d)).

Since $\mathcal{T}(g)$ is invariant under R (by Proposition 6.1(h)) and finite dimensional ($\dim \mathcal{T}(g) = 4$ by Lemma 8.1(d)), it contains an irreducible subspace V (take an invariant subspace of minimal dimension ≥ 1). Define $\pi := R|_V$. If $\dim V = 1$, then $\pi = \gamma I_V$ for some multiplicative function $\gamma \in k[S]$. From the formula (6.1) we see that $g = (\gamma + \gamma^*)/2$, so that g is abelian, which contradicts the assumption on g. Hence $\dim V \geq 2$. Combining (6.1) and Theorem 5.2 we see that $\dim V = 2$, that $\mu(x)\pi(x^*) = \operatorname{adj}(\pi(x))$ for all $x \in S$, and that $g = \frac{1}{2}\chi_\pi$.

If S is a monoid, then $R(e) = I$, so $\pi(e) = R(e)|_V = I$.

Let π' be a finite dimensional, irreducible representation of S, such that $g = \frac{1}{2}\chi_{\pi'}$. Assume to arrive at a contradiction that π and π' are not equivalent. Then the subspaces $k\chi_\pi$ and $k\chi_{\pi'}$ of $k[S]$ form a direct sum by standard knowledge (see for instance Bourbaki [2], Proposition 2 of Chap. VIII, §13, no. 3). Since they furthermore agree, because $\chi_\pi = \chi_{\pi'} = 2g$, they are $\{0\}$. In particular $g = \frac{1}{2}\chi_\pi = 0$. But $g \neq 0$, being a d'Alembert function. □

We next show that $\mathcal{T}(g)$ is the space of matrix coefficients of the representation π from Theorem 8.2, and we characterize the subspace $W(g)$ of $\mathcal{T}(g)$ in a similar way.

Theorem 8.3. *Let $g = \frac{1}{2}\chi_\pi$, where π is an irreducible representation of S on a 2-dimensional vector space V such that $\mu(x)\pi(x^*) = adj(\pi(x))$ for all $x \in S$. Then*

(a) $W(g) = \{\operatorname{tr}(A\pi(\cdot)) \in k[S] \mid A \in \mathcal{L}(V)$ *with* $\operatorname{tr} A = 0\}$.
(b) $\mathcal{T}(g) = kg + W(g) = \{\operatorname{tr}(A\pi(\cdot)) \in k[S] \mid A \in \mathcal{L}(V)\}$.

Proof. (a) Let $w \in W(g)$. According to Lemma 8.1(b) it is a linear combination of the functions $g_x \in W(g)$, so to get the inclusion $\subseteq$ we may assume that $w = g_x$ for some $x \in S$. For any $y \in S$ we find that

$$g_x(y) = g(xy) - g(x)g(y) = \frac{1}{2}\operatorname{tr}\left([\pi(x) - g(x)I]\pi(y)\right),$$

which is the desired expression, since $A := \frac{1}{2}(\pi(x) - g(x)I)$ has $\operatorname{tr} A = 0$. The converse inclusion is Proposition 4.4(c).

(b) $\mathcal{T}(g) = kg + W(g)$ according to Lemma 8.1(c). The rest follows from (a). □

Davison observed [4, Proposition 4.2], that the only central function in $W(g)$ is the zero function when g is a non-abelian, complex valued d'Alembert function on a monoid. Proposition 8.4 extends his result to any field of characteristic $\neq 2$ and any semigroup. It is not necessary that g is a d'Alembert function. We do not need the result.

Proposition 8.4. *Let k_1 be any field k of characteristic $\neq 2$, and let Σ be a semigroup. If $g_1 \in k_1[\Sigma]$ is non-abelian, then the set of central functions in $W(g_1)$ is $\{0\}$.*

Proof. Let $w \in k_1[\Sigma]$ be central. That $w \in W(g_1)$ then reduces to the sine addition law $w(xy) = w(x)g_1(y) + g_1(x)w(y)$, $x, y \in \Sigma$. If $w \neq 0$, then g_1 is abelian (by [12, Theorem 4.1(e)]), contradicting the assumption on g_1. Hence $w = 0$. □

9. Bounded and continuous d'Alembert functions

Section 9 studies bounded and continuous d'Alembert functions. Earlier works have been done on such functions on topological monoids or groups. Here we extend the theory to functions on topological semigroups.

Proposition 9.1. *Let $g \in \mathbb{C}[S]$ be a bounded d'Alembert function. Then $|g(x)| \leq 1$ for all $x \in S$.*

(a) Let g be abelian, and write it as $g = \frac{1}{2}(\chi + \chi^*)$, where $\chi \in \mathbb{C}[S]$ is multiplicative (by Theorem 7.1). Then $|\chi(x)| \leq 1$ and $|\chi^*(x)| \leq 1$ for all $x \in S$.

(b) Let g be non-abelian, and write it as $g = \frac{1}{2}\chi_\pi$, where π is the representation from Theorem 8.2. Then π is bounded.

Proof. We prove that $|g(x)| \leq 1$ under each of the points (a) and (b).

(a) It follows from general principles (see [12, Theorem 3.18(c)]), that χ and χ^* are bounded. A bounded multiplicative function is bounded by 1. Indeed, let $C > 0$ be such that $|\chi(x)| \leq C$ for all $x \in S$. Letting $n \to \infty$ in $|\chi(x)| = |\chi(x^n)|^{1/n} \leq C^{1/n}$ we get that $|\chi(x)| \leq 1$. Similarly for χ^*. Consequently $|g(x)| = \frac{1}{2}|\chi(x) + \chi^*(x)| \leq 1$.

(b) That π is bounded means that each matrix coefficient is a bounded function. According to Theorem 8.3(b) the space of matrix coefficients is $\mathcal{T}(g)$. But any translate of g is bounded, because g is bounded. Hence π is bounded.

Let $\lambda(x) \in \mathbb{C}$ be an eigenvalue of $\pi(x)$ with a corresponding eigenvector $f \in V$. From $\pi(x^n)f = \pi(x)^n f = \lambda(x)^n f$ we see that $\{\lambda(x)^n \in \mathbb{C} \mid n = 1, 2, \ldots\}$ is bounded, so that $|\lambda(x)| \leq 1$. Let $\{\lambda_1(x), \lambda_2(x)\}$ be the eigenvalues of $\pi(x)$, counted with multiplicity. Then $\operatorname{tr}\pi(x) = \lambda_1(x) + \lambda_2(x)$, and so $|g(x)| = \frac{1}{2}|\lambda_1(x) + \lambda_2(x)| \leq \frac{1}{2}(|\lambda_1(x)| + |\lambda_2(x)|) \leq \frac{1}{2}(1 + 1) = 1$. □

Definition 9.2. A *topological semigroup* is a semigroup Σ with a topology such that the composition map $(x, y) \mapsto xy$ of $\Sigma \times \Sigma$ into Σ is continuous.

Definition 9.3. A representation π of a topological semigroup Σ on a topological vector space V over $\mathbb{C}$ is said to be *continuous* if the map $(x, v) \mapsto \pi(x)v$ of $\Sigma \times V$ into V is continuous. If $\dim V < \infty$, then the continuity of π is equivalent to its matrix coefficients being continuous.

Definition 9.4. If X is a topological space, we let $C(X)$ denote the complex vector space of continuous, complex valued functions on X.

Let S be a topological semigroup. As we have seen in Theorems 7.1 and 8.2 any d'Alembert function g on S can be expressed in terms of multiplicative functions and 2-dimensional representations. Theorem 9.5 states that these ingredients are continuous when g is continuous.

Theorem 9.5. *Let S be a topological semigroup, and let $g \in C(S)$ be a d'Alembert function.*

(a) Let g be abelian. Such a g can be written as $g = (\chi + \chi^*)/2$, where $\chi \in \mathbb{C}[S]$ is a non-zero multiplicative function, and $\chi, \chi^* \in C(S)$. This decomposition of g is essentially unique: If $g = (\chi_1 + \chi_2)/2$, where $\chi_1, \chi_2 \in \mathbb{C}[S]$ are multiplicative functions, then $\chi_1 = \chi$ and $\chi_2 = \chi^*$, or $\chi_1 = \chi^*$ and $\chi_2 = \chi$.

(b) Let g be non-abelian. Such a g can be written as $g = \frac{1}{2}\chi_\pi$, where π is a continuous, irreducible representation of S on a 2-dimensional complex vector space such that $\mu(x)\pi(x^*) = adj(\pi(x))$ for all $x \in S$. Any finite dimensional, irreducible representation π' of S on a complex vector space, such that $g = \frac{1}{2}\chi_{\pi'}$, is equivalent to π. Finally $W(g) \subseteq C(S)$.

Proof. (a) That g can be written in the form $g = (\chi + \chi^*)/2$, where $\chi \in \mathbb{C}[S]$ is a non-zero multiplicative function was derived in Theorem 7.1. It is known from general principles that χ and χ^* are continuous ([12, Theorem 3.18(d)]) when g is. The uniqueness statement also follows from general principles (see for instance [12, Corollary 3.19]).

(b) Apart from the continuity statement Theorem 8.2 gives that g can be written in the desired form. The equivalence is also found in Theorem 8.2.

Left is the continuity of π, which means that its matrix coefficients are continuous. According to Theorem 8.3(b) the space of matrix coefficients of π is $\mathcal{T}(g)$. Any translate of g is continuous, because S is a topological semigroup and $g \in C(S)$, so $\mathcal{T}(g) \subseteq C(S)$. Hence π is continuous.

Finally $W(g) \subseteq kg + W(g) = \mathcal{T}(g) \subseteq C(S)$, in which Lemma 8.1(b) gives the equality sign. □

Corollary 9.6 contains [14, Theorem 3.1], because it considers the general form (1.1) of d'Alembert's functional equation and not just the classic form (2.1).

A *character* of a group G is a homomorphism $\chi : G \to \mathbb{C}^*$. It is *unitary*, if $|\chi(x)| = 1$ for all $x \in G$.

Corollary 9.6. *Assume that G is a compact group, that $\mu \in C(G)$ and that the involution $x \mapsto x^*$ of G into G is continuous.*

(a) Let $g \in C(G)$ be an abelian d'Alembert function. Then g can be written as $g = (\chi + \chi^*)/2$, where $\chi \in C(G)$ is a unitary character. This decomposition of g is essentially unique: If $g = (\chi_1 + \chi_2)/2$, where $\chi_1, \chi_2 \in \mathbb{C}[S]$ are multiplicative functions, then $\chi_1 = \chi$ and $\chi_2 = \chi^*$, or $\chi_1 = \chi^*$ and $\chi_2 = \chi$.

(b) Let $g \in C(G)$ be a non-abelian d'Alembert function. Then g can be written as $g = \frac{1}{2}\chi_\pi$, where π is a continuous, unitary, irreducible representation of G on a 2-dimensional complex vector space with the property that $\mu(x)\pi(x^*) = adj(\pi(x))$ for all $x \in G$.
Any finite dimensional, unitary, irreducible representation π' of G on a complex vector space such that $g = \frac{1}{2}\chi_{\pi'}$ is unitarily equivalent to π.

Proof. We use the notation and results of Theorem 9.5.

(a) Any non-zero multiplicative function on a group is a character (by [12, Lemma 3.4(a)]), so χ is a character. It is bounded, being a continuous function on a compact set, and so it is unitary (by [12, Lemma 3.4(b)]).

(b) g is bounded, being a continuous function on a compact set, so π is bounded (by Proposition 9.1). But a bounded representation of a group on $\mathbb{C}^n$ is equivalent to a unitary representation (see [7, 22.23(c)]), so π may be assumed unitary. The uniqueness claim follows from the well known fact that unitary representations are unitarily equivalent if they are equivalent. □

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Received: February 8, 2020
Revised: August 2, 2020