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Characterizations of inner product spaces via angular distances and Cauchy–Schwarz inequality

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Abstract. We study some interesting characterizations of real inner product spaces expressed in terms of angular distances. We first discuss the equivalence of characterizing an inner product space via the usual angular distance and the p-angular distance. Then, we establish a parametric family of upper bounds for the usual angular distance which also serves as a characterization of an inner product space. As an application, bounds for the usual angular distance are utilized in obtaining improvements of the real Cauchy–Schwarz inequality. Finally, we give several comparative relations for angular distances in inner product spaces.

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Keywords. Inner product space, Normed space, Angular distance, Characterization, Cauchy–Schwarz inequality.

1. Introduction

The problem of finding necessary and sufficient conditions for a normed space to be an inner product space has been studied by numerous authors (see, e.g. [4,6,19] and the references therein). In the present article we deal with a class of norm inequalities closely connected with characterizations of inner product spaces. One of the most interesting characterizations has been based on the so-called Dunkl–Wiliams inequality. In 1964, Dunkl and Wiliams [12], proved that the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{4\,\|x-y\|}{\|x\| + \|y\|} \tag{1}$$

holds for all nonzero vectors x, y in a real normed linear space X. Moreover, the authors also showed that the constant 4 can be replaced by 2 if X is an inner product space. On the other hand, Kirk and Smiley [14], showed that the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|} \tag{2}$$

characterizes inner product spaces.

The above inequalities (1) and (2) provide an upper bound for the angular distance

$$\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$$

between nonzero vectors x and y. This quantity, also called the Clarkson distance, was introduced by Clarkson [8], while studying the triangle inequality in uniformly convex spaces.

Recently, numerous interesting improvements and generalizations of bounds for angular distance $\alpha[x,y]$ have been established (see e.g. [3,9,16–18,23,25] and the references therein). In particular, Al-Rashed [3], generalized the characterization of an inner product space given by (2), in the following way: if $q > 0$, then a normed linear space $X = (X, \|\cdot\|)$ is an inner product space if and only if the inequality

$$\alpha[x,y] \leq \frac{2^{\frac{1}{q}}\|x-y\|}{(\|x\|^q + \|y\|^q)^{\frac{1}{q}}} \tag{3}$$

holds for all nonzero vectors $x, y \in X$.

In [16], Maligranda introduced the notion of p-angular distance between nonzero elements x, y in a normed linear space X as

$$\alpha_p[x,y] = \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|, \ p \in \mathbb{R},$$

as a generalization of the concept of angular distance (note that $\alpha_0[x,y] = \alpha[x,y]$).

There are several recent extensions of the characterization given by (3). In 2019, Rooin et al. [24], proved that if $p, q, r \in \mathbb{R}$, $0 \leq \frac{p}{q} < 1$, $q \neq 0$, then a normed linear space X is an inner product space if and only if the inequality

$$\alpha_p[x,y] \leq \frac{2^{\frac{1}{r}}\alpha_q[x,y]}{\left(\|x\|^{r(q-p)} + \|y\|^{r(q-p)}\right)^{\frac{1}{r}}}$$

holds for all nonzero elements $x, y \in X$. Similarly, Amini-Harandi et al. [5], extended the concept of p-angular distance to θ-angular distance, i.e.

$$\alpha_\theta[x,y] = \left\| \frac{x}{\theta(\|x\|)} - \frac{y}{\theta\|y\|)} \right\|,$$

where $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $t \mapsto \frac{t}{\theta(t)}$ is nondecreasing on $\mathbb{R}^+$. In this setting, they proved that if $q > 0$, then a normed linear space X is an inner product space if and only if the inequality

$$\alpha_\theta[x,y] \leq \frac{2^{\frac{1}{q}}\,\|x-y\|}{\left(\theta\left(\|x\|\right)^q + \theta\left(\|x\|\right)^q\right)^{\frac{1}{q}}}$$

holds for all nonzero vectors $x, y \in X$ (see also [9]).

In 2018, Rooin et al. [25], established a characterization of an inner product space by giving an explicit formula for p-angular distance. More precisely, they proved that if $p \neq 1$, then a normed space X is an inner product space if and only if the relation

$$\alpha_p^2[x,y] = \|x\|^{p-1}\|y\|^{p-1}\|x-y\|^2 + \left(\|x\|^{p-1} - \|y\|^{p-1}\right)\left(\|x\|^{p+1} - \|y\|^{p+1}\right) \quad (4)$$

holds for all nonzero elements $x, y \in X$. In the same paper, the authors introduced a concept of skew p-angular distance between nonzero elements x, y in a normed linear space X as

$$\beta_p[x,y] = \left\| \|y\|^{p-1}x - \|x\|^{p-1}y \right\| \;\; p \in \mathbb{R}. \quad (5)$$

We set $\beta[x,y]$ for $\beta_p[x,y]$ when $p=0$ and call it simply skew angular distance between x and y. It is easy to see that p-angular distance and skew p-angular distance are related by

$$\beta_p[x,y] = \|x\|^{p-1}\|y\|^{p-1}\alpha_{2-p}[x,y]. \quad (6)$$

In 2013, Dehghan [10], established an interesting characterization of an inner product space relying on the relationship between $\alpha[x,y]$ and $\beta[x,y]$. More precisely, he proved that a normed space X is an inner space if and only if $\alpha[x,y] \leq \beta[x,y]$ holds for all nonzero elements $x, y \in X$.

On the other hand, one of the most important inequalities in inner product spaces is the Cauchy–Schwarz inequality which asserts that if $X = (X, \langle\cdot,\cdot\rangle)$ is a real or complex inner product space, then

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad (7)$$

holds for all vectors $x, y \in X$. In addition, equality in (7) holds if and only if x and y are linearly dependent. For diverse applications, improvements and generalizations of the Cauchy–Schwarz inequality, the reader is referred, for example, to [1,11,20,21] and the references therein.

The main objective of the present paper is a study of characterizations of inner product spaces closely connected to (2), (3) and (4). The paper is divided into six sections as follows: after this Introduction, in Sect. 2 we first discuss a classical geometric background of the above relations. In Sect. 3, we discuss the equivalence of characterizing an inner product space via the usual angular distance $\alpha[x,y]$ and the p-angular distance $\alpha_p[x,y]$, $p \in \mathbb{R}, |p| \neq 1$. As a consequence, we obtain an alternative proof of the fact that (4) characterizes an inner product space. In Sect. 4, we interpolate relations (2) and (3), which yields a new characterization of an inner product space. Further, by virtue of some known bounds for angular distance, closely connected to our results from

Sect. 4, in Sect. 5 we give the corresponding refinements and reverses of the real Cauchy–Schwarz inequality. As an application, in Sect. 6, we give several comparative relations for angular distances in inner product spaces.

If nothing else is explicitly stated, in this paper we deal with real normed and real inner product spaces. Further, in order to simplify our discussion, the quantities $\alpha[x,y]$, $\beta[x,y]$, $\alpha_p[x,y]$ and $\beta_p[x,y]$ will sometimes be referred to simply as angular distances, for brevity.

2. Classical geometric interpretation and motivation

For the reader's convenience, we first discuss a geometric background of relations (2) and (4). Namely, the notion of the angular distance $\alpha[x,y]$ is closely connected to the concept of an angle between vectors x and y. To see this, we first give a classical geometric interpretation of inequality (2).

Let ABC be a triangle in $\mathbb{R}^2$ with sidelengths $a=|BC|, b=|CA|, c=|AB|$, and let h_a and w_a be the lengths of the altitude and internal angle bisector opposite $\overline{BC}$, respectively. Further, let α stand for a measure of the angle $\angle CAB$.

In [7], one can find an old geometric inequality of Ballieu (1949) which asserts that

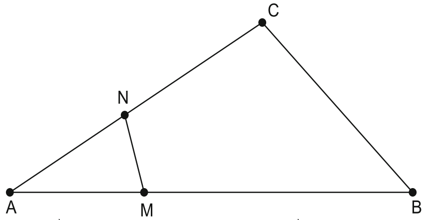
$$2^{t-1}\sin^t\frac{\alpha}{2}\le\frac{a^t}{b^t+c^t},\quad 0\le t\le 1,$$

holds in a triangle ABC. In particular, if $t=1$, the inequality of Ballieu reduces to

$$\sin\frac{\alpha}{2}\le\frac{a}{b+c}. \tag{8}$$

The classical proof of inequality (8) is quite simple. Namely, multiplying (8) by $\frac{2bc}{a}\cos\frac{\alpha}{2}$ and utilizing the well-known trigonometric formulas $h_a=\frac{bc\sin\alpha}{a}$ and $w_a=\frac{2bc}{b+c}\cos\frac{\alpha}{2}$, it turns out that (8) is equivalent to the inequality $h_a\le w_a$, which is obvious.

Now, we show that (8) is a special case of relation (2) that characterizes an inner product space. Namely, let $x=\overrightarrow{AB}$, $y=\overrightarrow{AC}$, and let $\overrightarrow{AM},\overrightarrow{AN}$ be the unit vectors in directions x,y, respectively.



Then, $\overrightarrow{AM}=\frac{x}{\|x\|}$, $\overrightarrow{AN}=\frac{y}{\|y\|}$, and so $|\overrightarrow{MN}|=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|$ represents the angular distance between vectors x and y. Consequently, since the triangle

AMN is isosceles, it follows that $\sin\frac{\alpha}{2} = \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$, so inequality (8) can be rewritten as relation (2).

To give a geometric interpretation of relation (4), let us apply the law of cosines to triangles AMN and ABC. It follows that

$$\cos\alpha = \frac{2 - |MN|^2}{2} = \frac{b^2 + c^2 - a^2}{2bc}.$$

In accordance with our previous discussion, the corresponding vector form of the last relation reads

$$\alpha^2[x, y] = \frac{\|x - y\|^2 - (\|x\| - \|y\|)^2}{\|x\|\,\|y\|}. \tag{9}$$

It should be noted here that identity (9) is a special case of (4), when $p = 0$. Of course, identities (4) and (9) are equivalent since they characterize an inner product space. In the next section we will discuss this equivalence without presuming that (4) and (9) characterize an inner product space.

3. Equivalence of characterizations via usual angular and *p*-angular distances

In this section, we discuss the equivalence of characterizing an inner product space via the usual angular distance $\alpha[x, y]$ and the p-angular distance $\alpha_p[x, y]$, $p \in \mathbb{R}, p \neq 1$. We start with showing that relations (4) and (9) are equivalent, without presuming that they characterize an inner product space. Then we will show that relation (9) is characteristic for an inner product space. As a consequence, we will establish an alternative proof of the fact that (4) characterizes an inner product space for $|p| \neq 1$.

Theorem 1. *Let $X = (X, \|\cdot\|)$ be a normed linear space and let $p \in \mathbb{R}$ be such that $|p| \neq 1$. Then relation (9) holds for all nonzero vectors $x, y \in X$, if and only if (4) holds for all nonzero vectors $x, y \in X$.*

Proof. Let $p \in \mathbb{R}$, $p \neq 1$, and suppose that (9) holds for all nonzero vectors $x, y \in X$. Then, considering (9) with $\|x\|^{p-1}x, \|y\|^{p-1}y$ instead of x, y respectively, we obtain

$$\alpha^2\left[\|x\|^{p-1}x, \|y\|^{p-1}y\right] = \frac{\alpha_p^2[x, y] - (\|x\|^p - \|y\|^p)^2}{\|x\|^p\|y\|^p},$$

that is,

$$\|x\|^p\|y\|^p\alpha^2[x, y] = \alpha_p^2[x, y] - (\|x\|^p - \|y\|^p)^2\,, \tag{10}$$

since $\alpha[x, y] = \alpha[ax, by]$, for $a, b \in \mathbb{R}$ such that $ab > 0$. Finally, substituting (9) in the last equality and making several simple calculations we obtain (4), as claimed.

Conversely, suppose that (4) holds for a fixed $p \in \mathbb{R}$, $|p| \neq 1$. Then, considering (4) with $\|x\|^{\frac{1}{p}-1}x, \|y\|^{\frac{1}{p}-1}y$ instead of x, y respectively, we obtain

$$\|x-y\|^2 = \|x\|^{1-\frac{1}{p}}\|y\|^{1-\frac{1}{p}}\alpha^2_{\frac{1}{p}}[x,y] + \left(\|x\|^{1-\frac{1}{p}} - \|y\|^{1-\frac{1}{p}}\right)\left(\|x\|^{1+\frac{1}{p}} - \|y\|^{1+\frac{1}{p}}\right),$$

that is,

$$\alpha^2_{\frac{1}{p}}[x,y] = \|x\|^{\frac{1}{p}-1}\|y\|^{\frac{1}{p}-1}\|x-y\|^2 + \left(\|x\|^{\frac{1}{p}-1} - \|y\|^{\frac{1}{p}-1}\right)\left(\|x\|^{\frac{1}{p}+1} - \|y\|^{\frac{1}{p}+1}\right),$$

after rewriting. This means that if (4) holds for p-angular distance, then it also holds for $\frac{1}{p}$-angular distance. Hence, without loss of generality we can suppose that $|p| < 1$. In addition, considering (4) with $\|x\|^{p-1}x, \|y\|^{p-1}y$ instead of x, y, it follows that

$$\begin{aligned}\alpha^2_{p^2}[x,y] &= \|x\|^{p(p-1)}\|y\|^{p(p-1)}\alpha^2_p[x,y] \\ &\quad + \left(\|x\|^{p(p-1)} - \|y\|^{p(p-1)}\right)\left(\|x\|^{p(p+1)} - \|y\|^{p(p+1)}\right).\end{aligned}$$

Now, substituting (4) in the last relation, we obtain

$$\alpha^2_{p^2}[x,y] = \|x\|^{p^2-1}\|y\|^{p^2-1}\|x-y\|^2 + \left(\|x\|^{p^2-1} - \|y\|^{p^2-1}\right)\left(\|x\|^{p^2+1} - \|y\|^{p^2+1}\right),$$

which means that if (4) holds for p-angular distance, then it also holds for p^2-angular distance. Consequently, the identity

$$\alpha^2_{p_n}[x,y] = \|x\|^{p_n-1}\|y\|^{p_n-1}\|x-y\|^2 + \left(\|x\|^{p_n-1} - \|y\|^{p_n-1}\right)\left(\|x\|^{p_n+1} - \|y\|^{p_n+1}\right)$$

holds for each term of the sequence $p_n = p^{2^n}$, $n \in \mathbb{N}$. Finally, since the norm is continuous, the last relation also holds for $\lim_n p_n = \lim_n p^{2^n} = 0$, which yields relation (9). The proof is now complete. □

Now, our next step is to show that relation (9) characterizes an inner product space.

Theorem 2. *Let $X = (X, \|\cdot\|)$ be a normed linear space. Then, X is an inner product space if and only if relation* (9) *holds for all nonzero vectors $x, y \in X$.*

Proof. Suppose that X is an inner product space. Then, for $x, y \in X$ and $a, b \in \mathbb{R}$, we have

$$\|ax - by\|^2 = \langle ax - by, ax - by\rangle = a^2\|x\|^2 - 2ab\langle x, y\rangle + b^2\|y\|^2.$$

Furthermore, it follows that

$$\begin{aligned}(b-a)\left(a\|x\|^2 - b\|y\|^2\right) + \|ax - by\|^2 &= ab\left(\|x\|^2 - 2\langle x,y\rangle + \|y\|^2\right) \\ &= ab\|x-y\|^2,\end{aligned} \tag{11}$$

and consequently, we obtain (9) after substituting $a = \frac{1}{\|x\|}$ and $b = \frac{1}{\|y\|}$ in the last equality.

Conversely, suppose that X is a normed space with a norm satisfying relation (9). Our intention is to show that this hypothesis implies that inequality

(2) holds for all nonzero elements x and y. Namely, by squaring the triangle inequality $\|x-y\| \le \|x\|+\|y\|$, we have that $\|x-y\|^2 \le (\|x\|+\|y\|)^2$. In addition, multiplying the previous inequality by $(\|x\|-\|y\|)^2$ both-sidedly, we obtain the inequality

$$(\|x\|-\|y\|)^2 \|x-y\|^2 \le (\|x\|-\|y\|)^2 (\|x\|+\|y\|)^2 .$$

Clearly, the last inequality implies the relation

$$(\|x\|+\|y\|)^2 \|x-y\|^2 - (\|x\|-\|y\|)^2 (\|x\|+\|y\|)^2 \le 4\|x\|\|y\|\|x-y\|^2,$$

which holds for all vectors $x, y \in X$. Therefore, dividing the previous inequality by $\|x\|\|y\| (\|x\|+\|y\|)^2$, provided that x and y are nonzero vectors, we obtain the inequality

$$\frac{\|x-y\|^2 - (\|x\|-\|y\|)^2}{\|x\| \cdot \|y\|} \le \frac{4\|x-y\|^2}{(\|x\|+\|y\|)^2},$$

which is equivalent to $\alpha^2[x,y] \le \dfrac{4\|x-y\|^2}{(\|x\|+\|y\|)^2}$, due to hypothesis (9). Finally, by taking a square root, the previous relation reduces to (2), which implies that X is an inner product space. □

Remark 1. In conclusion, Theorems 1 and 2 provide an alternative proof of the fact that relation (4) characterizes an inner product space for $|p| \ne 1$. The corresponding proof established in [25] (see Theorem 4.3) relies on characterizations due to Ficken [13] and Lorch [15]. It should be noted here that this proof also covers the case when $p=-1$. If $p=-1$, relation (4) reduces to $\alpha_{-1}[x,y] = \frac{\|x-y\|}{\|x\|\|y\|}$. Since this relation also characterizes an inner product space, it is equivalent to (9), although we do not have direct equivalence as it has been done in Theorem 1. □

Remark 2. It should be noted here that if $a+b=0$, $a, b \ne 0$, then relation (11) reduces to the parallelogram law, which induces the inner product space via the polarization identity. □

In order to conclude this section, we summarize our previous discussion.

Corollary 1. *Let $X = (X, \|\cdot\|)$ be a normed linear space. Then, the following statements are equivalent:*

(i) X is an inner product space;
(ii) relation (9) holds for all nonzero vectors $x, y \in X$;
(iii) relation (4) holds for all nonzero vectors $x, y \in X$ and $p \ne 1$.

4. Characterization of an inner product space obtained via interpolation

In this section, we give a characterization of an inner product space that follows by interpolating inequalities (2) and (3). We have already discussed that inequality (3), obtained by Al-Rashed [3], generalizes characterization (2) obtained by Kirk and Smiley. Since the function $f(s) = s^t$ is concave (convex) for $0 < t < 1$ ($t > 1$), it follows that inequality (3) is weaker than (2) when $0 < t < 1$, while for $t > 1$, inequality (3) is stronger than (2).

We start with interpolating relations (2) and (3). This can be done via the parametric family of functions $H_{\lambda,t} : X \times X \to \mathbb{R}$ defined by

$$H_{\lambda,t}(x, y) = \left((1-\lambda)\|x\| + \lambda\|y\|\right)^t + \left(\lambda\|x\| + (1-\lambda)\|y\|\right)^t, \tag{12}$$

where $0 \leq \lambda \leq 1$ and $t > 0$. The following interpolating series of inequalities will also be utilized for characterizing an inner product space.

Theorem 3. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space with a norm $\|\cdot\|$ induced by an inner product $\langle\cdot,\cdot\rangle$, and let $H_{\lambda,t} : X \times X \to \mathbb{R}$ be defined by (12). If $0 < t \leq 1$, then the series of inequalities*

$$\alpha[x, y] \leq \frac{2\,\|x-y\|}{\|x\| + \|y\|} \leq \frac{2^{\frac{1}{t}}\,\|x-y\|}{H_{\lambda,t}^{\frac{1}{t}}(x, y)} \leq \frac{2^{\frac{1}{t}}\,\|x-y\|}{\left(\|x\|^t + \|y\|^t\right)^{\frac{1}{t}}} \tag{13}$$

holds for $0 \leq \lambda \leq 1$ and for all nonzero vectors $x, y \in X$. Further, if $t > 1$, then

$$\alpha[x, y] \leq \frac{2^{\frac{1}{t}}\,\|x-y\|}{\left(\|x\|^t + \|y\|^t\right)^{\frac{1}{t}}} \leq \frac{2^{\frac{1}{t}}\,\|x-y\|}{H_{\lambda,t}^{\frac{1}{t}}(x, y)} \leq \frac{2\,\|x-y\|}{\|x\| + \|y\|} \tag{14}$$

holds for $0 \leq \lambda \leq 1$ and for all nonzero vectors $x, y \in X$.

Proof. If $f : I \to \mathbb{R}$ is a concave function, then the series of inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{(1-\lambda)a + \lambda b + \lambda a + (1-\lambda)b}{2}\right) \\ &\geq \frac{f((1-\lambda)a + \lambda b) + f(\lambda a + (1-\lambda)b)}{2} \\ &\geq \frac{f(a) + f(b)}{2} \end{aligned} \tag{15}$$

holds for all $a, b \in I$ and $0 \leq \lambda \leq 1$. Considering the above relation with a concave function $f(s) = s^t$, $s \geq 0$, $0 < t \leq 1$, and $a = \|x\|$, $b = \|y\|$, the above set of inequalities becomes

$$\left(\frac{\|x\| + \|y\|}{2}\right)^t \geq \frac{H_{\lambda,t}(x, y)}{2} \geq \frac{\|x\|^t + \|y\|^t}{2},$$

which is equivalent to

$$\frac{2^t}{(\|x\|+\|y\|)^t} \leq \frac{2}{H_{\lambda,t}(x,y)} \leq \frac{2}{\|x\|^t+\|y\|^t}.$$

Then, combining the last relation and (2), we obtain (13), as claimed.

On the other hand, if $f : I \to \mathbb{R}$ is a convex function, then the signs of inequalities in (15) are reversed. Therefore, since $f(s) = s^t$, $s \geq 0$, $t > 1$ is a convex function, similarly to the first case, we have

$$\frac{2^t}{(\|x\|+\|y\|)^t} \geq \frac{2}{H_{\lambda,t}(x,y)} \geq \frac{2}{\|x\|^t+\|y\|^t}.$$

Finally, combining the last relation with (3) we get (14), as claimed. □

Clearly, due to the interpolating series (13) and (14) established in Theorem 3, we obtain a parametric family of upper bounds for angular distance which can also be interpreted as a characterization of an inner product space.

Corollary 2. *Let $X = (X, \|\cdot\|)$ be a normed linear space, let $t > 0$, $0 \leq \lambda \leq 1$, and let $H_{\lambda,t} : X \times X \to \mathbb{R}$ be defined by (12). Then, X is an inner product space if and only if the inequality*

$$\alpha[x,y] \leq \frac{2^{\frac{1}{t}}\,\|x-y\|}{H_{\lambda,t}^{\frac{1}{t}}(x,y)}$$

holds for all nonzero vectors $x, y \in X$. □

5. More accurate Cauchy–Schwarz inequality in inner product spaces

Upper bounds for angular distances $\alpha[x,y]$ discussed in the previous two sections can be utilized in establishing reverses of the Cauchy–Schwarz inequality in a real inner product space. A crucial step in this direction is a representation of inner product via angular distance, established by Aldaz [2] (see also [22]). Namely, if $X = (X, \langle\cdot,\cdot\rangle)$ is a real inner product space with norm $\|\cdot\|$ induced by an inner product, then

$$\langle x,y\rangle = \|x\|\,\|y\|\left(1-\frac{1}{2}\alpha^2[x,y]\right) \tag{16}$$

holds for all nonzero vectors $x, y \in X$. Consequently, this representation enables us to establish a reverse of the Cauchy–Schwarz inequality.

Theorem 4. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$. Then inequalities*

$$\|x\|\|y\| - |\langle x,y\rangle| \leq \|x\|\|y\| - \langle x,y\rangle \leq \frac{2\|x\|\|y\|\|x-y\|^2}{(\|x\|+\|y\|)^2} \tag{17}$$

hold for all nonzero vectors $x, y \in X$. □

Proof. Relation (16) can be rewritten as

$$\|x\|\|y\| - \langle x, y\rangle = \frac{1}{2}\,\|x\|\,\|y\|\,\alpha^2[x, y].$$

Now, the result follows from (2). □

Remark 3. It should be noted here that (17) represents a reverse of the Cauchy–Schwarz inequality (7). However, in order to simplify our further discussion, in the sequel we consider only estimates for the difference $\|x\|\|y\| - \langle x, y\rangle$.

Remark 4. The second inequality in (17) is a simple consequence of inequality (2). It should be noted here that this inequality can be refined by virtue of the series of inequalities in (14). In particular, if $t > 1$, then

$$\|x\|\|y\| - \langle x, y\rangle \le \frac{2^{\frac{2-t}{t}}\|x\|\|y\|\|x-y\|^2}{(\|x\|^t + \|y\|^t)^{\frac{2}{t}}} \le \frac{2^{\frac{2-t}{t}}\|x\|\|y\|\|x-y\|^2}{H_{\lambda,t}^{\frac{2}{t}}(x, y)} \le \frac{2\|x\|\|y\|\|x-y\|^2}{(\|x\| + \|y\|)^2}.$$

However, the second inequality in (17) will be relevant in our further discussion.

We have just shown that the characterizations of inner product spaces described via inequalities in (2), (3), (13) and (14) provide reverses of the Cauchy–Schwarz inequality. Our next intention is to establish the corresponding refinements of the Cauchy–Schwarz inequality. To do this, we first discuss some mutual bounds for the angular distance $\alpha[x, y]$, known from the literature, which are closely connected to our characterization of an inner product space established in Theorem 2.

More precisely, Maligranda [16], established the following refinement and reverse of a triangle inequality in an arbitrary normed linear space $X = (X, \|\cdot\|)$:

$$\|x + y\| \le \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\min\{\|x\|, \|y\|\} \tag{18}$$

and

$$\|x + y\| \ge \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\max\{\|x\|, \|y\|\}. \tag{19}$$

In addition, it has been proved that if either $\|x\| = \|y\| = 1$ or $y = cx$, $c > 0$, then equality holds in both relations. The above two relations can be utilized in establishing mutual bounds for the angular distance. Namely, by replacing y with $-y$ in inequalities (18) and (19), and noting that

$$2\min\{\|x\|, \|y\|\} = \|x\| + \|y\| - |\|x\| - \|y\||,$$
$$2\max\{\|x\|, \|y\|\} = \|x\| + \|y\| + |\|x\| - \|y\||,$$

one obtains mutual bounds for angular distance

$$\frac{\|x - y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \le \alpha[x, y] \le \frac{\|x - y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}, \tag{20}$$

which hold for all nonzero vectors $x, y \in X$. It is easy to see that the upper bound in (20) interpolates in between the right-hand sides of inequalities (1) and (2).

Remark 5. The proof of inequalities (18) and (19) can be found in [16]. However, the proof of (20) is quite simple in an inner product space X. Namely, by virtue of Theorem 2, we have $\alpha[x,y] = \sqrt{AB}$, where $A = \frac{\|x-y\|-|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}$ and $B = \frac{\|x-y\|+|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}$. Now, it is easy to see that $A \le B$, and so $A \le \sqrt{AB} \le B$, which implies (20).

Remark 6. Motivated by (18) and (19), Dehghan [10], established yet another pair of refinement and reverse of the triangle inequality which provided mutual bounds for skew-angular distance $\beta[x,y]$ (for more details, see [10]).

Mutual bounds in (20) can be parameterized via a pair of nonzero real parameters of the same sign. More precisely, since $\alpha[x,y] = \alpha[ax, by]$, where $a, b \in \mathbb{R}$, $ab > 0$, we have the following slight extension of (20).

Corollary 3. *Let $X = (X, \|\cdot\|)$ be a normed linear space and let $a, b \in \mathbb{R}$ be such that $ab > 0$. Then the inequalities*

$$\frac{\|ax-by\| - |a\|x\| - b\|y\||}{\min\{|a|\|x\|, |b|\|y\|\}} \le \alpha[x,y] \le \frac{\|ax-by\| + |a\|x\| - b\|y\||}{\max\{|a|\|x\|, |b|\|y\|\}} \tag{21}$$

hold for all nonzero vectors $x, y \in X$. □

Remark 7. Similarly to Remark 5, it is interesting that if the square of angular distance is equal to the product of bounds in (21), then we deal with an inner product space. More precisely, if we replace x, y, a, b in (11) by $\frac{x}{\|x\|}, \frac{y}{\|y\|}, a\|x\|, b\|y\|$, respectively, we get

$$\alpha^2[x,y] = \frac{\|ax-by\|^2 - (a\|x\| - b\|y\|)^2}{ab\|x\|\|y\|}, \tag{22}$$

provided that $ab \neq 0$. On the other hand, if $ab > 0$, then by putting $\frac{1}{a}x$ and $\frac{1}{b}y$ in (22) instead of x and y respectively, we obtain relation (9). This means that if $ab > 0$, then relation (22) also characterizes an inner product space. □

Remark 8. Since $\frac{2\|x-y\|}{\|x\|+\|y\|} \le \frac{\|x-y\|+|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}$, inequality (2) is more accurate than the second inequality in (20). On the other hand, inequality (2) can also be extended in view of Corollary 3. Namely, a normed linear space X is an inner product space if and only if the relation

$$\alpha[x,y] \le \frac{2\|ax-by\|}{|a|\|x\| + |b|\|y\|} \tag{23}$$

holds for all nonzero vectors $x, y \in X$, provided that $ab > 0$. In the same way, inequality (23) is more accurate than the second inequality in (21). □

However, the left inequality signs in (20) and (21) provide refinements of the Cauchy–Schwarz inequality. Our next result is an interpolating set of inequalities which yields a more accurate version of the Cauchy–Schwarz inequality.

Theorem 5. *Let $X=(X,\langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$. If $a,b\in\mathbb{R}$ are such that $ab>0$, then the series of inequalities*

$$\begin{aligned}
&\frac{1}{2ab}\left(\|ax-by\|-|a\|x\|-b\|y\||\right)^2\\
&\quad\le \frac{\max\{|a|\|x\|,|b|\|y\|\}}{2ab\min\{|a|\|x\|,|b|\|y\|\}}\left(\|ax-by\|-|a\|x\|-b\|y\||\right)^2\\
&\quad\le \|x\|\|y\|-\langle x,y\rangle\\
&\quad\le \frac{2\|x\|\|y\|\|ax-by\|^2}{\left(|a|\|x\|+|b|\|y\|\right)^2}\le\frac{1}{2ab}\|ax-by\|^2
\end{aligned}\tag{24}$$

holds for all nonzero vectors $x,y\in X$.

Proof. Since $\|x\|\|y\|-\langle x,y\rangle=\frac{1}{2}\|x\|\,\|y\|\,\alpha^2[x,y]$, applying the first inequality in (21), we get that

$$\begin{aligned}
\|x\|\|y\|-\langle x,y\rangle &\ge \frac{\|x\|\|y\|}{2}\left(\frac{\|ax-by\|-|a\|x\|-b\|y\||}{\min\{|a|\|x\|,|b|\|y\|\}}\right)^2\\
&=\frac{\max\{|a|\|x\|,|b|\|y\|\}}{2ab\min\{|a|\|x\|,|b|\|y\|\}}\left(\|ax-by\|-|a\|x\|-b\|y\||\right)^2,
\end{aligned}$$

which yields the second inequality sign in (24). Moreover, since $\frac{\max\{|a|\|x\|,|b|\|y\|\}}{\min\{|a|\|x\|,|b|\|y\|\}}\ge 1$, we have the first inequality sign in (24).

In the same way, utilizing relation (23), we obtain the third inequality sign in (24). Finally, the last inequality sign in (24) holds by the arithmetic-geometric mean inequality since

$$\frac{2\|x\|\|y\|}{\left(|a|\|x\|+|b|\|y\|\right)^2}\le\frac{2\|x\|\|y\|}{4ab\|x\|\|y\|}=\frac{1}{2ab}.$$

□

Clearly, the first two inequality signs in (24) provide a refinement, while the last two signs yield a reverse of the Cauchy–Schwarz inequality when $ab>0$.

Remark 9. If we had utilized the upper bound in (21) instead of (23) in the proof of Theorem 5, we would have got the following reverse of the Cauchy–Schwarz inequality:

$$\begin{aligned}
\|x\|\|y\|-\langle x,y\rangle &\le \frac{\min\{|a|\|x\|,|b|\|y\|\}}{2ab\max\{|a|\|x\|,|b|\|y\|\}}\left(\|ax-by\|+|a\|x\|-b\|y\||\right)^2\\
&\le\frac{1}{2ab}\left(\|ax-by\|+|a\|x\|-b\|y\||\right)^2.
\end{aligned}$$

Although the above relation is, in some way, symmetric to the corresponding refinement in (24), it is obviously weaker than the reverse given in (24).

Remark 10. In particular, if $a = b$, then the set of inequalities in (24) implies the relation

$$\frac{1}{2}\left(\|x-y\| - |\|x\| - \|y\||\right)^2 \leq \|x\|\|y\| - \langle x, y\rangle \leq \frac{1}{2}\|x-y\|^2. \tag{25}$$

It should be noted here that the reverse of the Cauchy–Schwarz inequality in the above relation is more accurate than the corresponding one established in [20] (see also Remarks 8 and 9). □

Our next consequence of Theorem 5 represents the refinement and reverse of the Cauchy–Schwarz inequality expressed in terms of p-angular distance $\alpha_p[x, y]$.

Corollary 4. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$. If $p \geq 0$, then the series of inequalities*

$$\begin{aligned}
&\frac{1}{2\|x\|^{p-1}\|y\|^{p-1}}\left(\alpha_p[x,y] - |\|x\|^p - \|y\||^p\right)^2 \\
&\leq \frac{\|x\|\|y\|}{2\min\{\|x\|^{2p}, \|y\|^{2p}\}}\left(\alpha_p[x,y] - |\|x\|^p - \|y\||^p\right)^2 \\
&\leq \|x\|\|y\| - \langle x, y\rangle \\
&\leq \frac{2\|x\|\|y\|}{\left(\|x\|^p + \|y\|^p\right)^2}\alpha_p^2[x,y] \leq \frac{1}{2\|x\|^{p-1}\|y\|^{p-1}}\alpha_p^2[x,y]
\end{aligned} \tag{26}$$

holds for all nonzero vectors $x, y \in X$.

Remark 11. If $p = 0$, then the series of inequalities in (26) reduces to equality (16) obtained by Aldaz [2]. Furthermore, if $p = 1$, then (26) implies inequalities in (25).

So far, we have shown that the Cauchy–Schwarz inequality is closely connected to bounds for angular distance. Namely, upper bounds for $\alpha[x, y]$ provide reverses, while lower bounds yield refinements of the Cauchy–Schwarz inequality. Due to (16), it turns out that this problem can also be considered in the opposite direction. Namely, refinements and reverses of the Cauchy–Schwarz inequality provide lower and upper bounds for angular distance. In order to illustrate this equivalence, we give the following result with which we conclude this section.

Theorem 6. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space with norm $\|\cdot\|$ induced by an inner product $\langle\cdot,\cdot\rangle$, and let $x, y \in X$ be nonzero vectors such that $\|x\| \neq \|y\|$. Then,*

$$\|x\|\|y\| + \min\{\|x\|^2, \|y\|^2\} \leq 2\langle x, y\rangle \tag{27}$$

holds if and only if

$$\alpha[x,y] \le \frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}. \tag{28}$$

Proof. Without loss of generality, we can assume that $\|x\| < \|y\|$. Then, due to (16), it follows that (28) is equivalent to

$$\alpha^2[x,y] = \frac{2\|x\|\|y\| - 2\langle x,y\rangle}{\|x\|\|y\|} \le \frac{\|x-y\|^2}{\|y\|^2} = \frac{\|x\|^2 - 2\langle x,y\rangle + \|y\|^2}{\|y\|^2}.$$

After rearranging, the last inequality can be rewritten as

$$(\|y\| - \|x\|)\left(-\|x\|\|y\| - \|x\|^2 + 2\langle x,y\rangle\right) \ge 0,$$

wherefrom we get $\|x\|\|y\| + \|x\|^2 \le 2\langle x,y\rangle$, i.e. we obtain (27), due to our hypothesis. □

Remark 12. It should be noted here that the estimate in (28) is more accurate than the upper bounds given by (2) and (20). Further, considering relations (27) and (28) with reversed signs of inequality, we get lower bounds for angular distance.

6. Comparative relations for angular distances in inner product spaces

Besides characterizations of an inner product space, paper [25] also deals with some geometric aspects of angular distances. In particular, the authors established several results on the comparison of angular distances $\alpha_p[x,y]$ and $\alpha_q[x,y]$ for arbitrary values $p,q \in \mathbb{R}$. For example, they proved that the relation

$$\alpha_p[x,y] \ge \frac{|p|}{|p| + |p-q|} \min\left\{\|x\|^{p-q}, \|y\|^{p-q}\right\} \alpha_q[x,y] \tag{29}$$

holds for arbitrary $p,q \in \mathbb{R}$ and for all nonzero vectors x,y in a normed space X.

The main objective of this section is to show applications of relations (4) and (9) in obtaining some new comparative relations for angular distances. Since we will utilize relations that characterize inner product spaces, the estimates that follow will hold in an inner product space.

The following comparative relations between both p-angular distances ($\alpha_p[x,y]$, $\beta_p[x,y]$) and the usual angular distance $\alpha[x,y]$ are immediate consequences of the characterizing relation (9).

Corollary 5. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$. If $p \in \mathbb{R}$, then the inequalities*

$$\alpha_p[x,y] \ge \|x\|^{\frac{p}{2}} \|y\|^{\frac{p}{2}} \alpha[x,y] \tag{30}$$

and

$$\beta_p[x,y] \geq \|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}\alpha[x,y] \tag{31}$$

hold for all nonzero vectors $x, y \in X$.

Proof. Inequality (30) is an immediate consequence of relation (10). Further, considering identity (9) with x and y respectively replaced by $\|y\|^{p-1}x$ and $\|x\|^{p-1}y$, we arrive at the relation

$$\alpha^2\left[\|y\|^{p-1}x, \|x\|^{p-1}y\right] = \frac{\beta_p^2[x,y] - \left(\|x\|\|y\|^{p-1} - \|x\|^{p-1}\|y\|\right)^2}{\|x\|^p\|y\|^p}.$$

In addition, since $\alpha[x,y] = \alpha[ax,by]$, for $a,b \in \mathbb{R}$, the above relation can be rewritten in the following form:

$$\beta_p^2[x,y] - \|x\|^p\|y\|^p\alpha^2[x,y] = \|x\|^2\|y\|^2\left(\|x\|^{p-2} - \|y\|^{p-2}\right)^2. \tag{32}$$

Clearly, inequality (31) holds since the right-hand side of (32) is nonnegative. □

Remark 13. If $q = 0$, inequality (29) reduces to $\alpha_p[x,y] \geq \min\{\|x\|^p, \|y\|^p\}$ $\alpha[x,y]$. Now, since $\min\{\|x\|^p, \|y\|^p\} \leq \sqrt{\|x\|^p\|y\|^p} = \|x\|^{\frac{p}{2}}\|y\|^{\frac{p}{2}}$, it follows that inequality (30) is more accurate than (29). Of course, this is meaningful since (30) holds in an inner product space.

Remark 14. Combining relations (10) and (32), we arrive at the following relationship between p-angular distance and skew p-angular distance (see also [25]):

$$\begin{aligned}\alpha_p^2[x,y] - \beta_p^2[x,y] &= \left(\|x\|^p - \|y\|^p\right)^2 - \|x\|^2\|y\|^2\left(\|x\|^{p-2} - \|y\|^{p-2}\right)^2 \\ &= \left(\|x\|^2 - \|y\|^2\right)\left(\|x\|^{2p-2} - \|y\|^{2p-2}\right).\end{aligned}$$

By a more precise analysis, we can derive comparative relations for quantities $\alpha_p[x,y]$ and $\alpha_q[x,y]$, as well as for $\beta_p[x,y]$ and $\beta_q[x,y]$, where p and q are arbitrary real parameters. Our next result is an extension of inequality (30).

Theorem 7. *Let* $X = (X, \langle\cdot,\cdot\rangle)$ *be an inner product space and let* $\|\cdot\|$ *be the norm induced by an inner product* $\langle\cdot,\cdot\rangle$. *If* $|p| \geq |q|$, *then the inequality*

$$\alpha_p[x,y] \geq \|x\|^{\frac{p-q}{2}}\|y\|^{\frac{p-q}{2}}\alpha_q[x,y] \tag{33}$$

holds for all nonzero vectors $x, y \in X$. *Otherwise, the sign of inequality* (33) *is reversed. If* $|p-2| \geq |q-2|$, *then the inequality*

$$\beta_p[x,y] \geq \|x\|^{\frac{p-q}{2}}\|y\|^{\frac{p-q}{2}}\beta_q[x,y] \tag{34}$$

holds for all nonzero vectors $x, y \in X$. *Otherwise, the sign of inequality* (34) *is reversed.*

Proof. Obviously, inequality (33) holds for $\|x\| = \|y\|$. Hence, without loss of generality we can suppose that $\|x\| > \|y\|$, due to symmetry.

Now, consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(p) = \frac{\alpha_p^2[x,y]}{\|x\|^p\|y\|^p}$, where $x, y \in X$ are nonzero vectors. Then, by (10), it follows that

$$f(p) = \frac{\left(\|x\|^p - \|y\|^p\right)^2}{\|x\|^p\|y\|^p} + \alpha^2[x, y].$$

Therefore, we have

$$\begin{aligned} f(p) - f(q) &= \frac{\left(\|x\|^p - \|y\|^p\right)^2}{\|x\|^p\|y\|^p} - \frac{\left(\|x\|^q - \|y\|^q\right)^2}{\|x\|^q\|y\|^q} \\ &= \frac{\|x\|^{2p+q}\|y\|^q + \|x\|^q\|y\|^{2p+q} - \|x\|^{p+2q}\|y\|^p - \|x\|^p\|y\|^{p+2q}}{\|x\|^{p+q}\|y\|^{p+q}} \\ &= \frac{\left(\|x\|^p\|y\|^q - \|x\|^q\|y\|^p\right)\left(\|x\|^{p+q} - \|y\|^{p+q}\right)}{\|x\|^{p+q}\|y\|^{p+q}} \\ &= \left(\left(\frac{\|x\|}{\|y\|}\right)^p - \left(\frac{\|x\|}{\|y\|}\right)^q\right)\left(1 - \left(\frac{\|y\|}{\|x\|}\right)^{p+q}\right). \end{aligned}$$

Moreover, with an abbreviation $t = \frac{\|x\|}{\|y\|}$, we have

$$f(p) - f(q) = \frac{\left(t^p - t^q\right)\left(t^{p+q} - 1\right)}{t^{p+q}}, \tag{35}$$

where $t > 1$. Now, according to (35), it follows that $f(p) \geq f(q)$ if and only if $p \geq q$, $p+q \geq 0$ or $p \leq q$, $p+q \leq 0$. Clearly, this set of conditions is equivalent to $|p| \geq |q|$, which yields inequality (33).

To prove inequality (34), we utilize the relationship between p-angular and skew p-angular distance given by (6). Then, our assertion follows from the proof of inequality (33), after several elementary calculations. □

Remark 15. According to (35), equality in (33) holds if and only if $\|x\| = \|y\|$ or $p = q$ or $p = -q$. Similarly, equality in (34) holds if and only if $\|x\| = \|y\|$ or $p = q$ or $p + q = 4$.

Remark 16. Similarly to Remark 13, inequality (33) is more accurate than (29) since

$$\frac{|p|}{|p| + |p-q|} \min\left\{\|x\|^{p-q}, \|y\|^{p-q}\right\} \leq \sqrt{\|x\|^{p-q}\|y\|^{p-q}} = \|x\|^{\frac{p-q}{2}}\|y\|^{\frac{p-q}{2}}.$$

Clearly, this is consistent with the fact that (33) holds in an inner product space. □

Remark 17. In 2019, Rooin et al. [24], established more accurate versions of inequalities (30) and (33). Namely, by a more precise analysis, they proved that the inequality

$$\alpha_p[x,y] \geq \frac{\|x\|^p + \|y\|^p}{2}\alpha[x,y] \tag{36}$$

holds for all $p \in \mathbb{R}$, while the inequality

$$\alpha_p[x,y] \geq \frac{\|x\|^{p-q} + \|y\|^{p-q}}{2}\alpha_q[x,y] \tag{37}$$

holds for $\frac{p}{q} \geq 1$. Clearly, inequalities (36) and (37) are more accurate than (30) and (33), by the arithmetic-geometric mean inequality. Therefore, our aim is to establish improved forms of inequalities (30) and (33) and compare them with the above estimates.

Inequalities (30) and (31) have been derived as immediate consequences of identities (10) and (32), by neglecting the corresponding terms in these identities. By a more precise analysis of identities (10) and (32), we obtain more accurate forms of inequalities (30) and (31).

Theorem 8. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$. If $x, y \in X$ are nonzero vectors and $p \geq 1$, then*

$$\begin{aligned} &p^2\left(\|x\| - \|y\|\right)^2\left(\min\{\|x\|,\|y\|\}\right)^{2p-2} \\ &\quad\leq \alpha_p^2[x,y] - \|x\|^p\|y\|^p\alpha^2[x,y] \\ &\quad\leq p^2\left(\|x\| - \|y\|\right)^2\left(\max\{\|x\|,\|y\|\}\right)^{2p-2}. \end{aligned} \tag{38}$$

If $p < 1$, then the inequalities in (38) are reversed. In addition, if $p \geq 3$, then

$$\begin{aligned} &(p-2)^2\|x\|^2\|y\|^2\left(\|x\| - \|y\|\right)^2\left(\min\{\|x\|,\|y\|\}\right)^{2p-6} \\ &\quad\leq \beta_p^2[x,y] - \|x\|^p\|y\|^p\alpha^2[x,y] \\ &\quad\leq (p-2)^2\|x\|^2\|y\|^2\left(\|x\| - \|y\|\right)^2\left(\max\{\|x\|,\|y\|\}\right)^{2p-6}, \end{aligned} \tag{39}$$

while for $p < 3$ the inequalities in (39) are reversed.

Proof. From relation (10) we have $\alpha_p^2[x,y] - \|x\|^p\|y\|^p\alpha^2[x,y] = (\|x\|^p - \|y\|^p)^2$. On the other hand, by the Lagrange mean value theorem, it follows that there exists $\phi \in (\|x\|,\|y\|)$ such that $\|x\|^p - \|y\|^p = (\|x\| - \|y\|)\, p\phi^{p-1}$. Clearly, by squaring the last relation we obtain inequalities in (38) since $\min\{\|x\|,\|y\|\} < \phi < \max\{\|x\|,\|y\|\}$.

Similarly, by virtue of the Lagrange mean value theorem, there exists $\phi \in (\|x\|,\|y\|)$ such that $\|x\|^{p-2} - \|y\|^{p-2} = (\|x\| - \|y\|)\,(p-2)\phi^{p-3}$. Now, our assertion follows from (32).

Remark 18. The first inequality in (38) represents an improvement of inequality (30), while the second one is the corresponding reverse. The first inequalities in (38) and (36) are not comparable in general. To see this, consider the first inequality in (38), i.e.

$$\alpha_p^2[x,y] \geq \|x\|^p\|y\|^p\alpha^2[x,y] + p^2\left(\|x\| - \|y\|\right)^2\left(\min\{\|x\|,\|y\|\}\right)^{2p-2}.$$

Now, if we subtract $\left(\frac{\|x\|^p+\|y\|^p}{2}\right)^2\alpha^2[x,y]$ from both sides of the previous inequality, we have

$$\begin{aligned}&\alpha_p^2[x,y] - \left(\frac{\|x\|^p + \|y\|^p}{2}\right)^2\alpha^2[x,y]\\ &\quad\geq -\frac{1}{4}\alpha^2[x,y]\left(\|x\|^p - \|y\|^p\right)^2 + p^2\left(\|x\| - \|y\|\right)^2\left(\min\{\|x\|,\|y\|\}\right)^{2p-2}.\end{aligned} \tag{40}$$

Now, if $p = 1$, then the right-hand side of (40) reduces to $\left(1-\frac{1}{4}\alpha^2[x,y]\right)\left(\|x\| - \|y\|\right)^2$, which is nonnegative since $\alpha[x,y] \leq 2$, by the triangle inequality. This means that if $p = 1$, our inequality (38) is more precise than (36).

On the other hand, let $X = \mathbb{R}^2$, with the usual Euclidean inner product defined to be the sum of component-wise multiplication, and let $x = (2,0)$, $y = (0,1)$. Then, $\|x\| = 2$, $\|y\| = 1$ and $\alpha[x,y] = \sqrt{2}$. Consequently, if $p = 2$, then the right-hand side of (40) takes the value $-\frac{1}{2}$, which means that the first inequality in (38) is weaker than (36) in this setting. □

In order to conclude the paper, we discuss a special case of inequality (33) when $q = 1$. In this case, inequality (33) reduces to

$$\alpha_p[x,y] \geq \|x\|^{\frac{p-1}{2}}\|y\|^{\frac{p-1}{2}}\|x-y\|, \tag{41}$$

which holds for $|p| \geq 1$, while for $|p| < 1$ the inequality is reversed. The same conclusion can be drawn from identity (4) by neglecting the term

$$\left(\|x\|^{p-1} - \|y\|^{p-1}\right)\left(\|x\|^{p+1} - \|y\|^{p+1}\right).$$

Now, similarly to Theorem 8, we give a more accurate form of inequality (41).

Theorem 9. *Let $X = (X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the norm induced by an inner product $\langle\cdot,\cdot\rangle$. If $x, y \in X$ are nonzero vectors and $p \geq 2$, then*

$$\begin{aligned}&\left(p^2-1\right)\left(\|x\| - \|y\|\right)^2\left(\min\{\|x\|,\|y\|\}\right)^{2p-2}\\ &\quad\leq \alpha_p^2[x,y] - \|x\|^{p-1}\|y\|^{p-1}\|x-y\|^2\\ &\quad\leq \left(p^2-1\right)\left(\|x\| - \|y\|\right)^2\left(\max\{\|x\|,\|y\|\}\right)^{2p-2}.\end{aligned} \tag{42}$$

In addition, if $p < -1$ then the inequalities in (42) are reversed.

Proof. By identity (4) and the Lagrange mean value theorem, it follows that there exist $\phi, \psi \in (\|x\|, \|y\|)$ such that

$$\alpha_p^2[x,y] - \|x\|^{p-1}\|y\|^{p-1}\|x-y\|^2 = \left(p^2-1\right)\left(\|x\| - \|y\|\right)^2\phi^{p-2}\psi^p,$$

which proves our assertion. □

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