Aequat. Math. 94 (2020), 137–149

0001-9054/20/010137-13
published online December 5, 2019
https://doi.org/10.1007/s00010-019-00695-0

Aequationes Mathematicae

Approximately orthogonality preserving maps in Krein spaces

Ateyeh Saraei and Maryam Amyari

Abstract. In this paper, we investigate approximately orthogonality preserving maps in the setting of Krein spaces. More precisely, suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are two Krein spaces and that $T : \mathcal{K}_1 \to \mathcal{K}_2$ is a nonzero linear ε-orthogonality preserving map for some $\varepsilon \in [0, 1)$ such that $T(\mathcal{K}_1^{\pm}) \subseteq \mathcal{K}_2^{\pm}$. We show that T is injective and continuous and there exists $\gamma > 0$ such that $|[T(x), T(y)] - \gamma^2[x, y]| \leq \delta \min\{\gamma^2\|x\|\|y\|, \|T(x)\|\|T(y)\|\}$, for $x, y \in \mathcal{K}_1$ with $\delta = 12\varepsilon\left(\frac{1}{1-\varepsilon} + \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right)$. We also give some conditions under which the Pythagorean equality holds true in a Krein space.

Mathematics Subject Classification. Primary 46H05, Secondary 46C05.

Keywords. Orthogonality preserving map, Indefinite inner product space, Krein space.

1. Introduction

Let X and Y be two inner product spaces, a map $T : X \to Y$ is called orthogonality preserving if Tx and Ty are orthogonal for any orthogonal vectors x and y in X; see [6,18].

For a given $\varepsilon \in [0, 1)$, two vectors $x, y \in X$ are said to be approximately orthogonal or ε-orthogonal, denoted by $x \perp^{\varepsilon} y$, if $|\langle x, y\rangle| \leq \varepsilon\|x\|\|y\|$, where $\langle \cdot, \cdot\rangle$ denotes the inner product in X and $\|\cdot\|$ denotes the norm in X induced by $\langle \cdot, \cdot\rangle$. A map $T : X \to Y$ is said to be approximately orthogonality preserving if $x \perp y \Rightarrow Tx \perp^{\varepsilon} Ty$ for $x, y \in X$.

Approximately orthogonality preserving maps have been studied in several settings; see [1,5,8–13,17].

Chmieliński in [9, Theorem 1] proved that a linear map $T : X \to Y$ is orthogonality preserving if and only if there exists $\gamma > 0$ such that $\langle T(x), T(y)\rangle = \gamma^2\langle x, y\rangle$, for $x, y \in X$. Furthermore, he showed that for a nonzero linear ε-orthogonality preserving map $T : X \to Y$ for some $\varepsilon \in [0, 1)$ there is $\gamma > 0$ such that

$$|\langle T(x), T(y)\rangle - \gamma^2\langle x, y\rangle| \leq \delta \min\{\gamma^2\|x\|\|y\|, \|T(x)\|\|T(y)\|\}, \text{for } x, y \in X,$$

with $\delta = 4\varepsilon\big(\frac{1}{1-\varepsilon} + \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\big)$.

In the present paper, we generalize these results to the setting of Krein spaces. We first recall some basic facts on these structures; more details can be found, for example, in [3,4,7,14–16].

Definition 1.1. Suppose that $\mathcal{K}$ is a linear vector space equipped with a map $[\cdot,\cdot] : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ such that

$$[x, y] = \overline{[y, x]},$$
$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$$

for each $x, y, z \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$. Then $(\mathcal{K},\ [\cdot,\cdot])$ is called an indefinite inner product space.

Recall that an indefinite inner product has all the properties of the usual inner product except positive definiteness.

Vector spaces equipped with indefinite inner products were used for the first time in the quantum field theory in physics and mechanics.

We denote the sets of all positive, negative, and neutral elements of $\mathcal{K}$, by $\mathcal{K}^{++} \equiv \{x \in \mathcal{K} : [x, x] > 0\}$, $\mathcal{K}^{--} \equiv \{x \in \mathcal{K} : [x, x] < 0\}$, and $\mathcal{K}^0 \equiv \{x \in \mathcal{K} : [x, x] = 0\}$, respectively.

Let $x, y \in \mathcal{K}$. We say that x is orthogonal to y, denoted by $x[\perp]y$, when $[x, y] = 0$. If $\mathcal{L}$ is a subspace of $\mathcal{K}$, then the subspace $\mathcal{L}^{[\perp]} \equiv \{x \in \mathcal{K} : [x, y] = 0,$ for all $y \in \mathcal{L}\}$, is called the orthogonal complement of $\mathcal{L}$ with respect to $[\ ,\ ]$, and $x_0 \in \mathcal{L}$ is an isotropic element of $\mathcal{L}$ if $x_0 \neq 0$ and $x_0[\perp]\mathcal{L}$. Also $\mathcal{L}^0 \equiv \mathcal{L} \bigcap \mathcal{L}^{[\perp]}$ is the isotropic part of $\mathcal{L}$. If $\mathcal{L}^0 = \{0\}$, then $\mathcal{L}$ is called nondegenerate.

Azizov in [4, Theorem 1.24], proved that if $\mathcal{L}$ is a subspace of $\mathcal{K}$ such that $\mathcal{L}$ admits a decomposition $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^+ \subseteq \mathcal{K}^{++} \cup \{0\}$ and $\mathcal{L}^- \subseteq \mathcal{K}^{--} \cup \{0\}$ into the direct sum of the subspaces, then $\mathcal{L}$ is nondegenerate. In addition, if $\mathcal{L}^+[\perp]\mathcal{L}^-$, then we write

$$\mathcal{L} = \mathcal{L}^+[\oplus]\mathcal{L}^-.$$

This decomposition is called a canonical decomposition of the subspace $\mathcal{L}$.

Definition 1.2. An indefinite inner product space $(\mathcal{K}, [\cdot,\cdot])$ is called a Krein space, if the vector space $\mathcal{K}$ admits a canonical decomposition $\mathcal{K} = \mathcal{K}^+[\oplus]\mathcal{K}^-$ such that $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^-, -[\cdot,\cdot])$ are Hilbert spaces relative to the norms $\|x\| = [x, x]^{\frac{1}{2}}$ $(x \in \mathcal{K}^+)$ and $\|x\| = (-[x, x]^{\frac{1}{2}})$ $(x \in \mathcal{K}^-)$.

Suppose that $(\mathcal{K}, [\cdot,\cdot])$ is a Krein space with a canonical decomposition $\mathcal{K} = \mathcal{K}^+[\oplus]\mathcal{K}^-$ such that $P^+ : \mathcal{K} \to \mathcal{K}^+$ and $P^- : \mathcal{K} \to \mathcal{K}^-$ are two mutually orthogonal projection operators generated by the canonical decomposition above so that $P^+ + P^- = I$, where I is the identity operator on $\mathcal{K}$. Thus for any $x \in \mathcal{K}$, we have $x = x^+ + x^-$, where $x^+ = P^+x$ and $x^- = P^-x$. The linear operator $J : \mathcal{K} \to \mathcal{K}$, defined by $J = P^+ - P^-$, is called the canonical symmetry operator of the Krein space $\mathcal{K}$. Thus J is a bounded self-adjoint operator such that $J^2 = I$ and $J^{-1} = J^* = J$.

By using the canonical decomposition $\mathcal{K} = \mathcal{K}^+[\oplus]\mathcal{K}^-$ on the Krein space $(\mathcal{K}, [\cdot,\cdot])$, we can define an inner product as follows:

$$\langle x, y\rangle = [x^+, y^+] - [x^-, y^-], \tag{1.1}$$

where $x = x^+ + x^-$ and $y = y^+ + y^-$ are elements in $\mathcal{K}$.
Then $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ is a Hilbert space with respect to the norm $\|x\|^2 = \langle x, x\rangle$. In fact, $\langle x, y\rangle = [x, y]$, for vectors $x, y \in \mathcal{K}^+$, and $\langle x, y\rangle = -[x, y]$ for $x, y \in \mathcal{K}^-$.

Relation (1.1) between the indefinite inner product $[\cdot,\cdot]$ and the definite inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{K}$ implies that

$$\langle x, y\rangle = [Jx, y], \qquad [x, y] = \langle Jx, y\rangle$$

for each $x, y \in \mathcal{K}$, where J is the canonical symmetry operator and $J(x^+ + x^-) = J(x^+) + J(x^-) = x^+ - x^-$. In addition, $\|x\|^2 = \langle x, x\rangle = [Jx, x]$. A straightforward computation shows that

$$[x, y] = [x^+, y^+] + [x^-, y^-], \quad [x, y] = \langle x^+, y^+\rangle - \langle x^-, y^-\rangle,$$
$$[x, x] = \|x^+\|^2 - \|x^-\|^2.$$

Hence,

$$\begin{aligned}\|y\|^2 &= [Jy, y] = [J(y^+ + y^-), y^+ + y^-] = [y^+ - y^-, y^+ + y^-]\\ &= [y^+, y^+] + [-y^-, y^-].\end{aligned}$$

By the definition of J, we have $J(y^+) = y^+$, $J(y^-) = -y^-$. Therefore

$$\|y\|^2 = [Jy^+, y^+] + [Jy^-, y^-] = \|y^+\|^2 + \|y^-\|^2.$$

2. Approximate orthogonality in Krein spaces

In this section, we define approximate orthogonality in the framework of Krein spaces. We start our work with the following lemmata and prove them by similar strategies to those of [9].

Lemma 2.1. *Let $(\mathcal{K}, [\cdot,\cdot], J)$ be a Krein space, let $x, y \in \mathcal{K}$, and let $\varepsilon \geq 0$. If*

$$|[Jx + y, x - Jy]| \leq \varepsilon\|Jx + y\|\|x - Jy\|,$$

then

$$|\|x\|^2 - \|y\|^2| \leq \varepsilon(\|x\|^2 + \|y\|^2).$$

Proof. By a straightforward computation, we get

$$\begin{aligned}|[Jx + y, x - Jy]|^2 &= [Jx + y, x - Jy][x - Jy, Jx + y]\\ &= (\|x\|^2 - [x, y] + [y, x] - \|y\|^2)(\|x\|^2 + [x, y] - [y, x] - \|y\|^2)\\ &= \|x\|^4 + 2i\|x\|^2\mathrm{Im}\ [x, y] - \|x\|^2\|y\|^2 - 2i\|x\|^2\mathrm{Im}[x, y]\end{aligned}$$

$$\begin{aligned}
&+ 4(\mathrm{Im}[x,y])^2 + 2i\mathrm{Im}[x,y]\|y\|^2 - \|x\|^2\|y\|^2\\
&- 2i\mathrm{Im}[x,y]\|y\|^2 + \|y\|^4\\
&= \|x\|^4 + \|y\|^4 - 2\|x\|^2\|y\|^2 + 4(\mathrm{Im}[x,y])^2\\
&= (\|x\|^2 - \|y\|^2)^2 + 4(\mathrm{Im}[x,y])^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|Jx+y\|^2\|x-Jy\|^2 &= ([x+Jy,Jx+y])([Jx-y,x-Jy])\\
&= (\|x\|^2 + [x,y] + [y,x]\\
&\quad + \|y\|^2)(\|x\|^2 - [x,y] - [y,x] + \|y\|^2)\\
&= \|x\|^4 + \|y\|^4 + 2\|x\|^2\|y\|^2 - 4(\mathrm{Re}[x,y])^2\\
&= (\|x\|^2 + \|y\|^2)^2 - 4(\mathrm{Re}[x,y])^2.
\end{aligned}$$

The first inequality of the assumption implies that

$$\begin{aligned}
(\|x\|^2 - \|y\|^2)^2 + 4(\mathrm{Im}[x,y])^2 &\le \varepsilon^2((\|x\|^2 - \|y\|^2)^2 - 4(\mathrm{Re}[x,y])^2)\\
&\le \varepsilon^2(\|x\|^2 + \|y\|^2)^2.
\end{aligned}$$

Hence $\left|\|x\|^2 - \|y\|^2\right| \le \varepsilon(\|x\|^2 + \|y\|^2)$. □

Lemma 2.2. *Suppose that $(\mathcal{K}_1, [\cdot,\cdot], J_1)$ and $(\mathcal{K}_2, [\cdot,\cdot], J_2)$ are two Krein spaces and that $T : \mathcal{K}_1 \to \mathcal{K}_2$ is a map such that for each $x \in \mathcal{K}_1$, $T(x^{\pm}) = (Tx)^{\pm}$ and for some $\delta \ge 0$ and $\gamma > 0$, the functional inequality $\left|[T(x),T(y)] - \gamma^2[x,y]\right| \le \delta\gamma^2\|x\|\|y\|$ holds. Then*

$$\left|[J_2T(x),T(y)] - \gamma^2[J_1x,y]\right| \le 2\delta\gamma^2\|x\|\|y\| \qquad x,y \in \mathcal{K}_1.$$

Proof. From the properties of canonical symmetry operators, and from $\|x^+\| \le \|x\|$ and $\|x^-\| \le \|x\|$, we get

$$\begin{aligned}
\left|[J_2T(x),T(y)] - \gamma^2[J_1x,y]\right| &= \left|[T(x)^+ - T(x)^-, T(y)] - \gamma^2([x^+ + x^-, y])\right|\\
&= \Big|[T(x^+),T(y)] - [T(x^-),T(y)]\\
&\quad - \gamma^2([x^+,y] - [x^-,y])\Big|\\
&\le \Big|[T(x^+),T(y)]\\
&\quad - \gamma^2[x^+,y]\Big| + \left|[T(x^-),T(y)] - \gamma^2[x^-,y]\right|\\
&\le \delta\gamma^2\|x^+\|\|y\|\\
&\quad + \delta\gamma^2\|x^-\|\|y\| \le 2\delta\gamma^2\|x\|\|y\|.
\end{aligned}$$

□

Lemma 2.3. *Under the assumptions of Lemma* 2.2, T is a quasi-linear map, that is, T is quasiadditive,

$$\|T(x+y)-T(x)-T(y)\| \le \sqrt{8}\sqrt{\delta}\gamma(\|x\|+\|y\|), \quad (x,y \in \mathcal{K}_1)$$

and quasihomogeneous

$$\|T(\lambda x)-\lambda T(x)\| \le \sqrt{8}\sqrt{\delta}\gamma|\lambda|\|x\|. \quad (x \in \mathcal{K}_1, \lambda \in \mathbb{C}).$$

Proof. Let $x, y \in \mathcal{K}_1$ be arbitrary. Then

$$\begin{aligned}\|T(x+y)-T(x)-T(y)\|^2 =& [J_2(T(x+y)-T(x)-T(y)), T(x+y)-T(x)-T(y)]\\ \le& \Big|[J_2T(x+y), T(x+y)] - \gamma^2[J_1(x+y), x+y]\Big|\\ &+ \Big| -[J_2T(x+y), T(x)] + \gamma^2[J_1(x+y), x]\Big|\\ &+ \Big| -[J_2T(x+y), T(y)] + \gamma^2[J_1(x+y), y]\Big|\\ &+ \Big| -[J_2T(x), T(x+y)] + \gamma^2[J_1(x), x+y]\Big|\\ &+ \Big|[J_2T(x), T(x)] - \gamma^2[J_1(x), x]\Big|\\ &+ \Big|[J_2T(x), T(y)] - \gamma^2[J_1(x), y]\Big|\\ &+ \Big| -[J_2T(y), T(x+y)] + \gamma^2[J_1(y), x+y]\Big|\\ &+ \Big|[J_2T(y), T(x)] - \gamma^2[J_1(y), x]\Big|\\ &+ \Big|[J_2T(y), T(y)] - \gamma^2[J_1(y), y]\Big|.\end{aligned}$$

By using the above lemma and the assumption, we obtain

$$\begin{aligned}\|T(x+y)-T(x)-T(y)\|^2 \le& 2\delta\gamma^2(\|x+y\|^2 + 2\|x\|\|x+y\|\\ &+ 2\|y\|\|x+y\| + 2\|x\|\|y\| + \|x\|^2 + \|y\|^2)\\ \le& 8\delta\gamma^2(\|x\|+\|y\|)^2.\end{aligned}$$

In a similar way, for $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\|T(\lambda x)-\lambda T(x)\|^2 \le& \Big|[J_2T(\lambda x), T(\lambda x)] - \gamma^2[J_1(\lambda x), \lambda x]\Big|\\ &+ \Big| -\overline{\lambda}[J_2T(\lambda x), T(x)] + \gamma^2\overline{\lambda}[J_1(\lambda x), x]\Big|\\ &+ \Big| -\lambda[J_2T(x), T(\lambda x)] + \gamma^2\lambda[J_1x, \lambda x]\Big|\\ &+ \Big||\lambda|^2[J_2T(x), T(x)] + \gamma^2|\lambda|^2[J_1x, x]\Big|.\end{aligned}$$

Hence

$$\begin{aligned}\|T(\lambda x)-\lambda T(x)\|^2 &\le 2\delta\gamma^2(\|\lambda x\|^2 + 2|\lambda|\|x\|\|\lambda x\| + |\lambda|^2\|x\|^2)\\ &= 8\delta\gamma^2\|\lambda x\|^2.\end{aligned}$$

□

Lemma 2.4. *[4, Lemma 7.7] Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let $\mathcal{L}$ be a subspace of $\mathcal{K}$. Then $\overline{\mathcal{L}+\mathcal{L}^{[\perp]}} = \mathcal{K}$ if and only if $\mathcal{L}$ is nondegenerate.*

Recall that $\langle y\rangle$ is the closure of the linear span $\{y\}$. The following lemma states the condition under which $\langle y\rangle$ is nondegenerate.

Proposition 2.5. *Let $(\mathcal{K}, [\cdot,\cdot], J)$ be a Krein space and let y be a nonzero element of $\mathcal{K}$. Then $\langle y\rangle$ is nondegenerate if and only if y is not a neutral vector.*

Proof. Let $\langle y\rangle$ be nondegenerate. On the contrary, if y is neutral, then

$$[y, y] = 0 \Rightarrow ([y, \lambda y] = 0 \text{ for some } \lambda \in \mathbb{C}) \Rightarrow y \in \langle y\rangle^{[\perp]}. \tag{2.1}$$

Therefore $y \in \langle y\rangle^{[\perp]} \cap \langle y\rangle$. Then $\langle y\rangle^{[\perp]} \cap \langle y\rangle \neq \{0\}$, that is a contradiction, since $\langle y\rangle$ is nondegenerate.
Conversely, let y not be neutral. On the contrary, if $\langle y\rangle^{[\perp]} \cap \langle y\rangle \neq \{0\}$, then there exists a nonzero scalar number $\lambda \in \mathbb{C}$ such that $\lambda y \in \langle y\rangle^{[\perp]} \cap \langle y\rangle$. Then for all $\mu \in \mathbb{C}$, we reach $0 = [\lambda y, \mu y] = \lambda\bar{\mu}[y, y]$. Thus $[y, y] = 0$, that is, y is a neutral vector which is a contradiction. □

Corollary 2.6. *Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space. If y is not a neutral vector in $\mathcal{K}$, then $\langle y\rangle + \langle y\rangle^{[\perp]} = \mathcal{K}$.*

Proof. Let y not be a neutral vector. Then Proposition 2.5 ensures that $\langle y\rangle$ is nondegenerate and Lemma 2.4 yields $\overline{\langle y\rangle + \langle y\rangle^{[\perp]}} = \mathcal{K}$. On the other hand, since $\langle y\rangle$ is a finite-dimensional subspace and $\langle y\rangle^{[\perp]}$ is closed, so $\langle y\rangle + \langle y\rangle^{[\perp]}$ is closed and we can write $\langle y\rangle + \langle y\rangle^{[\perp]} = \mathcal{K}$. □

In Corollary 2.6, the condition that y is not a neutral vector is a necessary condition. The following example shows that if y is a neutral vector, then it may happen that $\mathcal{K} \neq \langle y\rangle + \langle y\rangle^{[\perp]}$.

Example 2.7. Consider the Krein space $(\mathbb{C}_2, [\cdot,\cdot])$ with the sesquilinear form $[x, y] = x_1y_1 - 4x_2y_2$. Take $y = (4, 2)$. Then $[(4, 2),(4, 2)] = 0$, and so y is a neutral vector and $y^{[\perp]} = \{k(2, 1);\ k \in \mathbb{R}\}$. If $x = (5, 6)$, then there are no $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $x = \lambda_1y_1 + \lambda_2y_2$ for some $y_1 \in \langle y\rangle$ and $y_2 \in \langle y\rangle^{[\perp]}$.

The main theorem is based on the property of Hilbert spaces that states that, if $x_1 \perp x_2$ and $x = x_1 + x_2$, then $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$, but we do not have this property in Krein spaces [2,11]. This means that we may have $x = x_1 + x_2$ and $x_1[\perp]x_2$ but $\|x\|^2 \neq \|x_1\|^2 + \|x_2\|^2$. In the following lemma, we investigate some conditions under which we can write $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$ for an element x in a Krein space.

Lemma 2.8. *Let $(\mathcal{K}, [\cdot,\cdot], J)$ be a Krein space. If $x, y \in \mathcal{K}$ are linearly independent so that $y \in \mathcal{K}^+$ (or $y \in \mathcal{K}^-$) is not neutral, then we can write*

$$x = x_1 + x_2, \qquad \|x\|^2 = \|x_1\|^2 + \|x_2\|^2,$$

where $x_1 \in \langle y\rangle$ and $x_2 \in \langle y\rangle^{[\perp]}$.

Proof. Let $y \in \mathcal{K}^+$ and y is not neutral. By Corollary 2.6, we can write $\langle y\rangle + \langle y\rangle^{[\perp]} = \mathcal{K}$. Note that in this case $\langle y\rangle \subseteq \mathcal{K}^+$. Thus $x = x_1 + x_2$, where $x_1 \in \langle y\rangle$ and $x_1[\perp]x_2$. Then

$$\begin{aligned}\|x\|^2 &= \|x_1 + x_2\|^2 = [Jx_1 + Jx_2, x_1 + x_2]\\ &= [Jx_1, x_1] + [Jx_1, x_2] + [Jx_2, x_1] + [Jx_2, x_2]\\ &= \|x_1\|^2 + \|x_2\|^2 + 2\mathrm{Re}[Jx_1, x_2]\\ &= \|x_1\|^2 + \|x_2\|^2 + 2\mathrm{Re}[x_1, x_2].\end{aligned}$$

Thus $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$. Similarly, it is true, when $y \in \mathcal{K}^-$. □

The next theorem shows that an approximately orthogonality preserving linear map T between two Krein spaces such that $T(\mathcal{K}_1^\pm) \subseteq \mathcal{K}_2^\pm$ is injective, continuous, and satisfies (2.2). To achieve the next result we adopt some arguments from [9, Theorem 3.1]. We also need the following facts about the polarization formula for $[x, y]$ in the Krein space. A straightforward computation shows that

$$\begin{aligned}[x, y] &= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) - \frac{1}{2}(\|x^- + y^-\|^2 - \|x^- - y^-\|^2)\\ &\quad + \frac{i}{4}(\|x + iy\|^2 - \|x - iy\|^2) - \frac{i}{2}(\|x^- + iy^-\|^2 - \|x^- - iy^-\|^2).\end{aligned}$$

Theorem 2.9. *Let $(\mathcal{K}_1, [\cdot,\cdot]_1, J_1)$ and $(\mathcal{K}_2, [\cdot,\cdot]_2, J_2)$ be two Krein spaces, let $T : \mathcal{K}_1 \to \mathcal{K}_2$ be a nonzero linear map such that $T(\mathcal{K}_1^\pm) \subseteq \mathcal{K}_2^\pm$, and let T be an approximately orthogonality preserving map for some $\varepsilon \in [0, 1)$. Then T is injective and continuous, and there exists $\gamma > 0$ such that for $x, y \in \mathcal{K}_1$,*

$$|[T(x), T(y)] - \gamma^2[x, y]| \le \delta \min\{\gamma^2\|x\|\|y\|, \|T(x)\|\|T(y)\|\} \tag{2.2}$$

with

$$\delta = 12\varepsilon\left(\frac{1}{1-\varepsilon} + \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right).$$

Conversely, if $T : \mathcal{K}_1 \to \mathcal{K}_2$ satisfies (2.2) with some $\delta \ge 0$ and $\gamma > 0$ and $T(x^\pm) = (Tx)^\pm$, then T is a quasilinear approximately orthogonality preserving map and

$$x[\perp]y \Rightarrow T(x)[\perp]^\delta T(y) \quad \textit{and} \quad T(x)[\perp]T(y) \Rightarrow x[\perp]^\delta y$$

for x, y in $\mathcal{K}_1$.

Proof. If $\dim \mathcal{K}_1 = 1$, then the assertion trivially holds. We assume that $\dim \mathcal{K}_1 \ge 2$.

Let x and y be two nonzero elements in $\mathcal{K}_1$, and we want to show that

$$\frac{1}{\delta_1}\lambda(y) \le \lambda(x) \le \delta_1\lambda(y), \tag{2.3}$$

where $\lambda(x) = \frac{\|T(x)\|}{\|x\|}$ with $\delta_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon} + 2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}} \geq 1$.

Let $y = y^+ + y^-$ be a decomposition of y, and put

$$y' = \begin{cases} y^+, & \lambda(y^+) < \lambda(y^-), \\ y^-, & \lambda(y^-) < \lambda(y^+); \end{cases}$$

therefore y' is not neutral and $\|y\|^2 = [Jy, y] = [Jy^+, y^+] + [Jy^-, y^-] = \|y^+\|^2 + \|y^-\|^2$ (recall that $Jy^+ = y^+$ and $Jy^- = -y^-$). Also, we have $\|Ty\|^2 = \|Ty^+\|^2 + \|Ty^-\|^2$.

There are three cases for x and y'.

(i) Suppose that x and y' are linearly dependent, that is, $x = \mu y'$ for some $\mu \in \mathbb{C}$. Then

$$\lambda(y') = \lambda(x). \tag{2.4}$$

If $\lambda(y^+) < \lambda(y^-)$, then $\frac{\|T(y^+)\|}{\|y^+\|} < \frac{\|T(y^-)\|}{\|y^-\|}$.

Similarly, if $\lambda(y^-) < \lambda(y^+)$, then $\frac{\|T(y^-)\|}{\|y^-\|} < \frac{\|T(y^+)\|}{\|y^+\|}$.Thus

$$\frac{\|T(y^+)\|^2}{\|y^+\|^2} \leq \frac{\|T(y^+)\|^2 + \|T(y^-)\|^2}{\|y^+\|^2 + \|y^-\|^2} \leq \frac{\|T(y^-)\|^2}{\|y^-\|^2}$$

or

$$\frac{\|T(y^-)\|^2}{\|y^-\|^2} \leq \frac{\|T(y^+)\|^2 + \|T(y^-)\|^2}{\|y^+\|^2 + \|y^-\|^2} \leq \frac{\|T(y^+\|^2}{\|y^+\|^2}.$$

That means $\lambda(y')^2 \leq \lambda(y)^2$, so equivalently from (2.4), $\lambda(x) \leq \lambda(y)$. Put $\delta_1 = 1$.

(ii) Suppose that x and y' are linearly independent and $x[\perp]y'$.

Define $u = \frac{x}{\|x\|}$, $v = \frac{y'}{\|y'\|}$. Then $\|u\| = \|v\| = 1$, $\|T(u)\| = \lambda(x)$ and $\|T(v)\| = \lambda(y')$; also $[u, v] = 0$ and $J_1u + v[\perp]u - J_1v$. By the assumption

$$J_2T(u) + T(v)[\perp]^{\varepsilon}T(u) - J_2T(v).$$

Thus $|[J_2T(u) + T(v), T(u) - J_2T(v)]| \leq \varepsilon\|J_2T(u) + T(v)\|\|T(u) - J_2T(v)\|$. Lemma 2.1 implies that

$$|\|T(u)\|^2 - \|T(v)\|^2| \leq \varepsilon(\|T(u)\|^2 + \|T(v)\|^2),$$

that is, $|\lambda(x)^2 - \lambda(y')^2| \leq \varepsilon(\lambda(x)^2 + \lambda(y')^2)$ and we reach $\lambda(x) \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\lambda(y')$. So by (2.4), we obtain

$$\lambda(x) \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\lambda(y') \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\lambda(y). \tag{2.5}$$

Then $\lambda(x) \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\lambda(y)$. Put $\delta_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$.

(iii) Now, we assume that x and y' are linearly independent but not orthogonal. By Lemma 2.8, we can choose two nonzero elements x_1 and x_2 in $\mathcal{K}_1$ such that $x = x_1 + x_2,\ \ x_1 \in \langle y' \rangle$, and $x_2[\perp]x_1$. $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$ and therefore

$$\|x_1\| \le \|x\|, \quad \|x_2\| \le \|x\|, \quad \|x_1\|\|x_2\| \le \|x\|^2.$$

Since $x_1 = \mu y'$ for some $\mu \in \mathbb{C}$ by (i), $\lambda(x_1) = \lambda(y')$ and since $x_2[\perp]x_1$ by (ii), we have

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\lambda(y') \le \lambda(x_2) \le \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\lambda(y').$$

We prove that $|[J_2T(x_1), T(x_2)]| = |[T(x_1), T(x_2)]|$

$$\begin{aligned} &T(x_1) \in \mathcal{K}_2^- \Rightarrow [J_2T(x_1), T(x_2)] = [-T(x_1), T(x_2)] = -[T(x_1), T(x_2)] \\ &T(x_1) \in \mathcal{K}_2^+ \Rightarrow [J_2T(x_1), T(x_2)] = [T(x_1), T(x_2)]. \end{aligned}$$

Therefore

$$[J_2T(x_1), T(x_2)] = \pm[T(x_1), T(x_2)] \Rightarrow |[J_2T(x_1), T(x_2)]| = |[T(x_1), T(x_2)]|$$

and also since $T(x_1) = T(\lambda y')$ that is $T(x_1) \in \mathcal{K}_2^+$ or $T(x_1) \in \mathcal{K}_2^-$, we get $|[J_2T(x_1), T(x_2)]| = |[T(x_1), T(x_2)]|$

$$\begin{aligned} \mathrm{Re}[J_2T(x_1), T(x_2)] &\le |[J_2T(x_1), T(x_2)]| = |[T(x_1), T(x_2)]| \le \varepsilon\|T(x_1)\|\|T(x_2)\| \\ &= \varepsilon\lambda(y')\|x_1\|\lambda(x_2)\|x_2\| \le \varepsilon\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\lambda(y')^2\|x\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \|T(x)\|^2 &= \|T(x_1) + T(x_2)\|^2 \\ &= \big[J_2\big(T(x_1)\big) + J_2\big(T(x_2)\big), T(x_1) + T(x_2)\big] \\ &= \|T(x_1)\|^2 + \|T(x_2)\|^2 + 2\mathrm{Re}[J_2T(x_1), T(x_2)] \\ &\le \lambda(y')^2\|x_1\|^2 + \frac{1+\varepsilon}{1-\varepsilon}\lambda(y')^2\|x_2\|^2 + 2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\lambda(y')^2\|x\|^2 \\ &= \lambda(y')^2\left(\|x_1\|^2 + \|x_2\|^2 + \big(\frac{1+\varepsilon}{1-\varepsilon} - 1\big)\|x_2\|^2 + 2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\|x\|^2\right) \\ &\le \lambda(y')^2\Big(\|x\|^2 + \big(\frac{1+\varepsilon}{1-\varepsilon} - 1\big)\|x\|^2 + 2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\|x\|^2\Big) \\ &= \lambda(y')^2\Big(\frac{1+\varepsilon}{1-\varepsilon} + 2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\Big)\|x\|^2, \end{aligned}$$

and we get

$$\begin{aligned}\lambda(x) &\le \lambda(y')\sqrt{\frac{1+\varepsilon}{1-\varepsilon}+2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}}\\ &\le \lambda(y)\sqrt{\frac{1+\varepsilon}{1-\varepsilon}+2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}}. \end{aligned}\tag{2.6}$$

Put $\delta_1 = \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}+2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right)}$ and hence $\lambda(x) \le \delta_1\lambda(y)$.

If x and y are arbitrary, then $\delta_1 = \max\{1, \sqrt{\frac{1+\varepsilon}{1-\varepsilon}+2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}}, \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\}$ and we reach (2.3).

Let $\ker T \neq \{0\}$. Then there exists a nonzero element $y \in \ker T$ such that $T(y) = 0$, so $\lambda(y) = 0$. By (2.3), $\lambda(x) = 0$ for all $x \neq 0$, that is $T \equiv 0$, which is contrary to the assumptions. Therefore $\ker T = \{0\}$ and T is injective.

Now, we show that T is continuous. Fix a nonzero element y_0 of $\mathcal{K}_1$, and take $\gamma = \lambda(y_0) > 0$. From (2.3), we have

$$\frac{1}{\delta_1}\gamma\|x\| \le \|T(x)\| \le \delta_1\gamma\|x\|, \quad x \in \mathcal{K}_1. \tag{2.7}$$

This inequality gives the continuity of T. Moreover, (2.7) gives that

$$\big|\|T(x)\| - \gamma\|x\|\big| \le (\delta_1 - 1)\gamma\|x\|, \quad x \in \mathcal{K}_1, \tag{2.8}$$

$$\big|\|T(x)\|^2 - \gamma^2\|x\|^2\big| \le (\delta_1^2 - 1)\gamma^2\|x\|^2, \quad x \in \mathcal{K}_1, \tag{2.9}$$

and

$$\frac{1}{\delta_1}\|T(x)\| \le \gamma\|x\| \le \delta_1\|T(x)\|, \quad x \in \mathcal{K}_1.$$

In a similar fashion, from inequalities (2.8) and (2.9), we get

$$\big|\|T(x)\| - \gamma\|x\|\big| \le (\delta_1 - 1)\gamma\|T(x)\|, \quad x \in \mathcal{K}_1, \tag{2.10}$$

$$\big|\|T(x)\|^2 - \gamma^2\|x\|^2\big| \le (\delta_1^2 - 1)\gamma^2\|T(x)\|^2, \quad x \in \mathcal{K}_1. \tag{2.11}$$

Let $x, y \in \mathcal{K}_1$ be arbitrary. Then

$$\begin{aligned}\Big|[T(x), T(y)] - \gamma^2[x, y]\Big| = \Big|&\frac{1}{4}(\|T(x) + T(y)\|^2 - \|T(x) - T(y)\|^2)\\ &-\frac{1}{2}(\|T(x^-) + T(y^-)\|^2 - \|T(x^-) - T(y^-)\|^2)\\ &+\frac{i}{4}(\|T(x) + iT(y)\|^2 - \|T(x) - iT(y)\|^2)\\ &-\frac{i}{2}(\|(Tx)^- + i(Ty)^-\|^2 - \|(Tx)^- - i(Ty)^-\|^2)\\ &-\gamma^2\Big(\frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)\\ &-\frac{1}{2}(\|x^- + y^-\|^2 - \|x^- - y^-\|^2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{4}(\|x+iy\|^2-\|x-iy\|^2)\\
&-\frac{i}{2}(\|x^-+iy^-\|^2-\|x^--iy^-\|^2)\Big)\Big|\\
&\le \frac{1}{4}\Big|\|T(x)+T(y)\|^2-\gamma^2\|x+y\|^2\Big|\\
&+\frac{1}{4}\Big|\|T(x)-T(y)\|^2-\gamma^2\|x-y\|^2\Big|\\
&+\frac{1}{2}\Big|\|T(x^-)+T(y^-)\|^2-\gamma^2\|x^-+y^-\|^2\Big|\\
&+\frac{1}{2}\Big|\|T(x^-)-T(y^-)\|^2-\gamma^2\|x^--y^-\|^2\Big|\\
&+\Big|\frac{i}{4}\Big(\|T(x)+iT(y)\|^2-\gamma^2\|x+iy\|^2\Big)\Big|\\
&+\Big|-\frac{i}{4}\Big(\|T(x)-iT(y)\|^2-\gamma^2\|x-iy\|^2\Big)\Big|\\
&+\Big|-\frac{i}{2}\Big(\|T(x^-)+iT(y^-)\|^2-\gamma^2\|x^-+iy^-\|^2\Big)\Big|\\
&+\Big|\frac{i}{2}\Big(\|T(x^-)-iT(y^-)\|^2-\gamma^2\|x^--iy^-\|^2\Big)\Big|.
\end{aligned}
$$

From inequality (2.9), we get

$$
\begin{aligned}
\Big|[T(x),T(y)]-\gamma^2[x,y]\Big| &\le \frac{1}{4}\gamma^2(\delta_1^2-1)(\|x+y\|^2+\|x-y\|^2)\\
&+\frac{1}{2}\gamma^2(\delta_1^2-1)(\|x^-+y^-\|^2+\|x^--y^-\|^2)\\
&+\frac{1}{4}\gamma^2(\delta_1^2-1)(\|x-iy\|^2+\|x+iy\|^2)\\
&+\frac{1}{2}\gamma^2(\delta_1^2-1)(\|x^--iy^-\|^2+\|x^-+iy^-\|^2)\\
&\le \gamma^2(\delta_1^2-1)\frac{1}{4}\Big(4\|x\|^2+4\|y\|^2\Big)+\frac{1}{2}\Big(4\|x^-\|^2+4\|y^-\|^2\Big)\\
&=\gamma^2(\delta_1^2-1)(\|x\|^2+\|y\|^2+2\|x^-\|^2+2\|y^-\|^2\\
&\le \gamma^2(\delta_1^2-1)(\|x\|^2+\|y\|^2+2\|x\|^2+2\|y\|^2)\\
&\le 3\gamma^2(\delta_1^2-1)(\|x\|^2+\|y\|^2). \qquad (2.12)
\end{aligned}
$$

Similarly, from inequality (2.11), we obtain

$$
\Big|[T(x),T(y)]-\gamma^2[x,y]\Big| \le 3(\delta_1^2-1)\Big(\|T(x)\|^2+\|T(y)\|^2\Big).
$$

Thus

$$
\begin{aligned}
\Big|[T(x),T(y)]-\gamma^2[x,y]\Big| &\le 3(\delta_1^2-1)\min\{\gamma^2\|x\|^2\\
&+\gamma^2\|y\|^2,\ \|T(x)\|^2+\|T(y)\|^2\}. \qquad (2.13)
\end{aligned}
$$

Now, suppose that x and y are two nonzero elements in $\mathcal{K}_1$. By applying (2.12) to vectors $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$, we get

$$\begin{aligned}\Big|[\frac{T(x)}{\|x\|},\frac{T(y)}{\|y\|}]-\gamma^2\Big[\frac{x}{\|x\|},\frac{y}{\|y\|}\Big]\Big| &\le 3(\delta_1^2-1)\Big(\gamma^2\|\frac{x}{\|x\|}\|^2+\gamma^2\frac{y}{\|y\|}\|^2\Big)\\ &\le 6\gamma^2(\delta_1^2-1).\end{aligned}$$

Hence

$$\Big|[T(x),T(y)]-\gamma^2[x,y]\Big|\le 6(\delta_1^2-1)\gamma^2\|x\|\|y\|.$$

Applying (2.13) to vectors $\frac{x}{\|T(x)\|}$ and $\frac{y}{\|T(y)\|}$, we arrive at

$$\Big|[T(x),T(y)]-\gamma^2[x,y]\Big|\le 6(\delta_1^2-1)\|T(x)\|\|T(y)\|. \tag{2.14}$$

Since $6(\delta_1^2-1)=\delta$, equation (2.2) follows for all $x,y\in\mathcal{K}_1$.

Conversely, if $T:\mathcal{K}_1\to\mathcal{K}_2$ satisfies (2.2) with some $\delta\ge 0$ and $\gamma>0$, then by Lemma 2.3, T is quasilinear.

If $x[\perp]y$, then $[x,y]=0$, and by (2.2), we reach $|[Tx,Ty]|\le\delta\|Tx\|\|Ty\|$. This means that

$Tx[\perp]^\delta Ty$. Thus T is an approximately orthogonality preserving map. □

Acknowledgements

The authors would like to thank the referees for their comments that helped us improve this article.

References

[1] Alsina, C., Sikorska, J., Santos Tomás, M.: Norm Derivatives and Characterizations of Inner Product Spaces. World Scientific Publishing Co. Pte. Ltd., Hackensack (2010)
[2] Amir, D.: Characterization of Inner Product Spaces. Birkhäuser Verlage, Basel-Boston-Stuttgart (1986)
[3] Ando, T.: Linear Operators on Krein space. Hokkaido University, Sappor (1979)
[4] Azizov, T.Ya., Iokhvidov, I.S.: Linear Operators in Spaces with an Indefinite Metric. Pure and Applied Mathematics, New York (1989)
[5] Badora, R., Chmieliński, J.: Decomposition of mappings approximately inner product preserving. Nonlinear Anal. **62**, 1015–1023 (2005)
[6] Blanco, A., Turnšek, A.: On maps that preserve orthogonality in normed spaces. Proc. Roy. Soc. Edinburgh Sect. A **136**(4), 709–716 (2006)
[7] Bognar, J.: Indefinite Inner Product Spaces. Springer, Berlin (1974)
[8] Chen, C., Lu, F.: Linear maps preserving orthogonality. Ann. Funct. Anal. **6**(4), 70–76 (2015)

[9] Chmieliński, J.: Linear mappings approximately preserving orthogonality. J. Math. Anal. Appl. **304**, 158–169 (2005)
[10] Chmieliński, J., Łukasik, R., Wójcik, P.: On the stability of the orthogonality equation and the orthogonality-preserving property with two unknown functions. Banach J. Math. Anal. **10**(4), 828–847 (2016)
[11] Chmieliński, J.: Stability of the orthogonality preserving property in finite-dimensional inner product spaces. J. Math. Anal. Appl. **318**, 433–443 (2006)
[12] Ilišević, D., Turnšek, A.: Approximately orthogonality preserving mappings on C^*-modules. J. Math. Anal. Appl. **341**, 298–308 (2008)
[13] Moslehian, M.S., Zamani, A.: Mappings preserving approximate orthogonality in Hilbert C^*-modules. Math. Scand. **122**(2), 257–276 (2018)
[14] Moslehian, M.S., Dehghani, M.: Operator convexity in Krein spaces. New York J. Math. **20**, 133–144 (2014)
[15] Pontrjagin, L.: Hermitian operators in spaces with indefinite metric. Izv. Akad. Nauk SSSR, Ser. Mat. **8**, 243–280 (1994)
[16] Saraei, A., Amyari, M.: Orthogonality preserving mappings in Krein spaces. J. Math. Anal. **3**(10), 112–122 (2019)
[17] Turnšek, A.: On mappings approximately preserving orthogonality. J. Math. Anal. Appl. **336**, 625–631 (2007)
[18] Zamani, A., Moslehian, M.S., Frank, M.: Angle preserving mappings. Z. Anal. Anwend. **34**(4), 485–500 (2015)

Ateyeh Saraei and Maryam Amyari
Department of Mathematics, Mashhad Branch
Islamic Azad University
Mashhad
Iran
e-mail: sarayee2010@yahoo.com

Maryam Amyari
e-mail: maryam_amyari@yahoo.com

Received: March 1, 2019
Revised: November 25, 2019