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An extension of Levin–Stečkin's theorem to uniformly convex and superquadratic functions

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Abstract. In this paper, some integral inequalities for uniformly convex functions are studied by using unordered submajorization for cumulative functions. Strongly convex functions and superquadratic functions are considered, too. A Levin–Stečkin like theorem is obtained for such functions. As applications, some bounds for the Fejér functional are derived. A result on the Schur-convexity of averages of convex functions is extended to uniformly convex functions. Some specifications for symmetric functions are also given. A corollary for symmetric probability density functions is established. A Levin–Stečkin type inequality for generalized ψ-uniformly convex functions is provided. Some interpretations for Simpson distributions are presented.

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1. Introduction and summary

Throughout $I \subset \mathbb{R}$ is an interval. For $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$ and $i = 1, 2, \dots, n$, the symbol $z_{[i]}$ stands for the ith largest entry of $\mathbf{z}$.

An n-tuple $\mathbf{y} = (y_1, y_2, \dots, y_n) \in I^n$ is said to be *weakly majorized* by an n-tuple $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$, written as $\mathbf{y} \prec_w \mathbf{x}$, if

$$\sum_{i=1}^{k} y_{[i]} \le \sum_{i=1}^{k} x_{[i]} \quad \text{for all } k = 1, 2, \dots, n \tag{1}$$

(see [9, p. 12]). If, in addition,

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i,$$

then $\mathbf{y}$ is said to be *majorized* by $\mathbf{x}$, written as $\mathbf{y} \prec \mathbf{x}$ (see [9, p. 8]).

A function $F : I^n \to \mathbb{R}$ is said to be *Schur-convex* (resp. *Schur-concave*) on I^n if

$$\mathbf{y} \prec \mathbf{x} \quad \text{implies} \quad F(\mathbf{y}) \le (\text{resp.} \ge) F(\mathbf{x}),$$

provided $\mathbf{x}, \mathbf{y} \in I^n$ (see [9, p. 80]).

Let $g_1, g_2 : [a, b] \to \mathbb{R}$ be two integrable real functions. The function g_2 is said to be *unordered submajorized* by g_1, written as $g_2 \prec_w^u g_1$, if

$$\int_a^s g_2(t)\, dt \le \int_a^s g_1(t)\, dt \quad \text{for } s \in [a, b].$$

If, moreover,

$$\int_a^b g_2(t)\, dt = \int_a^b g_1(t)\, dt,$$

then g_2 is said to be *(unordered) majorized* by g_1, written as $g_2 \prec^u g_1$ (see [4], cf. [9, p. 22]).

By a *cumulative* function induced by an integrable function $g : [a, b] \to \mathbb{R}$, we mean the integral function

$$G(s) = \int_a^s g(t)\, dt, \quad s \in [a, b]. \tag{2}$$

In what follows, we assume that there exist all integrals under consideration.

Elezović and Pečarić in [5] established the following result.

Theorem A. [5] *Let f be a continuous function on an interval I. Then the function*

$$F(x, y) = \begin{cases} \frac{1}{y-x} \int_x^y f(t)\, dt & \text{for } x, y \in I,\ x \ne y, \\ f(x) & \text{for } x = y \in I, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 iff f is convex (concave) on I.

It is well-known that if $f : I \to \mathbb{R}$ is a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$, then the following Hermite–Hadamard inequality holds:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a) + f(b)}{2} \tag{3}$$

(see [3, p. 137]).

A more general result is incorporated in the following [1,7,11].

Theorem B. [1] *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$, and let $p : [a, b] \to \mathbb{R}$ be a non-negative integrable weight on I. Assume that p is symmetric about $\frac{a+b}{2}$. Then the following Fejér inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)p(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{4}$$

Throughout, we denote by G_1 and G_2 the *cumulative functions* of g_1 and g_2 on $[a, b]$, respectively, in the sense that

$$G_1(s) = \int_a^s g_1(t)\,dt \text{ and } G_2(s) = \int_a^s g_2(t)\,dt \text{ for } s \in [a, b]. \tag{5}$$

Likewise, we denote by $\mathcal{G}_1$ and $\mathcal{G}_2$ the *cumulative functions* of G_1 and G_2 on $[a, b]$, respectively, that is

$$\mathcal{G}_1(s) = \int_a^s G_1(t)\,dt \text{ and } \mathcal{G}_2(s) = \int_a^s G_2(t)\,dt \text{ for } s \in [a, b]. \tag{6}$$

We now present the Levin–Stečkin theorem [8].

Theorem C. [8] *Let $g_1, g_2 : [a, b] \to \mathbb{R}$ be integrable functions and $G_1, G_2, \mathcal{G}_1$, $\mathcal{G}_2 : [a, b] \to \mathbb{R}$ defined by (5), (6) be functions satisfying the condition*

$$G_1(b) = G_2(b) \quad \text{and} \quad \mathcal{G}_1(b) = \mathcal{G}_2(b). \tag{7}$$

If

$$G_2 \prec_w^u G_1,$$

then

$$\int_a^b f(t)g_2(t)\,dt \leq \int_a^b f(t)g_1(t)\,dt \tag{8}$$

for all continuously twice differentiable convex functions $f : [a, b] \to \mathbb{R}$.

In this paper, we study integral inequalities of type (8) for uniformly convex functions, strongly convex functions and superquadratic functions. Our purpose is to establish some further results related to Theorems A, B and C. Similar problems for real convex functions f are well-known (see [8,12–14,16–19]).

The paper is arranged as follows. In Sect. 2, first we point out that for a given generalized uniformly convex function $f : [a, b] \to \mathbb{R}$, the unordered submajorization of cumulative functions G_1 and G_2 induced by g_1 and g_2, respectively, implies a refinement of inequality (8) (see Theorem 1).

Next, we provide some sufficient conditions under which the cumulative functions are unordered submajorized (see Lemma 2). In consequence, we are able to demonstrate sufficient conditions on two given functions g_1 and g_2 so that the refinement of inequality (8) holds (see Theorem 2). As an application, for uniformly convex functions we refine a result due to Elezović and Pečarić [5] (see Theorem A). This corresponds to the case of Theorem 1 when g_1 and g_2 represent two pdf's of uniform distribution.

In Sect. 3 we focus on symmetric functions. This leads to some simplifications of the results of Sect. 2. After giving some properties of cumulative functions (see Lemma 3), we interpret the previous results for symmetric functions (see Theorem 3). We establish a Levin–Stečkin type inequality with uniformly convex f. We also specify the obtained results for symmetric probability density functions (see Corollary 3). Finally, we show applications for Simpson distributions.

2. Results

Let $I = [a, b]$ be an interval and $\psi : [0, b - a] \to \mathbb{R}$ be a function. A function $f : [a, b] \to \mathbb{R}$ is said to be *generalized ψ-uniformly convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - t(1-t)\psi(|x-y|) \quad \text{for } x, y \in I \text{and } t \in [0,1] \tag{9}$$

(cf. [2]). If in addition $\psi \ge 0$, then f is said to be *ψ-uniformly convex* (see [15,20]).

Observe that the case $\psi = 0$ corresponds to usual convex functions. Moreover, a ψ-uniformly convex function f (so, $\psi \ge 0$) is necessarily convex. Conversely, if $\psi \le 0$, then a (usual) convex function f is generalized ψ-uniformly convex.

In general, if $\psi_1 \le \psi_2$, then generalized ψ_2-uniform convexity implies generalized ψ_1-uniform convexity.

We are now in a position to prove a Levin–Stečkin type theorem for generalized ψ-uniformly convex functions. Some simplifications of conditions (10) and (11) will be discussed after the end of the proof of Theorem 1. A similar approach for convex or n-convex functions can be found in [17–19].

Theorem 1. *Let $I = [a, b]$ be an interval and $\psi : [0, b-a] \to \mathbb{R}$ be a function. Let $f : [a, b] \to \mathbb{R}$ be a continuously twice differentiable generalized ψ-uniformly convex function on $[a, b]$. Denote $\varphi(t) = \frac{\psi(t)}{t^2}$ for $t \in (0, b-a]$ and $\varphi(0) = \lim_{t \to 0^+} \varphi(t)$.*

Let $g_1, g_2 : [a, b] \to \mathbb{R}$ be integrable functions and $G_1, G_2, \mathcal{G}_1, \mathcal{G}_2 : [a, b] \to \mathbb{R}$ defined by (5), (6) be functions satisfying the condition

$$f(b)[G_1(b) - G_2(b)] - f'(b)[\mathcal{G}_1(b) - \mathcal{G}_2(b)] \ge 0. \tag{10}$$

If

$$G_2 \prec_w^u G_1 \tag{11}$$

then

$$R + \int_a^b f(t)g_2(t)\,dt \le \int_a^b f(t)g_1(t)\,dt, \tag{12}$$

where $R = 2\varphi(0)\int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\,dt$. *In particular,* $R \ge 0$ *whenever* f *is a* ψ*-uniformly convex function on* $[a,b]$.

Proof. Inequality (11) means that

$$\int_a^s G_2(t)\,dt \le \int_a^s G_1(t)\,dt \quad \text{for } s \in [a,b]. \tag{13}$$

By using (6) and (13) we obtain

$$\mathcal{G}_2(t) \le \mathcal{G}_1(t) \quad \text{for } t \in [a,b]. \tag{14}$$

By integrating by parts twice [6, p. 129], we have (see (5) and (6))

$$\begin{aligned}\int_a^b f(t)[g_1(t) - g_2(t)]\,dt &= f(t)[G_1(t) - G_2(t)]\big|_a^b - \int_a^b f'(t)[G_1(t) - G_2(t)]\,dt \\ &= f(t)[G_1(t) - G_2(t)]\big|_a^b - f'(t)[\mathcal{G}_1(t) - \mathcal{G}_2(t)]\big|_a^b \\ &\quad + \int_a^b f''(t)[\mathcal{G}_1(t) - \mathcal{G}_2(t)]\,dt. \end{aligned}\tag{15}$$

It is easily seen from (5), (6) that

$$G_1(a) = G_2(a) = 0 \quad \text{and} \quad \mathcal{G}_1(a) = \mathcal{G}_2(a) = 0. \tag{16}$$

In consequence, by (16) and (10),

$$\begin{aligned}& f(t)[G_1(t) - G_2(t)]\big|_a^b - f'(t)[\mathcal{G}_1(t) - \mathcal{G}_2(t)]\big|_a^b \\ &\quad = f(b)[G_1(b) - G_2(b)] - f'(b)[\mathcal{G}_1(b) - \mathcal{G}_2(b)] \ge 0. \end{aligned}\tag{17}$$

It follows from (9) that

$$(f'(x) - f'(y))(x - y) \ge 2\psi(|x - y|) \quad \text{for } x, y \in I = [a,b]. \tag{18}$$

In fact, for $x, y \in I$ and $t \in [0,1]$, (9) gives

$$f(y + t(x - y)) - f(y) \le t(f(x) - f(y)) - t(1 - t)\psi(|x - y|) \tag{19}$$

and further for $t \in (0,1]$,

$$\frac{f(y + t(x - y)) - f(y)}{t} \le f(x) - f(y) - (1 - t)\psi(|x - y|). \tag{20}$$

Hence for $x, y \in I$, $x \neq y$,

$$\lim_{t \to 0^+} \frac{f(y + t(x-y)) - f(y)}{t(x-y)} (x-y) \leq \lim_{t \to 0^+} (f(x) - f(y) - (1-t)\psi(|x-y|)). \tag{21}$$

Therefore,

$$f'(y)(x-y) \leq f(x) - f(y) - \psi(|x-y|) \quad \text{for } x, y \in I, x \neq y. \tag{22}$$

For $x = y$ inequality (22) also holds, because $\psi(0) \leq 0$ is satisfied by (9).

By replacing the roles of x and y in (22), we get

$$f'(x)(y-x) \leq f(y) - f(x) - \psi(|x-y|) \quad \text{for } x, y \in I. \tag{23}$$

By multiplying both sides by -1, we obtain

$$f'(x)(x-y) \geq f(x) - f(y) + \psi(|x-y|) \quad \text{for } x, y \in I. \tag{24}$$

Now, subtracting inequalities (24) and(22) by sides yields (18), as claimed.

It holds that

$$f''(y) \geq 2\varphi(0) \quad \text{for } y \in I = [a, b]. \tag{25}$$

To see this, observe that (18) implies

$$\frac{f'(x) - f'(y)}{x - y} \geq 2\frac{\psi(x-y)}{(x-y)^2} = 2\varphi(x-y) \quad \text{for } x, y \in I, x > y, \tag{26}$$

because $\psi(x-y) = (x-y)^2\varphi(x-y)$.

Consequently,

$$f''(y) = \lim_{x \to y^+} \frac{f'(x) - f'(y)}{x - y} \geq 2 \lim_{x \to y^+} \varphi(x-y) = 2\varphi(0) \quad \text{for } y \in I, \tag{27}$$

which gives (25).

In conclusion, we get

$$\int_a^b f''(t)[\mathcal{G}_1(t) - \mathcal{G}_2(t)]\, dt \geq 2\varphi(0) \int_a^b [\mathcal{G}_1(t) - \mathcal{G}_2(t)]\, dt = R. \tag{28}$$

Therefore, by (15), (17) and (28), we deduce that

$$\int_a^b f(t)[g_1(t) - g_2(t)]\, dt \geq R.$$

In addition, $R \geq 0$ provided that f is ψ-uniformly convex, because $\mathcal{G}_1(t) - \mathcal{G}_2(t) \geq 0$ for $t \in [a, b]$ by (14), and $\psi \geq 0$ implies $\varphi \geq 0$.

This completes the proof of (12). □

Let $m \geq 0$ be a nonnegative number. A function $f : I = [a, b] \to \mathbb{R}$ is said to be *m-strongly convex* if it is ψ-uniformly convex for $\psi(t) = \frac{m}{2}t^2$, i.e.,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\frac{m}{2}(x-y)^2 \quad \text{for } x, y \in I \text{ and } t \in [0,1]. \tag{29}$$

Note that m-strongly convex functions with $m = 0$ are simply convex.

Corollary 1. *Under the hypothesis of Theorem 1, let $f : [a, b] \to \mathbb{R}$ be a continuously twice differentiable m-uniformly convex function on $[a, b]$ with $m \geq 0$. If conditions (10), (11) are fulfilled, then inequality (12) holds with $R = m\int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\, dt$.*

Proof. It is enough to use Theorem 1 with $\psi(t) = \frac{m}{2}t^2$ and $\varphi(t) = \frac{m}{2}$ for $t \in [0, b-a]$. □

Let $f : [0, b] \to \mathbb{R}$ be a differentiable function. The function f is said to be *superquadratic* on $[0, b]$ if

$$f(x) - f(y) \geq f'(y)(x-y) + f(|x-y|) \quad \text{for } x, y \in I = [0, b]. \tag{30}$$

Corollary 2. *Under the hypothesis of Theorem 1, let $f : [0, b] \to \mathbb{R}$ be a continuously twice differentiable superquadratic function on $[0, b]$. If conditions (10), (11) are fulfilled, then inequality (12) holds with $R = 2\varphi(0)\int_0^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\, dt$, and $\varphi(t) = \frac{f(t)}{t^2}$ for $t \in (0, b]$ and $\varphi(0) = \lim_{t\to 0^+} \varphi(t)$.*

Proof. Proceeding as in the proof of Theorem 1 with $a = 0$, $\psi(t) = f(t)$ for $t \in [0, b]$, and $\varphi(t) = \frac{f(t)}{t^2}$ for $t \in (0, b]$, we can see that the superquadracity of f on $[0, b]$ leads to the validity of inequality (12).

Indeed, property (30) guarantees that inequalities (22) and (24) are met with $\psi = f$, which implies (18) and (25) with $\psi = f$ and $\varphi(t) = \frac{f(t)}{t^2}$ for $t \in (0, b]$ and $\varphi(0) = \lim_{t\to 0^+} \varphi(t)$. Hence (28) is satisfied.

Finally, by compiling (15), (17) and (28) we get

$$\int_a^b f(t)[g_1(t) - g_2(t)]\, dt \geq R.$$

This completes the proof of (12) for a superquadratic function f. □

We now discuss sufficient conditions for majorization inequalities (11) and (13) to be valid.

The following lemma is based on a discrete result due to Marshall et al. (see [9, Proposition B.1., p. 186]). It is also inspired by Ohlin's Lemma [13], see also [14, Lemma 1].

Lemma 1. *Let $g_1, g_2 : [a,b] \to \mathbb{R}$ be integrable functions such that*

$$\int_a^b g_2(t)\,dt \le \int_a^b g_1(t)\,dt, \tag{31}$$

and, in addition, there exists $c \in [a,b]$ satisfying

$$g_2(t) \le g_1(t) \quad \text{for } t \in [a,c), \quad \text{and} \quad g_1(t) \le g_2(t) \quad \text{for } t \in [c,b]. \tag{32}$$

Then

$$\int_a^s g_2(t)\,dt \le \int_a^s g_1(t)\,dt \tag{33}$$

for $s \in [a,b]$.

Proof. It follows from the first inequality in (32) that (33) holds for $s \in [a,c)$.

Assume that $s \in [c,b]$. Due to (31) we can see that

$$\begin{aligned}\int_a^s g_2(t)\,dt &= \int_a^b g_2(t)\,dt - \int_s^b g_2(t)\,dt \le \int_a^b g_1(t)\,dt - \int_s^b g_2(t)\,dt \\ &\le \int_a^b g_1(t)\,dt - \int_s^b g_1(t)\,dt = \int_a^s g_1(t)\,dt,\end{aligned}$$

the last inequality being a consequence of the second inequality in (32).

Summarizing all of this, inequality (33) holds true for all $s \in [a,b]$. □

In the next lemma we utilize *interlaced* functions g_1 and g_2 (see (35), (36)). In consequence we obtain the required inequalities (11) and (13) for the corresponding cumulative functions G_1 and G_2 (see (38)).

Lemma 2. *Let $g_1, g_2 : [a,b] \to \mathbb{R}$ be integrable functions and $G_1, G_2 : [a,b] \to \mathbb{R}$ be functions defined by (5). Assume that there exists $c \in [a,b]$ satisfying*

$$\int_a^c g_2(t)\,dt = \int_a^c g_1(t)\,dt \quad \text{and} \quad \int_c^b g_1(t)\,dt = \int_c^b g_2(t)\,dt, \tag{34}$$

and, in addition, there exist $d_1 \in [a,c)$ and $d_2 \in [c,b]$ satisfying (a.e.)

$$g_2(t) \le g_1(t) \quad \text{for } t \in [a,d_1), \quad \text{and} \quad g_1(t) \le g_2(t) \quad \text{for } t \in [d_1,c], \tag{35}$$

$$g_1(t) \le g_2(t) \quad \text{for } t \in [c,d_2), \quad \text{and} \quad g_2(t) \le g_1(t) \quad \text{for } t \in [d_2,b]. \tag{36}$$

If

$$\int_a^b G_2(t)\,dt \le \int_a^b G_1(t)\,dt, \tag{37}$$

then

$$\int_a^s G_2(t)\,dt \le \int_a^s G_1(t)\,dt \quad \textit{for } s \in [a,b]. \tag{38}$$

Proof. We consider the restrictions of g_1 and g_2 to the interval $[a,c]$. In light of Lemma 1 applied to the interval $[a,c]$, by using (35) and the first part of (34), we find that

$$G_2(t) \le G_1(t) \quad \text{for } t \in [a,c], \tag{39}$$

with equality for $t = c$ (see (34)).

Likewise, consider the restrictions of g_1 and g_2 to the interval $[c,b]$. Denote

$$\widetilde{G}_1(t) = \int_c^t g_1(s)\,ds \quad \text{for } t \in [c,b], \quad \text{and} \quad \widetilde{G}_2(t) = \int_c^t g_2(s)\,ds \quad \text{for } t \in [c,b].$$

Hence

$$G_1(t) = G_1(c) + \widetilde{G}_1(t) \quad \text{and} \quad G_2(t) = G_2(c) + \widetilde{G}_2(t) \;\; \text{for } t \in [c,b]. \tag{40}$$

By making use of Lemma 1, applied to the interval $[c,b]$ via (36) and the second part of (34), we derive

$$\widetilde{G}_1(t) \le \widetilde{G}_2(t) \quad \text{for } t \in [c,b], \tag{41}$$

with equality for $t = b$ (see (34)).

By combining (40) and (41), with $G_1(c) = G_2(c)$ (see (34)), we obtain

$$G_1(t) \le G_2(t) \quad \text{for } t \in [c,b]. \tag{42}$$

According to Lemma 1 applied to the functions G_1 and G_2 on the interval $[a,b]$, properties (39), (42) and (37) imply (38), as desired. □

Remark 1. The conditions (35), (36) say that the pair (g_2, g_1) crosses two times (see [14, Definition 1]).

Remark 2. In Lemma 2, conditions (34), (35), (36) and (37) ensure that

$$g_2 \prec^u g_1 \quad \text{on } [a,c], \qquad g_1 \prec^u g_2 \quad \text{on } [c,b], \quad \text{and} \quad G_2 \prec^u_w G_1 \quad \text{on } [a,b].$$

Theorem 2. *Let $I = [a,b]$ be an interval and $\psi : [0, b-a] \to \mathbb{R}$ be a function. Let $f : [a,b] \to \mathbb{R}$ be a continuously twice differentiable generalized ψ-uniformly convex function on $[a,b]$. Denote $\varphi(t) = \frac{\psi(t)}{t^2}$ for $t \in (0, b-a]$ and $\varphi(0) = \lim\limits_{t\to 0^+} \varphi(t)$.*

Let $g_1, g_2 : [a,b] \to \mathbb{R}$ be integrable functions and $G_1, G_2, \mathcal{G}_1, \mathcal{G}_2 : [a,b] \to \mathbb{R}$ be functions defined by (5), (6). Assume that there exist $c \in [a,b]$, $d_1 \in [a,c)$ and $d_2 \in [c,b]$ satisfying conditions (34), (35), (36) and (37).

If

$$f'(b)[\mathcal{G}_1(b) - \mathcal{G}_2(b)] \le 0, \tag{43}$$

then

$$R + \int_a^b f(t)g_2(t)\,dt \le \int_a^b f(t)g_1(t)\,dt, \tag{44}$$

where $R = 2\varphi(0)\int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\,dt$. *In particular,* $R \ge 0$ *whenever* f *is a* ψ*-uniformly convex function on* $[a,b]$.

Proof. In light of (34) one has $G_1(b) = G_2(b)$, so $f(b)[G_1(b) - G_2(b)] = 0$. Therefore (10) reduces to (43).

Simultaneously, conditions (34), (35), (36) and (37) of Lemma 2 ensure that (38) is satisfied. Therefore (11) is fulfilled. Now, it is sufficient to apply Theorem 1 to get (44). □

2.1. Uniform distributions

In order to illustrate the above results, we now show how to use Theorem 2 to extend the sufficiency part of Theorem A [5] to uniformly convex functions.

Let $I = [a,b]$ be an interval, $\psi : [0, b-a] \to \mathbb{R}$ be a function, $\varphi(t) = \frac{\psi(t)}{t^2}$ for $t \in (0, b-a]$ and $\varphi(0) = \lim_{t\to 0^+} \varphi(t)$. Take $f : [a,b] \to \mathbb{R}$ to be a continuously twice differentiable generalized ψ-uniformly convex function on $[a,b]$.

Assume that $x_1, x_2, y_1, y_2 \in [a,b]$ such that $(x_2, y_2) \prec (x_1, y_1)$ and $a \le x_1 \le x_2 < \frac{a+b}{2} < y_2 \le y_1 \le b$, with $c = \frac{a+b}{2} = \frac{x_1+y_1}{2} = \frac{x_2+y_2}{2}$. Set

$$g_1(t) = \begin{cases} \frac{1}{y_1 - x_1} & \text{for } t \in [x_1, y_1], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g_2(t) = \begin{cases} \frac{1}{y_2 - x_2} & \text{for } t \in [x_2, y_2], \\ 0 & \text{otherwise.} \end{cases}$$

By putting $d_1 = x_2$ and $d_2 = y_2$, we see that conditions (35), (36) are satisfied. Furthermore, (34) holds in the form

$$\int_a^c g_2(t)\,dt = \int_a^c g_1(t)\,dt = \frac{1}{2} = \int_c^b g_1(t)\,dt = \int_c^b g_2(t)\,dt.$$

In this way, we have $G_1(b) = G_2(b)$. We also find by a straightforward calculation that

$$\mathcal{G}_1(b) = \int_a^b G_1(t)\,dt = \frac{1}{2}(b-a) \quad \text{and} \quad \mathcal{G}_2(b) = \int_a^b G_2(t)\,dt = \frac{1}{2}(b-a).$$

So, we infer that (37) is valid.

Since $\mathcal{G}_1(b)=\mathcal{G}_2(b)$, condition (43) is satisfied trivially. Taking Theorem 2 into consideration, we obtain (44) with the above g_1 and g_2, as follows:

$$R+\frac{1}{y_2-x_2}\int_{x_2}^{y_2} f(t)\,dt \le \frac{1}{y_1-x_1}\int_{x_1}^{y_1} f(t)\,dt, \tag{45}$$

where $R=2\varphi(0)\int_a^b(\mathcal{G}_1(t)-\mathcal{G}_2(t))\,dt$ (see (46)).

By direct computations, we find that

$$G_1(t)=\begin{cases}0 & \text{for } t\in[a,x_1)\\ \frac{t-x_1}{y_1-x_1} & \text{for } t\in[x_1,y_1]\\ 1 & \text{for } t\in(y_1,b]\end{cases} \quad\text{and}\quad G_2(t)=\begin{cases}0 & \text{for } t\in[a,x_2)\\ \frac{t-x_2}{y_2-x_2} & \text{for } t\in[x_2,y_2]\\ 1 & \text{for } t\in(y_2,b]\end{cases}.$$

Hence we derive

$$\mathcal{G}_1(u)=\begin{cases}0 & \text{for } u\in[a,x_1)\\ \frac{(u-x_1)^2}{2(y_1-x_1)} & \text{for } u\in[x_1,y_1]\\ u-\frac{x_1+y_1}{2} & \text{for } u\in(y_1,b]\end{cases} \text{ and } \mathcal{G}_2(u)=\begin{cases}0 & \text{for } u\in[a,x_2)\\ \frac{(u-x_2)^2}{2(y_2-x_2)} & \text{for } u\in[x_2,y_2]\\ u-\frac{x_2+y_2}{2} & \text{for } u\in(y_2,b]\end{cases}.$$

Therefore we have

$$\mathcal{G}_1(u)-\mathcal{G}_2(u)=\begin{cases}0 & \text{for } u\in[a,x_1)\\ \frac{(u-x_1)^2}{2(y_1-x_1)} & \text{for } u\in[x_1,x_2)\\ \frac{(u-x_1)^2}{2(y_1-x_1)}-\frac{(u-x_2)^2}{2(y_2-x_2)} & \text{for } u\in[x_2,y_2]\\ \frac{(u-x_1)^2}{2(y_1-x_1)}-u+\frac{x_2+y_2}{2} & \text{for } u\in[y_2,y_1]\\ 0 & \text{for } u\in(y_1,b]\end{cases}.$$

Because $x_1+y_1=x_2+y_2$, a bit of algebra gives

$$\int_a^b(\mathcal{G}_1(u)-\mathcal{G}_2(u))\,du=\frac{1}{6}\left[(y_1-y_2)(x_1+y_1)-(x_2-x_1)(x_1+x_2)\right].$$

So, we deduce from (45) that

$$\frac{1}{3}\varphi(0)\left[(y_1-y_2)(x_1+y_1)-(x_2-x_1)(x_1+x_2)\right]+\frac{1}{y_2-x_2}\int_{x_2}^{y_2} f(t)\,dt$$
$$\le\frac{1}{y_1-x_1}\int_{x_1}^{y_1} f(t)\,dt. \tag{46}$$

In particular, for an m-strongly convex function f we obtain the inequality

$$\frac{1}{6}m\left[(y_1-y_2)(x_1+y_1)-(x_2-x_1)(x_1+x_2)\right]+\frac{1}{y_2-x_2}\int_{x_2}^{y_2} f(t)\,dt$$
$$\le \frac{1}{y_1-x_1}\int_{x_1}^{y_1} f(t)\,dt.$$

Also, for a superquadratic function f inequality (46) holds valid with $\varphi(0)=\lim_{t\to 0^+}\frac{f(t)}{t^2}$. If, moreover, f is positive, then f must be convex, and in this case (46) refines the original inequality of Theorem A due to [5].

3. Applications for symmetric functions

We are interested in simplifying the assumptions of the results in the previous section. To this end we employ symmetric functions.

A function $g:[a,b]\to\mathbb{R}$ is said to *symmetric* about $c=\frac{a+b}{2}$ if

$$g(c-u)=g(c+u) \quad \text{for } u\in[0,\tfrac{b-a}{2}]. \tag{47}$$

Lemma 3. *Let $g:[a,b]\to\mathbb{R}$ be an integrable symmetric function about $c=\frac{a+b}{2}$, and $G:[a,b]\to\mathbb{R}$ be the cumulative function of g defined by (2).*

Then

(i) *G is rotational symmetric around the point $(c,G(c))$, i.e.,*

$$G(c)-G(c-u)=G(c+u)-G(c) \quad \textit{for } u\in\left[0,\frac{b-a}{2}\right], \tag{48}$$

(ii) *the following equality holds:*

$$\int_a^b G(t)\,dt=(b-a)G(c). \tag{49}$$

Proof. (i) Fix any $u\in[0,\frac{b-a}{2}]$. It is not hard to check that

$$G(c-u)=\int_a^{c-u} g(t)\,dt=\int_a^c g(t)\,dt+\int_c^{c-u} g(t)\,dt=G(c)-\int_0^u g(c-v)\,dv,$$
$$G(c+u)=\int_a^{c+u} g(t)\,dt=\int_a^c g(t)\,dt+\int_c^{c+u} g(t)\,dt=G(c)+\int_0^u g(c+v)\,dv.$$

Therefore, by (47), we derive

$$G(c)-G(c-u)=\int_0^u g(c-v)\,dv=\int_0^u g(c+v)\,dv=G(c+u)-G(c),$$

which proves (48).

(ii) It follows that

$$\int_a^c G(t)\,dt = \int_a^c G(c)\,dt - \left(\int_a^c (G(c) - G(t))\,dt\right)$$
$$= \int_a^c G(c)\,dt - P_1 = (c-a)G(c) - P_1, \tag{50}$$

and

$$\int_c^b G(t)\,dt = \int_c^b G(c)\,dt + \left(\int_c^b (G(t) - G(c))\,dt\right)$$
$$= \int_c^b G(c)\,dt + P_2 = (b-c)G(c) + P_2, \tag{51}$$

where

$$P_1 = \int_a^c (G(c) - G(t))\,dt = \int_0^{b-c} (G(c) - G(c-v))\,dv$$

and

$$P_2 = \int_c^b (G(t) - G(c))\,dt = \int_0^{b-c} (G(c+v) - G(c))\,dv.$$

In view of (48) we find that $P_1 = P_2$. Hence, by (50) and (51),

$$\int_a^b G(t)\,dt = \int_a^c G(t)\,dt + \int_c^b G(t)\,dt = (c-a+b-c)G(c) - P_1 + P_2 = (b-a)G(c).$$

Thus we see that (49) holds valid.

□

Theorem 3. (Symmetric functions.) *Let $I = [a,b]$ be an interval and $\psi : [0, b-a] \to \mathbb{R}$ be a function. Let $f : [a,b] \to \mathbb{R}$ be a continuously twice differentiable generalized ψ-uniformly convex function on $[a,b]$. Denote $\varphi(t) = \frac{\psi(t)}{t^2}$ for $t \in (0, b-a]$ and $\varphi(0) = \lim_{t\to 0^+} \varphi(t)$.*

Let $g_1, g_2 : [a,b] \to \mathbb{R}$ be integrable symmetric functions about $c = \frac{a+b}{2}$, and $G_1, G_2 : [a,b] \to \mathbb{R}$ be the cumulative functions of g_1 and g_2 defined by (5), respectively, and $\mathcal{G}_1, \mathcal{G}_2 : [a,b] \to \mathbb{R}$ be the cumulative functions of G_1 and G_2 defined by (6), respectively.

Assume that

$$G_2(c) = G_1(c) \tag{52}$$

and, in addition, there exists $d_2 \in [c, b]$ *satisfying (a.e.)*

$$g_1(t) \le g_2(t) \quad \text{for } t \in [c, d_2), \quad \text{and} \quad g_2(t) \le g_1(t) \quad \text{for } t \in [d_2, b]. \tag{53}$$

Then

$$R + \int_a^b f(t) g_2(t)\, dt \le \int_a^b f(t) g_1(t)\, dt, \tag{54}$$

where $R = 2\varphi(0) \int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\, dt$.

Proof. Because of (52), we have $G_1(b) = 2G_1(c) = 2G_2(c) = G_2(b)$. For symmetric functions conditions (34), (35), (36) are reduced to (52) and (53). To see (37), we apply $G_2(c) = G_1(c)$ via Lemma 3, part **(ii)**, and we derive

$$\mathcal{G}_2(b) = \int_a^b G_2(t)\, dt = (b-a)G_2(c) = (b-a)G_1(c) = \int_a^b G_1(t)\, dt = \mathcal{G}_1(b)$$

(see (6)). Moreover, condition (43) is fulfilled, too. We appeal now to Theorem 2 to get the desired result. □

A result for symmetric probability density functions is given as follows.

Corollary 3. (Symmetric p.d.f.) *Under the assumptions of Theorem 3 with deleted condition (52), let* $g_1, g_2 : [a, b] \to \mathbb{R}$ *be probability density functions symmetric about* $c = \frac{a+b}{2}$.

Then inequality (54) holds.

Proof. For symmetric p.d. functions g_1 and g_2, condition (52) holds, because

$$G_1(c) = \int_a^c g_1(t)\, dt = \frac{1}{2} = \int_a^c g_2(t)\, dt = G_2(c).$$

So, the result is true according to Theorem 3. □

3.1. Levin–Stečkin type inequalities for uniformly convex functions

We now demonstrate the use of Theorem 3 to derive a Levin–Stečkin type inequality with uniformly convex f.

Let $I = [a, b]$ be an interval and $\psi : [0, b-a] \to \mathbb{R}$ be a function. We denote $\varphi(t) = \frac{\psi(t)}{t^2}$ for $t \in (0, b-a]$ with $\varphi(0) = \lim_{t \to 0^+} \varphi(t)$.

Let $f : I \to \mathbb{R}$ be a continuously twice differentiable generalized ψ-uniformly convex function on I. Let $p : [a, b] \to \mathbb{R}$ be a non-negative integrable weight on I. Suppose that p is symmetric about $c = \frac{a+b}{2}$.

We also introduce

$$C = \frac{1}{b-a} \int_a^b p(t)\, dt \quad \text{for } t \in [a, b]. \tag{55}$$

In the case when there exists $d_2 \in [c, b]$ satisfying (a.e.)

$$C \le p(t) \quad \text{for } t \in [c, d_2), \quad \text{and} \quad p(t) \le C \quad \text{for } t \in [d_2, b], \tag{56}$$

we set

$$g_1(t) = C \qquad \text{and} \qquad g_2(t) = p(t) \quad \text{for } t \in [a, b]. \tag{57}$$

Thus (53) is fulfilled.

By referring to the symmetry of p about $c = \frac{a+b}{2}$ we can write $b-c = \frac{1}{2}(b-a)$ and

$$\int_a^c p(t)\, dt = \int_c^b p(t)\, dt = \frac{1}{2} \int_a^b p(t)\, dt.$$

From this, by (55) and (57),

$$g_1(t) = C = \frac{1}{b-c} \int_c^b p(t)\, dt = \frac{1}{b-c} \int_c^b g_2(t)\, dt \quad \text{for } t \in [a, b],$$

which easily leads to (52) as follows

$$\int_c^b g_1(t)\, dt = C(b-c) = \int_c^b g_2(t)\, dt.$$

To sum up, inequality (54) in Theorem 3 quarantees that

$$R + \int_a^b f(t)p(t)\, dt \le \frac{1}{b-a} \int_a^b f(t)\, dt \int_a^b p(t)\, dt, \tag{58}$$

which is a Levin–Stečkin type inequality for a generalized ψ-uniformly convex function f (cf. [10]). Here $R = 2\varphi(0) \int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\, dt$ (see below).

Additionally, we have

$$G_1(s) = \int_a^s g_1(u)\, du = \int_a^s C\, du = C(s-a) \quad \text{for } s \in [a, b].$$

Hence

$$\mathcal{G}_1(t) = \int_a^t G_1(s)\,ds = \int_a^t C(s-a)\,ds = C\frac{(t-a)^2}{2} \quad \text{for } t \in [a,b].$$

So, we infer that

$$R = 2\varphi(0)\int_a^b \left(C\frac{(t-a)^2}{2} - \mathcal{G}_2(t) \right) dt = 2\varphi(0)\left(C\frac{(b-a)^3}{6} - \int_a^b \mathcal{G}_2(t)\,dt \right).$$

On the other hand, in the case when there exists $d_2 \in [c,b]$ satisfying (a.e.)

$$p(t) \le C \quad \text{for } t \in [c,d_2), \quad \text{and} \quad C \le p(t) \quad \text{for } t \in [d_2,b], \tag{59}$$

we put

$$g_1(t) = p(t) \quad \text{and} \quad g_2(t) = C \quad \text{for } t \in [a,b]. \tag{60}$$

For this reason (53) is satisfied.

As previously, by the symmetry of p about $c = \frac{a+b}{2}$, and thanks to (55) and (60) we can write

$$g_2(t) = C = \frac{1}{b-c}\int_c^b p(t)\,dt = \frac{1}{b-c}\int_c^b g_1(t)\,dt \quad \text{for } t \in [a,b].$$

This forces (52), because

$$\int_c^b g_2(t)\,dt = C(b-c) = \int_c^b g_1(t)\,dt.$$

Finally, we deduce from inequality (54) in Theorem 3 that

$$R + \frac{1}{b-a}\int_a^b f(t)\,dt \int_a^b p(t)\,dt \le \int_a^b f(t)p(t)\,dt, \tag{61}$$

with $R = 2\varphi(0)\int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\,dt$ (see below). This is a Levin–Stečkin type inequality for a generalized ψ-uniformly convex function f (cf. [10]).

Furthermore,

$$G_2(s) = \int_a^s g_2(u)\,du = \int_a^s C\,du = C(s-a) \quad \text{for } s \in [a,b],$$

and

$$\mathcal{G}_2(t) = \int_a^t G_2(s)\,ds = \int_a^t C(s-a)\,ds = C\frac{(t-a)^2}{2} \quad \text{for } t \in [a,b].$$

Therefore, we conclude that

$$R = 2\varphi(0) \int_a^b \left(\mathcal{G}_1(t) - C\frac{(t-a)^2}{2} \right) dt = 2\varphi(0) \left(\int_a^b \mathcal{G}_1(t)\, dt - C\frac{(b-a)^3}{6} \right).$$

3.2. Simpson distributions

Recall that Theorem A corresponds to uniform distribution on an interval $[a,b]$. We shall establish a similar result corresponding to the Simpson (triangle) distribution on an interval $[a,b]$.

As usual, $f : [a,b] \to \mathbb{R}$ is a continuously twice differentiable generalized ψ-uniformly convex function, where $\psi : [0, b-a] \to \mathbb{R}$ is a function. Also, $\varphi(t) = \frac{\psi(t)}{t^2}$ for $t \in (0, b-a]$ with $\varphi(0) = \lim_{t \to 0^+} \varphi(t)$.

We put $c = \frac{a+b}{2}$ and take $x_1, x_2, y_1, y_2 \in [a,b]$ with $(x_2, y_2) \prec (x_1, y_1)$ and $a \leq x_1 < x_2 < c < y_2 < y_1 \leq b$.

We define g_1 and g_2 to be probability density functions of Simpson distributions on $[a,b]$ with triangles based on intervals $[x_1, y_1]$ and $[x_2, y_2]$, respectively. That is,

$$g_1(t) = \begin{cases} \frac{4(t-x_1)}{(y_1-x_1)^2} & \text{for } t \in [x_1, c], \\ \frac{4(y_1-t)}{(y_1-x_1)^2} & \text{for } |t \in [c, y_1], \\ 0 & \text{for } t \in [a, x_1] \cup [y_1, b], \end{cases}$$

$$\text{and} \quad g_2(t) = \begin{cases} \frac{4(t-x_2)}{(y_2-x_2)^2} & \text{for } t \in [x_2, c], \\ \frac{4(y_2-t)}{(y_2-x_2)^2} & \text{for } t \in [c, y_2], \\ 0 & \text{for } t \in [a, x_2] \cup [y_2, b]. \end{cases}$$

By setting $d_2 = \frac{\xi y_2 - y_1}{\xi - 1}$ with $\xi = \left(\frac{y_1 - x_1}{y_2 - x_2} \right)^2$, we see that condition (53) is satisfied. Taking Corollary 3 into account, we can rewrite (54) as

$$R + \frac{4}{(y_2-x_2)^2} \left(\int_{x_2}^{c} f(t)(t-x_2)\, dt + \int_{c}^{y_2} f(t)(y_2-t)\, dt \right)$$
$$\leq \frac{4}{(y_1-x_1)^2} \left(\int_{x_1}^{c} f(t)(t-x_1)\, dt + \int_{c}^{y_1} f(t)(y_1-t)\, dt \right),$$

where $R = 2\varphi(0) \int_a^b (\mathcal{G}_1(t) - \mathcal{G}_2(t))\, dt$.

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