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Aequationes Mathematicae

An inequality for the length of isoptic chords of convex bodies

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Abstract. In this note we prove the following result: for every $\alpha \in (0, \pi)$ and for a given convex body K in the plane, with minimal width w, there exists a chord $[x, y]$ with length larger than or equal to $w \cos \frac{\alpha}{2}$ such that there are support lines of K through x and y which form an angle α. Moreover, if there is no such chord with length exceeding $w \cos \frac{\alpha}{2}$, then K is a Euclidean disc.

Mathematics Subject Classification. 52A10, 52A40.

Keywords. Isoptic chords, Isoptic curves, Euclidean disc.

1. Introduction

Let B denote a Euclidean circle with radius $w/2$ in the plane and let B' be a circle concentric to B with radius $r > w/2$. It is easy to see that B is seen under a constant angle α from every point $p \in B'$. Moreover, if x and y are the points of contact with the tangent lines to B from p, then the length of the segment $[x, y]$ is $\|x - y\| = w \cos \frac{\alpha}{2}$ (Fig. 1).

Now, consider a convex body K in the plane, i.e., a compact and convex set with non-empty interior. We say that a line ℓ is a support line of K if $\ell \cap K \neq \emptyset$ and K is contained in one of the half-planes bounded by ℓ. Given a unit vector $u \in \mathbb{S}^1$, we denote by $w(u)$ the width of K in direction u, i.e., the distance between the two support lines of K orthogonal to u. The minimum of $w(u)$, with $u \in \mathbb{S}^1$, is called the width of K. It is natural to ask the following question: Is there a chord $[x, y]$ of K such that the support lines through x and y intersects each other at an angle α and $\|x - y\| \geq w \cos \frac{\alpha}{2}$?

The first result for this problem was given by Green in [3], who proved that in every convex body K, with minimum width w, there is at least one chord $[x, y]$ with length at least $\frac{w}{\sqrt{2}}$ and such that the support lines of K through x and y intersect at an angle $\pi/2$. Moreover, he also proved that if for a convex body K there is no such chord with length strictly larger than $\frac{w}{\sqrt{2}}$, then it is a

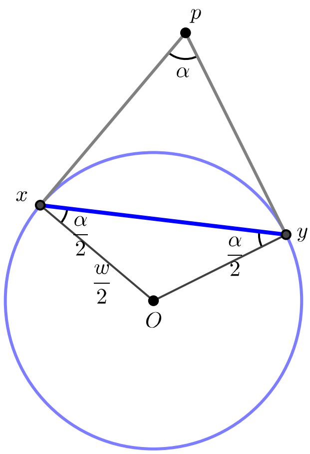

Figure 1. $[x, y]$ chord in the Euclidean circle B

Euclidean disc. The analogous case for the angle $\pi/3$ was recently proved in [8]. Let $[x, y]$ be a chord of a convex body K such that there are lines supporting K at x and y and intersecting at an angle α. For convenience, we will call every such chord an *α-chord*. There are some more results about α-chords, for instance, Green [4] gave inequalities between the length of the α-chords and the width or the diameter of K. Also, Santaló [7] has studied the problem and proved more inequalities relating to some other aspects of K, such as the minimum of the radius of curvature of ∂K or the length of ∂K.

The main result in this note is the following which was conjectured in [8].

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a convex body of minimal width w and let $\alpha \in (0, \pi)$ be a fixed angle. Then there exists an α-chord with length larger than or equal to*

$$w \cos \frac{\alpha}{2}.$$

Moreover, if there is not an α-chord with length exceeding $w \cos \frac{\alpha}{2}$, then K is a Euclidean disc.

We also give a proof of the following result.

Theorem 2. *Let $K \subset \mathbb{R}^2$ be a strictly convex body with differentiable boundary ∂K, and let $\alpha \in (0, \pi)$ be a fixed angle. Then there exists an α-chord $[x, y]$ of K such that the length of the arc $\widehat{xy}$ is larger than or equal to*

$$\frac{\pi - \alpha}{2\pi} L(K),$$

where $L(K)$ denotes the perimeter of K.

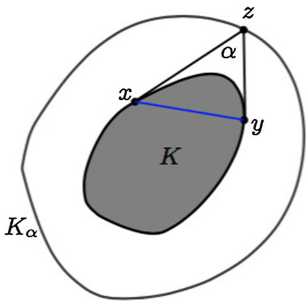

FIGURE 2. K_α is the isoptic of K

2. Auxiliary Lemmas

The two main results of this article are related to a special kind of curves associated to any given convex body, the so called *isoptic curves* (see for instance [2]). For a given number $\alpha \in (0, \pi)$ and a given convex body K, the *isoptic of angle* α or α-*isoptic*, denoted by K_α, is defined as the set of points in the plane from which K is seen under the constant angle α. With this notion of isoptic curves an α-chord is nothing more than a chord $[x, y] \subset K$ such that there is a point $z \in K_\alpha$ such that the lines zx and zy are support lines of K (see Fig. 2).

The isoptic of a given convex set is not necessarily a closed curve, for instance, if K is the convex hull of a parabola and $\alpha = \pi/2$, then its isoptic is a straight line (the directrix of the parabola). However, if K is a convex body, then its isoptic (for any angle) is a closed and periodic curve with period 2π (see [2]). Moreover, if K is a strictly convex body (without segments in its boundary) then K_α is of class C^2, i.e., with non vanishing curvature (also proved in [2]).

For every $t \in [0, 2\pi]$ denote by $\ell(t)$ the support line of K with outward normal vector $u(t) = (\cos t, \sin t)$, and let $p(t)$ denote the distance with sign from the origin O to $\ell(t)$. Using the support function, ∂K can be parameterized by (Fig. 3)

$$\gamma(t) = p(t)u(t) + p'(t)u'(t), \text{ for } t \in [0, 2\pi]. \tag{1}$$

The perimeter and area of K can be computed in terms of $p(t)$ by the formulas of Cauchy and Blaschke (see for instance [9]), respectively, by

$$L(\gamma) = \int_0^{2\pi} p(t)dt, \tag{2}$$

and

$$A(\gamma) = \frac{1}{2}\int_0^{2\pi} [p(t)^2 - p'(t)^2]dt. \tag{3}$$

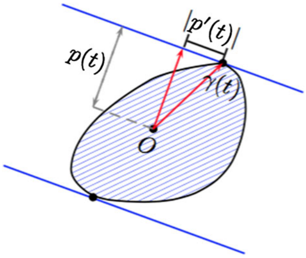

FIGURE 3. Parameterization of a convex curve using the support function

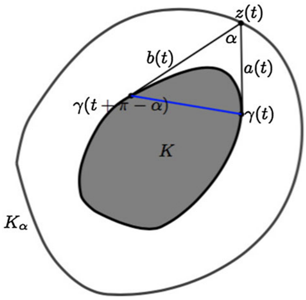

FIGURE 4. $a(t) = \frac{1}{\sin\alpha}\left[p(t+\pi-\alpha) + p(t)\cos\alpha - p'(t)\sin\alpha\right]$

Moreover, for a strictly convex body K the length of the arc $\widehat{\gamma(t_0)\gamma(t_1)}$ can be calculated by

$$\int_{t_0}^{t_1} [p(t) + p''(t)]dt. \tag{4}$$

Suppose the support line of K, $\ell(t)$, intersects K_α at the point $z(t)$ (as shown in Fig. 2). The other support line of K through $z(t)$ intersects ∂K at the point $\gamma(t+\pi-\alpha)$. Denote $a(t) = \|z(t)-\gamma(t)\|$ and $b(t) = \|z(t)-\gamma(t+\pi-\alpha)\|$. By a standard calculation (see for instance [2]) we have that (Fig. 4)

$$a(t) = \frac{1}{\sin\alpha}\left[p(t+\pi-\alpha) + p(t)\cos\alpha - p'(t)\sin\alpha\right], \tag{5}$$

$$b(t) = \frac{1}{\sin\alpha}\left[p(t+\pi-\alpha)\cos\alpha + p'(t+\pi-\alpha)\sin\alpha + p(t)\right]. \tag{6}$$

Lemma 1. *The mean value of $c(t) = a(t) + b(t)$ is equal to*

$$\overline{c_\alpha} = \frac{1+\cos\alpha}{\pi\sin\alpha} L(K).$$

Proof. From (5) and (6) we have that

$$c(t) = \frac{1}{\sin\alpha}\left[(p(t) + p(t+\pi-\alpha))\,(1+\cos\alpha) + (p'(t+\pi-\alpha) - p'(t))\sin\alpha\right].$$

From this we have that

$$\overline{c_\alpha} = \frac{1+\cos\alpha}{2\pi\sin\alpha}\int_0^{2\pi}[\,p(t) + p(t+\pi-\alpha)]\,dt + \frac{1}{2\pi}\int_0^{2\pi}[\,p'(t+\pi-\alpha) - p'(t)]\,dt.$$

Since $p(t)$ is a periodic function with period 2π, we have that the second integral is equal to 0. By Cauchy's formula we have that

$$\overline{c_\alpha} = \frac{1+\cos\alpha}{2\pi\sin\alpha}(2L(K)),$$

hence $\overline{c_\alpha} = \frac{1+\cos\alpha}{\pi\sin\alpha}L(K)$. □

Lemma 2. *Let $\triangle abc$ be a triangle such that $|ab| + |ac| = \lambda$, for a constant value λ, and such that $\measuredangle bac = \alpha$. Then $|bc| \geq \lambda\sin\frac{\alpha}{2}$, and it is minimal when $|ab| = |ac|$.*

Proof. For a proof the reader may consult the marvelous book by Niven [6, Problem C4]. □

Proof of Theorem 1. From Cauchy's formula for the perimeter we have that $L(K) \geq \pi w$. Now, by Lemma 1 we have that $\overline{c_\alpha} \geq \left(\frac{1+\cos\alpha}{\sin\alpha}\right)w$. Set $\lambda = \left(\frac{1+\cos\alpha}{\sin\alpha}\right)w$ and let $t_0 \in [0, 2\pi]$ such that $c(t_0) \geq \lambda$. By Lemma 2 we have that the minimum of $|\gamma(t_0)\gamma(t_0+\pi-\alpha)|$ is

$$\lambda\sin\frac{\alpha}{2} = w\cos\frac{\alpha}{2}.$$

Now, if there is no $t \in [0, 2\pi]$ such that $|\gamma(t)\gamma(t+\pi-\alpha)| > w\cos\frac{\alpha}{2}$ then $a(t) = b(t)$ and $|\gamma(t)\gamma(t+\pi-\alpha)| = w\cos\frac{\alpha}{2}$, for all $t \in [0, 2\pi]$. Since all the α-chords have the same length, we have that there exists a circle with center $\zeta(t)$ and radius $r(t)$ which shares tangent lines with the curve γ at the points $\gamma(t)$, $\gamma(t+\pi-\alpha)$, and $\gamma(t+2\pi-2\alpha)$. We then have that (Fig. 5)

$$\gamma(t) = \zeta(t) + r(t)u(t),$$

and since $\langle\gamma'(t), u(t)\rangle = 0$, we obtain that

$$\langle\zeta'(t) + r'(t)u(t) + r(t)u'(t), u(t)\rangle = \langle\zeta'(t), u(t)\rangle + r'(t) = 0. \tag{7}$$

Since $r(t+\pi-\alpha) = r(t+2\pi-2\alpha) = r(t)$, and $\zeta(t+\pi-\alpha) = \zeta(t+2\pi-2\alpha) = \zeta(t)$, for every $t \in [0, 2\pi]$, we also have that

$$\langle\zeta'(t), u(t+\pi-\alpha)\rangle + r'(t) = 0, \tag{8}$$

and

$$\langle\zeta'(t), u(t+2\pi-2\alpha)\rangle + r'(t) = 0. \tag{9}$$

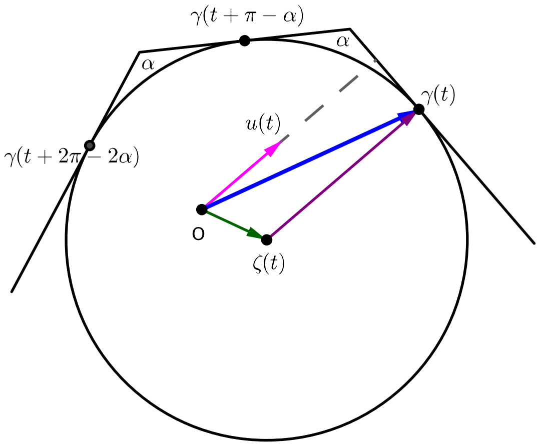

Figure 5. A circle sharing tangent lines with K

From Eqs. (7), (8), and (9) we have that

$$\langle \zeta'(t), u(t) - u(t+\pi-\alpha)\rangle = 0$$

and

$$\langle \zeta'(t), u(t) - u(t+2\pi-2\alpha)\rangle = 0.$$

Now, since the vectors $u(t)-u(t+\pi-\alpha)$ and $u(t)-u(t+2\pi-2\alpha)$ are linearly independent we have $\zeta'(t)=0$, i.e., $\zeta(t)=\zeta$ for a constant vector ζ. Finally, using that $\zeta'(t)=0$ for every $t\in[0,2\pi]$ in Eq. (7) we obtain that $r'(t)=0$. Since the centers $\zeta(t)$ and the radii $r(t)$ are constant, we conclude that γ is a circle. □

We wonder if there are convex bodies, besides the disc, such that the length of every α-chord, for some $\alpha\in(0,\pi)$, is constant. Obviuosly, the length of such α-chords must be larger than $w\cos\frac{\alpha}{2}$, where w is the minimum width. We don't know the answer to this question yet, however, if we consider non convex curves then there are some examples of curves with this property. For instance: all the chords of the cardioid $r=a(1+\cos\theta)$, through its pole, have length equal to $2a$, which is equal to $\frac{8}{9}w$, and the tangents to the cardioid at the endpoints of such chords intersect at an angle $\pi/2$ (Fig. 6).

On the other side, Santaló proved in [7] that the length of every α-chord of a C^2 convex body K is larger than or equal to $2r_0\cos\frac{\alpha}{2}$, where r_0 is the minimum radius of curvature of ∂K. He also proved that if the length of every α-chord is equal to $2r_0\cos\frac{\alpha}{2}$, then K is a Euclidean disc. It is worth mentioning that Santalo's result does not imply our Theorem 1 for the following reasons: for convex bodies of constant width w, the sum of the radii of curvature at every pair of opposite points (two points are opposite if there are parallel

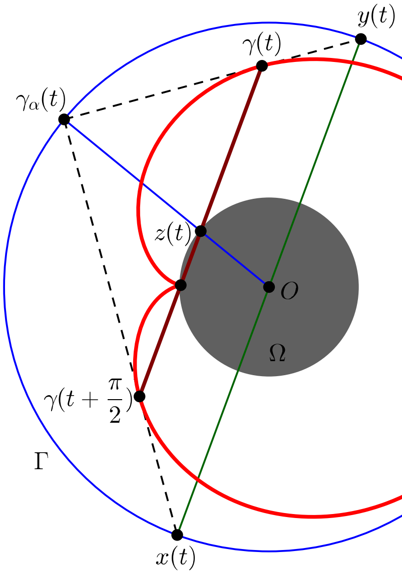

FIGURE 6. A curve with a $\pi/2$-chord of constant length

supporting lines through them) is equal to w (see for instance [1,9]), which implies that $r_0 < w/2$ and hence Santalo's result does not imply the existence of an α-chord with length larger than or equal to $w\cos\frac{\alpha}{2}$.

Proof of Theorem 2. Suppose ∂K is parameterized by its support function $p(t)$, for $t \in [0, 2\pi]$. According to (4) the length of the arc from $\gamma(t)$ to $\gamma(t+\pi-\alpha)$, denoted $L(t)$, is

$$L(t) = \int_t^{t+\pi-\alpha} [\, p(\theta) + p''(\theta)\,]\, d\theta,$$

for every $\theta \in [0, 2\pi)$. The mean value of $L(t)$ is

$$\overline{L_\alpha} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\int_t^{t+\pi-\alpha} [\, p(\theta) + p''(\theta)\,]\, d\theta\right\} dt.$$

We apply Fubini's theorem and then (Fig. 7)

$$\overline{L_\alpha} = \frac{1}{2\pi}\int_0^{\pi-\alpha}\left\{\int_0^{\theta}[p(\theta)+p''(\theta)]\,dt\right\}d\theta + \frac{1}{2\pi}\int_{\pi-\alpha}^{2\pi}\left\{\int_{\theta-\pi+\alpha}^{\theta}[p(\theta)+p''(\theta)]\,dt\right\}d\theta$$

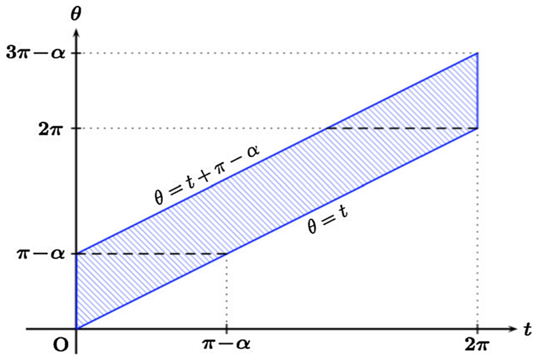

FIGURE 7. Integral region

$$+\frac{1}{2\pi}\int\limits_{2\pi}^{3\pi-\alpha}\left\{\int\limits_{\theta-\pi+\alpha}^{2\pi}[\,p(\theta)+p''(\theta)]\,dt\right\}d\theta$$

$$=\frac{1}{2\pi}\int\limits_{0}^{\pi-\alpha}\theta\cdot[\,p(\theta)+p''(\theta)]\,d\theta+\frac{1}{2\pi}\int\limits_{\pi-\alpha}^{2\pi}(\pi-\alpha)\,[\,p(\theta)+p''(\theta)]\,d\theta$$

$$+\frac{1}{2\pi}\int\limits_{2\pi}^{3\pi-\alpha}(3\pi-\alpha-\theta)\,[\,p(\theta)+p''(\theta)]\,d\theta$$

$$=\frac{1}{2\pi}\int\limits_{0}^{\pi-\alpha}\theta\cdot[\,p(\theta)+p''(\theta)]\,d\theta+\frac{1}{2\pi}\int\limits_{\pi-\alpha}^{2\pi}(\pi-\alpha)\,[\,p(\theta)+p''(\theta)]\,d\theta$$

$$+\frac{1}{2\pi}\int\limits_{0}^{\pi-\alpha}(\pi-\alpha-\theta)\,[\,p(\theta)+p''(\theta)]\,d\theta$$

$$=\frac{1}{2\pi}\int\limits_{0}^{2\pi}(\pi-\alpha)\,[\,p(\theta)+p''(\theta)]\,d\theta$$

$$=\frac{\pi-\alpha}{2\pi}\,L(K).$$

Therefore, there exists an α-arc of K such that its length is larger than or equal to $\frac{\pi-\alpha}{2\pi}L(K)$. □

Corollary 1. *Let $K \subset \mathbb{R}^2$ be a convex body of minimal width w and let $\alpha \in (0, \pi)$ be a fixed angle. Then there exists an α-arc with length larger than or equal to*

$$\left(\frac{\pi - \alpha}{2}\right) w.$$

Remark 1. That the length of all the α-arcs is constant is not enough to characterize the Euclidean disc, as was proved by Nakajima [5]: Let K be a convex body in the plane, let $n \geq 2$ be an integer, and $\alpha \in [0, \pi)$ such that all the α-arcs of K have the same length. If α is an irrational multiple of π, then K is an Euclidean disc. If α is a rational multiple of π and if α is represented in the form $\alpha = \left(1 - \frac{2m}{n}\right)\pi$, with m and n relatively prime, then K is strictly convex and has n-fold rotational symmetry. Conversely, if K is strictly convex and n-fold rotationally symmetric, then all the α-arcs, for $\alpha = \left(1 - \frac{2m}{n}\right)\pi$, have the same length.

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