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Aequationes Mathematicae

Derivations and Leibniz differences on rings

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Abstract. Let R be a commutative ring and n a positive integer. We show that the composition of n derivations of order 1 results in a derivation of order n on R. If in addition R is an integral domain of characteristic 0, then the composition of n nontrivial derivations of order 1 forms a nontrivial derivation of order n. This is also true for integral domains of characteristic larger than $n!$, but not for integral domains of characteristic $n!$, nor for commutative rings (even of characteristic 0) which are not integral domains. We prove our results by the use of Leibniz differences. One corollary is that nontrivial derivations of all orders exist on $\mathbb{R}$.

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1. Introduction

The research in this paper was motivated by the question of whether nontrivial derivations of all orders exist on $\mathbb{R}$. The affirmative answer to this question is a consequence of more general results presented herein. We will prove that if $d_1, \ldots, d_n$ are nontrivial derivations (of order 1) on a commutative ring R, then the composite function $d_1 \circ \cdots \circ d_n$ is a nontrivial derivation of order n provided that R is an integral domain of characteristic 0 or of characteristic greater than $n!$.

Given a commutative ring S with subring R, a *derivation* of R into S is a function $d : R \to S$ which satisfies *additivity* and the *product rule* (or *Leibniz rule*), respectively

$$d(x + y) = d(x) + d(y) \qquad \text{and} \qquad d(xy) = xd(y) + d(x)y,$$

for all $x, y \in R$. A mapping $B : R\times R \to S$ is a *bi-derivation* if B is a derivation in each variable when the other variable is fixed. Examples of bi-derivations are functions such as $(x, y) \mapsto \sum_i \phi_i(x)\psi_i(y)$ where each ϕ_i and ψ_i is a derivation.

We define derivations of other orders inductively as follows.

Definition 1.1. Let S be a commutative ring with subring R. The zero function is the only *derivation of order 0* on R into S. For each $n \in \mathbb{N}$, suppose we have defined derivations of order $n-1$. If $f : R \to S$ is additive, then f is said to be a *derivation of order* n if there exists a function $B : R \times R \to S$ which is a derivation of order $n-1$ in each variable such that

$$f(xy) - xf(y) - f(x)y = B(x, y), \qquad x, y \in R. \tag{1}$$

We will refer to such a function B as a *bi-derivation of order* $n-1$. We use $\mathcal{D}_n(R, S)$ to denote the set of all derivations of order n on R into S. In case $R = S$ we define $\mathcal{D}_n(R) := \mathcal{D}_n(R, R)$.

It is evident that the definition of derivation of order 1 agrees with the previous definition of derivation.

We say that a derivation d of order 1 is *nontrivial* if $d \neq 0$, that is if $d \in \mathcal{D}_1(R, S) \setminus \mathcal{D}_0(R, S)$. Similarly, a derivation of order n is *nontrivial* if it is not a derivation of order $n-1$. So the existence of a nontrivial derivation of order n on R into S is equivalent to the statement $\mathcal{D}_n(R, S) \setminus \mathcal{D}_{n-1}(R, S) \neq \emptyset$.

Throughout this paper we work only with rings that are commutative. An integral domain is a nontrivial ($R \neq \{0\}$) commutative ring such that $x, y \in R \setminus \{0\}$ implies $xy \neq 0$.

Our main results are the following. In Proposition 3.1(ii) we prove that if $d_1, \ldots, d_n \in \mathcal{D}_1(R)$, then $d_1 \circ \cdots \circ d_n \in \mathcal{D}_n(R)$. In Theorem 3.6 we show that if R is an integral domain of characteristic larger than $n!$ (including characteristic 0) and $d_1, \ldots, d_n \in \mathcal{D}_1(R) \setminus \mathcal{D}_0(R)$, then $d_1 \circ \cdots \circ d_n \in \mathcal{D}_n(R) \setminus \mathcal{D}_{n-1}(R)$.

The organization of the paper is as follows. We begin with some notation and preliminaries in the next section, including the introduction of Leibniz differences. Section 3 contains the main results mentioned above. Then we conclude with a short section consisting of two examples demonstrating the sharpness of our results. Specifically, we provide counterexamples to the conclusion of Theorem 3.6 in case R either has characteristic $n!$ or is not an integral domain.

2. Notation and preliminaries

Using methods developed in the author's paper [1], it can be shown that our main results hold for the special case $d_1 = \cdots = d_n$. That is, if d^n denotes the n-th iterate of a function $d : R \to R$, then d^n is a derivation of order n for any derivation d of order 1; furthermore, d^n is nontrivial if $d \neq 0$ and R is an integral domain of characteristic 0. We do not give separate proofs for these statements since they will follow from our main results.

Our primary tool is the Leibniz difference operator, which plays a key role in these investigations. It seems to be an important tool for the study of derivations of higher order.

Definition 2.1. Let S be a commutative ring with subring R. For any function $f : R \to S$ and any $x \in R$ we define the function $\Lambda_x f : R \to S$, called the *Leibniz difference (of order 1) of f with increment x*, by

$$\Lambda_x f(y) := f(xy) - xf(y) - f(x)y \quad \text{for all} \quad y \in R.$$

We call Λ_x a *Leibniz difference operator of order 1* on S^R. We also define *Leibniz difference operators of order n* on S^R by

$$\Lambda_{y_1,\ldots,y_n} := \Lambda_{y_n} \circ \cdots \circ \Lambda_{y_1} \quad \text{for all} \quad y_1, \ldots, y_n \in R,$$

and for any $f \in S^R$ we say $\Lambda_{y_1,\ldots,y_n} f$ is a *Leibniz difference of order n of f*.

Note that if f is a derivation (of any order), then the function $(x, y) \mapsto \Lambda_x f(y)$ is identical to the function B appearing in (1).

Now we show that the expressions defined above are completely symmetric functions of all the variables. For each positive integer n we use the notation S_n to denote the *symmetric group* of all permutations on the set $\{1, \ldots, n\}$.

Lemma 2.2. *Let S be a commutative ring with subring R, let $n \in \mathbb{N}$ and let $f : R \to S$. For any $y_1, \ldots, y_{n+1} \in R$ and any permutation $\pi \in S_{n+1}$ we have*

$$\Lambda_{y_1,\ldots,y_n} f(y_{n+1}) = \Lambda_{\pi(y_1),\ldots,\pi(y_n)} f(\pi(y_{n+1})).$$

Proof. It is clear from the definition that this is true for $n = 1$. For larger values of n the statement follows from a calculation of the explicit form of the left hand side. It is not difficult to show by a simple inductive argument that

$$\begin{aligned}\Lambda_{y_1,\ldots,y_n} f(y_{n+1}) = & f(y_1 y_2 \cdots y_{n+1}) - \sum_{i=1}^{n+1} y_i f(y_1 \cdots \widehat{y_i} \cdots y_{n+1}) \\ & + \sum_{1 \le i < j \le n+1} y_i y_j f(y_1 \cdots \widehat{y_i} \cdots \widehat{y_j} \cdots y_{n+1}) \\ & + \ldots + (-1)^n \sum_{i=1}^{n+1} y_1 \cdots \widehat{y_i} \cdots y_{n+1} f(y_i),\end{aligned}$$

where we use the hat symbol $\widehat{\ }$ over a variable to indicate that that variable is to be omitted. □

Now suppose $f : R \to S$ is an additive function. Then $f \in \mathcal{D}_n(R, S)$ if and only if $\Lambda_x f \in \mathcal{D}_{n-1}(R, S)$ for each $x \in R$. By taking further Leibniz differences we have the following.

Proposition 2.3. *Let S be a commutative ring with subring R, let $f : R \to S$ be additive, let $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$. Then $f \in \mathcal{D}_n(R, S)$ if and only if $\Lambda_{y_1,\dots,y_j} f \in \mathcal{D}_{n-j}(R, S)$ for all $y_1, \dots, y_j \in R$. In particular, $f \in \mathcal{D}_n(R, S)$ if and only if all Leibniz differences of f of order n vanish.*

From this follows the nesting property

$$\{0\} = \mathcal{D}_0(R, S) \subseteq \mathcal{D}_1(R, S) \subseteq \cdots \subseteq \mathcal{D}_n(R, S) \subseteq \cdots$$

for spaces of derivations on commutative rings. Indeed, if $f \in \mathcal{D}_n(R, S)$ then $\Lambda_{y_1,\dots,y_n} f = 0$ for any $y_1, \dots, y_n \in R$, so any further Leibniz differences of f also vanish. Thus $f \in \mathcal{D}_m(R, S)$ for all $m > n$.

3. General compositions of derivations

We show now that the composition of n derivations of order 1 forms a derivation of order n.

Proposition 3.1. *Let R be a commutative ring, and let $n \in \mathbb{N}$.*

(i) If $h \in \mathcal{D}_n(R)$ and $d \in \mathcal{D}_1(R)$, then

$$\Lambda_x(h \circ d)(y) = h(x)d(y) + d(x)h(y) + (\Lambda_x h)(d(y)) + (\Lambda_y h)(d(x)) \qquad (2)$$

for all $x, y \in R$. It follows that $h \circ d \in \mathcal{D}_{n+1}(R)$.

(ii) If $d_1, \dots, d_n \in \mathcal{D}_1(R)$, then $d_1 \circ \cdots \circ d_n \in \mathcal{D}_n(R)$.

Proof. We prove statement (i) by induction. For $n = 1$ we assume that $h, d \in \mathcal{D}_1(R)$, then

$$\begin{aligned}
\Lambda_x(h \circ d)(y) &= (h \circ d)(xy) - x(h \circ d)(y) - (h \circ d)(x)y \\
&= h(xd(y) + d(x)y) - xh(d(y)) - h(d(x))y \\
&= h(x)d(y) + xh(d(y)) + h(d(x))y + d(x)h(y) \\
&\quad - xh(d(y)) - h(d(x))y \\
&= h(x)d(y) + d(x)h(y),
\end{aligned}$$

for all $x, y \in R$. Since $\Lambda_x h = \Lambda_y h = 0$ for $h \in \mathcal{D}_1(R)$, this proves (2) for $n = 1$. Furthermore, the right hand side is a bi-derivation of order 1, so $h \circ d \in \mathcal{D}_2(R)$.

Now suppose statement (i) is true for some $n \geq 1$, and let $h \in \mathcal{D}_{n+1}(R)$ and $d \in \mathcal{D}_1(R)$. Starting out as before we compute

$$\begin{aligned}
\Lambda_x(h \circ d)(y) &= h(xd(y)) + h(d(x)y) - xh(d(y)) - h(d(x))y \\
&= [h(xd(y)) - xh(d(y)) - h(x)d(y)] + h(x)d(y) \\
&\quad + [h(d(x)y) - h(d(x))y - d(x)h(y)] + d(x)h(y) \\
&= \Lambda_x h(d(y)) + h(x)d(y) + \Lambda_y h(d(x)) + d(x)h(y),
\end{aligned} \qquad (3)$$

which is (2) again for n augmented by 1. Now consider the terms on the right hand side of (3) as functions of y. For each fixed x in R we see that

$h(x)d \in \mathcal{D}_1(R)$, $d(x)h \in \mathcal{D}_{n+1}(R)$, and $y \mapsto \Lambda_y h(d(x)) = \Lambda_{d(x)} h(y) \in \mathcal{D}_n(R)$. The term $(\Lambda_x h) \circ d$ is the composition of the derivation $\Lambda_x h$ of order n with the derivation d of order 1, hence it is a derivation of order $n+1$ by the inductive hypothesis. Since each $\mathcal{D}_j(R)$ is a linear space and $\mathcal{D}_1(R) \subseteq \mathcal{D}_n(R) \subseteq \mathcal{D}_{n+1}(R)$, we find that $\Lambda_x(h \circ d) \in \mathcal{D}_{n+1}(R)$ for each $x \in R$. Thus $h \circ d \in \mathcal{D}_{n+2}(R)$ and part (i) is proved.

Part (ii) follows directly from part (i) by a simple inductive argument. □

What remains is to answer the question of whether the composition of nontrivial derivations produces nontrivial derivations of higher orders. In order to answer this question, we need to understand more about the interaction between Leibniz differences and compositions of derivations. This will be accomplished through a series of steps, beginning with the following reduction formula.

Lemma 3.2. *Let R be a commutative ring, let $n \in \mathbb{N}$ with $n \geq 2$, and suppose $d_1, \ldots, d_{n+1} \in \mathcal{D}_1(R)$. Let $z, y_1, \ldots, y_n \in R$ and define $Y := \{y_1, \ldots, y_n\}$. Then for each $k \in \{1, \ldots, n-1\}$ we have*

$$\begin{aligned} &\Lambda_{y_{k+1},\ldots,y_n}[\Lambda_{y_1,\ldots,y_k}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) \\ &\quad = \Lambda_{y_{k+2},\ldots,y_n} d_{n+1}(z) \Lambda_{y_1,\ldots,y_k}(d_1 \circ \cdots \circ d_n)(y_{k+1}) \\ &\qquad + d_{n+1}(y_{k+1}) \Lambda_{Y \setminus \{y_{k+1}\}}(d_1 \circ \cdots \circ d_n)(z) \\ &\qquad + \Lambda_{y_{k+2},\ldots,y_n}[\Lambda_{y_1,\ldots,y_{k+1}}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z). \end{aligned} \tag{4}$$

An empty operator (such as $\Lambda_{y_{k+2},\ldots,y_n}$ when $k = n-1$) is to be interpreted as the identity operator.

Proof. The proof proceeds by induction on k, and we shall use Proposition 3.1 (i) repeatedly. For $k = 1$ we take $h := \Lambda_{y_1}(d_1 \circ \cdots \circ d_n)$, $d := d_{n+1}$, and $x := y_2$. Then Eq. (2) yields

$$\begin{aligned} &\Lambda_{y_2}[\Lambda_{y_1}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) \\ &\quad = d_{n+1}(z)\Lambda_{y_1}(d_1 \circ \cdots \circ d_n)(y_2) + d_{n+1}(y_2)\Lambda_{y_1}(d_1 \circ \cdots \circ d_n)(z) \\ &\qquad + \Lambda_{y_1,y_2}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}(z) + \Lambda_{z,y_1}(d_1 \circ \cdots \circ d_n)(d_{n+1}(y_2)). \end{aligned}$$

If $n = 2$ then this equation reduces to

$$\Lambda_{y_2}[\Lambda_{y_1}(d_1 \circ d_2) \circ d_3](z) = d_3(z)\Lambda_{y_1}(d_1 \circ d_2)(y_2) + d_3(y_2)\Lambda_{y_1}(d_1 \circ d_2)(z),$$

since $\Lambda_{u,v}(d_1 \circ d_2) = 0$ for any $u, v \in R$, and this agrees with (4).

If $n \geq 3$ then we have

$$\begin{aligned} &\Lambda_{y_2,\ldots,y_n}[\Lambda_{y_1}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) = \Lambda_{y_3,\ldots,y_n}\{\Lambda_{y_2}[\Lambda_{y_1}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}]\}(z) \\ &\quad = \Lambda_{y_3,\ldots,y_n} d_{n+1}(z)\Lambda_{y_1}(d_1 \circ \cdots \circ d_n)(y_2) + d_{n+1}(y_2)\Lambda_{y_3,\ldots,y_n}\Lambda_{y_1}(d_1 \circ \cdots \circ d_n)(z) \\ &\qquad + \Lambda_{y_3,\ldots,y_n}[\Lambda_{y_1,y_2}(d_1 \circ \cdots \circ d_n) \circ d_{n+1})](z) \end{aligned}$$

$$+\Lambda_{y_3,\dots,y_n}\Lambda_{z,y_1}(d_1\circ\cdots\circ d_n)(d_{n+1}(y_2))$$
$$= d_{n+1}(y_2)\Lambda_{y_1,y_3,\dots,y_n}(d_1\circ\cdots\circ d_n)(z)+\Lambda_{y_3,\dots,y_n}[\Lambda_{y_1,y_2}(d_1\circ\cdots\circ d_n)\circ d_{n+1})](z),$$

since $d_1\circ\cdots\circ d_n\in\mathcal{D}_n(R)$ is annihilated by the operator $\Lambda_{z,y_1,y_3,\dots,y_n}$ of order n, and d_{n+1} is annihilated by the operator $\Lambda_{y_3,\dots,y_n}$ of order at least 1. This again agrees with (4), which is now established for $k=1$.

Now suppose (4) is valid for some $k\in\{1,\dots,n-2\}$, and consider the expression $\Lambda_{y_{k+2},\dots,y_n}[\Lambda_{y_1,\dots,y_{k+1}}(d_1\circ\cdots\circ d_n)\circ d_{n+1}]$. Notice that this can be considered only if $n\geq 3$. We apply Proposition 3.1 (i) with $h:=\Lambda_{y_1,\dots,y_{k+1}}(d_1\circ\cdots\circ d_n)$, $d:=d_{n+1}$, and $x:=y_{k+2}$, obtaining

$$\begin{aligned}
&\Lambda_{y_{k+2},\dots,y_n}[\Lambda_{y_1,\dots,y_{k+1}}(d_1\circ\cdots\circ d_n)\circ d_{n+1}](z)\\
&\quad=\Lambda_{y_{k+3},\dots,y_n}\{\Lambda_{y_{k+2}}[\Lambda_{y_1,\dots,y_{k+1}}(d_1\circ\cdots\circ d_n)\circ d_{n+1}]\}(z)\\
&\quad=\Lambda_{y_{k+3},\dots,y_n}d_{n+1}(z)\Lambda_{y_1,\dots,y_{k+1}}(d_1\circ\cdots\circ d_n)(y_{k+2})\\
&\qquad+d_{n+1}(y_{k+2})\Lambda_{Y\setminus\{y_{k+2}\}}(d_1\circ\cdots\circ d_n)(z)\\
&\qquad+\Lambda_{y_{k+3},\dots,y_n}[\Lambda_{y_1,\dots,y_{k+2}}(d_1\circ\cdots\circ d_n)\circ d_{n+1}](z)\\
&\qquad+\Lambda_{\{z\}\cup Y\setminus\{y_{k+2}\}}(d_1\circ\cdots\circ d_n)(d_{n+1}(y_{k+2}))\\
&\quad=\Lambda_{y_{k+3},\dots,y_n}d_{n+1}(z)\Lambda_{y_1,\dots,y_{k+1}}(d_1\circ\cdots\circ d_n)(y_{k+2})\\
&\qquad+d_{n+1}(y_{k+2})\Lambda_{Y\setminus\{y_{k+2}\}}(d_1\circ\cdots\circ d_n)(z)\\
&\qquad+\Lambda_{y_{k+3},\dots,y_n}[\Lambda_{y_1,\dots,y_{k+2}}(d_1\circ\cdots\circ d_n)\circ d_{n+1}](z),
\end{aligned}$$

since $d_1\circ\cdots\circ d_n\in\mathcal{D}_n(R)$ is annihilated by the Leibniz difference operator $\Lambda_{\{z\}\cup Y\setminus\{y_{k+2}\}}$ of order n. This is exactly Eq. (4) for k increased to $k+1$ and that completes the proof. □

The next result describes what happens when we apply Lemma 3.2 successively for $k=1$ through $k=n-1$.

Lemma 3.3. *Let R be a commutative ring, let $n\in\mathbb{N}$ with $n\geq 2$, and let $d_1,\dots,d_{n+1}\in\mathcal{D}_1(R)$. Suppose $z,y_1,\dots,y_n\in R$ and let $Y=\{y_1,\dots,y_n\}$. Then we have*

$$\begin{aligned}
&\Lambda_{y_2,\dots,y_n}[\Lambda_{y_1}(d_1\circ\cdots\circ d_n)\circ d_{n+1}](z)\\
&\qquad=\sum_{i=2}^{n}d_{n+1}(y_i)\Lambda_{Y\setminus\{y_i\}}(d_1\circ\cdots\circ d_n)(z)+d_{n+1}(z)\Lambda_{Y\setminus\{y_n\}}(d_1\circ\cdots\circ d_n)(y_n).
\end{aligned}\tag{5}$$

Proof. We start by proving (5). As observed above, for $n=2$ Eq. (4) with $k=1$ gives

$$\Lambda_{y_2}[\Lambda_{y_1}(d_1\circ d_2)\circ d_3](z)=d_3(z)\Lambda_{y_1}(d_1\circ d_2)(y_2)+d_3(y_2)\Lambda_{y_1}(d_1\circ d_2)(z),$$

confirming (5). Now suppose $n \geq 3$. Then successive applications of Lemma 3.2 for $k = 1, 2, \ldots, n-1$ yield

$$\begin{aligned}
&\Lambda_{y_2,\ldots,y_n}[\Lambda_{y_1}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) \\
&\quad = \Lambda_{y_3,\ldots,y_n} d_{n+1}(z)\Lambda_{y_1}(d_1 \circ \cdots \circ d_n)(y_2) + d_{n+1}(y_2)\Lambda_{Y\setminus\{y_2\}}(d_1 \circ \cdots \circ d_n)(z) \\
&\qquad + \Lambda_{y_3,\ldots,y_n}[\Lambda_{y_1,y_2}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) \\
&\quad = d_{n+1}(y_2)\Lambda_{Y\setminus\{y_2\}}(d_1 \circ \cdots \circ d_n)(z) + \Lambda_{y_4,\ldots,y_n} d_{n+1}(z)\Lambda_{y_1,y_2}(d_1 \circ \cdots \circ d_n)(y_3) \\
&\qquad + d_{n+1}(y_3)\Lambda_{Y\setminus\{y_3\}}(d_1 \circ \cdots \circ d_n)(z) + \Lambda_{y_4,\ldots,y_n}[\Lambda_{y_1,\ldots,y_3}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) \\
&\quad = \ldots \\
&\quad = \sum_{i=2}^{n-1} d_{n+1}(y_i)\Lambda_{Y\setminus\{y_i\}}(d_1 \circ \cdots \circ d_n)(z) + \Lambda_{y_n}[\Lambda_{y_1,\ldots,y_{n-1}}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}](z) \\
&\quad = \sum_{i=2}^{n-1} d_{n+1}(y_i)\Lambda_{Y\setminus\{y_i\}}(d_1 \circ \cdots \circ d_n)(z) + d_{n+1}(z)\Lambda_{Y\setminus\{y_n\}}(d_1 \circ \cdots \circ d_n)(y_n) \\
&\qquad + d_{n+1}(y_n)\Lambda_{Y\setminus\{y_n\}}(d_1 \circ \cdots \circ d_n)(z) + \Lambda_{y_1,\ldots,y_n}(d_1 \circ \cdots \circ d_n) \circ d_{n+1}(z).
\end{aligned}$$

Since the last term vanishes, this proves the statement. □

The following result provides another key to answering our main question about nontrivial derivations.

Proposition 3.4. *Let R be a commutative ring, let $m \in \mathbb{N}$ with $m \geq 2$, and suppose $d_1, \ldots, d_m \in \mathcal{D}_1(R)$. Then*

$$\Lambda_{y_1,\ldots,y_{m-1}}(d_1 \circ \cdots \circ d_m)(y_m) = \sum_{\pi \in S_m} d_{\pi(1)}(y_1) \cdots d_{\pi(m)}(y_m), \tag{6}$$

for all $y_1, \ldots, y_m \in R$.

Proof. The proof goes by induction on m. For $m = 2$ the statement is

$$\Lambda_{y_1}(d_1 \circ d_2)(y_2) = d_1(y_1)d_2(y_2) + d_2(y_1)d_1(y_2)$$

for all $y_1, y_2 \in R$, which is just the case $n = 1$ of Proposition 3.1 (i).

Now suppose Eq. (6) holds for some $m \geq 2$, and let $Y = \{y_1, \ldots, y_m\}$. By Proposition 3.1 (i) with $h := d_1 \circ \cdots \circ d_m$, $d := d_{m+1}$, and $x := y_1$, we know that

$$\begin{aligned}
&\Lambda_{y_1}(d_1 \circ \cdots \circ d_m \circ d_{m+1})(z) \\
&\quad = d_{m+1}(z)(d_1 \circ \cdots \circ d_m)(y_1) + d_{m+1}(y_1)(d_1 \circ \cdots \circ d_m)(z) \\
&\qquad + \Lambda_{y_1}(d_1 \circ \cdots \circ d_m) \circ d_{m+1}(z) + \Lambda_z(d_1 \circ \cdots \circ d_m)(d_{m+1}(y_1)).
\end{aligned}$$

Using this formula and Eq. (5) with n replaced by m, together with the inductive hypothesis, we calculate that

$$\begin{aligned}
&\Lambda_{y_1,\ldots,y_m}(d_1 \circ \cdots \circ d_{m+1})(z) = \Lambda_{y_2,\ldots,y_m}[\Lambda_{y_1}(d_1 \circ \cdots \circ d_m \circ d_{m+1})](z) \\
&\quad = \Lambda_{y_2,\ldots,y_m}\{d_{m+1}(\cdot)(d_1 \circ \cdots \circ d_m)(y_1) + d_{m+1}(y_1)(d_1 \circ \cdots \circ d_m)(\cdot)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Lambda_{y_1}(d_1 \circ \cdots \circ d_m) \circ d_{m+1}(\cdot) \\
&\quad + \Lambda_{(\cdot)}(d_1 \circ \cdots \circ d_m)(d_{m+1}(y_1))]\}(z) \\
&= \Lambda_{y_2,\ldots,y_m} d_{m+1}(z)(d_1 \circ \cdots \circ d_m)(y_1) + d_{m+1}(y_1)\Lambda_{y_2,\ldots,y_m}(d_1 \circ \cdots \circ d_m)(z) \\
&\quad + \Lambda_{y_2,\ldots,y_m}[\Lambda_{y_1}(d_1 \circ \cdots \circ d_m) \circ d_{m+1}](z) + \Lambda_{z,y_2,\ldots,y_m}(d_1 \circ \cdots \circ d_m)(d_{m+1}(y_1)) \\
&= d_{m+1}(y_1)\Lambda_{y_2,\ldots,y_m}(d_1 \circ \cdots \circ d_m)(z) + \sum_{i=2}^{m} d_{m+1}(y_i)\Lambda_{Y\setminus\{y_i\}}(d_1 \circ \cdots \circ d_m)(z) \\
&\quad + d_{m+1}(z)\Lambda_{Y\setminus\{y_m\}}(d_1 \circ \cdots \circ d_m)(y_m) \\
&= \sum_{i=1}^{m} d_{m+1}(y_i) \sum_{\pi \in S_m} d_{\pi(1)}(y_1) \cdots \widehat{d_{\pi(i)}(y_i)} \cdots d_{\pi(m)}(y_m) d_{\pi(i)}(z) \\
&\quad + d_{m+1}(z) \sum_{\pi \in S_m} d_{\pi(1)}(y_1) \cdots d_{\pi(m)}(y_m),
\end{aligned}
$$

where in the fourth step we utilized

$$\Lambda_{y_2,\ldots,y_m} d_{m+1} = \Lambda_{z,y_2,\ldots,y_m}(d_1 \circ \cdots \circ d_m) = 0.$$

Evaluating at $z = y_{m+1}$ we get

$$\Lambda_{y_1,\ldots,y_m}(d_1 \circ \cdots \circ d_{m+1})(y_{m+1}) = \sum_{\pi \in S_{m+1}} d_{\pi(1)}(y_1) \cdots d_{\pi(m+1)}(y_{m+1}),$$

which is Eq. (6) with m augmented by 1. This completes the proof. □

Here we single out one useful benefit of the preceding result: It shows that the function $d_i \mapsto \Lambda_{y_1,\ldots,y_{m-1}}(d_1 \circ \cdots \circ d_m)$ is linear for each $i \in \{1,\ldots,n\}$. Additivity is obvious, but it is not necessarily true that $(d_1 \circ \cdots \circ d_{j-1} \circ kd_j \circ d_{j+1} \circ \cdots \circ d_m) = k(d_1 \circ \cdots \circ d_m)$ for a given element $k \in R$. Nevertheless it is true that

$$
\begin{aligned}
&\Lambda_{y_1,\ldots,y_{m-1}}(d_1 \circ \cdots \circ d_{j-1} \circ kd_j \circ d_{j+1} \circ \cdots \circ d_m)(y_m) = k \sum_{\pi \in S_m} d_{\pi(1)}(y_1) \cdots d_{\pi(m)}(y_m) \\
&\quad = k\Lambda_{y_1,\ldots,y_{m-1}}(d_1 \circ \cdots \circ d_m)(y_m) \qquad (7)
\end{aligned}
$$

for any $k \in R$ and any $j \in \{1,\ldots,m\}$. We will use this linearity in the proof of Theorem 3.6 below.

Before proving our main theorem, we prove the following special result. Now additional conditions must be imposed on the ring R. We use the notation $char(R)$ for the characteristic of R.

Proposition 3.5. *Let $m \in \mathbb{N}$ and let R be an integral domain with $char(R) > m!$ or $char(R) = 0$. If $d \in \mathcal{D}_1(R) \setminus \mathcal{D}_0(R)$ and $k_1,\ldots,k_m \in R \setminus \{0\}$, then $(k_1 d) \circ (k_2 d) \circ \cdots \circ (k_m d) \in \mathcal{D}_m(R) \setminus \mathcal{D}_{m-1}(R)$.*

Proof. The case $m = 1$ is obviously true. For $m \geq 2$ the proof is by contradiction. Suppose $(k_1 d) \circ (k_2 d) \circ \cdots \circ (k_m d) \in \mathcal{D}_{m-1}(R)$. Then all Leibniz differences of $(k_1d)\circ(k_2d)\circ\cdots\circ(k_md)$ of order $m-1$ vanish. Hence by Proposition 3.4 we have

$$0=\sum_{\pi\in S_m}k_{\pi(1)}d(y_1)\cdots k_{\pi(m)}d(y_m),$$

for all $y_1,\ldots,y_m\in R$. In particular, for $y_1=\cdots=y_m=x$ we get

$$0=m!k_1\cdots k_m[d(x)]^m,\quad x\in R,$$

which is impossible according to our hypotheses. □

We are now ready to state and prove our main theorem. The case $char(R)=0$ of the following result has been obtained recently by G. Kiss and M. Laczkovich [2] using different methods.

Theorem 3.6. *Let $m\in\mathbb{N}$ and let R be an integral domain with $char(R)>m!$ or $char(R)=0$. If $d_1,\ldots,d_m\in\mathcal{D}_1(R)\setminus\mathcal{D}_0(R)$, then $d_1\circ\cdots\circ d_m\in\mathcal{D}_m(R)\setminus\mathcal{D}_{m-1}(R)$.*

Proof. The proof is by induction on m. The statement is trivially true for $m=1$. Now suppose it is true for some $m\geq 1$ and let $d_1,\ldots,d_{m+1}\in\mathcal{D}_1(R)\setminus\mathcal{D}_0(R)$, where R is an integral domain with $char(R)>(m+1)!$ or $char(R)=0$.

For a contradiction, suppose $d_1\circ\cdots\circ d_{m+1}\in\mathcal{D}_m(R)$. Then any Leibniz difference of order m of $d_1\circ\cdots\circ d_{m+1}$ vanishes, so by Proposition 3.4 we have for all $y_1,\ldots,y_{m+1}\in R$ that

$$\begin{aligned}0&=\sum_{\pi\in S_{m+1}}d_1(y_{\pi(1)})\cdots d_{m+1}(y_{\pi(m+1)})\\&=\sum_{k=1}^{m+1}d_k(y_{m+1})\sum_{\pi\in S_{m+1},\pi(k)=m+1}d_1(y_{\pi(1)})\cdots\widehat{d_k(y_{\pi(k)})}\cdots d_{m+1}(y_{\pi(m+1)})\\&=\sum_{k=1}^{m+1}d_k(y_{m+1})\Lambda_{y_1,\ldots,y_{m-1}}(d_1\circ\cdots\circ\widehat{d_k}\circ\cdots\circ d_{m+1})(y_m).\end{aligned}\tag{8}$$

If we assume that $\{d_1,\ldots,d_{m+1}\}$ is linearly independent, then in the preceding equation the coefficient of $d_k(y_{m+1})$ for each $1\leq k\leq m+1$ must be 0. In particular for $k=m+1$ we would get

$$\Lambda_{y_1,\ldots,y_{m-1}}(d_1\circ\cdots\circ d_m)(y_m)=0$$

for all $y_1,\ldots,y_m\in R$. This would entail $d_1\circ\cdots\circ d_m\in\mathcal{D}_{m-1}(R)$, contradicting the induction hypothesis.

Therefore $\{d_1,\ldots,d_{m+1}\}$ must be linearly dependent. Relabeling if necessary, we can write

$$d_{m+1}=\sum_{j=1}^{m}c_jd_j$$

for some constants $c_1, \dots, c_m \in R$ with $c_m \neq 0$. Using this in (8) we find that

$$\begin{aligned}0 &= d_{m+1}(y_{m+1})\Lambda_{y_1,\dots,y_{m-1}}(d_1 \circ \cdots \circ d_m)(y_m)\\ &\quad + \sum_{k=1}^{m} d_k(y_{m+1})\Lambda_{y_1,\dots,y_{m-1}}(d_1 \circ \cdots \circ \widehat{d_k} \circ \cdots \circ d_{m+1})(y_m)\\ &= \sum_{j=1}^{m} c_j d_j(y_{m+1})\Lambda_{y_1,\dots,y_{m-1}}(d_1 \circ \cdots \circ d_m)(y_m)\\ &\quad + \sum_{k=1}^{m} d_k(y_{m+1})\Lambda_{y_1,\dots,y_{m-1}}\left(d_1 \circ \cdots \circ \widehat{d_k} \circ \cdots \circ d_m \circ \left(\sum_{j=1}^{m} c_j d_j\right)\right)(y_m).\end{aligned} \tag{9}$$

If $m = 1$ this equation states that

$$0 = c_1 d_1(y_2) d_1(y_1) + d_1(y_2) c_1 d_1(y_1) = 2c_1 d_1(y_1) d_1(y_2),$$

which is impossible by our hypotheses. Thus we have verified the inductive step if $m = 1$.

Now suppose that $m \geq 2$ and $\{d_1, \dots, d_m\}$ is linearly independent. Then in Eq. (9) the coefficient of $d_j(y_{m+1})$ for each $1 \leq j \leq m$ must be 0. In particular for $j = m$ we have

$$\begin{aligned}0 &= c_m \Lambda_{y_1,\dots,y_{m-1}}(d_1 \circ \cdots \circ d_m)(y_m)\\ &\quad + \Lambda_{y_1,\dots,y_{m-1}}\left(d_1 \circ \cdots \circ d_{m-1} \circ \left(\sum_{j=1}^{m} c_j d_j\right)\right)(y_m)\\ &= \Lambda_{y_1,\dots,y_{m-1}}\left(d_1 \circ \cdots \circ d_{m-1} \circ \left(2c_m d_m + \sum_{j=1}^{m-1} c_j d_j\right)\right)(y_m),\end{aligned}$$

where we have used the linearity property (7). From this we see that

$$d_1 \circ \cdots \circ d_{m-1} \circ \left(2c_m d_m + \sum_{j=1}^{m-1} c_j d_j\right) \in \mathcal{D}_{m-1}(R).$$

By the inductive hypothesis, this can be true only if

$$2c_m d_m + \sum_{j=1}^{m-1} c_j d_j = 0,$$

and since $2c_m \neq 0$ this contradicts the hypothesis that $\{d_1, \dots, d_m\}$ is linearly independent.

Therefore $\{d_1, \ldots, d_m\}$ must be linearly dependent. Again re-labeling if needed, we have

$$d_m = \sum_{j=1}^{m-1} b_j d_j$$

for some constants $b_1, \ldots, b_{m-1} \in R$ with $b_{m-1} \neq 0$. In case $m = 2$ it follows that both d_2 and d_3 are constant multiples of d_1, say

$$d_2 = k_2 d_1 \quad \text{and} \quad d_3 = k_3 d_1,$$

with $k_2 k_3 \neq 0$ since d_2 and d_3 are nontrivial. By Proposition 3.5 this is impossible.

If $m > 2$ we continue the reduction process until we arrive at

$$d_2 = k_2 d_1,\ d_3 = k_3 d_1,\ \ldots,\ d_m = k_m d_1,$$

with $k_2 \cdots k_m \neq 0$. Applying Proposition 3.5 we again have a contradiction, and this completes the proof. □

We recall (see [3], Chapter XIV, section 2, Theorem 2) that there exist nontrivial derivations (of order 1) of $\mathbb{R}$. In fact the same is true for any transcendental extension field of $\mathbb{Q}$ (see [3], Chapter XIV, section 2, Theorem 1). Let $\delta \in \mathcal{D}_1(\mathbb{R}) \setminus \mathcal{D}_0(\mathbb{R})$. Then by Theorem 3.6 the iterate δ^n is a nontrivial derivation of $\mathbb{R}$ of order n. Thus we have proved the following.

Corollary 3.7. *There exist nontrivial derivations of $\mathbb{R}$ of every order. The same is true if we replace $\mathbb{R}$ by any field extension of $\mathbb{Q}$ which contains at least one element transcendental over $\mathbb{Q}$.*

4. Examples

We close the paper with two examples illustrating the sharpness of the results in Theorem 3.6. The first shows the necessity of assuming that the characteristic of R exceeds $n!$ in order to guarantee that the n-th iterate of a nonzero derivation is a nontrivial derivation of order n.

Example 4.1. Let $R = \mathbb{Z}_{n!}[x]$ be the polynomial ring with coefficients in $\mathbb{Z}_{n!}$. Here the derivative function $d : R \to R$ defined by $d(p) := p'$ is a nontrivial derivation of order 1. Yet it is easy to verify that the n-th iterate ($= n$-th derivative) d^n is identically zero. Indeed, for any polynomial $p \in R$ we have

$$d^n(p) = d^n\left(\sum_j a_j x^j\right) = \sum_{j \geq n} j(j-1)\cdots(j-n+1)a_j x^{j-n} = \sum_{j \geq n} n!\binom{j}{n} a_j x^{j-n} = 0.$$

The second example shows the necessity of R being an integral domain. The following R has characteristic 0, but it has nontrivial divisors of 0: $x \neq 0$ and $y \neq 0$ yet $xy = 0$. A different example has been given in the paper [2] by Kiss and Laczkovich.

Example 4.2. Let R be the quotient ring $\mathbb{Z}[x, y]/(xy)$, where (xy) is the ideal generated by xy. (Multiplication of elements in R is done as usual for polynomials except that any term which is a multiple of xy is deleted.)

We define $d_1, d_2 : R \to R$ by $d_1(p) := \frac{\partial p}{\partial x}$, respectively $d_2(p) := \frac{\partial p}{\partial y}$. It is easy to see that d_1 and d_2 are nontrivial derivations, yet $d_1 \circ d_2 = 0$.

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