



## On the Skitovich–Darmois theorem for some locally compact Abelian groups

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**Abstract.** Let  $X$  be a locally compact Abelian group,  $\alpha_j, \beta_j$  be topological automorphisms of  $X$ . Let  $\xi_1, \xi_2$  be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. It is known that if  $X$  contains no subgroup topologically isomorphic to the circle group  $\mathbb{T}$ , then the independence of the linear forms  $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$  and  $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$  implies that  $\mu_j$  are Gaussian distributions. We prove that if  $X$  contains no subgroup topologically isomorphic to  $\mathbb{T}^2$ , then the independence of  $L_1$  and  $L_2$  implies that  $\mu_j$  are either Gaussian distributions or convolutions of Gaussian distributions and signed measures supported in a subgroup of  $X$  generated by an element of order 2. The proof is based on solving the Skitovich–Darmois functional equation on some locally compact Abelian groups.

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### 1. Introduction

One of the most well-known characterization theorems in mathematical statistics is the following theorem which characterizes the Gaussian distribution on the real line.

**The Skitovich–Darmois theorem** ([19, Ch. 3]). *Let  $\xi_j$ ,  $j = 1, 2, \dots, n$ ,  $n \geq 2$ , be independent random variables,  $\alpha_j, \beta_j$  be nonzero real numbers. If the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  are independent, then all  $\xi_j$  are Gaussian.*

The Skitovich–Darmois theorem was generalized by S.G. Ghurye and I. Olkin to the case when instead of random variables random vectors  $\xi_j$  in the space  $\mathbb{R}^a$  are considered and coefficients of the linear forms  $L_1$  and  $L_2$  are nonsingular matrices. They proved that in this case the independence of  $L_1$  and  $L_2$  implies that all  $\xi_j$  are also Gaussian ([19, Ch. 3]). The Skitovich–Darmois theorem was generalized in different directions (see e.g. [4, 18, 22, 24]).

Especially many publications have been devoted to group analogues of the Skitovich–Darmois theorem in the case, when independent random variables take values in a locally compact Abelian group, and coefficients of the forms are topological automorphisms of the group (see e.g. [7, 9, 13–15, 21, 23], and also [10, 11] where one can find additional references). In this paper we continue this research. It should be noted that the study of group analogues of the Skitovich–Darmois theorem on a locally compact Abelian group  $X$  is based on the study of solutions of the Skitovich–Darmois functional equation on the character group of  $X$ .

Denote by  $\mathbb{T}$  the circle group (the one dimensional torus), i.e.  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ . The following theorem was proved in [9].

**Theorem A.** *Let  $X$  be a second countable locally compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}$ . Let  $\alpha_j, \beta_j, j = 1, 2, \dots, n, n \geq 2$ , be topological automorphisms of the group  $X$ . Let  $\xi_j$  be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Then the independence of the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  implies that all  $\mu_j$  are Gaussian distributions.*

As noted for example in [6], if a locally compact Abelian group  $X$  contains a subgroup topologically isomorphic to  $\mathbb{T}$ , Theorem A fails even for the simplest linear forms  $L_1 = \xi_1 + \xi_2$  and  $L_2 = \xi_1 - \xi_2$ . Thus Theorem A gives a complete description of locally compact Abelian groups for which the Skitovich–Darmois theorem holds in its classical formulation under the assumption that the characteristic functions of  $\mu_j$  do not vanish. We will formulate now the following general problem.

**Problem 1.** *Let  $X$  be a second countable locally compact Abelian group,  $\alpha_j, \beta_j, j = 1, 2, \dots, n, n \geq 2$ , be topological automorphisms of  $X$ . Let  $\xi_j$  be independent random variables with values in the group  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Assume that the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  are independent. Describe the possible distributions  $\mu_j$ .*

Taking Theorem A into account, it is sufficient to solve Problem 1 for groups  $X$  containing a subgroup topologically isomorphic to  $\mathbb{T}$ . Problem 1 for arbitrary  $n \geq 2$  is complicated enough. For  $n = 2$  a partial solution of Problem 1 for the group  $X = \mathbb{T}^2$  was obtained in [23], and a complete solution for the groups  $X = \mathbb{R} \times \mathbb{T}$  and  $X = \Sigma_a \times \mathbb{T}$ , where  $\Sigma_a$  is an  $a$ -adic solenoid, was obtained in [15].

The main result of the article is a complete solution of Problem 1 for  $n = 2$  for an arbitrary locally compact Abelian group  $X$ , containing a subgroup topologically isomorphic to  $\mathbb{T}$ , but not containing a subgroup topologically isomorphic to  $\mathbb{T}^2$ . We prove that in this case the distributions  $\mu_j$  are not

only Gaussian, but convolutions of Gaussian distributions and some signed measures (Theorem 3). The proof is based on the fact that the solution of Problem 1 for an arbitrary locally compact Abelian group can be reduced to the case when a group is of the form  $\mathbb{R}^n \times \mathbb{T}^m$ , where  $n \leq \aleph_0$ ,  $m \leq \aleph_0$  (Theorems 1 and 2). As to the case when a locally compact Abelian group  $X$  contains a subgroup topologically isomorphic to  $\mathbb{T}^2$ , taking into account results of [23], we can hardly hope for a complete solution of Problem 1.

We will use in the article the standard results of abstract harmonic analysis (see [17]). Recall some definitions and agree on notation. Let  $X$  be a locally compact Abelian group,  $Y = X^*$  be its character group,  $(x, y)$  be the value of a character  $y \in Y$  at an element  $x \in X$ ,  $\text{Aut}(X)$  be the group of topological automorphisms of  $X$ . A Borel subgroup of the group  $X$  is called characteristic if it is invariant with respect to any automorphism  $\delta \in \text{Aut}(X)$ . Let  $A$  be a closed subgroup of  $X$ , and  $\delta$  be a continuous endomorphism of  $X$ . Denote by  $\delta|_A$  the restriction of  $\delta$  to  $A$ . If  $H$  is a subgroup of the group  $Y$ , denote by  $A(X, H) = \{x \in X : (x, y) = 1 \text{ for all } y \in H\}$  its annihilator. Let  $X_1$  and  $X_2$  be locally compact Abelian groups with character groups  $Y_1$  and  $Y_2$ , respectively. For any continuous homomorphism  $f : X_1 \rightarrow X_2$  define an adjoint homomorphism  $\bar{f} : Y_2 \rightarrow Y_1$  by the formula  $(x_1, \bar{f}y_2) = (fx_1, y_2)$  for all  $x_1 \in X_1$ ,  $y_2 \in Y_2$ . Denote by  $c_X$  the connected component of zero of the group  $X$ . Denote by  $\mathbb{Z}$  the group of integers, by  $\mathbb{Z}(2)$  the subgroup of  $\mathbb{T}$  consisting of elements  $\pm 1$ . Denote by  $\mathbb{T}^m$ , where  $m \leq \aleph_0$ , the finite-dimensional or infinite-dimensional torus. Denote by  $\mathbb{Z}^{m*}$  the free group of rank  $m$ . Denote by  $I$  the identity automorphism of a group. Put  $Y^{(2)} = \{2y : y \in Y\}$ .

Denote by  $\mathbb{R}^{\aleph_0}$  the space of all sequences of real numbers in the topology of the projective limit of spaces  $\mathbb{R}^n$  (in the product topology), and by  $\mathbb{R}^{\aleph_0*}$  the space of all finite sequences of real numbers with the topology of strictly inductive limit of spaces  $\mathbb{R}^n$ . We note that the groups  $\mathbb{R}^{\aleph_0}$  and  $\mathbb{R}^{\aleph_0*}$  are the character groups of one another.

Let  $\psi(y)$  be a function on  $Y$ , and  $h$  be an arbitrary element of  $Y$ . Denote by  $\Delta_h$  the finite difference operator

$$\Delta_h \psi(y) = \psi(y + h) - \psi(y), \quad y \in Y.$$

Denote by  $\langle \cdot, \cdot \rangle$  the scalar product in the space  $\mathbb{R}^b$ .

Let  $M^1(X)$  be the convolution semigroup of probability distributions on  $X$  and  $\mu \in M^1(X)$ . Denote by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x)$$

the characteristic function (Fourier transform) of  $\mu$ , and by  $\sigma(\mu)$  the support of  $\mu$ . For  $\mu \in M^1(X)$  define the distribution  $\bar{\mu} \in M^1(X)$  by the formula  $\bar{\mu}(B) = \mu(-B)$  for any Borel set  $B$ . We note that  $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)}$ . Denote by  $E_x$  the degenerate distribution concentrated at a point  $x \in X$ .

A distribution  $\gamma$  on the group  $X$  is called Gaussian in the sense of Parthasarathy ([25, Ch. IV], if its characteristic function is of the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\},$$

where  $x \in X$ , and  $\varphi(y)$  is a continuous nonnegative function on  $Y$ , satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \tag{1}$$

Since we will deal only with Gaussian distributions in the sense of Parthasarathy we will call them Gaussian. Denote by  $\Gamma(X)$  the set of Gaussian distributions on the group  $X$ .

**2. Reduction of Problem 1 for arbitrary groups to groups of the form  $\mathbb{R}^n \times \mathbb{T}^m$**

It is convenient for us to formulate as a lemma the following statement.

**Lemma 1.** ([10, §10.1]). *Let  $X$  be a second countable locally compact Abelian group,  $\alpha_j, \beta_j \in \text{Aut}(X)$ . Let  $\xi_j, j = 1, 2, \dots, n, n \geq 2$ , be independent random variables with values in  $X$  and distributions  $\mu_j$ . The linear forms  $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n, L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$  are independent if and only if the characteristic functions  $\hat{\mu}_j(y)$  satisfy the Skitovich–Darmois functional equation*

$$\prod_{j=1}^n \hat{\mu}_j(\tilde{\alpha}_j u + \tilde{\beta}_j v) = \prod_{j=1}^n \hat{\mu}_j(\tilde{\alpha}_j u) \prod_{j=1}^n \hat{\mu}_j(\tilde{\beta}_j v), \quad u, v \in Y. \tag{2}$$

Taking into account that the characteristic function of the distribution  $\mu_j$  is the mathematical expectation  $\hat{\mu}_j(y) = \mathbf{E}[(\xi_j, y)]$ , the proof of Lemma 1 is the same as in the classical case  $X = \mathbb{R}$ .

We need the following K. Stein theorem, which we formulate as a lemma.

**Lemma 2.** ([16, §19.3]). *Let  $H$  be a countable Abelian group. Then  $H$  is represented in the form  $H = N \times M$ , where  $M$  is free, and  $N$  has no free factor-groups. The group  $N$  is uniquely determined by the group  $H$ .*

Lemma 2 implies the following statement.

**Lemma 3.** *Let  $X$  be a second countable connected locally compact Abelian group. Then the group  $X$  is topologically isomorphic to a group of the form  $\mathbb{R}^a \times K \times \mathbb{T}^m$ , where  $a \geq 0, K$  is a connected compact Abelian group containing no subgroup topologically isomorphic to  $\mathbb{T}, m \leq \aleph_0$ . Furthermore,  $a$  and  $m$  are uniquely determined by the group  $X$ , and the group  $K$  is uniquely determined by the group  $X$  up to topological isomorphism.*

*Proof.* By the structure theorem for connected locally compact Abelian groups  $X = L \times F$ , where  $L \cong \mathbb{R}^a$ ,  $a \geq 0$ , and  $F$  is a second countable connected compact Abelian group. Furthermore,  $a$  and  $F$  are uniquely determined by the group  $X$ . Put  $H = F^*$ . Then  $H$  is a countable discrete torsion free Abelian group. By Lemma 2  $H$  is represented in the form  $H = N \times M$ , where  $M$  is a free Abelian group,  $N$  is a torsion free Abelian group without free factor-groups. Furthermore, the group  $N$  is uniquely determined by the group  $H$ . This implies that  $F = K \times G$ , where  $K \cong N^*$  is a connected compact group containing no subgroup topologically isomorphic to  $\mathbb{T}$ ,  $G \cong M^* \cong \mathbb{T}^m$ , where  $m \leq \aleph_0$ . Since  $G = A(F, N)$ , the group  $G$  is uniquely determined by the group  $F$ , and hence by the group  $X$ , too. It follows from this that  $m$  is uniquely determined by the group  $X$ , and the subgroup  $K$ , which is topologically isomorphic to the factor-group  $F/G$ , is uniquely determined up to topological isomorphism.  $\square$

We also need the following standard lemma.

**Lemma 4.** (see e.g. [8, §2.5]). *Let  $X$  be a topological Abelian group,  $B$  be a Borel subgroup of  $X$ ,  $\mu$  be a distribution on  $X$  concentrated on  $B$ . Let  $\mu = \mu_1 * \mu_2$ , where  $\mu_j$  are distributions on  $X$ . Then the distributions  $\mu_j$  can be replaced by their shifts  $\mu'_j$  in such a manner that  $\mu = \mu'_1 * \mu'_2$  and  $\mu'_j$  are concentrated on  $B$ .*

**Lemma 5.** ([9]). *Let  $X$  be a second countable locally compact Abelian group,  $\alpha_j, \beta_j \in \text{Aut}(X)$ . Let  $\xi_j, j = 1, 2, \dots, n, n \geq 2$ , be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Then the independence of the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  implies that there exist elements  $x_j \in X, j = 1, 2, \dots, n$  such that all distributions  $\mu'_j$  of the random variables  $\xi'_j = \xi_j + x_j$  are supported in the subgroup  $c_X$ .*

Taking into account that  $c_X$  is a characteristic subgroup, it follows from Lemma 5 that the study of distributions of independent random variables with non-vanishing characteristic functions which are characterized by the independence of the linear forms  $L_1$  and  $L_2$  is reduced to the case when  $X$  is a connected group. Although the structure of connected locally compact Abelian groups is still complicated (each such group is topologically isomorphic to a group of the form  $\mathbb{R}^a \times F, a \geq 0, F$  is a connected compact Abelian group), it is much simpler than the structure of arbitrary locally compact Abelian groups.

In this section we considerably strengthen Lemma 5. Namely, we prove that the study of distributions of independent random variables with non-vanishing characteristic functions which are characterized by the independence of linear forms  $L_1$  and  $L_2$  is reduced to the case when the group  $X$  is topologically isomorphic to a group of the form  $\mathbb{R}^n \times \mathbb{T}^m$ , where  $n \leq \aleph_0, m \leq \aleph_0$ . It should be noted that if  $n = \aleph_0$ , then  $X$  is not a locally compact group, although its structure is simple enough.

Let  $X$  be a second countable locally compact Abelian group. By Lemma 3  $c_X \cong \mathbb{R}^a \times K \times \mathbb{T}^m$ , where  $a \geq 0$ ,  $K$  is a connected compact Abelian group containing no subgroup topologically isomorphic to  $\mathbb{T}$ ,  $m \leq \aleph_0$ . Since the group  $K$  is uniquely determined by the group  $X$  up to topological isomorphism, the dimension of the group  $K$  is uniquely determined by the group  $X$ . First consider the case when  $K$  has a finite dimension.

**Theorem 1.** *Let  $X$  be a second countable locally compact Abelian group such that its connected component of zero  $c_X \cong \mathbb{R}^a \times K \times \mathbb{T}^m$ , where  $a \geq 0$ ,  $K$  be a connected compact Abelian group containing no subgroup topologically isomorphic to  $\mathbb{T}$ ,  $m \leq \aleph_0$ . Assume that  $K$  has a finite dimension  $l$ . Let  $\alpha_j, \beta_j \in \text{Aut}(X)$ . Let  $\xi_j, j = 1, 2, \dots, n, n \geq 2$ , be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Assume that the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  are independent. Then there exist a continuous monomorphism  $p : G \mapsto X$ , where  $G = \mathbb{R}^b \times \mathbb{T}^m, b = a + l$ , and elements  $x_j \in X, j = 1, 2, \dots, n$ , such that all distributions  $\mu_j * E_{x_j}$  are concentrated on the subgroup  $p(G)$ . Furthermore,  $p(G)$  is a characteristic subgroup.*

*Proof.* It follows from Lemma 5 that we can assume from the beginning that  $X$  is a connected group, i.e.  $X = c_X$ . Then  $Y \cong \mathbb{R}^a \times D \times \mathbb{Z}^{m*}$ , where  $D = K^*$  is a countable discrete torsion free Abelian group containing no free factor-groups. To avoid introducing additional notation we assume that  $X = \mathbb{R}^a \times K \times \mathbb{T}^m$  and  $Y = \mathbb{R}^a \times D \times \mathbb{Z}^{m*}$ . Since the group  $K$  has dimension  $l$ , the rank of the group  $D$  is also  $l$ . Put  $G = \mathbb{R}^b \times \mathbb{T}^m, b = a + l$ . Then  $H = G^* \cong \mathbb{R}^b \times \mathbb{Z}^{m*}$ . We also assume that  $H = \mathbb{R}^b \times \mathbb{Z}^{m*}$ . Denote by  $(s, d, k), s \in \mathbb{R}^a, d \in D, k \in \mathbb{Z}^{m*}$ , elements of the group  $Y$ , and by  $(w, k), w \in \mathbb{R}^b, k \in \mathbb{Z}^{m*}$ , elements of the group  $H$ . Construct a natural embedding of the group  $Y$  to the group  $H$  (see e.g. [8, §5.6]). Choose in  $D$  a maximal independent system of elements  $d_1, \dots, d_l$ . Then for every  $d \in D$  there exist integers  $q \neq 0, q_1, \dots, q_l$ , such that

$$qd = q_1 d_1 + \dots + q_l d_l. \tag{3}$$

The independence of the set  $\{d_1, \dots, d_l\}$  implies that the rational numbers  $\{q_j/q\}$  are uniquely determined by  $d$ . Define the mapping  $f_0 : \mathbb{R}^a \times D \mapsto \mathbb{R}^b$  by the formula  $f_0(s, d) = (s, q_1/q, \dots, q_l/q), s \in \mathbb{R}^a, d \in D$ . Since  $D$  is a torsion free group,  $f_0$  is a continuous monomorphism of the group  $\mathbb{R}^a \times D$  to  $\mathbb{R}^b$ . Extend  $f_0$  from  $\mathbb{R}^a \times D$  to the continuous monomorphism  $f : Y \mapsto H$  putting

$$fy = f(s, d, k) = (s, q_1/q, \dots, q_l/q, k), \quad y = (s, d, k), \quad s \in \mathbb{R}^a, d \in D, k \in \mathbb{Z}^{m*}. \tag{4}$$

Since the group  $K$  has no subgroups topologically isomorphic to  $\mathbb{T}$ , we have  $\overline{f_0(\mathbb{R}^a \times D)} = \mathbb{R}^b$  ([5]). Then obviously  $\overline{f(Y)} = H$ . Set  $p = \tilde{f}$ . Since  $\overline{f(Y)} = H$ , it follows from the properties of adjoint homomorphisms that  $p : G \mapsto X$  is a continuous monomorphism.

Note now that the inequality

$$|\psi(u) - \psi(v)|^2 \leq 2(1 - \operatorname{Re} \psi(u - v)), \quad u, v \in Y, \tag{5}$$

holds true for any positive definite function  $\psi(y)$  on an arbitrary Abelian group  $Y$ .

By Lemma 1 the characteristic functions  $\hat{\mu}_j(y)$  satisfy the Skitovich–Darmois functional Eq. (2). Consider the distributions  $\nu_j = \mu_j * \bar{\mu}_j$ ,  $j = 1, 2, \dots, n$ . Then  $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$ . Obviously, the characteristic functions  $\hat{\nu}_j(y)$  also satisfy the Skitovich–Darmois functional Eq. (2).

Since  $\mathbb{R}^a$  is a connected component of zero of the group  $Y$ , it follows that  $\mathbb{R}^a$  is a characteristic subgroup of the group  $Y$ . For this reason  $K \times \mathbb{T}^m = A(X, \mathbb{R}^a)$  is a characteristic subgroup of the group  $X$ . It follows from Lemma 2 that  $D$  is a characteristic subgroup of the group  $D \times \mathbb{Z}^{m*}$ . This implies that  $\mathbb{T}^m = A(K \times \mathbb{T}^m, D)$  is a characteristic subgroup of the group  $K \times \mathbb{T}^m$ , and hence,  $\mathbb{T}^m$  is a characteristic subgroup of the group  $X$ . It follows from this that  $\mathbb{R}^a \times D = A(Y, \mathbb{T}^m)$  is a characteristic subgroup of the group  $Y$ , and we can consider the restriction of the Skitovich–Darmois functional Eq. (2) for the characteristic functions  $\hat{\nu}_j(y)$  to this subgroup. Since  $\mathbb{R}^a \times D \cong (\mathbb{R}^a \times K)^*$ , and the group  $\mathbb{R}^a \times K$  has no subgroups topologically isomorphic to  $\mathbb{T}$ , it follows from Lemma 1 and Theorem A that these restrictions are the characteristic functions of some Gaussian distributions on the group  $\mathbb{R}^a \times K$ . Taking into account that  $\hat{\nu}_j(y) > 0$ , we have the representations

$$\hat{\nu}_j(y) = \exp\{-\varphi_j(y)\}, \quad y \in \mathbb{R}^a \times D, \quad j = 1, 2, \dots, n, \tag{6}$$

where  $\varphi_j(y)$  are continuous nonnegative functions on  $\mathbb{R}^a \times D$  satisfying Eq. (1). As has been proved in [8, §5.6], it follows from the properties of the functions  $\varphi_j(y)$  that there exist symmetric positive semidefinite  $(b \times b)$ -matrices  $Q_j$ , such that

$$\varphi_j(y) = \langle Q_j f_0 y, f_0 y \rangle, \quad y \in \mathbb{R}^a \times D, \quad j = 1, 2, \dots, n. \tag{7}$$

Assume that  $(w, k) \in f(Y)$ , i.e.  $(w, k) = f(y)$ ,  $y \in Y$ . Consider on the subgroup  $f(Y)$  the functions  $h_j(w, k) = \hat{\nu}_j(f^{-1}(w, k))$ ,  $j = 1, 2, \dots, n$ . Since  $f(y) = f_0(y)$  for  $y \in \mathbb{R}^a \times D$ , it follows from (6) and (7) that

$$h_j(w, 0) = \exp\{-\langle Q_j w, w \rangle\}, \quad (w, 0) \in f(\mathbb{R}^a \times D), \quad j = 1, 2, \dots, n. \tag{8}$$

Taking (5) into account it follows from (8) that the positive definite functions  $h_j(w, k)$  are uniformly continuous on the subgroup  $f(Y)$  in the topology induced on  $f(Y)$  by the standard topology of  $H$ . Since  $\overline{f(Y)} = H$ , the functions  $h_j(w, k)$  can be extended by continuity to some continuous functions  $\bar{h}_j(w, k)$ ,  $w \in \mathbb{R}^b$ ,  $k \in \mathbb{Z}^{m*}$ , on the group  $H$ . Obviously,  $\bar{h}_j(w, k)$  are also positive definite functions. By the Bochner theorem there exist distributions  $\lambda_j$  on  $G$  such that  $\hat{\lambda}_j(w, k) = \bar{h}_j(w, k)$ ,  $w \in \mathbb{R}^b$ ,  $k \in \mathbb{Z}^{m*}$ . Since  $\hat{\lambda}_j(f(y)) = \hat{\nu}_j(y)$  for all  $y \in Y$ ,  $\nu_j = p(\lambda_j)$ . Hence, the distributions  $\nu_j$  are concentrated on  $p(G)$ . It is obvious that  $p(G)$  is a Borel subgroup. By Lemma 4 this implies that there

exist elements  $x_j \in X$ ,  $j = 1, 2, \dots, n$ , such that all distributions  $\mu_j * E_{x_j}$  are concentrated on  $p(G)$ .

It remains to prove that  $p(G)$  is a characteristic subgroup. Let  $\delta \in \text{Aut}(X)$ . Put  $A_\delta = f\tilde{\delta}f^{-1}$ . Then  $A_\delta$  is an algebraic automorphism of the subgroup  $f(Y)$ . The automorphism  $\tilde{\delta}$  is determined by its restriction on the subgroup  $\mathbb{R}^a$  and its values on the maximal independent system elements of the subgroup  $D \times \mathbb{Z}^{m*}$ . For this reason the automorphism  $\tilde{\delta}$  determines in a natural way a matrix  $A(\delta)$ . It is easily seen that if  $fy = (s, q_1/q, \dots, q_l/q, k)$ ,  $y = (s, d, k)$ ,  $s \in \mathbb{R}^a$ ,  $d \in D$ ,  $k \in \mathbb{Z}^{m*}$ , then  $A_\delta fy = A(\delta)fy$ , where the expression in the right-hand side of this equality is the product of the matrix  $A(\delta)$  and the vector  $fy$ . It is clear that the automorphism  $A_\delta$  can be uniquely extended to the topological automorphism of the group  $H$ . Denote by  $\bar{A}_\delta$  this extended automorphism. We have  $\tilde{\delta}y = f^{-1}A_\delta fy$ ,  $A_\delta fy = \bar{A}_\delta fy$ ,  $y \in Y$ . Let  $g \in G$ ,  $y \in Y$ . Then  $(\delta pg, y) = (pg, \tilde{\delta}y) = (pg, f^{-1}A_\delta fy) = (g, A_\delta fy) = (g, \bar{A}_\delta fy) = (p\bar{A}_\delta g, y)$ . It follows from this that  $\delta pg = p\bar{A}_\delta g$ . Hence  $p(G)$  is a characteristic subgroup.  $\square$

*Remark 1.* We keep the notation used in the proof of Theorem 1. Let  $X = \mathbb{R}^a \times K \times \mathbb{T}^m$ . Denote by  $L$  the arcwise connected component of zero of the group  $X$ . We verify that  $p(G) = L$ . By the Dixmier theorem  $L$  is a union of all one-parametric subgroups of the group  $X$  ([1, §8.19]). Let  $p_1 : \mathbb{R} \mapsto X$  be a continuous homomorphism. Put  $f_1 = \tilde{p}_1$ . Taking into account Dixmier's theorem the required statement will be proved if we check that  $p_1 t \in p(G)$  for all  $t \in \mathbb{R}$ . To this end it suffices to show that there exists an element  $g_t \in G$  such that  $(y, p_1 t) = (y, pg_t)$  for all  $y \in Y$ . Let  $\{e_j\}_{j=1}^a$  be a natural basis in  $\mathbb{R}^a$ , and  $\{b_j\}$  be a natural basis in  $\mathbb{Z}^{m*}$ . Taking (3) into account, we obtain

$$f_1 y = f_1(s, d, k) = \sum_{j=1}^a s_j f_1 e_j + \sum_{j=1}^l \frac{q_j}{q} f_1 d_j + \sum_j k_j f_1 b_j,$$

$$s = (s_1, \dots, s_a) \in \mathbb{R}^a, d \in D, k = (k_1, k_2, \dots) \in \mathbb{Z}^{m*}.$$

This implies that

$$(y, p_1 t) = (f_1 y, t) = \exp \left\{ it \left( \sum_{j=1}^a s_j f_1 e_j + \sum_{j=1}^l \frac{q_j}{q} f_1 d_j + \sum_j k_j f_1 b_j \right) \right\}. \tag{9}$$

Put

$$g_t = (t f_1 e_1, \dots, t f_1 e_a, t f_1 d_1, \dots, t f_1 d_l, e^{it f_1 b_1}, e^{it f_1 b_2}, \dots) \in G.$$

Let  $y = (s_1, \dots, s_a, d, k_1, k_2, \dots) \in Y$ . Taking (4) into account we get

$$(y, pg_t) = (fy, g_t) = \exp \left\{ it \left( \sum_{j=1}^a s_j f_1 e_j + \sum_{j=1}^l \frac{q_j}{q} f_1 d_j + \sum_j k_j f_1 b_j \right) \right\}. \tag{10}$$



The required statement follows from (9) and (10).

Consider now the case when  $K$  has infinite dimension. Let  $N = \mathbb{R}^{\aleph_0} \times \mathbb{Z}^{m*}$ ,  $m \leq \aleph_0$ . Put  $M = \mathbb{R}^{\aleph_0} \times \mathbb{T}^m$ . The groups  $N$  and  $M$  are the character groups of one another.

We need some properties of nuclear and strongly reflexive topological Abelian groups (see [2]). We use them in the proof of Theorems 2 and 3.

The group  $N$  is nuclear ([2, (7.8), (7.10)]). For such groups the Bochner theorem about one-to-one correspondence between the family of all continuous positive definite functions on the group  $N$  and the family of all regular finite Borel measures on its character group holds. We note that the factor-group of a nuclear group with respect to a closed subgroup is also nuclear.

The groups  $M$  and  $N$  are strongly reflexive ([2, (17.3)]). These groups have, in particular, the following properties analogous to those of locally compact Abelian groups. The Pontryagin duality theorem holds for such groups. Let  $P$  be a closed subgroup of the group  $M$ . For any  $x \in M \setminus P$  there exists a character  $y \in A(Y, P)$  such that  $(x, y) \neq 1$ . Any character of the subgroup  $P$  can be extended to a character of the group  $M$ . Moreover, the natural homomorphisms  $N/A(N, P) \mapsto P^*$  and  $(M/P)^* \mapsto A(N, P)$  are topological isomorphisms.

**Theorem 2.** *Let  $X$  be a second countable locally compact Abelian group such that its connected component of zero  $c_X \cong \mathbb{R}^a \times K \times \mathbb{T}^m$ , where  $a \geq 0$ ,  $K$  be a connected compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}$ ,  $m \leq \aleph_0$ . Assume that  $K$  has infinite dimension. Let  $\alpha_j, \beta_j \in \text{Aut}(X)$ . Let  $\xi_j, j = 1, 2, \dots, n, n \geq 2$ , be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Assume that the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  are independent. Then there exist a continuous monomorphism  $p : G \mapsto X$ , where  $G = \mathbb{R}^{\aleph_0} \times \mathbb{T}^m$ , and elements  $x_j \in X, j = 1, 2, \dots, n$ , such that all distributions  $\mu_j * E_{x_j}$  are concentrated on the subgroup  $p(G)$ . Furthermore,  $p(G)$  is a characteristic subgroup.*

*Proof.* We prove Theorem 2 in analogy with the proof of Theorem 1. As in the proof of Theorem 1 we can assume that  $X = \mathbb{R}^a \times K \times \mathbb{T}^m$ , where  $K$  is a connected compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}$ , and  $Y = \mathbb{R}^a \times D \times \mathbb{Z}^{m*}$ , where  $D = K^*$  is a countable discrete torsion free Abelian group containing no free factor-groups. Since the group  $K$  has infinite dimension, the rank of the group  $D$  is  $\aleph_0$ . Put  $G = \mathbb{R}^{\aleph_0} \times \mathbb{T}^m$ . Then  $H = G^* \cong \mathbb{R}^{\aleph_0} \times \mathbb{Z}^{m*}$ . We assume that  $H = \mathbb{R}^{\aleph_0} \times \mathbb{Z}^{m*}$ . Construct the natural embedding of the group  $Y$  to the group  $H$  (see e.g. ([8, §5.9])). For this purpose we choose in  $D$  a maximal independent system of elements  $d_1, \dots, d_l, \dots$  and, as we constructed in the proof of Theorem 1 the continuous homomorphism  $f_0 : \mathbb{R}^a \times D \mapsto \mathbb{R}^b$ , construct the continuous homomorphism  $f_0 : \mathbb{R}^a \times D \mapsto \mathbb{R}^{\aleph_0}$ . Next extend it to the continuous monomorphism  $f : Y \mapsto$

$H$ . Since the group  $K$  has no subgroups topologically isomorphic to  $\mathbb{T}$ , we have  $\overline{f_0(\mathbb{R}^a \times D)} = \mathbb{R}^{\aleph_0^*}$  ([8, §5.17]). This obviously implies that  $\overline{f(Y)} = H$ . Put  $p = \tilde{f}$ . Since  $G$  is a strongly reflexive group, it follows from  $\overline{f(Y)} = H$  that the homomorphism  $p : G \mapsto X$  is a monomorphism.

Consider the distributions  $\nu_j = \mu_j * \bar{\mu}_j$  and reason as in the proof of Theorem 1. We show that the restrictions of the characteristic functions  $\hat{\nu}_j(y)$  to the subgroup  $\mathbb{R}^a \times D$  are the characteristic functions of some Gaussian distributions on the group  $\mathbb{R}^a \times K$ . Hence, the representations (6) and (7) hold, where  $Q_j$  are infinite symmetric positive semidefinite matrices. Next, reasoning as in the proof of Theorem 1 we come to the continuous positive definite functions  $\bar{h}_j(w, k)$ ,  $w \in \mathbb{R}^{\aleph_0^*}$ ,  $k \in \mathbb{Z}^{m^*}$ , on the group  $H$ . Since  $H$  is a nuclear group, we can correspond to each function  $\bar{h}_j(w, k)$  a distribution  $\lambda_j$  on the group  $G$  such that  $\hat{\lambda}_j(w, k) = \bar{h}_j(w, k)$ ,  $w \in \mathbb{R}^{\aleph_0^*}$ ,  $k \in \mathbb{Z}^{m^*}$ . We have  $\hat{\lambda}_j(f(y)) = \hat{\nu}_j(y)$   $y \in Y$ . Taking into account that  $G$  is a strongly reflexive group, it implies that  $\nu_j = p(\lambda_j)$ . The final part of the proof of Theorem 2 is the same as in Theorem 1. □

*Remark 2.* Reasoning as in Remark 1 we see that in the case when the subgroup  $K$  has infinite dimension, the subgroup  $p(G)$  is also the arcwise connected component of zero of the group  $X$ .

We need the following general statement.

**Proposition 1.** *Let  $X$  and  $G$  be complete separable metric Abelian groups,  $\alpha_j, \beta_j \in \text{Aut}(X)$ ,  $j = 1, 2, \dots, n$ ,  $n \geq 2$ , and  $p : G \mapsto X$  be a continuous monomorphism such that  $\alpha_j(p(G)) = \beta_j(p(G)) = p(G)$ ,  $j = 1, 2, \dots, n$ . Let  $\xi_j$  be independent random variables with values in the group  $X$  and distributions  $\mu_j$ . Assume that there exist elements  $x_j \in X$  such that all distributions  $\mu_j * E_{x_j}$  are concentrated in the subgroup  $p(G)$ . Assume that the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  are independent. Then  $\hat{\xi}_j = p^{-1}(\xi_j + x_j)$  are independent random variables with values in the group  $G$ ,  $\hat{\alpha}_j = p^{-1} \alpha_j p$ ,  $\hat{\beta}_j = p^{-1} \beta_j p$  are topological automorphisms of the group  $G$  and the linear forms  $\hat{L}_1 = \hat{\alpha}_1 \hat{\xi}_1 + \dots + \hat{\alpha}_n \hat{\xi}_n$  and  $\hat{L}_2 = \hat{\beta}_1 \hat{\xi}_1 + \dots + \hat{\beta}_n \hat{\xi}_n$  are independent.*

*Proof.* We use the following theorem by Suslin [20, §39, IV]: Let  $X_1$  be a complete separable metric space,  $X_2$  be a metric space,  $p : X_1 \mapsto X_2$  be a continuous one-to-one mapping. If  $B$  is a Borel set in  $X_1$ , then  $p(B)$  is a Borel set in  $X_2$ . By the Suslin theorem  $\hat{\xi}_j$  are independent random variables with values in the group  $G$ . It is obvious that the random variables  $\hat{\xi}_j$  are independent. Since  $p$  is a continuous monomorphism and  $G$  is a complete separable metric group, by the Suslin theorem images of Borel sets under the mapping  $p$  are also Borel. Hence  $\hat{\alpha}_j, \hat{\beta}_j$  are Borel automorphisms of the group  $G$ , and hence  $\hat{\alpha}_j, \hat{\beta}_j \in \text{Aut}(G)$  (see e.g. [26, §4.3.9]). To prove the

independence of the linear forms  $\hat{L}_1$  and  $\hat{L}_2$  it suffices to note that for any Borel subsets  $A_1$  and  $A_2$  of the group  $G$  we have:  $\{\omega : \hat{L}_i(\omega) \in A_i\} = \{\omega : L_i(\omega) \in p(A_i) - (\alpha_1x_1 + \dots + \alpha_nx_n)\}, i = 1, 2.$   $\square$

*Remark 3.* Let  $X = \mathbb{R}^a \times K \times \mathbb{T}^m$ , where  $a \geq 0$ ,  $K$  be a connected compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}$  and having dimension  $l$ ,  $m \leq \aleph_0$ . Let  $\xi_j, j = 1, 2, \dots, n, n \geq 2$ , be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Let  $\alpha_j, \beta_j \in \text{Aut}(X)$ . Assume that the linear forms  $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$  and  $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$  are independent. By Theorem 1 there exist a continuous monomorphism  $p : G \mapsto X$ , where  $G = \mathbb{R}^b \times \mathbb{T}^m$ ,  $b = a + l$ , and elements  $x_j \in X, j = 1, 2, \dots, n$ , such that all distributions  $\mu_j * E_{x_j}, j = 1, 2, \dots, n$ , are concentrated on the subgroup  $p(G)$ , and  $p(G)$  is a characteristic subgroup. It follows from Proposition 1 that the study of possible distributions of the independent random variables  $\xi_j$  is reduced to the study of possible distributions of independent random variables  $\hat{\xi}_j$  with values in a group of the form  $\mathbb{R}^b \times \mathbb{T}^m$ , where  $b \geq 0, m \leq \aleph_0$ .

If the subgroup  $K$  has infinite dimension, applying Theorem 2 instead of Theorem 1, we come to the conclusion that the study of possible distributions of the independent random variables  $\xi_j$  is reduced to the study of possible distributions of independent random variables  $\hat{\xi}_j$  with values in a group of the form  $\mathbb{R}^{\aleph_0} \times \mathbb{T}^m$ , where  $m \leq \aleph_0$ .

### 3. The Skitovich–Darmois theorem for groups containing no subgroup topologically isomorphic to $\mathbb{T}^2$

We will solve Problem 1 when  $n = 2$  for second countable locally compact Abelian groups  $X$  containing a subgroup topologically isomorphic to  $\mathbb{T}$  and containing no subgroups topologically isomorphic to  $\mathbb{T}^2$ . Taking Lemma 5 into account, we can assume that  $X$  is a connected a group. By Lemma 3 such group is topologically isomorphic to a group of the form  $\mathbb{R}^a \times K \times \mathbb{T}$ , where  $a \geq 0, K$  is a connected compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}$ . To avoid introducing additional notation we assume that  $X = \mathbb{R}^a \times K \times \mathbb{T}$ . As has been noted in the proof of Theorem 1,  $\mathbb{T}$  is a characteristic subgroup of the group  $X$ . Let  $\delta \in \text{Aut}(X)$ . Since  $\text{Aut}(\mathbb{T}) = \{\pm I\}$ , we have either  $\delta|_{\mathbb{T}} = I$ , or  $\delta|_{\mathbb{T}} = -I$ . Let  $\xi_1, \xi_2$  be independent random variables with values in  $X$  and distributions  $\mu_j$ . Let  $\alpha_j, \beta_j \in \text{Aut}(X)$ . It is easily seen that the study of possible distributions  $\mu_j$  provided that the linear forms  $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$  and  $L_2 = \beta_1\xi_1 + \beta_2\xi_2$  are independent is reduced to the case when  $L_1$  and  $L_2$  are of the form  $L_1 = \xi_1 + \xi_2$  and  $L_2 = \xi_1 + \delta\xi_2$ , where  $\delta \in \text{Aut}(X)$ . Then the Skitovich–Darmois functional Eq. (2) becomes

$$\hat{\mu}_1(u+v)\hat{\mu}_2(u+\varepsilon v) = \hat{\mu}_1(u)\hat{\mu}_1(v)\hat{\mu}_2(u)\hat{\mu}_2(\varepsilon v), \quad u, v \in Y, \tag{11}$$

where  $\varepsilon = \tilde{\delta}$ .

Let  $t = (t_1, \dots, t_n, \dots) \in \mathbb{R}^{\mathbb{N}_0}$  and  $s = (s_1, \dots, s_n, 0, \dots) \in \mathbb{R}^{\mathbb{N}_0^*}$ . Put

$$\langle t, s \rangle = \sum_{j=1}^{\infty} t_j s_j, \quad (t, s) = \exp\{i\langle t, s \rangle\}.$$

Let  $\mu$  be a distribution on the group  $\mathbb{R}^{\mathbb{N}_0}$ . The characteristic function of the distribution  $\mu$  is defined by the formula

$$\hat{\mu}(s) = \int_{\mathbb{R}^{\mathbb{N}_0}} (t, s) d\mu(t), \quad s \in \mathbb{R}^{\mathbb{N}_0^*}.$$

We recall that a distribution  $\gamma$  on the group  $\mathbb{R}^{\mathbb{N}_0}$  is called Gaussian if its characteristic function is represented in the form

$$\hat{\gamma}(s) = (t, s) \exp\{-\langle As, s \rangle\}, \quad s \in \mathbb{R}^{\mathbb{N}_0^*},$$

where  $t \in \mathbb{R}^{\mathbb{N}_0}$ , and  $A = (\alpha_{ij})_{i,j=1}^{\infty}$  is a symmetric positive semidefinite matrix (see e.g. [8, §5.8]).

We need some lemmas. It is convenient for us to formulate as a lemma the following standard statement.

**Lemma 6.** *Let  $X$  be a locally compact Abelian group,  $H$  be a closed subgroup of  $Y$  and  $\mu \in M^1(X)$ . If  $\hat{\mu}(y) = 1$  for  $y \in H$ , then the characteristic function  $\hat{\mu}(y)$  is  $H$ -invariant and  $\sigma(\mu) \subset A(X, H)$ .*

**Lemma 7.** ([15]). *Let  $Y$  be an Abelian group,  $\varepsilon$  be an automorphism of the group  $Y$ . Assume that the functions  $f_j(y)$  do not vanish, satisfy Eq. (11) and conditions  $f_1(0) = f_2(0) = 1$ . Then each function  $f_j(y)$  satisfies the equation*

$$f_j(u+v)f_j(u-v) = f_j^2(u)f_j(v)f_j(-v), \quad u \in (\varepsilon - I)Y, \quad v \in Y. \tag{12}$$

**Lemma 8.** ([6]). *Let  $X$  be a second countable locally compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}^2$ . Let  $\xi_1, \xi_2$  be independent identically distributed random variables with values in  $X$  and distribution  $\mu$ . Assume that the characteristic function  $\hat{\mu}(y)$  does not vanish. If the linear forms  $L_1 = \xi_1 + \xi_2$  and  $L_2 = \xi_1 - \xi_2$  are independent, then  $\mu \in \Gamma(X)$ .*

**Lemma 9.** ([12], see also [11, Lemma 12.4]). *Let  $Y$  be a topological Abelian group, let  $\psi(y)$  be a continuous function on  $Y$  satisfying the equation*

$$\Delta_{2k}\Delta_h^2\psi(y) = 0, \quad h, k, y \in Y, \tag{13}$$

and the conditions  $\psi(-y) = \psi(y)$ ,  $\psi(0) = 0$ . Then the function  $\psi(y)$  can be represented in the form

$$\psi(y) = \varphi(y) + c_\alpha, \quad y \in y_\alpha + \overline{Y^{(2)}}, \tag{14}$$

where  $\varphi(y)$  is a continuous function on  $Y$  satisfying Eq. (1), and

$$Y = \bigcup_{\alpha} (y_{\alpha} + \overline{Y^{(2)}}), \quad y_0 = 0,$$

is a decomposition of the group  $Y$  with respect to the subgroup  $\overline{Y^{(2)}}$ .

**Lemma 10.** *Let  $X = \mathbb{R}^{\mathbb{N}_0}$ ,  $\alpha_j, \beta_j, j = 1, 2, \dots, n, n \geq 2$ , be topological automorphisms of the group  $X$ . Let  $\xi_j$  be independent random variables with values in  $X$  and distributions  $\mu_j$ . Assume that the linear forms  $L_1 = \alpha_1 \xi_1 + \dots + \alpha_n \xi_n$  and  $L_2 = \beta_1 \xi_1 + \dots + \beta_n \xi_n$  are independent. Then all  $\mu_j$  are Gaussian distributions.*

*Proof.* Each element  $s \in \mathbb{R}^{\mathbb{N}_0*}$  defines a linear continuous functional in the space  $\mathbb{R}^{\mathbb{N}_0}$ , and hence for any distribution  $\mu$  on the group  $\mathbb{R}^{\mathbb{N}_0}$  we can consider its image  $s(\mu) \in M^1(\mathbb{R})$ . We shall say that a distribution  $\gamma$  on the group  $\mathbb{R}^{\mathbb{N}_0}$  is weak Gaussian if  $s(\gamma) \in \Gamma(\mathbb{R})$  for all  $s \in \mathbb{R}^{\mathbb{N}_0*}$ . It is easy to see that the definitions of a Gaussian distribution and a weak Gaussian distribution on the group  $\mathbb{R}^{\mathbb{N}_0}$  are equivalent. As has been noted in [22], the Skitovich–Darmois theorem holds true for weak Gaussian distributions on the group  $\mathbb{R}^{\mathbb{N}_0}$ .  $\square$

**Lemma 11.** *Let  $X = \mathbb{R}^{\mathbb{N}_0} \times \mathbb{T}$ . Let  $\xi_1, \xi_2$  be independent identically distributed random variables with values in  $X$  and distribution  $\mu$ . Assume that the characteristic function  $\hat{\mu}(y)$  does not vanish. If the linear forms  $L_1 = \xi_1 + \xi_2$  and  $L_2 = \xi_1 - \xi_2$  are independent, then  $\mu$  is a Gaussian distribution.*

*Proof.* The group  $X$  is strongly reflexive, and its character group is topologically isomorphic to the group  $\mathbb{R}^{\mathbb{N}_0*} \times \mathbb{Z}$ , and a Gaussian distribution on the group  $X$  is defined in the same way as for the group  $\mathbb{R}^{\mathbb{N}_0}$ . Since  $X$  is strongly reflexive, as is easily seen, Lemma 1 holds for the group  $X$ . In [10, Lemma 9.11] Lemma 11 was proved for second countable locally compact Abelian groups containing no more than one element of order 2. This proof is based on Lemma 1 and the properties of strongly reflexive topological Abelian groups listed before the formulation of Theorem 2. Hence, it is valid for the group  $X$ .  $\square$

**Theorem 3.** *Let  $X = \mathbb{R}^a \times K \times \mathbb{T}$ , where  $a \geq 0, K$  be a second countable connected compact Abelian group containing no subgroups topologically isomorphic to  $\mathbb{T}$ . Let  $\delta \in \text{Aut}(X)$ . Let  $\xi_1, \xi_2$  be independent random variables with values in  $X$  and distributions  $\mu_j$  with non-vanishing characteristic functions. Assume that the linear forms  $L_1 = \xi_1 + \xi_2$  and  $L_2 = \xi_1 + \delta \xi_2$  are independent. If  $\delta|_{\mathbb{T}} = I$ , then  $\mu_j \in \Gamma(X)$ . If  $\delta|_{\mathbb{T}} = -I$ , then  $\mu_j = \gamma_j * \pi_j$ , where  $\gamma_j \in \Gamma(X)$ , and  $\pi_j$  are signed measures on the subgroup  $\mathbb{Z}(2) \subset \mathbb{T}$ .*

*Proof.* There are two cases: either the dimension of the group  $K$  is finite or is infinite.

1. Assume that the group  $K$  has a finite dimension  $l$ . Then the rank of the group  $D$  is also  $l$ . Taking into account Proposition 1 and Remark 3, it suffices to prove Theorem 3 for groups of the form  $X = \mathbb{R}^b \times \mathbb{T}$ , where  $b \geq 0$ , because in the notation of Proposition 1  $\delta|_{\mathbb{T}} = \hat{\delta}|_{\mathbb{T}}$ . We have  $Y \cong \mathbb{R}^b \times \mathbb{Z}$ . To avoid introducing additional notation we assume that  $Y = \mathbb{R}^b \times \mathbb{Z}$ . Denote elements of the group  $X$  by  $(t, z)$ ,  $t \in \mathbb{R}^b$ ,  $z \in \mathbb{T}$ , and elements of the group  $Y$  by  $(s, n)$ ,  $s \in \mathbb{R}^b$ ,  $n \in \mathbb{Z}$ . It is easy to see that each automorphism  $\delta \in \text{Aut}(X)$  is determined by the matrix

$$\begin{pmatrix} \alpha & v \\ 0 & \pm 1 \end{pmatrix}, \quad \alpha \in \text{Aut}(\mathbb{R}^b), \quad v \in \mathbb{R}^b, \tag{15}$$

and the automorphisms  $\delta$  and  $\varepsilon = \tilde{\delta}$  act on the groups  $X$  and  $Y$ , respectively, by the formulas

$$\delta(t, z) = (\alpha t, e^{i\langle v, t \rangle} z^{\pm 1}), \quad (t, z) \in X, \quad \varepsilon(s, n) = (\alpha s + nv, \pm n), \quad (s, n) \in Y.$$

It is obvious that  $\mathbb{R}^b$  is a characteristic subgroup of the group  $Y$ , because  $\mathbb{R}^b$  is the connected component of zero of  $Y$ . Put  $L = \text{Ker}(I - \varepsilon)|_{\mathbb{R}^b}$  and first verify that the proof of Theorem 3 is reduced to the case when  $L = \{0\}$ .

By Lemma 1 the characteristic functions  $\hat{\mu}_j(y)$  satisfy Eq. (11). Put  $\nu_j = \mu_j * \bar{\mu}_j$ ,  $j = 1, 2$ . Then  $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$  and the characteristic functions  $\hat{\nu}_j(y)$  also satisfy Eq. (11). Consider the restriction of Eq. (11) for the functions  $\hat{\nu}_j(y)$  to the subgroup  $L$ . We have

$$\hat{\nu}_1(u + v)\hat{\nu}_2(u + v) = \hat{\nu}_1(u)\hat{\nu}_1(v)\hat{\nu}_2(u)\hat{\nu}_2(v), \quad u, v \in L. \tag{16}$$

Set  $h(y) = \hat{\nu}_1(y)\hat{\nu}_2(y)$ . It follows from (16) that the function  $h(y)$  on the group  $L$  satisfies the equation  $h(u + v) = h(u)h(v)$ , i.e.  $h(y)$  is a character of the group  $L$ . This implies that the restrictions of the characteristic functions  $\hat{\nu}_j(y)$  to  $L$  are also characters of the subgroup  $L$ . Since  $\hat{\nu}_j(y) > 0$ ,  $y \in Y$ , we have  $\hat{\nu}_1(y) = \hat{\nu}_2(y) = 1$  for  $y \in L$ . Applying Lemma 6, we get  $\sigma(\nu_j) \subset G = A(X, L)$ . Inasmuch as  $L$  is the kernel of a continuous linear operator in the space  $\mathbb{R}^b$ ,  $L$  is a closed subspace in  $\mathbb{R}^b$ . It follows from this that  $G = W \times \mathbb{T}$ , where  $W$  is a closed subspace in  $\mathbb{R}^b$ . It is obvious that  $\varepsilon(L) = L$ . Hence  $\delta(G) = G$ . It is clear that if  $L \neq \{0\}$ , then  $G$  is a proper subgroup of  $X$ .

Consider a family of subgroups  $B$  of the group  $X$  having the properties:

- (i)  $B = V \times \mathbb{T}$ , where  $V$  is a closed subspace in  $\mathbb{R}^b$ ;
- (ii)  $\delta(B) = B$ ;
- (iii)  $\sigma(\nu_j) \subset B$ ,  $j = 1, 2$ .

Let  $N$  be an intersection of all subgroups of the group  $X$  having properties (i)–(iii). Obviously, the subgroup  $N$  also possesses properties (i)–(iii) and  $N$  is the smallest subgroup having these properties. To avoid introducing additional notation we assume that  $N = \mathbb{R}^c \times \mathbb{T}$ ,  $c \leq b$ , and  $N^* = \mathbb{R}^c \times \mathbb{Z}$ . Put  $\beta = \delta|_N$ . If  $\text{Ker}(I - \tilde{\beta})|_{\mathbb{R}^c} \neq \{0\}$ , then the above reasoning shows that the subgroup  $A(N, \text{Ker}(I - \tilde{\beta})|_{\mathbb{R}^c})$  possesses properties (i)–(iii) and is a proper subgroup of

$N$ , contrary to the construction. Since  $\nu_j = \mu_j * \bar{\mu}_j$ , it follows from Lemma 4 that the distributions  $\mu_j$  can be replaced by their shifts  $\mu'_j$  in such a manner that  $\sigma(\mu'_j) \subset N$ . It follows from what has been said that we can assume from the beginning without loss of generality that  $L = \{0\}$ . Let us note that the condition  $L = \{0\}$  means that in (15)  $(I - \alpha) \in \text{Aut}(\mathbb{R}^b)$ .

**1a.** Assume that  $\delta|_{\mathbb{T}} = I$ . It means that the matrix which corresponds to the automorphism  $\delta$  is of the form  $\begin{pmatrix} \alpha & v \\ 0 & 1 \end{pmatrix}$ . Put  $M = \text{Ker}(I - \varepsilon)$  and consider the restriction of Eq. (11) for the functions  $\hat{\nu}_j(y)$  to the subgroup  $M$ . We come to Eq. (16), but when  $u, v \in M$ . Reasoning as above we conclude that  $\sigma(\nu_j) \subset F = A(X, M)$ . We have  $F = A(X, M) = A(X, \text{Ker}(I - \varepsilon)) = (I - \delta)(X)$ . It follows from  $X = \mathbb{R}^b \times \mathbb{T}$ , where  $b \geq 0$ , and  $L = \{0\}$  that  $F = (I - \delta)(X) \cong \mathbb{R}^b$ . Since  $\varepsilon(M) = M$ , the restriction of the automorphism  $\delta$  to the subgroup  $F$  is a topological automorphism of the group  $F$ . Let  $\eta_j$  be independent random variables with values in  $F$  and distributions  $\nu_j$ . Since the characteristic functions  $\hat{\nu}_j(y)$  satisfy Eq. (11), the linear forms  $\tilde{L}_1 = \eta_1 + \eta_2$  and  $\tilde{L}_2 = \eta_1 + \delta\eta_2$  by Lemma 1 are independent. This implies by the Ghurye–Olkin theorem that  $\nu_j \in \Gamma(F)$ . Next, applying the Cramér theorem about the decomposition of a Gaussian distribution in the space  $\mathbb{R}^b$  and Lemma 4, we get  $\mu_j \in \Gamma(X)$ .

**1b.** Assume that  $\delta|_{\mathbb{T}} = -I$ . It means that the matrix which corresponds to the automorphism  $\delta$  is of the form  $\begin{pmatrix} \alpha & v \\ 0 & -1 \end{pmatrix}$ . Put  $H = (I - \varepsilon)Y$ . Since  $L = \text{Ker}(I - \varepsilon)|_{\mathbb{R}^b} = \{0\}$ , we have  $H = Y^{(2)} = \mathbb{R}^b \times \mathbb{Z}^{(2)} \cong \mathbb{R}^b \times \mathbb{Z}$ . It follows from Lemma 7 that each characteristic function  $\hat{\mu}_j(y)$  satisfies Eq. (12) when  $u, v \in Y^{(2)}$ . Since  $H \cong \mathbb{R}^b \times \mathbb{Z}$ , we have  $H^* \cong \mathbb{R}^b \times \mathbb{T}$ . Taking into account that the group  $H^*$  contains no subgroups topologically isomorphic to  $\mathbb{T}^2$ , and applying Lemmas 1 and 8 we conclude that the restrictions of the characteristic functions  $\hat{\mu}_j(y)$  to the subgroup  $H$  are characteristic functions of Gaussian distributions. Thus, we have on the subgroup  $H$  the representation

$$\hat{\mu}_j(y) = m_j(y) \exp\{-\varphi_j(y)\}, \quad j = 1, 2,$$

where  $m_j(y)$  are characters of the subgroup  $H$ , and  $\varphi_j(y)$  are continuous non-negative functions on  $H$  satisfying Eq. (1). Replacing, if necessary, the distributions  $\mu_j$  by their shifts we can assume that

$$\hat{\mu}_j(y) = \exp\{-\varphi_j(y)\}, \quad y \in H, \quad j = 1, 2. \tag{17}$$

Next the proof of Theorem 3 is quite similar to the proof of this theorem for the group  $X = \mathbb{R} \times \mathbb{T}$  given in ([15]). Put

$$l_1(y) = \frac{\hat{\mu}_1(y)}{|\hat{\mu}_1(y)|}, \quad l_2(y) = \frac{\hat{\mu}_2(y)}{|\hat{\mu}_2(y)|}, \quad y \in Y,$$

and verify that  $l_j(y)$  are characters of the group  $Y$ . It follows from the equality  $\hat{\mu}_j(y) = |\hat{\mu}_j(y)|$  for  $y \in H$ , that  $l_j(y) = 1$  for  $y \in H$ . The functions  $l_j(y)$  satisfy Eq. (12), which takes the form

$$l_j(u + v)l_j(u - v) = 1, \quad u \in H, \quad v \in Y. \tag{18}$$

Substitute in (18)  $u = (s/2, 0)$ ,  $v = (s/2, n)$ . We obtain

$$l_j(s, n)l_j(0, -n) = 1, \quad s \in \mathbb{R}^b, \quad n \in \mathbb{Z}.$$

Multiplying both sides of this equation by  $l_j(0, n)$  and taking into account that  $|l_j(y)| = 1, l_j(-y) = \overline{l_j(y)}$ , we find

$$l_j(s, n) = l_j(0, n), \quad s \in \mathbb{R}^b, \quad n \in \mathbb{Z}. \tag{19}$$

Obviously, the functions  $l_j(y)$  satisfy Eq. (11). Taking (19) into account, we can write Eq. (11) for the functions  $l_j(y)$  in the form

$$l_1(0, m + n)l_2(0, m - n) = l_1(0, m)l_1(0, n)l_2(0, m)l_2(0, -n), \quad m, n \in \mathbb{Z}. \tag{20}$$

We can get by induction from Eq. (20) that  $l_j(0, n)$  are characters of the group  $\mathbb{Z}$ , and hence  $l_j(y)$  are characters of the group  $Y$ . Thus, there exist elements  $x_j \in X$  such that

$$l_j(y) = (x_j, y), \quad y \in Y, \quad j = 1, 2. \tag{21}$$

Now find the representations for  $|\hat{\mu}_j(y)|$ . Put  $\psi_j(y) = -\log |\hat{\mu}_j(y)|$ . Then (11) implies that the functions  $\psi_j(y)$  satisfy the equation

$$\psi_1(u + v) + \psi_2(u + \varepsilon v) = P(u) + Q(v), \quad u, v \in Y, \tag{22}$$

where  $P(u) = \psi_1(u) + \psi_2(u)$ ,  $Q(v) = \psi_1(v) + \psi_2(\varepsilon v)$ .

As has been proved in [10, Lemma 10.9], (22) implies that each of the functions  $\psi_j(y)$  satisfies the equation

$$\Delta_{(I-\varepsilon)k} \Delta_h^2 \psi_j(y) = 0, \quad y, k, h \in Y. \tag{23}$$

Since  $(I - \varepsilon)Y = Y^{(2)}$ , we conclude from (23) that each of the functions  $\psi_j(y)$  satisfies the equation

$$\Delta_{2k} \Delta_h^2 \psi_j(y) = 0, \quad y, k, h \in Y. \tag{24}$$

Since  $H = Y^{(2)} = \overline{Y^{(2)}}$ , the decomposition of the group  $Y$  with respect to the subgroup  $\overline{Y^{(2)}}$  is of the form  $Y = H \cup ((0, 1) + H)$ . It follows from (17) and Lemma 9 that there exist real constants  $c_1, c_2$  such that

$$\psi_j(y) = \begin{cases} \varphi_j(y), & y \in H, \\ \varphi_j(y) + c_j, & y \in (0, 1) + H, \end{cases} \tag{25}$$

$j = 1, 2$ .

It follows from (11) and (17) that the equality

$$\varphi_1(u + v) + \varphi_2(u + \varepsilon v) = \varphi_1(u) + \varphi_1(v) + \varphi_2(u) + \varphi_2(\varepsilon v) \tag{26}$$



holds true for any  $u, v \in H$ , and hence, for any  $u, v \in Y$ . Substituting in (22)  $u, v \in (0, 1) + H$  and taking into account (25) and (26), we find that  $c_1 = -c_2$ . Put  $c_1 = -c_2 = -2\kappa$ . It follows from (25) that the functions  $|\hat{\mu}_j(y)|$  are of the form

$$\begin{aligned} |\hat{\mu}_1(y)| &= \exp\{-\varphi_1(y) + \kappa(1 - (-1)^n)\}, & y = (s, n) \in Y, \\ |\hat{\mu}_2(y)| &= \exp\{-\varphi_2(y) - \kappa(1 - (-1)^n)\}, & y = (s, n) \in Y. \end{aligned}$$

Taking (21) into account we finally obtain

$$\hat{\mu}_1(y) = (x_1, y) \exp\{-\varphi_1(y) + \kappa(1 - (-1)^n)\}, \quad y = (s, n) \in Y, \quad (27)$$

$$\hat{\mu}_2(y) = (x_2, y) \exp\{-\varphi_2(y) - \kappa(1 - (-1)^n)\}, \quad y = (s, n) \in Y. \quad (28)$$

Consider the signed measures

$$\begin{aligned} \pi_1 &= \frac{1}{2}(1 + e^{2\kappa})E_{(0,1)} + \frac{1}{2}(1 - e^{2\kappa})E_{(0,-1)}, & \pi_2 &= \frac{1}{2}(1 + e^{-2\kappa})E_{(0,1)} \\ &+ \frac{1}{2}(1 - e^{-2\kappa})E_{(0,-1)} \end{aligned}$$

supported in  $\mathbb{Z}(2) \subset X$  (it is clear that one of  $\pi_j$  is a distribution). Obviously, the characteristic functions  $\hat{\pi}_j(y)$  are of the form

$$\hat{\pi}_1(y) = \exp\{\kappa(1 - (-1)^n)\}, \quad \hat{\pi}_2(y) = \exp\{-\kappa(1 - (-1)^n)\}, \quad y = (s, n) \in Y. \quad (29)$$

The statement of Theorem 3 follows from (27), (28) and (29). Note that  $\pi_1 * \pi_2 = E_{(0,1)}$ .

**2.** Assume that the group  $K$  has infinite dimension. It follows from Proposition 1 and Remark 3 that it suffices to prove Theorem 3 for the group of the form  $X = \mathbb{R}^{N_0} \times \mathbb{T}$ . Then  $Y \cong \mathbb{R}^{N_0^*} \times \mathbb{Z}$ . To avoid introducing additional notation we assume that  $Y = \mathbb{R}^{N_0^*} \times \mathbb{Z}$ . We follow the scheme of the proof of Theorem 3 in case 1. First note that Lemma 1 holds true for the group  $X$  because  $X$  is strongly reflexive. Moreover, Lemma 6 is also valid for the group  $X$  because  $Y$  is a nuclear group and  $X$  is strongly reflexive. As in case 1 we consider the subgroup  $L = \text{Ker}(I - \varepsilon)|_{\mathbb{R}^{N_0^*}}$  and reduce the proof of Theorem 3 to the case when  $L = \{0\}$ . In so doing we use the fact that any closed linear subspace in  $\mathbb{R}^{N_0}$  is either finite-dimensional or topologically isomorphic to  $\mathbb{R}^{N_0}$  ([3]), and that Lemma 6 holds true for the group  $X$ .

**2a.** Let  $\delta|_{\mathbb{T}} = I$ . Reason as in case 1a. We use the fact that Lemma 1 is valid for the group  $\mathbb{R}^{N_0}$ , and that Lemma 6 holds true for the group  $X$ . Instead of the Ghurys–Olkin theorem we apply Lemma 10, and also use the fact that the Cramér theorem about the decomposition of a Gaussian distribution holds true for the space  $X = \mathbb{R}^{N_0}$ .

**2b.** Let  $\delta|_{\mathbb{T}} = -I$ . Reason as in case 1b. We use the fact that Lemma 1 is valid for the group  $X$  and  $Y$  is a nuclear group. Instead of Lemma 8 we apply Lemma 11. In all other respects the proof repeats the proof in case 1b.  $\square$

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