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Generalized convolutions and the Levi-Civita functional equation

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Abstract. In Borowiecka et al. (Bernoulli 21(4):2513–2551, 2015) the authors show that every generalized convolution can be used to define a Markov process, which can be treated as a Lévy process in the sense of this convolution. The Bessel process is the best known example here. In this paper we present new classes of regular generalized convolutions enlarging the class of such Markov processes. We give here a full characterization of such generalized convolutions $\diamond$ for which $\delta_x \diamond \delta_1$, $x \in [0,1]$, is a convex linear combination of $n = 3$ fixed measures and only the coefficients of the linear combination depend on x. For $n = 2$ it was shown in Jasiulis-Goldyn and Misiewicz (J Theor Probab 24(3):746–755, 2011) that such a convolution is unique (up to the scale and power parameters). We show also that characterizing such convolutions for $n \geqslant 3$ is equivalent to solving the Levi-Civita functional equation in the class of continuous generalized characteristic functions.

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1. Motivations

Generalized convolutions were invented and studied by K. Urbanik (see [11–15]). The idea was taken from the paper of Kingman [5], who introduced and studied special cases of such convolutions now called Kingman convolutions or Bessel convolutions. In the simplest case Kingman's work was based on an obvious observation that rotationally invariant distributions in $\mathbb{R}^n$ form a convex weakly closed set with the extreme points $\{T_a\omega_n : a \geqslant 0\}$, where ω_n is the uniform distribution on the unit sphere $S_{n-1} \subset \mathbb{R}^n$, T_a is the rescaling operator, i.e. $T_a\lambda$ is the distribution of aX if λ is the distribution of X (abbreviation: $\lambda = \mathcal{L}(X)$).

Kingman was working on one-dimensional projections of ω_n and he found all distributions λ, $\lambda = \mathcal{L}(\theta)$ for which

$$a\theta + b\theta' \stackrel{d}{=} \sqrt{a^2 + b^2 + 2abR}\,\theta, \qquad a, b > 0,$$

for some fixed, but dependent on n, random variable R independent of θ. Here $\stackrel{d}{=}$ denotes equality of distributions and θ' is an independent copy of θ. In this case the generalized convolution $\diamond$ is defined by the formula

$$\delta_a \diamond \delta_b = \mathcal{L}\big(\sqrt{a^2+b^2+2abR}\big).$$

Urbanik noticed that the Kingman convolution is a special case of generalized convolutions, i.e. associative, symmetric, weakly continuous linear operators $\diamond : \mathcal{P}_+^2 \mapsto \mathcal{P}_+$ (here $\mathcal{P}_+$ denotes the set of all probability measures on $[0,\infty)$) for which $\lambda \diamond \delta_0 = \lambda$, $\lambda \diamond (p\lambda_1 + (1-p)\lambda_2) = p\lambda \diamond \lambda_1 + (1-p)\lambda \diamond \lambda_2$, $T_a(\lambda_1 \diamond \lambda_2) = (T_a\lambda_1) \diamond (T_a\lambda_2)$. For some technical reasons Urbanik assumed also that there exists a sequence of positive numbers (a_n) such that $T_{a_n}\delta_1^{\diamond n}$ converges weakly to some non-degenerate to δ_0 measure. This assumption is not necessary in most of the results.

We see that generalized convolutions extend in, the language of distributions, the idea of sums of independent random variables. It was shown in [1] that if we restrict our attention to generalized sums of independent random variables considering $\oplus$ as an associative, symmetric operation for which $a(X \oplus Y) = (aX) \oplus (aY)$, then we have only two possibilities:

- $X \oplus Y = \big(X^\alpha + Y^\alpha\big)^{1/\alpha}$ in the case of positive variables,
- $X \oplus Y = \big(X^{<\alpha>} + Y^{<\alpha>}\big)^{<1/\alpha>}$ for variables taking values in $\mathbb{R}$.

Here α can be any number from the set $(0,\infty]$ and $x^{<\alpha>} := |x|^\alpha \mathrm{sign(x)}$. Even the Kingman convolution cannot be written in this way as it requires the assistance of an extra variable R. Considering generalized convolutions instead of generalized sums enrich the theory significantly.

The introduction of generalized convolutions required very laborious and time consuming introductory studies before the theory was read to define stochastic processes in the sense of generalized convolutions and before they could be used in stochastic modeling and other applications. This was done in a series of papers by many authors, see e.g. [9,11–15]

In the paper [1] the authors defined, proved the existence of and studied properties of stochastic processes with independent increments in the sense of generalized convolutions and the corresponding stochastic integrals. Some of these constructions were given earlier by Thu [9,10] in a special case of Bessel convolutions.

In this paper we focus on constructing new examples of generalized convolutions with the special property

$$\delta_x \diamond \delta_1 = \sum_{k=0}^{n-1} p_k(x)\lambda_k, \qquad x \in [0,1],\ \sum_{k=0}^{n-1} p_k(x) = 1, \tag{*}$$

for some fixed choice of probability measures $\lambda_0, \ldots, \lambda_{n-1}$. For $n = 2$ it was shown in [3] that the only possible (up to the scale parameter) generalized convolution of this type is the Kendall convolution:

$$\delta_x \vartriangle_\alpha \delta_1 = x^\alpha \pi_{2\alpha} + (1 - x^\alpha)\delta_1, \qquad x \in [0, 1],$$

where α is a fixed positive number and $\pi_{2\alpha}$ is the Pareto distribution with density $2\alpha x^{-2\alpha-1}\mathbf{1}_{[1,\infty)}$. This convolution, thanks to its connections with the Williamson transform, turned out to be very convenient in calculations. Similar properties describe the Kucharczak-Urbanik convolution (see Example 1 in this paper) which is an example of convolutions with property $(*)$ with an arbitrary but fixed n. Using the Levi-Civita functional equation we show here that for $n \geqslant 2$ we do not have the uniqueness of the convolution with property $(*)$. This leads to new classes of generalized convolutions and also, to new classes of integral transforms uniquely identifying transformed measures.

2. Preliminaries

According to the Urbanik paper (see [11]) a commutative and associative $\mathcal{P}_+$-valued binary operation $\diamond$ defined on $\mathcal{P}_+^2$ is called a *generalized convolution* if for all $\lambda, \lambda_1, \lambda_2 \in \mathcal{P}_+$ and $a \geqslant 0$ we have:

(i) $\delta_0 \diamond \lambda = \lambda$;
(ii) $(p\lambda_1 + (1-p)\lambda_2) \diamond \lambda = p(\lambda_1 \diamond \lambda) + (1-p)(\lambda_2 \diamond \lambda)$ whenever $p \in [0, 1]$;
(iii) $T_a(\lambda_1 \diamond \lambda_2) = (T_a\lambda_1) \diamond (T_a\lambda_2)$;
(iv) if $\lambda_n \to \lambda$, then $\lambda_n \diamond \eta \to \lambda \diamond \eta$ for all $\eta \in \mathcal{P}$ and $\lambda_n \in \mathcal{P}_+$,
(v) there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of positive numbers such that the sequence $T_{c_n}\delta_1^{\diamond n}$ converges to a measure different from δ_0;

where $\to$ denotes the weak convergence of probability measures.

A pair $(\mathcal{P}_+, \diamond)$ is called a *generalized convolution algebra*. It has been proven in [15] (Theorem 4.1 and Corollary 4.4) that each generalized convolution admits a weak characteristic function, i.e. a one-to-one correspondence $\lambda \leftrightarrow \Phi_\lambda$ between measures λ from $\mathcal{P}_+$ and real-valued Borel functions Φ_λ from $L_\infty(m_0)$, $m_0 = \delta_0 + \ell$, where ℓ is the Lebesgue measure on $(0,\infty)$, so that

1. $\Phi_{p\lambda+q\nu} = p\Phi_\lambda + q\Phi_\nu$ for $p, q \geqslant 0$, $p + q = 1$;
2. $\Phi_{\lambda\diamond\nu} = \Phi_\lambda \cdot \Phi_\nu$;
3. $\Phi_{T_a\lambda}(t) = \Phi_\lambda(at)$;
4. the uniform convergence of Φ_{λ_n} on every bounded interval is equivalent to the weak convergence of λ_n.

The characteristic function is uniquely determined up to a scale coefficient. Moreover, if λ is absolutely continuous with respect to the measure m_0 then Φ_λ is continuous and (see Lemma 3.11, Propositions 3.3 and 3.4 and Theorem 4.1 in [15])

$$\lim_{t\to\infty} \Phi_\lambda(t) = \lambda(\{0\}).$$

The function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, defined by $\varphi(t) = \Phi_{\delta_t}(1) = \Phi_{\delta_1}(t)$ is called a *probability kernel* of $(\mathcal{P}_+, \diamond)$. The kernel φ is a Borel function, $\varphi(0) = 1$ and $|\varphi(t)| \leqslant 1$ for each $t \in [0, \infty)$. It is evident that

$$\Phi_\lambda(t) = \int_0^\infty \varphi(ts)\, \lambda(ds).$$

A generalized convolution algebra $(\mathcal{P}_+, \diamond)$ (and the corresponding generalized convolution $\diamond$) is said to be *regular* if its probability kernel φ is a continuous function. It is known by [11], p.219, that the *max-convolution* introduced by the operation $\max(X, Y)$ on independent random variables X and Y is not regular and its probability kernel is given by $\varphi(t) = \mathbf{1}_{[0,1]}(t)$.

The $\diamond$-generalized characteristic function in a generalized convolution algebra plays the same role as the classical Laplace or Fourier transform for convolutions defined by addition of independent random elements.

The following proposition shows how we can get a new generalized convolution using an already known one. This result is not especially deep however it will be useful in further considerations.

Proposition 1. *Assume that a non-trivial generalized convolution algebra $(\mathcal{P}_+, \diamond)$ admits a characteristic function Φ with the probability kernel φ. Then for every $\alpha > 0$ there exists a generalized convolution $\circledast$ on $\mathcal{P}_+$ with the generalized characteristic function*

$$\Psi_{\mathcal{L}(Y)}(t) \stackrel{def}{=} \Phi_{\mathcal{L}(Y^\alpha)}(t^\alpha),$$

where $\mathcal{L}(Y)$ denotes the distribution of the random variable Y.

Proof. It is enough to define the generalized convolution on the measures δ_x, δ_y for $x, y \geqslant 0$. Assume that $\delta_{x^\alpha} \diamond \delta_{y^\alpha} = \mathcal{L}(Z)$ for some nonnegative random variable Z. We see that

$$\begin{aligned}\Psi_{\delta_x}(t)\Psi_{\delta_y}(t) &= \Phi_{\delta_{x^\alpha}}(t^\alpha)\Phi_{\delta_{y^\alpha}}(t^\alpha) = \int_0^\infty \varphi(t^\alpha z)\, \delta_{x^\alpha} \diamond \delta_{y^\alpha}(dz)\\ &= \int_0^\infty \varphi(t^\alpha u^\alpha)\, \delta_{x^\alpha} \diamond \delta_{y^\alpha}(du^\alpha) = \int_0^\infty \Psi_{\delta_u}(t)\mathcal{L}(Z^{1/\alpha})(du).\end{aligned}$$

Now we are able to define the generalized convolution $\circledast$:

$$\delta_x \circledast \delta_y \stackrel{def}{=} \mathcal{L}(Z^{1/\alpha}) \quad \text{if} \quad \delta_{x^\alpha} \diamond \delta_{y^\alpha} = \mathcal{L}(Z).$$

Checking that $\circledast$ is a generalized convolution and that Ψ is the generalized characteristic function for the algebra $(\mathcal{P}_+, \circledast)$ is trivial and will be omitted. $\square$

3. Main problem

We want to characterize such general convolutions for which the convolution of two one-point measures δ_x, δ_1 is a convex linear combination of n fixed measures and only the coefficients of this linear combination depend on x. More exactly: there exist measures $\lambda_0, \dots, \lambda_{n-1} \in \mathcal{P}_+$, $\lambda_k \neq \lambda_j$ for $k \neq j$, such that for all $x \in [0,1]$

$$\delta_x \diamond \delta_1 = \sum_{k=0}^{n-1} p_k(x)\lambda_k, \tag{*}$$

for some functions $p_k\colon [0,1] \mapsto [0,1]$ such that $p_0(x) + \cdots + p_{n-1}(x) = 1$ for all $x \in [0,1]$.

Remark 1. Since the measures $\lambda_0, \dots, \lambda_{n-1}$ are different, δ_1 is an extreme point in the convex set of probability measures and

$$\delta_1 = \delta_0 \diamond \delta_1 = \sum_{k=0}^{n-1} p_k(0)\lambda_k,$$

we see that one of the measures, say λ_0, must be equal to δ_1 and then $p_0(0) = 1$, $p_k(0) = 0$ for $k \geqslant 1$.

Let φ be the kernel (unknown) of the considered generalized convolution and

$$\mathcal{D}(\varphi) = \left\{ \Phi\colon \Phi(t) = \int_0^\infty \varphi(ts)\lambda(ds) \text{ for some } \lambda \in \mathcal{P}_+ \right\}.$$

In the language of generalized characteristic functions our problem leads to the following functional equation

$$\begin{array}{c} \exists\, \Phi_1, \dots, \Phi_{n-1} \in \mathcal{D}(\varphi)\, \forall\, x \in [0,1]\ \forall\, t \geqslant 0 \\ \varphi(xt)\varphi(t) = p_0(x)\varphi(t) + \sum_{k=1}^{n-1} p_k(x)\Phi_k(t). \end{array} \tag{**}$$

Remark 2. Without loss of generality we can assume that $\lambda_1(\{1\}) = \cdots = \lambda_{n-1}(\{1\}) = 0$. If this is not the case then we put

$$\lambda_k = q_k\delta_1 + (1-q_k)\lambda_k', \quad k = 1, \dots n,$$

for some $q_k \in [0,1]$, $k = 1, \dots n$, such that $\lambda_k'(\{1\}) = 0$ and then we can write for $q_0 = 1$

$$\delta_x \diamond \delta_1 = \sum_{k=0}^{n} p_k(x)q_k\delta_1 + \sum_{k=1}^{n-1} p_k(x)(1-q_k)\lambda_k' =: p_0'(x)\delta_1 + \sum_{k=1}^{n-1} p_k'(x)\lambda_k'.$$

Remark 3. Notice that if the measures $\lambda_0, \ldots, \lambda_{n-1}$ are linearly dependent, i.e. one or more measures can be obtained as a convex linear combination of others then equality $(*)$ can be written as

$$\delta_x \diamond \delta_1 = \sum_{k=0}^{m} p_k(x) \lambda'_k$$

for some $m < n-1$ and some probability measures $\lambda'_1, \ldots, \lambda'_m$. From now on we will assume that $\lambda_0, \ldots, \lambda_{n-1}$ are linearly independent. It means also that their generalized characteristic functions $\varphi, \Phi_1, \ldots, \Phi_{n-1}$ are linearly independent.

We will show that under some additional assumptions equation $(**)$ can be written in the form of the multiplicative Levi-Civita functional equation, which is described in the following theorem (for details see e.g. [8]).

Theorem 1. *Let a complex-valued continuous function φ satisfy the equation*

$$\varphi(xy) = \sum_{k=0}^{n} p_k(x) \Psi_k(y), \qquad \textit{for all} \;\; x, y \in (0,1)$$

with some functions $\{p_k\}$, $\{\Psi_k\}$. Then

$$\varphi(x) = \widetilde{\varphi}(-\ln x) = \sum_{j=1}^{m} P_j(-\ln x) x^{-\lambda_j}, \qquad \sum_{j=1}^{m} (deg P_j + 1) = n+1,$$

where P_j are polynomials and $\lambda_j \in \mathbb{C}$.

Lemma 1. *If for a nontrivial, continuous probability kernel φ equation $(**)$ holds then* $\lim_{t\to\infty} \varphi(t) = 0$.

Proof. Let $t > 0$. If there exists a sequence (a_n), $a_n \to \infty$ for $n \to \infty$, such that $\lim_{n\to\infty} \varphi(ta_n) = c \neq 0$ then we have

$$\varphi(t)\varphi(ta_n) = p_0(a_n^{-1})\varphi(ta_n) + \sum_{k=1}^{n-1} p_k(a_n^{-1}) \Phi_k(ta_n).$$

We can choose n_0 large enough to have $|\varphi(ta_n)| > |c|/2$ for $n \geqslant n_0$. Then $|\Phi_k(ta_n)/\varphi(ta_n)| < 2/|c|$. Since

$$\varphi(t) = p_0(a_n^{-1}) + \sum_{k=1}^{n-1} p_k(a_n^{-1}) \Phi_k(ta_n)/\varphi(ta_n)$$

and $p_k(a_n^{-1}) \to 0$ for each $k \geqslant 1$, $p_0(a_n^{-1}) \to 1$, we would have $\varphi(t) = 1$ for each $t > 0$ in contradiction to our assumptions. $\square$

Lemma 2. *If for a nontrivial, continuous probability kernel φ equation $(**)$ holds then*

$$a := \inf\{t \geq 0 \colon \varphi(t) = 0\} < \infty.$$

Proof. Of course $a > 0$ since $\varphi(0) = 1$ and φ is a continuous function. Assume that $a = \infty$. We have that $\varphi(t) > 0$, $\varphi(xt) > 0$ for every $t > 0$ and $x \in [0, 1]$. In equation $(**)$ we can divide both sides by $\varphi(t)$ and obtain

$$\varphi(xt) = p_0(x) + \sum_{k=1}^{n-1} p_k(x)\frac{\Phi_k(t)}{\varphi(t)}, \quad x \in (0, 1),\ t > 0.$$

If we restrict the argument t to the interval $(0, 1)$ we get a version of the Levi-Civita functional equation with $\Psi_0 = 1$, $\Psi_k = \Phi_k/\varphi$.

$$\varphi(xt) = p_0(x) + \sum_{k=1}^{n-1} p_k(x)\frac{\Phi_k(t)}{\varphi(t)}, \quad x, t \in (0, 1). \tag{***}$$

We can apply now Theorem 1 and obtain that

$$\varphi|_{(0,1]}(t) = 1 + \sum_{j=1}^{M} P_j(-\ln t)t^{-\lambda_j},$$

for some $M \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$, and some polynomials P_j. Since our function φ is real as a generalized characteristic function, we have that $\lambda_1, \ldots, \lambda_M$ are real. Considering the function $\varphi_c(\cdot) := \varphi(c\cdot)$ for $c > 0$ we see that

$$\varphi_c(xt) = p_0(x) + \sum_{k=1}^{n-1} p_k(x)\frac{\Phi_k(ct)}{\varphi_c(t)}, \quad x \in [0, 1],\ t \geqslant 0,$$

thus, using Theorem 1 again, we obtain that

$$\varphi_c|_{(0,1]}(t) = 1 + \sum_{j=1}^{M_c} P_{j,\,c}(-\ln t)t^{-\lambda_{j,\,c}},$$

for some $M_c \in \mathbb{N}$, $\lambda_{j,\,c} \in \mathbb{R}$, and some polynomials $P_{j,\,c}$. Consequently

$$\varphi|_{(0,c^{-1}]}(t) = 1 + \sum_{j=1}^{M_c} P_{j,\,c}(-\ln(ct))(ct)^{-\lambda_{j,\,c}}.$$

The functions $\varphi|_{(0,1]}$ and $\varphi|_{(0,c^{-1}]}$ coincide on the interval $(0, 1]$ for $c < 1$, thus for every $c < 1$

$$\varphi(t) = 1 + \sum_{j=1}^{M} P_j(-\ln t)t^{-\lambda_j}, \quad t \in (0, c^{-1}].$$

Letting $c \searrow 0$ we obtain that for some $M \in \mathbb{N}$, $\lambda_j \in \mathbb{R}$, and some polynomials P_j, $j \in \{1, \ldots, M\}$

$$\varphi(t) = 1 + \sum_{j=1}^{M} P_j(-\ln t)t^{-\lambda_j}, \quad t > 0.$$

In order to discuss the limit behavior of the function φ around zero and infinity we substitute $t \to e^{-x}$ and obtain

$$\varphi(e^{-x}) = 1 + \sum_{j=1}^{M} P_j(x) e^{\lambda_j x}, \quad x \in \mathbb{R}.$$

Let $r = \max\{\lambda_j : j \leqslant M\}$ and $s = \min\{\lambda_j : j \leqslant M\}$. It is easy to see now that if $r > 0$ then $|\varphi(e^{-x})| \to \infty$ if $x \to \infty$, which is impossible since any generalized characteristic function is bounded. If $s < 0$ then $|\varphi(e^{-x})| \to \infty$ if $x \to -\infty$, which is impossible for the same reason. Thus we have that $r = s = 0$ and φ is a polynomial bounded on $(0, \infty)$. This however is possible only if φ is a constant function in contradiction to our assumptions. $\square$

Without loss of generality, rescaling eventually the functions $\varphi, \Phi_1, \ldots, \Phi_{n-1}$, we can assume that $a = 1$.

Example 1. The convolutions described in this example were introduced by J. Kucharczak and K. Urbanik in [6]. If $\varphi_n(t) = (1 - t^\alpha)_+^n$ then for all $x \in [0, 1]$ and $t \geqslant 0$

$$\begin{aligned} \varphi_n(xt)\varphi_n(t) &= ((1 - x^\alpha) + x^\alpha(1 - t^\alpha))_+^n \, (1 - t^\alpha)_+^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{\alpha k} (1 - x^\alpha)^{n-k} (1 - t^\alpha)_+^{n+k}. \end{aligned}$$

We see that φ_n is a solution of the Levi-Civita equation $(***)$, but in order to see that it is also a solution of equation $(**)$ we need to show that for each $k = 1, \ldots, n$ there exists a measure $\lambda_{k,n}$ with distribution function $F_{k,n}$ such that

$$(1 - t^\alpha)_+^{n+k} = \int_0^\infty (1 - s^\alpha t^\alpha)_+^n \; dF_{k,n}(ds).$$

It is easy to see that for $\lambda_{1,n} = \pi_{\alpha(n+1)}$, where π_c is the Pareto distribution with density function $g_c(s) = c s^{-c-1} \mathbf{1}_{[1,\infty)}(s)$, we have

$$\int_0^\infty (1 - s^\alpha t^\alpha)_+^n \; \pi_{\alpha(n+1)}(ds) = (1 - t^\alpha)_+^{n+1}.$$

Consequently

$$\begin{aligned} &\int_0^\infty \int_0^\infty (1 - s^\alpha y^\alpha t^\alpha)_+^n \; \pi_{\alpha(n+1)}(ds) \; \pi_{\alpha(n+2)}(dy) \\ &\quad = \int_0^\infty (1 - y^\alpha t^\alpha)_+^{n+1} \pi_{\alpha(n+2)}(dy) = (1 - t^\alpha)_+^{n+2}. \end{aligned}$$

We see now that $\lambda_{k,n} = \mathcal{L}(Z_{1,n} \ldots Z_{n,n})$ where $Z_{1,n} \ldots Z_{n,n}$ are independent and $\mathcal{L}(Z_{k,n}) = \pi_{\alpha(n+k)}$. It is only a matter of laborious calculations to show that $\lambda_{k,n}$, $k \geqslant 1$ has density function

$$f_{k,n}(s) = \alpha k \binom{n+k}{n} s^{-\alpha(n+1)-1} \left(1 - s^{-\alpha}\right)_+^{k-1}.$$

Of course $\lambda_{0,n} = \delta_1$. Consequently φ_n is a solution of equation ($**$). The formal definition of this convolution for $x \in [0,1]$ can be written in the following form:

$$\delta_x \diamond \delta_1(ds) = (1 - x^\alpha)^n \delta_1(ds) + \sum_{k=1}^n \binom{n}{k} x^{\alpha k}(1 - x^\alpha)^{n-k} f_{k,n}(s)ds.$$

Example 2. In [16] K. Urbanik gave an example of a not regular generalized convolution different from the max-convolution. It is called $(1,p)$-convolution with $p \in (0,1)$ and defined for $p \neq \frac{1}{2}$ by

$$\delta_x \diamond_p \delta_1(ds) = (1 - px)\delta_1(ds) + px(2p-1)^{-1}s^{-3}\left(2p - s^{-q}\right)\mathbf{1}_{[1,\infty)}(s)ds,$$

and for $p = \frac{1}{2}$

$$\delta_x \diamond_p \delta_1(ds) = \left(1 - \frac{1}{2}x\right)\delta_1(ds) + \frac{1}{2}xs^{-3}\left(1 + 2\ln s\right)\mathbf{1}_{[1,\infty)}(s)ds.$$

Notice that we have here a solution of equation ($**$) with $n = 2$, $p_0(x) = (1 - px)$, $p_1(x) = px$ and the probability kernel given by

$$\varphi(t) = (1 - pt)\mathbf{1}_{[0,1]}(t).$$

Notice that φ here is not continuous at 1 as a function on the whole $[0,\infty)$ and discontinuity appears only at this point.

4. Applying the solution of the Levi-Civita equation for $n = 3$

The main aim of this paper is to show that for $n > 2$ there exist more than one solution of equation ($**$) in the set of generalized characteristic functions. We show this in the case $n = 3$ under the following additional assumptions:

$$p_1(1) = 1, \quad \varphi(t) = 0 \text{ for each } t > 1, \quad \lim_{t\to 1^-} \frac{\Phi_2(t)}{\varphi(t)} = 0. \tag{A}$$

The assumption $p_1(1) = 1$ implies that $\Phi_1(t) = \varphi(t)^2$. Since $\varphi(t) \neq 0$ for each $t \in [0,1)$ equation ($**$) can be written in the following way:

$$\varphi(xt) = p_0(x) + p_1(x)\varphi(t) + p_2(x)\frac{\Phi_2(t)}{\varphi(t)}.$$

By the continuity of generalized characteristic functions we see that

$$\varphi(x) = p_1(x) + p_2(x) \lim_{t\to 1^-} \frac{\Phi_2(t)}{\varphi(t)},$$

thus the limit $g = \lim_{t\to 1^-} \frac{\Phi_2(t)}{\varphi(t)}$ exists anyway, but the additional assumption $g = 0$ is equivalent to the condition $p_0(x) = \varphi(x)$. Consequently, equation ($**$)

restricted to the set $[0,1]$ under the additional assumptions (A) can be written as the following version of the Levi-Civita functional equation:

$$\varphi(xt) = \varphi(x) + p_1(x)\varphi(t) + \big((1-\varphi(x)-p_1(x)\big)\frac{\Phi_2(t)}{\varphi(t)}, \qquad (^{***})$$

with $1-\varphi(x) \geqslant p_1(x) \geqslant 0$ for $x,t \in [0,1]$.

Proposition 2. *In the case $n=3$ every continuous function φ satisfying $(**)$, $(***)$ and assumptions (A) has to have one of the forms*

$$\varphi(t) = \big(1 - t^\alpha + ct^\alpha \ln t\big)\mathbf{1}_{[0,1]}(t), \quad \textit{for some} \quad \alpha > 0,$$

or, for some $\alpha,\beta > 0$, $c \geqslant -1$, $cp \leqslant c+1$ and $p = \beta/\alpha > 1$

$$\varphi(t) = \big(1-(c+1)t^\alpha + ct^\beta\big)\mathbf{1}_{[0,1]}(t).$$

Proof. By the Levi-Civita result we have only 3 possible solutions of equation $(***)$ for $n=3$:

$$\varphi(t) = \big(a\ln^2 t + b\ln t + c\big)t^\alpha \mathbf{1}_{[0,1]}(t),$$
$$\varphi(t) = \big((a\ln t + b)t^\alpha + ct^\beta\big)\mathbf{1}_{[0,1]}(t),$$
$$\varphi(t) = \big(at^\alpha + bt^\beta + ct^\gamma\big)\mathbf{1}_{[0,1]}(t),$$

for some constants $a,b,c,\alpha,\beta,\gamma$. Applying the information which we already have: $\varphi(0^+) = 1$, $\varphi(1) = 0$ we see that only two types of functions can be considered:

$$\varphi(t) = \big(1 - t^\alpha + ct^\alpha \ln t\big)\mathbf{1}_{[0,1]}(t), \quad \varphi(t) = \big(1-(c+1)t^\alpha + ct^\beta\big)\mathbf{1}_{[0,1]}(t),$$

for some $c \in \mathbb{R}$ and $\beta > \alpha > 0$. The condition $\inf\{t > 0\colon \varphi(t) = 0\} = 1$, in particular $\varphi(1^-) = 0$, implies the final restrictions for c. □

By Proposition 1, without loss of generality, we can assume that $\alpha = 1$ and $p > 1$, thus we shall discuss only the following functions:

$$\varphi(t) = 1 - t + ct\ln t, \quad \varphi(t) = 1-(c+1)t + ct^p.$$

It turns out that only one type of such functions is admissible for us.

Proposition 3. *If $c \neq 0$ then none of the functions $\varphi(t) = \big(1-t^\alpha+ct^\alpha \ln t\big)\mathbf{1}_{[0,1]}(t)$ can be a solution of equation $(**)$.*

Proof. We show that there is no cumulative distribution function F for which

$$(1-t+ct\ln t)_+^2\,\mathbf{1}_{[0,1]}(t) = \int_0^\infty (1-st+cst\ln(st))\,\mathbf{1}_{[0,1]}(st)dF(s).$$

Notice first that the function $\varphi(st)$ integrated on the right hand side is positive if and only if $s < 1/t$, thus the area of integration is included in $[0,1/t]$. The function $\varphi(t)^2$ on the left hand side of this equation is equal to zero for all $t > 1$, thus the integral on the right hand side disappears for all $t > 1$. This implies that $F(s) = 0$ for all $s < 1$. We see now that the function F satisfies the following equation:

$$(1 - t + ct\ln t)_+^2\, \mathbf{1}_{[0,1]}(t) = \int_1^{1/t} (1 - st + cst\ln(st))\, \mathbf{1}_{[0,1]}(st) dF(s),$$

where by the Stieltjes integral $\int_a^b g\,dF$ we understand $\int_{[a,b)} g\,dF$. Notice now that for a continuous differentiable function g and a cumulative distribution function F such that $F(1) = 0 < F(1^+)$ we have the following formula for integration by parts for every continuity (with respect to F) point $b > 1$

$$\begin{aligned}\int_1^b g(x)dF(x) &= g(1)F(1^+) + \int_1^b g(x)\,d(F(x) - F(1^+))\\ &= g(b)F(b) - \int_1^b g'(x)F(x)\,dx.\end{aligned}$$

Using this formula, dividing both sides of our equation by t and substituting $t^{-1} = u$ we get for almost every $u > 1$

$$\begin{aligned}&u\left(1 - u^{-1} - cu^{-1}\ln{}^u\right)_+^2 \mathbf{1}_{[1,\infty)}(u)\\ &= (c\ln u + 1 - c)\int_1^u F(s)ds - c\int_1^u \ln s\, F(s)ds.\end{aligned}$$

The left hand side of this equation is differentiable for each $u > 1$ thus also the right hand side is differentiable for $u > 1$ and we get

$$\begin{aligned}&(1-c)F(u) + cu^{-1}\int_1^u F(s)ds \qquad (B)\\ &= 1 - 2cu^{-1} - (1-2c)u^{-2} - 2c(1-c)u^{-2}\ln u - c^2u^{-2}\ln^2 u.\end{aligned}$$

Case 1. If $c = 1$ then we obtain

$$F(u) = 1 - u^{-2}\left(1 - \ln^2 u + 2\ln u\right)\mathbf{1}_{[1,\infty)}(u).$$

We see that $F(1^+) = 0$, $\lim_{s\to\infty} F(u) = 1$, as it shall be expected, but the corresponding density function can take negative values:

$$f(u) = F'(u) = 2u^{-3}\ln u\,(3 - \ln u)\,\mathbf{1}_{[1,\infty)}(u),$$

thus this function φ is not a solution of equation $(**)$.

Let $H(u) := \int_1^u F(t)dt$. To solve equation (B) we solve first the homogenous equation $(1-c)H'(u) + cu^{-1}H(u) = 0$ and obtain $H(u) = Au^{-\beta}$, where $\beta = \frac{c}{1-c}$. Coming back to the original equation (B) we assume that $A = A(u)$. Thus for $u > 1$

$$(1-c)A'(u)u^{-\beta} = 1 - 2cu^{-1} - (1-2c)u^{-2} - 2c(1-c)u^{-2}\ln u - c^2u^{-2}\ln^2 u.$$

Case 2. If $c = \frac{1}{2}$ then we have

$$\frac{1}{2}A'(u)u^{-1} = 1 - u^{-1} - \frac{1}{2}u^{-2}\ln u - \frac{1}{4}u^{-2}\ln^2 u,$$

thus for some constant K

$$F(u) = 1 - u^{-2}\left(K + \ln u - \frac{1}{6}\ln^3 u\right).$$

Since $F_1(1^+) = 0$, $K = 1$ and the eventual density function $f = F'$ would be the following:

$$f(u) = u^{-3}\ln^3 u\left(\ln^{-3} u + 2\ln^{-2} u - \frac{1}{2}\ln^{-1} u - \frac{1}{3}\right)\mathbf{1}_{[1,\infty)}(u).$$

This however is impossible since the expression in brackets is negative for u large enough.

Case 3. If $c \notin \{1, \frac{1}{2}\}$ then for some K we have

$$\begin{aligned} H(u) = A(u)u^{-\beta} &= u - 2 + u^{-1} - \frac{2c(1-c)}{2c-1}\, u^{-1}\left[\ln u - \frac{1}{\beta-1}\right] \\ &\quad + Ku^{-\beta} - \frac{c^2}{2c-1}\, u^{-1}\left[\ln^2 u - \frac{2}{\beta-1}\ln u + \frac{2}{(\beta-1)^2}\right]. \end{aligned}$$

Consequently

$$\begin{aligned} F(u) = H'(u) &= 1 - u^{-2} + \frac{2c(1-c)}{2c-1}\left[\ln u - \frac{c}{2c-1}\right]u^{-2} \\ &\quad - K\beta u^{-\beta-1} + \frac{c^2}{2c-1}\left[\ln^2 u - \frac{2c}{2c-1}\ln u + \frac{2c(1-c)}{(2c-1)^2}\right]\, u^{-2}. \end{aligned}$$

By Remark 2 we can assume that $F_1(1^+) = 0$, thus $K = \frac{2c(1-c)^3}{(2c-1)^3}$. Now we see that the eventual density function would be the following:

$$\begin{aligned} f(u) = F'(u) &= u^{-3}\ln^2 u\left[K\beta(\beta+1)\frac{u^{1-\beta}}{\ln^2 u} - \frac{c^2}{2c-1}\right. \\ &\quad \left. + \frac{2(1-c)^2(3c-1)}{(2c-1)^3}\frac{1}{\ln^2 u} + 2c\left(1 + \frac{c}{(2c-1)^2}\right)\frac{1}{\ln u}\right]. \end{aligned}$$

If $c > \frac{1}{2}$ then $1-\beta < 0$ thus the expression in the brackets is close to $-\frac{c^2}{2c-1} < 0$ for u large enough, thus f cannot be a density function for any probability distribution. If $c \in (0, \frac{1}{2})$ then $1 - \beta > 0$ and $K < 0$, thus the expression in the brackets has the same limit at infinity as

$$\lim_{u\to\infty} K\beta(\beta+1)\frac{u^{1-\beta}}{\ln^2 u} = -\infty,$$

which is also impossible for any probability density function. $\square$

Considering the probability kernel $\varphi(t) = (1-(c+1)t+ct^p)_+$ we will show that for $n = 3$ there exist generalized convolutions defined by equation $(*)$ other than the Kucharczak-Urbanik convolutions described in Example 1.

Lemma 3. *Assume that the function $\varphi(t) = (1-(c+1)t+ct^p)\mathbf{1}_{[0,1]}(t)$ satisfies both equations* $(**)$ *and* $(***)$, *i.e. it defines a generalized convolution $\diamond$ on $\mathcal{P}_+$. Then the cumulative distribution function F_x of the measure $\delta_x \diamond \delta_1$, $x \in [0,1]$ satisfies the following equation:*

$$\begin{aligned}(1+c-pc)F_x(u) &+ p(p-1)cu^{-p}\int_1^u s^{p-1}F_x(s)ds \\ &= 1 - c(p-1)(x^p+1)u^{-p} + c(1+c)p(x^p+x)u^{-p-1} \\ &-(1+c)^2xu^{-2} - c^2(2p-1)x^pu^{-2p}.\end{aligned}$$

Proof. Let $x \in [0,1]$. We need to calculate the function F_x for which the following equality holds:

$$\begin{aligned}L &:= \varphi(xt)\varphi(t) = \big(1-(c+1)xt + cx^pt^p\big)_+\big(1-(c+1)t+ct^p\big)_+ \\ &= \int_0^{1/t}\big(1-(c+1)st+cs^pt^p\big)\,dF_x(s) =: R.\end{aligned}$$

The function L is zero if $xt > 1$ or $t > 1$, thus the integral R vanishes if $1/t < 1$. This means that the distribution function F_x is supported on $[1,\infty)$. For $t < 1$ integrating by parts we obtain

$$R = (1+c)t\int_1^{1/t} F_x(s)ds - pct^p\int_1^{1/t} s^{p-1}F_x(s)ds.$$

Substituting $t = u^{-1} > 1$ we have that

$$\big(uR\big)' = (1+c-pc)F_x(u) + p(p-1)cu^{-p}\int_1^u s^{p-1}F_x(s)ds.$$

Applying the same operations to the function L we have

$$\begin{aligned}\big(uL\big)' &= 1 - c(p-1)(x^p+1)u^{-p} + c(1+c)p(x^p+x)u^{-p-1} \\ &-(1+c)^2xu^{-2} - c^2(2p-1)x^pu^{-2p}.\end{aligned}$$

$\square$

Proposition 4. *For every $p \geqslant 2$ and $c = (p-1)^{-1}$ the function $\varphi(t) = \varphi_{c,p}(t) = (1-(c+1)t+ct^p)\mathbf{1}_{[0,1]}(t)$ is the kernel of the generalized characteristic function for the convolution $\diamond$ on $\mathcal{P}_+$ defined for $x \in [0,1]$ by the formula:*

$$\delta_x \diamond \delta_1 = \varphi(x)\delta_1 + x^p\lambda_1 + (c+1)(x-x^p)\lambda_2,$$

where λ_1, λ_2 are probability measures with densities

$$\lambda_1(du) = \frac{2pu^{-3}}{(p-1)^2}\Big[(p+1)u^{1-p} + (p-2) - (2p-1)u^{2-2p}\Big]\mathbf{1}_{[1,\infty)}(u)du,$$

and

$$\lambda_2(du) = c\big[2(p-2) + (p+1)u^{-p+1}\big]u^{-3}\mathbf{1}_{[1,\infty)}(u)du.$$

If $c = (p-1)^{-1}$ and $p \in (1,2)$ then none of the functions $\varphi_{c,p}$ can be a probability kernel of any generalized characteristic function.

Proof. Applying Lemma 3 we see that for $c = (p-1)^{-1}$ we have $1+c-pc = 0$. Comparing $\big(u^p(uR)'\big)' = \big(u^p(uL)'\big)'$ we obtain

$$pu^{p-1}F_x(u) = pu^{p-1} - \frac{p^2}{(p-1)^2}(x^p + x)u^{-2} \\ - \frac{p^2(p-2)}{(p-1)^2}\, xu^{p-3} + \frac{p(2p-1)}{(p-1)^2}\, x^p u^{-p-1}.$$

Consequently for $u \geqslant 1$

$$F_x(u) = 1 - \frac{p(x^p + x)}{(p-1)^2} u^{-p-1} - \frac{p(p-2)x}{(p-1)^2}\, u^{-2} + \frac{(2p-1)x^p}{(p-1)^2}\, u^{-2p}.$$

The function F_x is a cumulative distribution function of some measure Λ_x. We see that $F_x(+\infty) = 1$ thus $\Lambda_x([1,\infty)) = 1$ and

$$F_x(1^+) = 1 - \frac{p}{p-1}\, x + \frac{1}{p-1}\, x^p = \varphi(x) > 0,$$

which means that the measure Λ_x has an atom at the point 1 of the weight $\varphi(x)$. Moreover,

$$(p-1)^2 p^{-1} F_x'(u) \\ = (p+1)(x^p + x)u^{-p-2} + 2(p-2)xu^{-3} - 2(2p-1)x^p u^{-2p-1}.$$

Case 1. If $p \geqslant 2$ it is enough to notice that $x > x^p$, $u^{-p-1} > u^{-2p}$ and $u^{-2} > u^{-2p}$, and we obtain

$$(p-1)^2 p^{-1} u F_x'(u) > \big[2(p+1) + 2(p-2) - 2(2p-1)\big]\, \frac{x^p}{u^{2p}} = 0,$$

which shows that λ_x is a positive measure. In order to get the final formulation of Proposition 4 it is enough to notice that for $u \geqslant 1$

$$F_x(u) = \varphi(x) + x^p\, F_1(u) + \frac{(x - x^p)}{(p-1)} F_2(u),$$

where

$$F_2(u) = \Big[1 - (p-1)^{-1}\big((p-2)\, u^{-2} + u^{-p-1}\big)\Big]\mathbf{1}_{[1,\infty)}(u)$$

is the cumulative distribution function of the measure

$$\lambda_2(du) = \frac{1}{p-1}\Big[2(p-2)u^{-3} + (p+1)u^{-p-2}\Big]\mathbf{1}_{[1,\infty)}(u)\, du,$$

and

$$F_1(u) = \left[1 - \frac{2p}{(p-1)^2}\, u^{-p-1} - \frac{p(p-2)}{(p-1)^2}\, u^{-2} + \frac{2p-1}{(p-1)^2}\, u^{-2p}\right] \mathbf{1}_{[1,\infty)}(u)$$

is the cumulative distribution function for the measure

$$\lambda_1(du) = \frac{2p\, u^{-p-3}}{(p-1)^2}\Big[(p+1)u + (p-2)u^p - (2p-1)u^{2-p}\Big]\mathbf{1}_{[1,\infty)}(u)\, du.$$

We need to check that λ_1 is a positive measure. To see this it is enough to notice that in the last formula the expression in the brackets for $u > 0$ is greater than $(p+1) + (p-2) - (p-1) = p > 0$. It is evident that $F_1(+\infty) = F_2(+\infty) = 1$ thus λ_1, λ_2 are probability measures, which ends the proof in the case $p \geqslant 2$.

Case 2. If $p \in (1,2)$ we can write

$$\begin{aligned}&(p-1)^2 p^{-1} u^3 F_x'(u)\\ &\quad = (p+1)\,(x + x^p)\, u^{-(p-1)} + 2(p-2)x - 2(2p-1)x^p u^{-2(p-1)}.\end{aligned}$$

This means that $\lim_{u\to\infty} u^3 F_x'(u) = \frac{2p(p-2)x}{(p-1)^2} < 0$ thus $F_x'(u)$ is negative at least for u large enough and $x \neq 0$, and it cannot be a density function for any positive measure. □

Lemma 4. *Let $c(p-1) \neq 1$ and assume that the function $\varphi\colon [0,1] \mapsto \mathbb{R}$, $\varphi(t) = \varphi_{c,p}(t) = (1-(c+1)t + ct^p)\mathbf{1}_{[0,1]}(t)$ defines a generalized convolution $\diamond$ on $\mathcal{P}_+$. Then the cumulative distribution function F_x of the measure $\delta_x \diamond \delta_1$, $x \in [0,1]$, satisfies the equation $H(u) = A(u)u^{-\gamma}$, where $H(u) = \int_1^u s^{p-1}F_x(s)ds$, $\gamma = \frac{cp(p-1)}{1+c-cp}$ and*

$$\begin{aligned}(1+c-cp)A'(u) &= u^{p+\gamma-1} - c(p-1)(x^p+1)u^{\gamma-1} \\ &+c(c+1)p(x^p+x)u^{\gamma-2} - (1+c)^2 x u^{p+\gamma-3} - c^2(2p-1)x^p u^{\gamma-p-1}.\end{aligned}\tag{C}$$

Proof. By Lemma 3 we need to solve the equation $\big(uR\big)' = \big(uL\big)'$. Substituting $H(u) = \int_1^u s^{p-1}F_x(s)ds$, thus $H'(u) = u^{p-1}F(u)$, we solve first the homogenous equation $\big(uR\big)' = 0$ written in the following form:

$$(1+c-pc)u^{1-p}H'(u) + p(p-1)cu^{-p}H(u) = 0.$$

The solution is $H(u) = Au^{-\gamma}$, where $\gamma = \frac{cp(p-1)}{1+c-cp}$. Substituting $A = A(u)$ we see that the equation $\big(uR\big)' = \big(uL\big)'$ can be reformulated now as equation (C). □

Now we shall consider a few special cases.

Proposition 5. *If $\gamma = 1$, i.e. $c = (p^2-1)^{-1}$ and $p \geqslant 2$ then the function $\varphi(t) = \varphi_{c,p}(x) = \big(1-(c+1)x + cx^p\big)\mathbf{1}_{[0,1]}(x)$ is the probability kernel for a generalized convolution $\diamond$ given by*

$$\delta_x \diamond \delta_1 = \varphi(x)\delta_1 + x^p\,\lambda_1 + (c+1)(x - x^p)\,\lambda_2, \qquad x \in [0,1],$$

where λ_1, λ_2 *are absolutely continuous measures supported on* $[1, \infty)$ *with*

$$\lambda_1(du) = \frac{2pu^{-2p-1}\,du}{(p^2-1)(p-1)^2} \times \Big[p(p+1)u^{p-1}\ln u^{p-1} - p(p-1)u^{p-1} + p^2(p-2)u^{2(p-1)} + (2p-1)\Big],$$

and

$$\lambda_2(du) = \frac{u^{-p-2}}{(p-1)^2}\Big[2(p+1)\ln u^{p-1} + (3-p) + 2p(p-2)u^{p-1}\Big]du.$$

If $\gamma = 1$, $c = (p^2-1)^{-1}$ *and* $p \in (1,2)$ *then none of the functions* $\varphi_{c,p}$ *can be the probability kernel of a generalized convolution.*

Proof. If $\gamma = 1$ then $c = (p^2-1)^{-1}$ and equation $(*)$ from Lemma 4 takes the form

$$\frac{p}{p+1}\,A'(u) = u^p - \frac{1}{p+1}\,(x^p+1) + \frac{p^3}{(p^2-1)^2}\,(x^p+x)u^{-1} - \frac{p^4}{(p^2-1)^2}\,xu^{p-2} - \frac{2p-1}{(p^2-1)^2}\,x^p u^{-p}.$$

Since $H(u) = A(u)u^{-1}$ and $H'(u) = u^{p-1}F_x(u)$, we obtain for some constant K:

$$F_x(u) = 1 - Ku^{-p-1} - \frac{p^2(x^p+x)}{(p^2-1)(p-1)}\,u^{-p-1}\ln u + \frac{p^2(x^p+x)}{(p^2-1)(p-1)}\,u^{-p-1} - \frac{p^3(p-2)x}{(p^2-1)(p-1)^2}\,u^{-2} - \frac{(2p-1)x^p}{(p^2-1)(p-1)^2}\,u^{-2p}.$$

Since $F_x(1^+) = \varphi(x)$, we obtain that $K = \frac{p^2(px+(p-2)x^p)}{(p^2-1)(p-1)^2}$, thus

$$F_x(u) = 1 - \frac{p^2(x^p+x)}{(p^2-1)(p-1)}\,u^{-p-1}\ln u - \frac{p^2(x-x^p)}{(p^2-1)(p-1)^2}\,u^{-p-1} - \frac{p^3(p-2)x}{(p^2-1)(p-1)^2}\,u^{-2} - \frac{(2p-1)x^p}{(p^2-1)(p-1)^2}\,u^{-2p}.$$

If for every $x \in [0,1]$ the measure λ_x with the distribution function F_x were a probability measure then in particular λ_1 were a probability measure and its density function F_1' were nonnegative. However

$$F_1'(u) = \frac{2pu^{-p-2}\ln u}{(p^2-1)(p-1)^2}\Big[p(p^2-1) - \frac{p(p-1)}{\ln u} + p^2(p-2)\,\frac{u^{p-1}}{\ln u} + \frac{(2p-1)}{u^{p-1}\ln u}\Big],$$

thus for $p \in (1,2)$ the expression in the brackets is negative for u large enough and in this case λ_1 is not a probability measure. If $p \geqslant 2$ then we can write

$$F_1'(u) = \frac{2pu^{-2p-1}}{(p^2-1)(p-1)^2} \times \Big[p(p+1)u^{p-1}\ln u^{p-1} - p(p-1)u^{p-1} + p^2(p-2)u^{2(p-1)} + (2p-1)\Big].$$

Substituting $u^{p-1} = e^t$, $t > 0$, we can write the expression in the brackets as $g(t) = p(p+1)te^t - p(p-1)e^t + p(p-2)e^{2t} + (2p-1)$. We see that $g(0) = p-1 > 1$ and $g'(t) > 0$ for $t > 0$, thus F_1' is a density of some probability measure. Now we can write for $u \geqslant 1$ and $x \in [0,1]$

$$\begin{aligned} F_x(u) &= \varphi(x) + x^p F_1(u) \\ &+ \frac{p^2}{p^2-1}(x-x^p)\left[1 - \frac{2}{p-1}\,u^{-p-1}\ln u - \frac{1}{(p-1)^2}\,u^{-p-1} - \frac{p(p-1)}{(p-1)^2}\,u^{-2}\right] \\ &=: \varphi(x) + x^p F_1(u) + \frac{p^2}{p^2-1}(x-x^p)F_2(u). \end{aligned}$$

It remains to show that F_2 is the distribution function for some probability measure. We see that $F_2(1^+) = 0$, $F_2(+\infty) = 1$ and

$$F_2'(u) = \frac{u^{-p-2}}{(p-1)^2}\Big[2(p+1)\ln u^{p-1} + (3-p) + 2p(p-2)u^{p-1}\Big].$$

Substituting $u^{p-1} = e^z$, $z > 0$ we can write the expression in the brackets as $g(z) = 2(p+1)z + (3-p) + 2p(p-2)e^z$. Since $g(0) = (p-1)(2p-3)$, which is positive for $p \geqslant 2$ and $g'(u) > 0$ for $u > 1$, we conclude that λ_2 with cumulative distribution function F_2 and density F_2' is a probability measure. $\square$

Proposition 6. *If $\gamma = 2-p$, then for each $p > 2$ the function $\varphi_{c,p}$ is the probability kernel of a generalized convolution $\diamond$ defined by*

$$\delta_x \diamond \delta_1 = \varphi_{c,p}(x)\,\delta_1 + x^p\,\lambda_1 + \frac{p}{2(p-1)}\,(x-x^p)\lambda_2,$$

where λ_1, λ_2 are probability measures supported on $(1,\infty)$ and

$$\begin{aligned} \lambda_1(du) &= \frac{(p-2)(p^2+6)+4}{2(p-1)^3}\,u^{-3}\,du \\ &\quad + \frac{p(p-2)}{2(p-1)^3}\Big[2(p+1)u^{-p-2} + 2u^{-3}\ln u + (p-2)(2p-1)u^{-2p-1}\Big]du, \\ \lambda_2(du) &= \left[\frac{(p-2)(p+1)}{(p-1)^2}\,u^{-p-2} + \frac{2(p-2)(p-1)}{(p-1)^2}u^{-3}\ln u \right. \\ &\quad \left. + \frac{(p-1)(p-2)+2}{(p-1)^2}\,u^{-3}\right]du. \end{aligned}$$

If $\gamma = 2-p$ and $p \in (1,2)$ then none of the functions $\varphi_{c,p}$ can be a probability kernel of any generalized convolution.

Proof. If $\gamma = 2-p$ then

$$c = \frac{2-p}{2(p-1)}, \quad 1+c = \frac{p}{2(p-1)}, \quad 1+c-cp = \frac{p}{2}.$$

Equation $(*)$ takes the form

$$A'(u) = \frac{2}{p}\,u - \frac{2-p}{p}\,(x^p+1)u^{1-p} + \frac{p(2-p)}{2(p-1)^2}\,(x^p+x)u^{-p} - \frac{p}{2(p-1)^2}xu^{-1} - \frac{(2p-1)(2-p)^2}{2p(p-1)^2}\,x^p u^{1-2p}.$$

Since $H(u) = A(u)u^{p-2}$ and $H'(u) = u^{p-1}F_x(u)$, we obtain

$$F_x(u) = 1 - \frac{p(p-2)}{2(p-1)^3}(x+x^p)u^{-p-1} - \frac{p(p-2)}{2(p-1)^2}xu^{-2}\ln u - \frac{(p-2)^2(2p-1)}{4(p-1)^3}x^p u^{-2p} + Ku^{-2},$$

for some constant K, which can be obtained from the relation $F_x(1^+) = \varphi_{c,p}(x)$. Finally we can write

$$F_x(u) = \varphi(x) + x^p F_1(u) + \frac{p}{2(p-1)}(x-x^p)F_2(u),$$

where

$$F_1(u) = 1 - \frac{2p(p^2-3p+3)+(p-2)(p+4)}{4(p-1)^3}\,u^{-2} - \frac{p(p-2)}{(o-1)^3}u^{-p-1} - \frac{p(p-2)}{2(p-1)^2}u^{-2}\ln u - \frac{(p-2)^2(2p-1)}{4(p-1)^3}u^{-2p},$$

$$F_1'(u) = \frac{(p-2)(p^2+6)+4}{2(p-1)^3}\,u^{-3} + \frac{p(p-2)}{2(p-1)^3}\Big[2(p+1)u^{p-1} + 2u^{2(p-1)}\ln u + (p-2)(2p-1)\Big]u^{-2p-1},$$

$$F_2(u) = 1 - \frac{p-2}{(p-1)^2}\,u^{-p-1} - \frac{p-2}{p-1}\,u^{-2} - \frac{p^2-3p+3}{(p-1)^2}\,u^{-2},$$

$$F_2'(u) = \frac{u^{-3}}{(p-1)^2}\left[\frac{(p-2)(p+1)}{u^{p-1}} + 2(p-2)(p-1)\ln u + (p-1)(p-2)+2\right].$$

Evidently $F_1'(u) > 0$ for all $u > 1$ if $p > 2$. If $p \in (1,2)$ then

$$F_1'(u)\,u^3 \sim \frac{(p-2)(p^2+6)+4}{2(p-1)^3} + \frac{p(p-2)}{(p-1)^3}\,\ln u,$$

thus it converges to $-\infty$ if $u \to \infty$, which is impossible. Now it is enough to notice that for $p > 2$ we have $F_2'(u) > 0$ for all $u > 1$. $\square$

Proposition 7. *If $\gamma = p \geqslant 2$ then $c = \frac{1}{2(p-1)}$ and $\varphi = \varphi_{c,p}$ is a probability kernel for the generalized convolution defined by*

$$\delta_x \diamond \delta_1 = \varphi_{c,p}(x)\delta_1 + x^p\lambda_1 + \frac{2p-1}{2(p-1)}(x - x^p)\lambda_2,$$

where λ_1, λ_2 are probability measures supported on $[1,\infty)$ and on this set

$$\begin{aligned}\lambda_1(du) = \frac{(2p-1)u^{-2p-1}}{2(p-1)^3}\Big[&2p(p+1)u^{p-1} + (2p-1)(p-2)u^{2(p-1)} \\ &-2p^2 \ln u^{p-1} - \frac{p(6p^2-4p+1)}{(2p-1)}\Big]du, \\ \lambda_2(du) = \frac{u^{-2p-1}}{(p-1)^2}\Big[&p(p+1)u^{p-1} + (2p-1)(p-2)u^{2(p-1)} - p^2\Big]du.\end{aligned}$$

If $\gamma = p$ and $p \in (1,2)$ then none of the functions $\varphi_{c,p}$ can be a probability kernel of a generalized convolution.

Proof. Since $\gamma = p$ then $c = \frac{1}{2(p-1)}$ and equation $(*)$ takes the form

$$\begin{aligned}A'(u) = {} & 2u^{2p-1} - (x^p+1)u^{p-1} \\ & + \frac{p(2p-1)}{2(p-1)^2}\,(x^p+x)u^{p-2} - \frac{(2p-1)^2}{2(p-1)^2}\,xu^{2p-3} - \frac{2p-1}{2(p-1)^2}\,x^p u^{-1}.\end{aligned}$$

Using the relations $H(u) = A(u)u^{-p}$, $H'(u) = u^{p-1}F_x(u)$ and $F_x(1^+) = \varphi(x)$, after laborious calculations we obtain

$$F_x(u) = \varphi_{c,p}(x) + x^p F_1(u) + \frac{2p-1}{2(p-1)}\,(x - x^p)F_2(u),$$

where for $u > 1$

$$\begin{aligned}F_1(u) = 1 & - \frac{p(2p-1)}{(p-1)^3}\,u^{-p-1} - \frac{(2p-1)^2(p-2)}{4(p-1)^3}\,u^{-2} \\ & + \frac{p(2p-1)}{2(p-1)^2}\,u^{-2p}\,\ln u + \frac{3p(2p-1)+2(p-1)^2}{4(p-1)^3}\,u^{-2p},\end{aligned}$$

and

$$F_2(u) = 1 - \frac{p}{(p-1)^2}\,u^{-p-1} - \frac{(2p-1)(p-2)}{2(p-1)^2}\,u^{-2} + \frac{p}{2(p-1)^2}\,u^{-2p}.$$

For $u > 1$ we have

$$\begin{aligned}F_1'(u) = \frac{(2p-1)u^{-2p-1}}{2(p-1)^3}\Big[&2p(p+1)u^{p-1} + (2p-1)(p-2)u^{2(p-1)} \\ &-2p^2 \ln u^{p-1} + \frac{p(4p^2-3p-3)}{(2p-1)}\Big].\end{aligned}$$

Substituting $e^z = u^{p-1}$, $z > 0$, the expression in the brackets can be written in the form $g(z) = 2p(p+1)e^z + (2p-1)(p-2)e^{2z} - 2p^2 z + p(4p^2-3p-3)(2p-1)^{-1}$. If $p > 2$ we have $g(0) > 0$ and

$$\begin{aligned} g'(z) &= 2p(p+1)e^z + 2(2p-1)(p-2)e^{2z} - 2p^2 \\ &> 2pe^z + (2p-1)(p-2)e^{2z} > 0, \end{aligned}$$

thus $F_1'(u)\mathbf{1}_{[1,\infty)}(u)$ is the density of a probability measure λ_1. If $p \in (1,2)$ then $\lim_{z\to\infty} g'(z) = -\infty$ thus $g(z)$ must be negative for z large enough, which is impossible.

Now we shall consider

$$F_2'(u) = \frac{u^{-2p-1}}{(p-1)^2}\Big[p(p+1)u^{p-1} + 2(2p-1)(p-2)u^{2(p-1)} - p^2\Big],$$

for $u > 1$ and $p > 2$. Substituting $z = u^{p-1} > 1$ the expression in the brackets can be written in the form $h(z) = 2(2p-1)(p-2)z^2 + p(p+1)z - p^2$. For $p > 2$ we have $h(1^+) = p + 2(2p-1)(p-2) > 0$ and $h'(z) > 0$ for $z > 1$ thus F_2 is the cumulative distribution function of the probability measure λ_2 with density F_2'. □

Proposition 8. *If $\gamma \notin \{1, 2-p, p\}$, $c \neq (p-1)^{-1}$ and $c \in ((p^2-1)^{-1}, (2(p-1))^{-1})$ then the function $\varphi = \varphi_{c,p}$ is the probability kernel of a generalized convolution defined by*

$$\delta_x \diamond \delta_1 = \varphi(x)\,\delta_1 + x^p\,\lambda_1 + (c+1)(x - x^p)\,\lambda_2,$$

where λ_1 has the distribution function $F_1(u) = G_1(u) - G_1(1)u^{-p-\gamma}$, for

$$\begin{aligned} G_1(u) = 1 &- \frac{2\gamma(c+1)}{(p-1)(\gamma-1)}\,u^{-p-1} - \frac{\gamma(p-2)(c+1)^2}{cp(p-1)(p+\gamma-2)}\,u^{-2} \\ &+ \frac{c\gamma(2p-1)(p+1)}{p(p-1)(\gamma-p)}\,u^{-2p}, \end{aligned}$$

and λ_2 has the distribution function $F_2(u) = G_2(u) - G_2(1)u^{-p-\gamma}$, for

$$G_2(u) = 1 - \frac{\gamma}{(p-1)(\gamma-1)}\,u^{-p-1} - \frac{\gamma(p-2)(c+1)}{cp(p-1)(p+\gamma-2)}\,u^{-2}.$$

Proof. If $\gamma \notin \{1, 2-p, p\}$ and $c \neq (p-1)^{-1}$ then coming back to equation $(*)$ we can calculate the function F_x for $u > 1$:

$$\begin{aligned} F_x(u) = 1 - Ku^{-p-\gamma} &- \frac{\gamma(c+1)}{(p-1)(\gamma-1)}\,(x^p + x)u^{-p-1} \\ &- \frac{\gamma(p-2)(c+1)^2}{cp(p-1)(p+\gamma-2)}\,xu^{-2} + \frac{c\gamma(2p-1)(p+1)}{p(p-1)(\gamma-p)}\,x^p u^{-2p}. \end{aligned}$$

Since $F_x(1^+) = \varphi_{c,p}(x)$, we can calculate the missing constant K:

$$K = (c+1)x\left[1 - \frac{\gamma}{(p-1)(\gamma-1)} - \frac{(c+1)(p-2)\gamma}{c(p-1)(p+\gamma-2)}\right] \\ +cx^p\left[-1 - \frac{(c+1)\gamma}{c(p-1)(\gamma-1)} + \frac{\gamma(2p-1)(p+1)}{p(p-1)(\gamma-p)}\right].$$

Finally we obtain

$$F_x(u) = \Big[\varphi(x) + x^p F_1(u) + (c+1)(x - x^p)F_2(u)\Big]\,\mathbf{1}_{(1,\infty)}(u),$$

where

$$\begin{aligned} F_1(u) &= G_1(u) - G_1(1)u^{-p-\gamma}, \\ G_1(u) &= 1 - \frac{2\gamma(c+1)}{(p-1)(\gamma-1)}\,u^{-p-1} - \frac{\gamma(p-2)(c+1)^2}{cp(p-1)(p+\gamma-2)}\,u^{-2} \\ &\quad + \frac{c\gamma(2p-1)(p+1)}{p(p-1)(\gamma-p)}\,u^{-2p}, \end{aligned}$$

and

$$\begin{aligned} F_2(u) &= G_2(u) - G_2(1)u^{-p-\gamma}, \\ G_2(u) &= 1 - \frac{\gamma}{(p-1)(\gamma-1)}\,u^{-p-1} - \frac{\gamma(p-2)(c+1)}{cp(p-1)(p+\gamma-2)}\,u^{-2}. \end{aligned}$$

Now it is enough to notice that F_x is a distribution function of a probability measure if $\gamma - 1 > 0$, $\gamma - p < 0$ and $(p-2)(p+\gamma-2) > 0$. This gives $c \in ((p^2-1)^{-1}, (2(p-1))^{-1})$. $\square$

The next remark can be easily derived from the previous results:

Remark 4. For every $c \in [\frac{1}{3}, \frac{1}{2}] \cup \{1\}$ (and only for such c) the function $\varphi(t) = (1 - (c+1)t + ct^2)\mathbf{1}_{[0,1]}(t)$ is the probability kernel for some generalized convolution.

Proof. The classical Kucharczak-Urbanik generalized convolution with the probability kernel

$$\varphi(t) = (1 - 2t + t^2)\mathbf{1}_{[0,1]}(t)$$

is a special case in Proposition 4 since in this case $c = 1$. By Proposition 5 for $c = \frac{1}{3}$ we obtain that the function

$$\varphi(t) = \left(1 - \frac{4}{3}\,t + \frac{1}{3}\,t^2\right)\mathbf{1}_{[0,1]}(t)$$

is the probability kernel for the generalized convolution defined by $\delta_x \diamond \delta_1 = \varphi(x)\delta_1 + x^2\lambda_1 + \frac{4}{3}(x-x^2)\lambda_2$, where

$$\lambda_1(du) = \frac{4}{3}\,(6u\ln u - 2u + 3)\,u^{-4}\mathbf{1}_{[1,\infty)}(u)\,du;$$
$$\lambda_2(du) = (6\ln u + 1)\,u^{-4}\mathbf{1}_{[1,\infty)}(u)\,du.$$

By Proposition 7 we have that for $c = \frac{1}{2}$ the function

$$\varphi(t) = \left(1 - \frac{3}{2}\,t + \frac{1}{2}\,t^2\right)\mathbf{1}_{[0,1]}(t)$$

is the probability kernel for the generalized convolution defined by $\delta_x \diamond \delta_1 = \varphi(x)\delta_1 + x^2\lambda_1 + \frac{3}{2}(x-x^2)\lambda_2$, where

$$\lambda_1(du) = (18u + 12\ln u + 7)\,u^{-5}\mathbf{1}_{[1,\infty)}(u)\,du;$$
$$\lambda_2(du) = 2\,(3u-2)\,u^{-5}\mathbf{1}_{[1,\infty)}(u)\,du.$$

Finally, by Proposition 8 for every $c \in (\frac{1}{3}, \frac{1}{2})$ the function

$$\varphi(t) = \big(1 - (c+1)\,t + c\,t^2\big)\mathbf{1}_{[0,1]}(t)$$

is the probability kernel for the generalized convolution defined by $\delta_x \diamond \delta_1 = \varphi(x)\delta_1 + x^2\lambda_1 + (c+1)(x-x^2)\lambda_2$, where λ_1, λ_2 are probability measures on $[1,\infty)$ with distribution functions

$$F_1(u) = G_1(u) - G_1(1^+)u^{-p-\gamma}, \quad F_2(u) = G_2(u) - G_2(1)u^{-p-\gamma},$$

for $\gamma = \frac{2c}{1-c}$ and

$$G_1(u) = \left(1 - \frac{4c(c+1)}{3c-1}\,u^{-3} - \frac{9c^2}{2(1-2c)}u^{-4}\right)\mathbf{1}_{[1,\infty)}(u);$$
$$G_2(u) = \left(1 - \frac{2c}{3c-1}\,u^{-3}\right)u^{-5}\,\mathbf{1}_{[1,\infty)}(u)\,du.$$

$\square$

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