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Aequationes Mathematicae

Some stochastic HH-divergences in information theory

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Abstract. In this paper, we introduce the concept of stochastic HH-divergences based on convex stochastic processes. As an application, we propose some inequalities related to stochastic HH-divergences for convex stochastic processes. Our result extends HH-divergence in the class of f-divergence to the class of convex stochastic processes.

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1. Introduction

In information divergence, the Kullback–Leibler divergence [10] is a well-known concept in difference problems, information theory and statistics. f-divergence is a class of generalized divergences for a convex function f [7]. In 1991, Lin [11] introduced a new divergence in the class of f-divergences. As a result of Lin' divergence, the Hermite–Hadamard (HH-) divergence was introduced in [13] based on convex functions. Then upper and lower bounds for HH-divergence were obtained by the Hermite–Hadamard inequality for convex functions.

Let χ be a set and μ be a σ-finite measure on χ. Consider

$$\Lambda := \left\{ p \mid p : \chi \to \mathbb{R},\ p(x) \geq 0,\ \int_{\chi} p(x) d\mu(x) = 1 \right\},$$

as the set of all probability densities on μ. f-divergence [7] is defined as

$$D_f(p,q) := \int_{\chi} q(x) f\left[\frac{p(x)}{q(x)}\right] d\mu(x), \qquad p,q \in \Lambda, \tag{1}$$

where f is a nonnegative convex function such that $f(1) = 0$.

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For arbitrary probability densities $p(x)$ and $q(x)$, the Kullback–Leibler divergence [6, pp. 342] is defined as follows:

$$D_{KL}(p,q) := \int_{\chi} p(x) \log \left[\frac{p(x)}{q(x)} \right] d\mu(x), \qquad p, q \in \Lambda,$$

where log is on base 2.

Next, in the class of f-divergences, Lin [11] introduced the following divergence

$$D_{Lin}(p,q) := \int_{\chi} p(x) \log \left[\frac{p(x)}{\frac{1}{2}p(x) + \frac{1}{2}q(x)} \right] d\mu(x), \qquad p, q \in \Lambda.$$

By the Kullback–Leibler divergence, Lin's divergence is given by:

$$D_{Lin}(p,q) = D_{KL}\left(p, \frac{1}{2}p + \frac{1}{2}q\right).$$

As a generalization of Lin's divergence, the following divergence, called Hermite–Hadamard (HH)-divergence was introduced [6, 13]:

$$D^{f}_{HH}(p,q) := \int_{\chi} q(x) \frac{\int_{1}^{\frac{p(x)}{q(x)}} f(t)dt}{\left(\frac{p(x)}{q(x)} - 1\right)} d\mu(x), \quad p, q \in \Lambda. \tag{2}$$

Some new inequalities for HH-divergence in information theory were proved by Barnett et al. [5].

Recently, the three concepts of fractional HH-divergence were studied in [3], which generalizes the HH-divergence (2). Recall that the Riemann–Liouville fractional HH f-divergence of order $\alpha > 0$ is defined as [3]:

$${}^{\alpha}\mathbb{D}^{f}_{HH}(p,q) := \int_{\chi} q(x) \frac{\left[\left(\mathbb{I}^{\alpha}_{1+} f\right)\left(\frac{p(x)}{q(x)}\right) + \left(\mathbb{I}^{\alpha}_{\frac{p(x)}{q(x)}-} f\right)(1)\right]}{2\left(\frac{p(x)}{q(x)} - 1\right)^{\alpha}} d\mu(x), \qquad p, q \in \Lambda \tag{3}$$

where the left- and the right-side Riemann-Liouville fractional integrals of order α >0 of a real function f are defined by:

$$\left(\mathbb{I}^{\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t)dt, \qquad (x > a),$$

and

$$\left(\mathbb{I}^{\alpha}_{b-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t)dt, \qquad (x < b),$$

respectively and $\Gamma(\alpha)$ is the Gamma function. Clearly when $\alpha = 1$, (3) coincides to the HH f-divergence (2).

Convex stochastic processes and some of their properties were presented by Nikodem [12] in 1980. Let Ω be a probability measure space. A stochastic process $X : I \times \Omega \longrightarrow \mathbb{R}$ is called convex, if for all $\lambda \in [0,1]$ and $a, b \in I$, the inequality

$$X(\lambda a + (1-\lambda)b, \cdot) \leqslant \lambda X(a, \cdot) + (1-\lambda)X(b, \cdot) \quad (a.e.), \tag{4}$$

is satisfied. The concept of stochastic divergence is based on convex stochastic processes. The following definition gives us the concept of stochastic divergence for convex stochastic processes.

Definition 1. Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process in the interval $I \subseteq (0, \infty)$ such that $X(1, \cdot) = 0$. Stochastic divergence for $\gamma, \delta \in \Lambda$ is defined as:

$$SD_X(\gamma, \delta) := \int_{\chi} \delta(\omega) X\left(\frac{\gamma(\omega)}{\delta(\omega)}, \cdot\right) d\mu(\omega).$$

Stochastic processes play important roles in different fields of mathematics. For example, Kotrys in [9] proposed the Hermite–Hadamard inequality for convex stochastic processes. In [1], some refinements of mean-square stochastic integral inequalities on convex stochastic processes were proved. Recently, some fractional stochastic inequalities for convex stochastic processes were proposed in [2]. Also, the authors introduced the concepts of generalized stochastic mean square fractional integrals and comonotonic stochastic processes in [4].

In this paper, we introduce the concept of stochastic HH-divergence based on convex stochastic processes. Then its upper and lower bounds are obtained. Moreover, we introduce fractional stochastic HH-divergence which is a generalization of stochastic HH-divergence.

Now, we recall some basic definitions that are needed to prove our results.

Definition 2. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E\left[X^2(t, \cdot)\right] < \infty$ for all $t \in I$, where $E[X(t, \cdot)]$ denotes the expectation value of $X(t, \cdot)$. Let $[a, b] \subset I$, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \ldots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square stochastic integral of the process X on $[a, b]$, if for all sequences of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \ldots, n$ we have

$$\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k, \cdot)(t_k - t_{k-1}) - Y\right)^2\right] = 0.$$

Then we write

$$Y(\cdot) = \int_a^b X(s, \cdot)\, ds \qquad (a.e.).$$

In 2004, Hafiz [8] introduced the following definition of stochastic mean-square fractional integral.

Definition 3. For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the concept of stochastic mean-square fractional integrals $\mathbb{SFI}_{a+}^{\alpha}$ and $\mathbb{SFI}_{b-}^{\alpha}$ of X of order $\alpha > 0$ is defined by

$$\mathbb{SFI}_{a+}^{\alpha}\left[X\right](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} X(s,.)ds \quad (a.e.), \quad t > a,$$

and

$$\mathbb{SFI}_{b-}^{\alpha}\left[X\right](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} X(s,.)ds \quad (a.e.), \quad t < b.$$

Here $\Gamma(\alpha)$ is the Gamma function.

The paper is organized as follows: In Sect. 2, stochastic HH-divergences are presented. Next, we prove some inequalities related to stochastic HH-divergences. In Sect. 3, we introduce the concept of fractional stochastic HH-divergence with some results. Finally, we add some conclusions.

2. Main results

In this section, we first propose the concept of stochastic Hermite–Hadamard (HH-) divergence. Then, we prove some inequalities for stochastic HH-divergence on convex stochastic processes. Throughout this paper, $X : I \times \Omega \to \mathbb{R}$ is a convex stochastic process in the interval $I \subseteq (0,\infty)$ such that $X(1,\cdot) = 0$.

Definition 4. Let $\gamma, \delta \in \Lambda$. For a convex stochastic process $X : I \times \Omega \to \mathbb{R}$, the stochastic Hermite–Hadamard (HH) divergence is defined as:

$$SD_{HH}^{X}(\gamma,\delta) := \int_{\chi} \delta(x) \frac{\int_1^{\frac{\gamma(x)}{\delta(x)}} X(t,\cdot)\,dt}{\left(\frac{\gamma(x)}{\delta(x)} - 1\right)} d\mu(x) \quad (a.e.). \tag{5}$$

Theorem 5. *Let $\gamma, \delta \in \Lambda$. Then we have the inequality*

$$SD_X\left(\frac{1}{2}\gamma + \frac{1}{2}\delta, \delta\right) \leqslant SD_{HH}^{X}(\gamma,\delta) \leqslant SD_X(\gamma,\delta) \quad \text{(a.e.)}. \tag{6}$$

Proof. First, we recall the following Hermite–Hadamard inequalities for convex stochastic processes [9]:

$$X\left(\frac{a+b}{2},\cdot\right) \leqslant \frac{1}{(b-a)} \int_a^b X(t,\cdot)\,dt \leqslant \frac{X(a,\cdot)+X(b,\cdot)}{2} \quad (a.e.). \tag{7}$$

Taking $a = 1, b = \frac{\gamma(x)}{\delta(x)}$ in (7), we obtain

$$X\left(\frac{1+\frac{\gamma(x)}{\delta(x)}}{2},\cdot\right) \leqslant \frac{1}{\left(\frac{\gamma(x)}{\delta(x)}-1\right)} \int_1^{\frac{\gamma(x)}{\delta(x)}} X(t,\cdot)\,dt \leqslant \frac{X(1,\cdot)+X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)}{2} \quad (a.e.).$$

Since $X(1,.)=0$, we get

$$X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)},\cdot\right)\leqslant\frac{1}{\left(\frac{\gamma(x)}{\delta(x)}-1\right)}\int_1^{\frac{\gamma(x)}{\delta(x)}}X\left(t,\cdot\right)dt\leqslant\frac{1}{2}X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)\quad\textit{(a.e.)}.\tag{8}$$

Multiplying both sides of (8) by $\delta(x)\geqslant 0$, $x\in\chi$ and integrating on χ, we have

$$\int_\chi\delta(x)X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)},\cdot\right)d\mu(x)\leqslant\int_\chi\delta(x)\frac{\int_1^{\frac{\gamma(x)}{\delta(x)}}X\left(t,\cdot\right)dt}{\left(\frac{\gamma(x)}{\delta(x)}-1\right)}d\mu(x)$$
$$\leqslant\frac{1}{2}\int_\chi\delta(x)X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)d\mu(x)\ \textit{(a.e.)}.$$

By Definitions 1 and 4, we obtain (6). □

Theorem 6. *Let $\gamma,\delta\in\Lambda$. Then for any $\lambda\in[0,1]$, the following inequality*

$$SD_X\left(\frac{1}{2}\gamma+\frac{1}{2}\delta,\delta\right)\leqslant l(\lambda)\leqslant\ SD_{HH}^X\left(\gamma,\delta\right)\leqslant L(\lambda)\leqslant\frac{1}{2}SD_X\left(\gamma,\delta\right)\quad\text{(a.e.)}\tag{9}$$

holds where

$$l(\lambda):=\lambda SD_X\left(\delta+\frac{\lambda}{2}\left(\gamma-\delta\right),\delta\right)+\left(1-\lambda\right)SD_X\left(\frac{\gamma+\delta}{2}+\frac{\lambda}{2}\left(\gamma-\delta\right),\delta\right)$$

and

$$L(\lambda):=\frac{1}{2}\left[SD_X\left(\lambda\gamma+\left(1-\lambda\right)\delta,\delta\right)+\left(1-\lambda\right)SD_X\left(\gamma,\delta\right)\right].$$

Proof. By applying the following refinement of Hermite–Hadamard's inequalities for convex stochastic processes in [1], we have

$$X\left(\frac{a+b}{2},\cdot\right)\leqslant\lambda X\left(\frac{\lambda b+\left(2-\lambda\right)a}{2},\cdot\right)+\left(1-\lambda\right)X\left(\frac{\left(1+\lambda\right)b+\left(1-\lambda\right)a}{2},\cdot\right)$$
$$\leqslant\frac{1}{(b-a)}\int_a^b X\left(t,\cdot\right)dt\leqslant\frac{1}{2}\left[X\left(\lambda b+\left(1-\lambda\right)a,\cdot\right)+\lambda X\left(a,\cdot\right)+\left(1-\lambda\right)X\left(b,\cdot\right)\right]$$
$$\leqslant\frac{X(a,\cdot)+X(b,\cdot)}{2}\quad\textit{(a.e.)}.\tag{10}$$

Put $a=1,b=\frac{\gamma(x)}{\delta(x)}$ in (10). Then

$$X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)},\cdot\right)\leqslant\lambda X\left(\frac{2\delta(x)+\lambda\left(\gamma\left(x\right)-\delta(x)\right)}{2\delta(x)},\cdot\right)$$
$$+\left(1-\lambda\right)X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)}+\frac{\lambda\left(\gamma(x)-\delta(x)\right)}{2\delta\left(x\right)},\cdot\right)$$

$$
\begin{aligned}
&\leqslant \frac{1}{\left(\frac{\gamma(x)}{\delta(x)}-1\right)} \int_{1}^{\frac{\gamma(x)}{\delta(x)}} X\left(t,\cdot\right) dt \\
&\leqslant \frac{1}{2}\left[X\left(\frac{\lambda\gamma(x)+(1-\lambda)\,\delta(x)}{\delta(x)},\cdot\right)+\lambda X\left(1,\cdot\right)+(1-\lambda)\,X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)\right] \\
&\leqslant \frac{1}{2}\left(X(1,\cdot)+X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)\right) \quad \textit{(a.e.)}. \qquad (11)
\end{aligned}
$$

Multiplying both sides of (11) by $\delta(x)>0$ and integrating on χ, since $X(1,\cdot)=0$, we obtain

$$
\begin{aligned}
&\int_{\chi} \delta(x) X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)},\cdot\right) d\mu(x) \\
&\quad \leqslant \lambda \int_{\chi} \delta(x) X\left(\frac{2\delta(x)+\lambda\left(\gamma\left(x\right)-\delta(x)\right)}{2\delta(x)},\cdot\right) d\mu(x) \\
&\qquad +(1-\lambda)\int_{\chi} \delta(x) X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)}+\frac{\lambda\left(\gamma(x)-\delta(x)\right)}{2\delta\left(x\right)},\cdot\right) d\mu(x) \\
&\quad \leqslant \int_{\chi} \delta(x) \frac{\int_{1}^{\frac{\gamma(x)}{\delta(x)}} X\left(t,\cdot\right) dt}{\left(\frac{\gamma(x)}{\delta(x)}-1\right)} d\mu(x) \\
&\quad \leqslant \frac{1}{2}\int_{\chi} \delta(x) X\left(\frac{\lambda\gamma(x)+(1-\lambda)\,\delta(x)}{\delta(x)},\cdot\right) d\mu(x) \\
&\qquad +\frac{1}{2}\left(1-\lambda\right)\int_{\chi} \delta(x) X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right) d\mu(x) \\
&\quad \leqslant \frac{1}{2}\int_{\chi} \delta(x) X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)\ d\mu(x)\ \textit{(a.e.)}.
\end{aligned}
$$

By Definitions 1 and 4, we can obtain inequality (9). □

3. Further discussions: fractional stochastic *HH*-divergence

First, we introduce the fractional stochastic Hermite–Hadamard (HH-) divergence on a convex stochastic process $X : I\times\Omega\to\mathbb{R}$ in the interval $I\subseteq(0,\infty)$, such that $X(1,\cdot)=0$, which is generalization of the stochastic HH-divergence obtained in Definition 4. Then, a general version of inequality (6) for fractional stochastic HH-divergence is given.

Definition 7. For a convex stochastic process $X : I\times\Omega\to\mathbb{R}$, the fractional stochastic Hermite–Hadamard (HH) divergence of order $\alpha>0$ is defined as:

$$^{\alpha}SD_{HH}^{X}(\gamma,\delta) := \Gamma(\alpha+1)\int_{\chi}\delta(x)\frac{\left[\mathbb{SFI}_{1+}^{\alpha}[X]\left(\frac{\gamma(x)}{\delta(x)}\right)+\mathbb{SFI}_{\frac{\gamma(x)}{\delta(x)}-}^{\alpha}[X](1)\right]}{2\left(\frac{\gamma(x)}{\delta(x)}-1\right)^{\alpha}}d\mu(x)\ \textit{(a.e.)},$$

where $\mathbb{SFI}^{\alpha}$ is defined in Definition 3.

Remark 8. If $\alpha = 1$, then Definition 7 becomes Definition 4.

Theorem 9. *Let $\gamma, \delta \in \Lambda$. Then we have the inequality*

$$SD_X\left(\frac{1}{2}\gamma+\frac{1}{2}\delta,\delta\right) \leqslant {}^{\alpha}SD_{HH}^{X}(\gamma,\delta) \leqslant \frac{1}{2}SD_X(\gamma,\delta)\quad \text{(a.e.)}. \tag{12}$$

Proof. We recall the following Hermite–Hadamard inequalities via stochastic mean-square fractional integrals of order α for a convex stochastic process $X : I \times \Omega \to \mathbb{R}$ in [2]:

$$\begin{aligned} X\left(\frac{a+b}{2},\cdot\right) &\leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathbb{SFI}_{a+}^{\alpha}[X](b)+\mathbb{SFI}_{b-}^{\alpha}[X](a)\right] \\ &\leqslant \frac{X(a,\cdot)+X(b,\cdot)}{2}\quad \textit{(a.e.)}, \end{aligned} \tag{13}$$

where $a, b \in I$. If we choose $a = 1$, $b = \frac{\gamma(x)}{\delta(x)}$ in (13), then we get

$$\begin{aligned} X\left(\frac{1+\frac{\gamma(x)}{\delta(x)}}{2},\cdot\right) &\leqslant \frac{\Gamma(\alpha+1)}{2\left(\frac{\gamma(x)}{\delta(x)}-1\right)^{\alpha}}\left[\mathbb{SFI}_{1+}^{\alpha}[X]\left(\frac{\gamma(x)}{\delta(x)}\right)+\mathbb{SFI}_{\frac{\gamma(x)}{\delta(x)}-}^{\alpha}[X](1)\right] \\ &\leqslant \frac{X(1,\cdot)+X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)}{2}\quad \textit{(a.e.)}, \end{aligned}$$

since $X(1,\cdot) = 0$, we have

$$\begin{aligned} X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)},\cdot\right) &\leqslant \frac{\Gamma(\alpha+1)}{2\left(\frac{\gamma(x)}{\delta(x)}-1\right)^{\alpha}}\left[\mathbb{SFI}_{1+}^{\alpha}[X]\left(\frac{\gamma(x)}{\delta(x)}\right)+\mathbb{SFI}_{\frac{\gamma(x)}{\delta(x)}-}^{\alpha}[X](1)\right] \\ &\leqslant \frac{1}{2}X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)\quad \textit{(a.e.)}. \end{aligned} \tag{14}$$

Multiplying both sides of (14) by $\delta(x) > 0$ and integrating on χ, we obtain

$$\begin{aligned} &\int_{\chi}\delta(x)X\left(\frac{\gamma(x)+\delta(x)}{2\delta(x)},\cdot\right)d\mu(x) \\ &\leqslant \Gamma(\alpha+1)\int_{\chi}\delta(x)\frac{\left[\mathbb{SFI}_{1+}^{\alpha}[X]\left(\frac{\gamma(x)}{\delta(x)}\right)+\mathbb{SFI}_{\frac{\gamma(x)}{\delta(x)}-}^{\alpha}[X](1)\right]}{2\left(\frac{\gamma(x)}{\delta(x)}-1\right)^{\alpha}}d\mu(x) \\ &\leqslant \frac{1}{2}\int_{\chi}\delta(x)X\left(\frac{\gamma(x)}{\delta(x)},\cdot\right)d\mu(x)\quad \textit{(a.e.)}. \end{aligned}$$

The proof is completed by Definitions 1 and 7. □

Remark 10. In Theorem 9, if $\alpha = 1$, then we obtain Theorem 5.

4. Conclusions

We have introduced the concept of stochastic Hermite–Hadamard (HH-) divergence based on convex stochastic processes. Then, the upper and lower bounds of stochastic HH-divergence are proposed. Next, we have improved the result of Theorem 5 for stochastic HH-divergence obtained in Theorem 6. Also, we have introduced fractional stochastic HH-divergence of order α which is a generalization of stochastic HH-divergence. Thus, we have extended some previous results on HH-divergence in the class of f-divergence to the class of convex stochastic processes.

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