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Spherical bodies of constant width

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Abstract. The intersection L of two different non-opposite hemispheres G and H of the d-dimensional unit sphere S^d is called a lune. By the thickness of L we mean the distance of the centers of the $(d-1)$-dimensional hemispheres bounding L. For a hemisphere G supporting a convex body $C \subset S^d$ we define $\mathrm{width}_G(C)$ as the thickness of the narrowest lune or lunes of the form $G \cap H$ containing C. If $\mathrm{width}_G(C) = w$ for every hemisphere G supporting C, we say that C is a body of constant width w. We present properties of these bodies. In particular, we prove that the diameter of any spherical body C of constant width w on S^d is w, and that if $w < \frac{\pi}{2}$, then C is strictly convex. Moreover, we check when spherical bodies of constant width and constant diameter coincide.

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1. Introduction

Consider the unit sphere S^d in the $(d+1)$-dimensional Euclidean space E^{d+1} for $d \geq 2$. The intersection of S^d with any two-dimensional subspace of E^{d+1} is called a *great circle* of S^d. By a $(d-1)$-*dimensional great sphere* of S^d we mean the common part of S^d with any hyper-subspace of E^{d+1}. The 1-dimensional great spheres of S^2 are called *great circles*. By a pair of *antipodes* of S^d we understand a pair of points of intersection of S^d with a straight line through the origin of E^{d+1}.

Clearly, if two different points $a, b \in S^d$ are not antipodes, there is exactly one great circle containing them. As the *arc* ab connecting a and b we define the shorter part of the great circle containing these points. The length of this arc is called the *spherical distance* $|ab|$ *of* a *and* b, or shortly *distance*. Moreover, we agree that the distance of coinciding points is 0, and that of any pair of antipodes is π.

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A *spherical ball* $B_\rho(x)$ *of radius* $\rho \in (0, \frac{\pi}{2}]$, or *a ball* for short, is the set of points of S^d at distances at most ρ from a fixed point x, which is called the *center* of this ball. An *open ball* (a *sphere*) is the set of points of S^d having distance smaller than (respectively, exactly) ρ from a fixed point. A spherical ball of radius $\frac{\pi}{2}$ is called a *hemisphere*. So it is the common part of S^d and a closed half-space of E^{d+1}. We denote by $H(m)$ the hemisphere with center m. Two hemispheres with centers at a pair of antipodes are called *opposite*.

A *spherical* $(d-1)$*-dimensional ball of radius* $\rho \in (0, \frac{\pi}{2}]$ is the set of points of a $(d-1)$-dimensional great sphere of S^d which are at distances at most ρ from a fixed point. We call it the *center* of this ball. The $(d-1)$-dimensional balls of radius $\frac{\pi}{2}$ are called $(d-1)$*-dimensional hemispheres*, and *semicircles* for $d = 2$.

A set $C \subset S^d$ is said to be *convex* if no pair of antipodes belongs to C and if for every $a, b \in C$ we have $ab \subset C$. A closed convex set on S^d with non-empty interior is called a *convex body*. Some basic references on convex bodies and their properties are [4], [9] and [10]. A short survey of other definitions of convexity on S^d is given in Section 9.1 of [2].

Since the intersection of every family of convex sets is also convex, for every set $A \subset S^d$ contained in an open hemisphere of S^d there is the smallest convex set $\mathrm{conv}(A)$ containing Q. We call it *the convex hull of* A.

Let $C \subset S^d$ be a convex body. Let $Q \subset S^d$ be a convex body or a hemisphere. We say that C *touches* Q *from inside* if $C \subset Q$ and $\mathrm{bd}(C) \cap \mathrm{bd}(Q) \neq \emptyset$. We say that C *touches* Q *from outside* if $C \cap Q \neq \emptyset$ and $\mathrm{int}(C) \cap \mathrm{int}(Q) = \emptyset$. In both cases, points of $\mathrm{bd}(C) \cap \mathrm{bd}(Q)$ are called *points of touching*. In the first case, if Q is a hemisphere, we also say that Q *supports* C, or *supports* C *at* t, provided t is a point of touching. If at every boundary point of C exactly one hemisphere supports C, we say that C is *smooth*. We call $e \in C$ an extreme point of C if $C \setminus \{e\}$ is convex.

If hemispheres G and H of S^d are different and not opposite, then $L = G \cap H$ is called *a lune* of S^d. This notion is considered in many books and papers (for instance, see [12]). The $(d-1)$-dimensional hemispheres bounding L and contained in G and H, respectively, are denoted by G/H and H/G.

Observe that $(G/H) \cup (H/G)$ is the boundary of the lune $G \cap H$. Denote by $c_{G/H}$ and $c_{H/G}$ the centers of G/H and H/G, respectively. By *corners* of the lune $G \cap H$ we mean points of the set $(G/H) \cap (H/G)$. In particular, every lune on S^2 has two corners. They are antipodes.

We define the *thickness* $\Delta(L)$ *of a lune* $L = G \cap H$ on S^d as the spherical distance of the centers of the $(d-1)$-dimensional hemispheres G/H and H/G bounding L. Clearly, it is equal to each of the non-oriented angles $\angle c_{G/H} r c_{H/G}$, where r is any corner of L.

Compactness arguments show that for any hemisphere K supporting a convex body $C \subset S^d$ there is at least one hemisphere K^* supporting C such that the lune $K \cap K^*$ is of the minimum thickness. In other words, there is a "narrowest" lune of the form $K \cap K'$ over all hemispheres K' supporting C. The thickness of the lune $K \cap K^*$ is called *the width of C determined by K.* We denote it by $\mathrm{width}_K(C)$.

We define the *thickness* $\Delta(C)$ of a spherical convex body C as the smallest width of C. This definition is analogous to the classical definition of thickness (also called minimal width) of a convex body in Euclidean space.

By *the relative interior* of a convex set $C \subset S^d$ we mean the interior of C with respect to the smallest sphere $S^k \subset S^d$ that contains C.

2. A few lemmas on spherical convex bodies

Lemma 1. *Let A be a closed set contained in an open hemisphere of S^d. Then* $\mathrm{conv}(A)$ *coincides with the intersection of all hemispheres containing A.*

Proof. First, let us show that $\mathrm{conv}(A)$ is contained in the intersection of all hemispheres containing A. Take any hemisphere H containing A and denote by J the open hemisphere from the formulation of our lemma. Recall that $A \subset J$ and $A \subset H$. Thus since $J \cap H$ is a convex set, we obtain $\mathrm{conv}(A) \subset \mathrm{conv}(J \cap H) = J \cap H \subset H$. Thus, since $\mathrm{conv}(A)$ is contained in any hemisphere that contains A, also $\mathrm{conv}(A)$ is a subset of the intersection of all those hemispheres.

Now we intend to show that the intersection of all hemispheres containing A is contained in $\mathrm{conv}(A)$. Assume the opposite, i.e., that there is a point $x \notin \mathrm{conv}(A)$ which belongs to every hemisphere containing A. Since A is closed, by Lemma 1 of [6] the set $\mathrm{conv}(A)$ is also closed. Hence there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap \mathrm{conv}(A) = \emptyset$. Since these two sets are convex, we may apply the following more general version of Lemma 2 of [6]: *any two convex disjoint sets on S^d are subsets of two opposite hemispheres* (which is true again by the separation theorem for convex cones in E^{d+1}). So $B_\varepsilon(x)$ and $\mathrm{conv}(A)$ are in some two opposite hemispheres. Hence x does not belong to the one which contains $\mathrm{conv}(A)$. Clearly, that one also contains A. This contradicts our assumption on the choice of x, and thus the proof is finished. □

We omit the simple proof of the next lemma, which is analogous to the situation in E^d and needed a few times later. Here our hemisphere plays the role of a closed half-space there.

Lemma 2. *Let C be a spherical convex body. Assume that a hemisphere H supports C at a point p of the relative interior of a convex set $T \subset C$. Then $T \subset \mathrm{bd}(H)$.*

Lemma 3. *Let K, M be hemispheres such that the lune $K \cap M$ is of thickness smaller than $\frac{\pi}{2}$. Denote by b the center of M/K. Every point of $K \cap M$ at distance $\frac{\pi}{2}$ from b is a corner of $K \cap M$.*

Proof. Denote the center of K/M by a. Take any point $p \in K \cap M$. Let us show that there are points $x \in (K/M) \cap (M/K)$ and $y \in ab$ such that $p \in xy$.

If $p = b$ then it is obvious. Otherwise there is a unique point $q \in K/M$ such that $p \in bq$. Moreover, there exists $x \in (K/M) \cap (M/K)$ such that $q \in ax$. The reader can easily show that points p, q belong to the triangle abx and thus observe that there exists $y \in ab$ such that $p \in xy$, which confirms the statement from the first paragraph of the proof.

We have $|by| \leq |ba| < \frac{\pi}{2}$. The inequality $|by| < \frac{\pi}{2}$ means that y is in the interior of $H(b)$. Of course, $|bx| = \frac{\pi}{2}$, which means that $x \in \mathrm{bd}(H(b))$. From the two preceding sentences we conclude that xy is a subset of $H(b)$ with x being its only point on $\mathrm{bd}(H(b))$. Thus, if $|pb| = \frac{\pi}{2}$, we conclude that $p \in \mathrm{bd}(H(b))$, and consequently $p = x$, which implies that p is a corner of $K \cap M$. The last sentence means that the statement of our lemma holds true. □

Lemma 4. *Let $o \in S^d$ and $0 < \mu < \frac{\pi}{2}$. For every $x \in S^d$ at distance $\frac{\pi}{2}$ from o denote by x' the point of the arc ox at distance μ from x. Consider two points x_1, x_2 at distance $\frac{\pi}{2}$ from o such that $|x_1x_2| < \pi - \mu$. Then for every $x \in x_1x_2$ we have*

$$B_\mu(x') \subset \mathrm{conv}(B_\mu(x'_1) \cup B_\mu(x'_2)).$$

Proof. Let o, m be points of S^d and ρ be a positive number less than $\frac{\pi}{2}$. Let us show that

$$B_\rho(o) \subset H(m) \text{ if and only if } |om| \leq \frac{\pi}{2} - \rho. \tag{1}$$

First assume that $B_\rho(o) \subset H(m)$. Let b be a boundary point of $B_\rho(o)$ such that $o \in mb$. We have: $|om| = |bm| - |ob| = |bm| - \rho \leq \frac{\pi}{2} - \rho$, which confirms the "only if" part of (1). Assume now that $|om| \leq \frac{\pi}{2} - \rho$. Let b be any point of $B_\rho(o)$. We have: $|bm| \leq |bo| + |om| \leq \rho + \left(\frac{\pi}{2} - \rho\right) = \frac{\pi}{2}$. Therefore every point of $B_\rho(o)$ is at a distance at most $\frac{\pi}{2}$ from m. Hence $B \subset H(m)$, which confirms the "if" part of (1). So (1) is shown.

Lemma 1 of [6] guarantees that $Y = \mathrm{conv}(B_\mu(x'_1) \cup B_\mu(x'_2))$ is a closed set as a convex hull of a closed set. Consequently, from Lemma 1 we see that Y is the intersection of all hemispheres containing Y. Moreover, observe that an arbitrary hemisphere contains a set if and only if it contains the convex hull of it. Hence Y is the intersection of all hemispheres containing $B_\mu(x'_1) \cup B_\mu(x'_2)$.

As a result of the preceding paragraph, in order to prove the statement of our lemma it is sufficient to show that every hemisphere $H(m)$ containing $B_\mu(x'_1) \cup B_\mu(x'_2)$ also contains $B_\mu(x')$. Thus, having (1) in mind we see that in order to verify this it is sufficient to show that for any $m \in S^d$

$$|x'_1m| \leq \frac{\pi}{2} - \mu \text{ and } |x'_2m| \leq \frac{\pi}{2} - \mu \text{ imply } |x'm| \leq \frac{\pi}{2} - \mu. \tag{2}$$

Let us assume the first two of these inequalities and show the third one.

Observe that x, x_1' and x_2' belong to the spherical triangle x_1x_2o. Therefore the arcs xo and $x_1'x_2'$ intersect. Denote the point of intersection by g.

In this paragraph we consider the intersection of S^d with the three-dimensional subspace of E^{d+1} containing x_1', x_2', m. Observe that this intersection is a two-dimensional sphere concentric with S^d. Denote this sphere by S^2. Denote by $\bar{o}$ the other unique point on S^2 such that the triangles $x_1'x_2'o$ and $x_1'x_2'\bar{o}$ are congruent. By the first two inequalities of (2) we obtain $m \in B_{\frac{\pi}{2}-\mu}(x_1') \cap B_{\frac{\pi}{2}-\mu}(x_2')$. Observe that $g\bar{o} \cup go$ dissects $B_{\frac{\pi}{2}-\mu}(x_1') \cap B_{\frac{\pi}{2}-\mu}(x_2')$ into two parts so that x_1' belongs to one of them and x_2' belongs to the other. Therefore at least one of the arcs $x_1'm$ and $x_2'm$, say $x_1'm$ intersects $g\bar{o}$ or go, say go. Denote this point of intersection by s. Taking the first assumption of (2) into account and using two times the triangle inequality we obtain $|og| = (|os| + |x_1's|) - |x_1's| + |sg| \geq |ox_1'| - |x_1's| + |sg| = \frac{\pi}{2} - \mu - |x_1's| + |sg| \geq |x_1'm| - |x_1's| + |sg| = |sm| + |sg| \geq |gm|$.

Applying the just obtained inequality and looking now again on the whole S^d we get $|x'm| \leq |x'g| + |gm| \leq |x'g| + |og| = |x'o| = \frac{\pi}{2} - \mu$ which is the required inequality in (2). Thus by (2) also our lemma holds true. $\square$

Lemma 5. *Let $C \subset S^d$ be a convex body. Every point of C belongs to the convex hull of at most $d+1$ extreme points of C.*

Proof. We apply induction with respect to d. For $d = 1$ the statement is trivial since every convex body on S^1 is a spherical arc. Let $d \geq 2$ be a fixed integer. Assume that for each $k = 1, 2, \ldots, d-1$ every boundary point of a spherical convex body $\widehat{C} \subset S^k$ belongs to the convex hull of at most $k+1$ extreme points of $\widehat{C}$.

Let x be a point of C. Take an extreme point e of C. If x is not a boundary point of C, take the boundary point f of C such that $x \in ef$. In the opposite case put $f = x$.

If f is an extreme point of C, the statement follows immediately. In the opposite case take a hemisphere K supporting C at f. Put $C' = \mathrm{bd}(K) \cap C$. Observe that every extreme point of C' is also an extreme point of C. Let Q be the intersection of the smallest linear subspace of E^{d+1} containing C' with S^d. Clearly, Q is isomorphic to S^k for $k < d$. Moreover, C' has non-empty relative interior with respect to Q, because otherwise there would exist a smaller linear subspace of E^{d+1} containing C'. Thus, by the inductive hypothesis we obtain that f is in the convex hull of a set F of at most d extreme points of C. Therefore $x \in \mathrm{conv}(\{e\} \cup F)$ which means that x belongs to the convex hull of $d+1$ extreme points of C. This finishes the inductive proof. $\square$

The proof of the following d-dimensional lemma is analogous to that of the two-dimensional Lemma 4.1 from [8] shown there for a wider class of reduced spherical convex bodies.

Lemma 6. *Let $C \subset S^d$ be a spherical convex body with $\Delta(C) > \frac{\pi}{2}$ and let $L \supset C$ be a lune such that $\Delta(L) = \Delta(C)$. Each of the centers of the $(d-1)$-dimensional hemispheres bounding L belongs to the boundary of C and both are smooth points of the boundary of C.*

Having the next lemma in mind, we note the obvious fact that the diameter of a convex body $C \subset S^d$ is realized only for some pairs of points of $\mathrm{bd}(C)$.

Lemma 7. *Assume that the diameter of a convex body $C \subset S^d$ is realized for points p and q. The hemisphere K orthogonal to pq at p and containing $q \in K$ supports C.*

Proof. Denote the diameter of C by δ.

Assume first that $\delta > \frac{\pi}{2}$. The set of points at distance at least δ from q is the ball $B_{\pi-\delta}(q')$, where q' is the antipode of q. Clearly, K has only p in common with $B_{\pi-\delta}(q')$.

Since the diameter δ of C is realized for pq, every point of C is at distance at most δ from q. Thus C has empty intersection with the interior of $B_{\pi-\delta}(q')$.

Assume that K does not contain C. Then C contains a point $b \notin K$. Observe that the arc bp has nonempty intersection with the interior of $B_{\pi-\delta}(q')$ [the reason: K is the only hemisphere touching $B_{\pi-\delta}(q')$ from outside at p]. On the other hand, by the convexity of C we have $bp \subset C$. This contradicts the fact from the preceding paragraph that C has empty intersection with the interior of $B_{\pi-\delta}(q')$. Consequently, K contains C.

Now consider the case when $\delta \leq \frac{\pi}{2}$. For every $y \notin K$ we have $|pq| < |yq|$ which by $|pq| = \delta$ implies $y \notin C$. Thus if $y \in C$, then $y \in K$. □

Let us apply our Lemma 7 for a convex body C of diameter larger than $\frac{\pi}{2}$. Having in mind that the center k of K is in pq and thus in C, by Part III of Theorem 1 in [6] we obtain that $\Delta(K \cap K^*) > \frac{\pi}{2}$. This gives the following corollary which implies the other one. The symbol $\mathrm{diam}(C)$ denotes the diameter of C.

Corollary 1. *Let $C \subset S^d$ be a convex body of diameter larger than $\frac{\pi}{2}$ and let* $\mathrm{diam}(C)$ *be realized for points $p, q \in C$. Take the hemisphere K orthogonal to pq at p which supports C. Then* $\mathrm{width}_K(C) > \frac{\pi}{2}$.

Corollary 2. *Let $C \subset S^d$ be a convex body of diameter larger than $\frac{\pi}{2}$ and let $\mathcal{K}$ denote the family of all hemispheres supporting C. Then* $\max_{K \in \mathcal{K}} \mathrm{width}_K(C) > \frac{\pi}{2}$.

3. Spherical bodies of constant width

If for every hemisphere supporting a convex body $W \subset S^d$ the width of W determined by K is the same, we say that W is a *body of constant width* (see [6] and for an application also [5]). In particular, spherical balls of radius smaller than $\frac{\pi}{2}$ are bodies of constant width. Also every spherical Reuleaux odd-gon (for the definition see [6], p. 557) is a convex body of constant width. Each of the 2^{d+1} parts of S^d dissected by $d+1$ pairwise orthogonal $(d-1)$-dimensional spheres of S^d is a spherical body of constant width $\frac{\pi}{2}$, which easily follows from the definition of a body of constant width. The class of spherical bodies of constant width is a subclass of the class of spherical reduced bodies considered in [6] and [8], and mentioned by [3] in a larger context, (recall that a convex body $R \subset S^d$ is called *reduced* if $\Delta(Z) < \Delta(R)$ for every body $Z \subset R$ different from R, see also [7] for this notion in E^d).

By the definition of width and by Claim 2 of [6], if $W \subset S^d$ is a body of constant width, then every supporting hemisphere G of W determines a supporting hemisphere H of W for which $G \cap H$ is a lune of thickness $\Delta(W)$ such that the centers of G/H and H/G belong to the boundary of W. Hence *every spherical body W of constant width is an intersection of lunes of thickness $\Delta(W)$ such that the centers of the $(d-1)$-dimensional hemispheres bounding these lunes belong to* $\mathrm{bd}(W)$. Recall the related question from p. 563 of [6] if a convex body $W \subset S^d$ is of constant width provided every supporting hemisphere G of W determines at least one hemisphere H supporting W such that $G \cap H$ is a lune with the centers of G/H and H/G in $\mathrm{bd}(W)$.

Here is an example of a spherical body of constant width on S^3.

Example. Take a circle $X \subset S^3$ (i.e., a set congruent to a circle in E^2) of a positive diameter $\kappa < \frac{\pi}{2}$, and a point $y \in S^3$ at distance κ from every point $x \in X$. Prolong every spherical arc yx by a distance $\sigma \le \frac{\pi}{2} - \kappa$ up to points a and b so that a, y, x, b are on one great circle in this order. All these points a form a circle A, and all points b form a circle B. On the sphere on S^3 of radius σ whose center is y take the "smaller" part A^+ bounded by the circle A. On the sphere on S^3 of radius $\kappa + \sigma$ with center y take the "smaller" part B^+ bounded by B. For every $x \in X$ denote by x' the point on X such that $|xx'| = \kappa$. Prolong every xx' up to points d, d' so that d, x, x', d' are in this order and $|dx| = \sigma = |x'd'|$. For every x provide the shorter piece C_x of the circle with center x and radius σ connecting b and d determined by x and also the shorter piece D_x of the circle with center x of radius $\kappa + \sigma$ connecting a and d' determined by x. Denote by W the convex hull of the union of A^+, B^+ and all the pieces C_x and D_x. It is a body of constant width $\kappa + 2\sigma$ (hint: for every hemisphere H supporting W and every H^* the centers of H/H^* and H^*/H belong to $\mathrm{bd}(W)$ and the arc connecting them passes through one of our points x, or through the point y).

Theorem 1. *At every boundary point p of a body $W \subset S^d$ of constant width $w > \pi/2$ we can inscribe a unique ball $B_{w-\pi/2}(p')$ touching W from inside at p. What is more, p' belongs to the arc connecting p with the center of the unique hemisphere supporting W at p, and $|pp'| = w - \frac{\pi}{2}$.*

Proof. Observe that if a ball touches W at p from inside, then there exists a unique hemisphere supporting W at p such that our ball touches this hemisphere at p. So for any $\rho \in (0, \frac{\pi}{2})$ there is at most one ball of radius ρ touching W from inside at p. Our aim is to show that we can always find one.

In the first part of the proof consider the case when p is an extreme point of W. By Theorem 4 of [6] there is a lune $L = K \cap M$ of thickness w containing W such that p is the center of K/M. Denote by m the center of M and by k the center of K. Clearly, $m \in pk$ and $|pm| = w - \frac{\pi}{2}$. Since $\mathrm{width}_M(W) = w$, by the third part of Theorem 1 of [6] the ball $B_{w-\pi/2}(m)$ touches W from inside. Moreover, it touches W from inside at the center of M^*/M. Since K is one of these hemispheres M^*, our ball touches W at p.

In the second part consider the case when p is not an extreme point of W. From Lemma 5 we see that p belongs to the convex hull of a finite set E of extreme points of W. We do not lose the generality assuming that E is a minimum set of extreme points of W with this property. Hence p belongs to the relative interior of $\mathrm{conv}(E)$.

Take a hemisphere K supporting W at p and denote by o the center of K. Since p belongs to the relative interior of $\mathrm{conv}(E)$, by Lemma 2 we obtain $\mathrm{conv}(E) \subset \mathrm{bd}(K)$. Moreover, $\mathrm{conv}(E)$ is a subset of the boundary of W.

We intend to show that for every $x \in \mathrm{conv}(E)$ the inclusion

$$B_{w-\frac{\pi}{2}}(x') \subset W \tag{3}$$

holds true, where x' denotes the point on ox at distance $w - \frac{\pi}{2}$ from x.

By Lemma 4 for $w = \mu$, if (3) holds true for $x_1, x_2 \in \mathrm{conv}(E)$, then (3) is also true for every point of the arc x_1x_2. Applying this lemma a finite number of times and considering the first part of this proof, we conclude that (3) is true for every point of $\mathrm{conv}(E)$, so in particular for p. Clearly, the ball $B_{w-\frac{\pi}{2}}(p')$ supports W at p from inside.

Both parts of the proof confirm the statement of our theorem. □

By Lemma 6 we obtain the following proposition generalizing Proposition 4.2 from [8] for arbitrary dimension d. We omit an analogous proof.

Proposition 1. *Every spherical body of constant width larger than $\frac{\pi}{2}$ (and more generally, every reduced body of thickness larger than $\frac{\pi}{2}$) of S^d is smooth.*

From Corollary 2 we obtain the following corollary which implies the two other ones.

Corollary 3. *If* $\mathrm{diam}(W) > \frac{\pi}{2}$ *for a body $W \subset S^d$ of constant width w, then* $w > \frac{\pi}{2}$.

Corollary 4. *For every body $W \subset S^d$ of constant width $w \le \frac{\pi}{2}$ we have* $\mathrm{diam}(W) \le \frac{\pi}{2}$.

Corollary 5. *Let p be a point of a body $W \subset S^d$ of constant width at most $\frac{\pi}{2}$. Then $W \subset H(p)$.*

The following theorem generalizes Theorem 5.2 of [8] proved there for $d = 2$ only.

Theorem 2. *Every spherical convex body of constant width smaller than $\frac{\pi}{2}$ on S^d is strictly convex.*

Proof. Take a body W of constant width $w < \frac{\pi}{2}$ and assume it is not strictly convex. Then there is a supporting hemisphere K of W that supports W at more than one point. By Claim 2 of [6] the centers a of K/K^* and b of K^*/K belong to $\mathrm{bd}(W)$. Since K supports W at more than one point, K/K^* contains also a boundary point $x \neq a$ of W. By the first statement of Lemma 3 of [6] we have $|xb| > |ab| = w$. Hence $\mathrm{diam}(W) > w$.

By Corollary 4 we have $\mathrm{diam}(W) \leq \frac{\pi}{2}$. By Theorem 3 of [6] we see that $w = \mathrm{diam}(W)$. This contradicts the inequality $\mathrm{diam}(W) > w$ from the preceding paragraph. The contradiction means that our assumption that W is not strictly convex must be false. Consequently, W is strictly convex. □

On p. 566 of [6] the question is put if for every reduced spherical body $R \subset S^d$ and for every $p \in \mathrm{bd}(R)$ there exists a lune $L \supset R$ fulfilling $\Delta(L) = \Delta(R)$ with p as the center of one of the two $(d-1)$-dimensional hemispheres bounding this lune. The following theorem gives a positive answer in the case of spherical bodies of constant width. It is a generalization of the version for S^2 given as Theorem 5.3 in [8]. The idea of the proof of our theorem below for S^d substantially differs from the one given for S^2.

Theorem 3. *For every body $W \subset S^d$ of constant width w and every $p \in \mathrm{bd}(W)$ there exists a lune $L \supset W$ fulfilling $\Delta(L) = w$ with p as the center of one of the two $(d-1)$-dimensional hemispheres bounding this lune.*

Proof. Part I for $w < \frac{\pi}{2}$.

By Theorem 2 the body W is strictly convex, which means that all its boundary points are extreme. Thus the statement follows from Theorem 4 of [6].

Part II for $w = \frac{\pi}{2}$.

If p is an extreme point of W we again apply Theorem 4 of [6].

Consider the case when p is not an extreme point. Take a hemisphere G supporting W at p. Applying Corollary 5 we see that $W \subset H(p)$. Clearly, the lune $H(p) \cap G$ contains W. The point p is at distance $\frac{\pi}{2}$ from every corner of this lune and also from every point of the opposite $(d-1)$-dimensional hemisphere bounding the lune. Hence this is a lune that we are looking for.

Part III, for $w > \frac{\pi}{2}$.

By Lemma 5 the point p belongs to the convex hull $\mathrm{conv}(E)$ of a finite set E of extreme points of W. We do not lose the generality by assuming that E is a minimum set of extreme points of W with this property. Hence p belongs to the relative interior of $\mathrm{conv}(E)$. By Proposition 1 we know that there is a unique hemisphere K supporting W at p. Since p belongs to the relative interior of $\mathrm{conv}(E)$, by Lemma 2 we have $\mathrm{conv}(E) \subset \mathrm{bd}(K)$. Moreover, $\mathrm{conv}(E)$ is a subset of the boundary of W.

By Theorem 4 of [6] for every $e \in E$ there exists a hemisphere K_e^* (it plays the part of K^* in Theorem 1 of [6]) supporting W such that the lune $K \cap K_e^*$ is of thickness $\Delta(W)$ with e as the center of K/K_e^*. By Proposition 1, for every e the hemisphere K_e^* is unique. For every $e \in E$ denote by t_e the center of K_e^*/K and by k_e the boundary point of K such that $t_e \in ok_e$, where o is the center of K. So e, k_e are antipodes. Denote the set of all these points k_e by Q.

Clearly, the ball $B = B_{\Delta(W)-\frac{\pi}{2}}(o)$ (as in Part III of Theorem 1 in [6]) touches W from inside at every point t_e. Moreover, from the proof of Theorem 1 of [6] and from the earlier established fact that every $e \in E$ is the center of K/K_e^* and every t_e is the center of K_e^*/K we obtain that o belongs to all the arcs of the form et_e.

Put $U = \mathrm{conv}(Q \cup \{o\})$. Denote by U_B the intersection of U with the boundary of B, and by U_W the intersection of U with the boundary of W. Having this construction in mind we see the following one-to-one correspondence between some pairs of points in U_B and U_W. Namely, between the pairs of points of U_B and U_W such that each pair is on the arc connecting o with a point of $\mathrm{conv}(Q)$.

Now, we will show that $U_W = U_B$. Assume the opposite. By the preceding paragraph, our opposite assumption means that there is a point x which belongs to U_W but not to U_B. Hence $|xo| > \Delta(W) - \frac{\pi}{2}$. Moreover, there is a boundary point y of the $(d-1)$-dimensional great sphere bounding K such that $o \in xy$ and a point $y' \in oy$ at distance $\Delta(W) - \frac{\pi}{2}$ from y.

We have $|xy'| = |xo| + |oy| - |yy'| > \left(\Delta(W) - \frac{\pi}{2}\right) + \frac{\pi}{2} - \left(\Delta(W) - \frac{\pi}{2}\right) = \frac{\pi}{2}$.

By Lemma 5 the point x belongs the convex hull of a finite set of extreme points of W. Assume for a while that all these extreme points are at distance at most $\frac{\pi}{2}$ from y'. Therefore all of them are contained in $H(y')$. Thus their convex hull is contained in $H(y')$ and so $x \in H(y')$. This contradicts the fact established in the preceding paragraph that $|xy'| > \frac{\pi}{2}$. The contradiction shows that at least one of these extreme points is at distance larger than $\frac{\pi}{2}$ from y'. Take such a point z for which $|zy'| > \frac{\pi}{2}$.

Since z is an extreme point of W, by Theorem 4 of [6] it is the center of one of the $(d-1)$-dimensional hemispheres bounding a lune L of thickness $\Delta(W)$ which contains W. Hence by the third part of Lemma 3 of [6] every point of L different from the center of the other $(d-1)$-dimensional hemisphere bounding L is at distance smaller than $\Delta(W)$ from z. Taking into account that the distance of these centers is $\Delta(W)$ we see that the distance of every point of L, and in particular of W, from z is at most $\Delta(W)$.

By Theorem 1 the ball $B_{\Delta(W)-\frac{\pi}{2}}(y')$ touches W from inside at y.

For the boundary point v of this ball such that $y' \in zv$ we have $|zv| = |zy'| + |y'v| > \frac{\pi}{2} + \big(\Delta(W) - \frac{\pi}{2}\big) = \Delta(W)$, which by $v \in W$ contradicts the result of the preceding paragraph. Consequently, $U_W = U_B$.

Since $U_W = U_B$, the ball B touches W from inside at every point of U_B, in particular at the point t_p such that $o \in pt_p$. Therefore by Part III of Theorem 1 in [6] there exists a hemisphere K_p^* supporting W at t_p, such that t_p is the center of K_p^*/K, p is the center of K/K_p^* and the lune $L = K \cap K_p^*$ is of thickness $\Delta(W)$. Consequently, L is a lune announced in our theorem. □

If the body W from Theorem 3 is of constant width greater than $\frac{\pi}{2}$, then by Proposition 1 it is smooth. Thus at every $p \in \mathrm{bd}(W)$ there is a unique supporting hemisphere of W, and so the lune L from the formulation of this theorem is unique. If the constant width of W is at most $\frac{\pi}{2}$, there are non-smooth bodies of constant width (e.g., a Reuleaux triangle on S^2) and then for non-smooth $p \in \mathrm{bd}(W)$ we may have more such lunes.

Our Theorem 3 and Claim 2 in [6] imply the first sentence of the following corollary. The second sentence follows from Proposition 1 and the last part of Lemma 3 in [6].

Corollary 6. *For every convex body $W \subset S^d$ of constant width w and for every $p \in \mathrm{bd}(W)$ there exists $q \in \mathrm{bd}(W)$ such that $|pq| = w$. If $w > \frac{\pi}{2}$, this q is unique.*

Theorem 4. *If $W \subset S^d$ is a body of constant width w, then* $\mathrm{diam}(W) = w$.

Proof. If $\mathrm{diam}(W) \leq \frac{\pi}{2}$, then the statement is an immediate consequence of Theorem 3 in [6].

Assume that $\mathrm{diam}(W) > \frac{\pi}{2}$. Take an arc pq in W such that $|pq| = \mathrm{diam}(W)$. By Corollary 1 this hemisphere K orthogonal to pq at p which contains q, contains also W. The center of K lies strictly inside pq and thus by Part III of Theorem 1 in [6] we have $w > \frac{\pi}{2}$

Having Theorem 3 in mind, consider a lune $L \supset W$ with $\Delta(L) = \Delta(W)$ such that p is the center of a $(d-1)$-dimensional hemisphere bounding L. Clearly, $q \in W \subset L$. Since W is of constant width $w > \frac{\pi}{2}$, we have $\Delta(L) > \frac{\pi}{2}$.

Thus from the last part of Lemma 3 of [6] it easily follows that the center of the other $(d-1)$-dimensional hemisphere bounding L is a farthest point of L from p. Since it is at distance w from p, we obtain $w \geq |pq| = \mathrm{diam}(W)$.

On the other hand, by Proposition 1 of [6] we have $w \leq \mathrm{diam}(W)$.

As a consequence, $\mathrm{diam}(W) = w$. □

4. Constant width and constant diameter

We say that a convex body $W \subset S^d$ is *of constant diameter* w if the following two conditions hold true

(i) $\mathrm{diam}(W) = w$,
(ii) for every boundary point p of W there exists a boundary point p' of W with $|pp'| = w$.

This definition is analogous to the Euclidean notion (compare the beginning of Part 7.6 of [11] for the Euclidean plane, and the bottom of p. 53 of [1] also for higher dimensions). Here is a theorem similar to the planar Euclidean version from [11] (see the beginning of Part 7.6).

Theorem 5. *Every spherical convex body $W \subset S^d$ of constant width w is of constant diameter w. Every spherical convex body $W \subset S^d$ of constant diameter $w \geq \frac{\pi}{2}$ is of constant width w.*

Proof. For the proof of the first statement of our theorem assume that W is of constant width w. Theorem 4 implies (i) and Corollary 6 implies (ii), which means that W is of constant diameter w.

Let us prove the second statement. Let $W \subset S^d$ be a spherical body of constant diameter $w \geq \frac{\pi}{2}$. We have to show that W is a body of constant width w.

Consider an arbitrary hemisphere K supporting W. As an immediate consequence of two facts from [6], namely Theorem 3 and Proposition 1, we obtain that

$$\mathrm{width}_K(W) \leq w. \tag{4}$$

Let us show that $\mathrm{width}_K(W) = w$.

Make the opposite assumption (that is $\mathrm{width}_K(W) \neq w$) in order to provide an indirect proof of this equality. By (4) this opposite assumption implies that $\mathrm{width}_K(W) < w$.

Consider two cases.

At first consider the case when $w > \pi/2$.

Put $w' = \mathrm{width}_K(W)$. There exists a hemisphere M supporting W such that $\Delta(K \cap M) = w'$. Denote the center of K/M by a and the center of M/K by b. From Corollary 2 of [6] we see that $b \in \mathrm{bd}(W)$.

We have $\frac{\pi}{2} < w'$ since the opposite means $w' \leq \frac{\pi}{2}$ and then every point of the lune $K \cap M$ is at distance at most $\frac{\pi}{2}$ from the center b of M/K (for $w' = \frac{\pi}{2}$ this is clear by $K \cap M \subset H(b)$, and consequently this is also true if $w' < \frac{\pi}{2}$). Since b is a boundary point of our body W of constant diameter $w > \pi/2$, we get a contradiction to (ii).

Since b is a boundary point of the body W of constant diameter, by the assumption (ii) there exists $b' \in \mathrm{bd}(W)$ such that $|bb'| = w$. By the definition of the thickness of a lune, we have $|ab| = w'$. Observe that the last part of Lemma 3 of [6] implies that $|uc_{H/G}| \leq |c_{G/H}c_{H/G}|$ for every point u of the lune $H \cap G$. This observation applies to our lune $K \cap M$ since $\Delta(K \cap M) > \frac{\pi}{2}$ (i.e, $w' > \frac{\pi}{2}$). Hence we obtain $|b'b| \leq |ab|$, which by the first two sentences of this paragraph gives $w \leq w'$. This contradicts the inequality $w' < w$ resulting from our opposite assumption that $\text{width}_K(W) \neq w$.

Consequently, $\text{width}_K(W) = w$.

Now consider the case when $w = \frac{\pi}{2}$.

From $\text{width}_K(W) < w$ (resulting from our opposite assumption) we obtain $\text{width}_K(W) < \pi/2$. Thus $\Delta(K \cap K^*) < \frac{\pi}{2}$. Denote by b the center of K^*/K. From Corollary 2 of [6] we see that $b \in \text{bd}(W)$.

The set $D = (K/K^*) \cap (K^*/K)$ of corner points of $K \cap K^*$ is isomorphic to S^{d-2}. Moreover, S^k contains at most $k+1$ points pairwise distant by $\frac{\pi}{2}$, which follows from the fact (which is easy to show) that *every set of at least* $k+2$ *points pairwise equidistant on* S^k *must be the set of vertices of a regular simplex inscribed in* S^k (still the distances of these vertices are not $\frac{\pi}{2}$). Putting $k = d-2$, we see that D contains at most $d-1$ points pairwise distant by $\frac{\pi}{2}$. Therefore there exists a set P_{max} of the maximum number (being at most $d-1$) of points of $W \cap D$ pairwise distant by $\frac{\pi}{2}$.

Put $T = \text{conv}(P_{max} \cup \{b\})$. Clearly, $T \subset W$, and since moreover $T \subset \text{bd}(K^*)$ and $W \subset K^*$, we obtain $T \subset \text{bd}(W)$. Take a point x from the relative interior of T. The inclusion $T \subset \text{bd}(W)$ implies that $x \in \text{bd}(W)$. Hence by (ii) there exists $y \in \text{bd}(W)$ such that $|xy| = \frac{\pi}{2}$. By Lemma 2 we have $T \subset \text{bd}(H(y))$. By this inclusion and $b \in T$ we obtain $|by| = \frac{\pi}{2}$. Thus by Lemma 3 we have $y \in D$. As a consequence, the set $P_{max} \cup \{y\}$ is a set of points of $W \cap D$ pairwise distant by $\frac{\pi}{2}$. This set has a greater number of points than the set P_{max}. This contradiction shows that our assumption $\text{width}_K(W) \neq w$ is wrong. So $\text{width}_K(W) = w$.

In both cases, from the arbitrariness of the hemisphere K supporting our convex body W we get that W is a body of constant width w. □

Problem. Is every spherical body of constant diameter $w < \frac{\pi}{2}$ a body of constant width w?

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