Aequat. Math. 92 (2018), 763–800

0001-9054/18/040763-38
published online May 4, 2018
https://doi.org/10.1007/s00010-018-0557-4

Aequationes Mathematicae

Elementary geometry on the integer lattice

Hiroshi Maehara and Horst Martini

Abstract. The n-dimensional integer lattice, denoted by $\mathbb{Z}^n$, is the subset of $\mathbb{R}^n$ consisting of those points whose coordinates are all integers. In this expository paper, many concrete, intuitive, and geometric results concerning the integer lattice $\mathbb{Z}^n$ are presented, most of them together with new elementary or streamlined proofs. Some of the presented results are new, and others are improved versions of old results.

Mathematics Subject Classification. 11H06, 11P21, 52B20, 52C05, 52C07.

Keywords. Integer lattice, Geometry of numbers, Pick's theorem, Lattice polygon, Steinhaus' lattice point problem.

1. Introduction

Lattice point problems have a long history connected with famous names like Gauss, Dirichlet, Minkowski, Weyl, Siegel, Mordell and many others, see the historical contributions of Hlawka [46] and Schwermer [90]. They nicely demonstrate how deep cross connections between basic mathematical fields (like, e.g., number theory, convexity, discrete geometry, algebraic geometry and, more specifically, the theory of positive quadratic forms, as observed by Gauss [31]) can be. There are also many applications (e.g., in numerical analysis, discrete optimization, discrete and computational geometry, stochastic geometry, image processing and pattern recognition, computer science, and crystallography), so that the development of suitable tools to handle lattice point problems as unitedly as possible has a natural motivation. To present such tools is one of our main aims.

The most important historical contribution regarding the alignment of our paper was given by Minkowski, namely by his "Geometrie der Zahlen" (see [71]); he proposed the application of geometric methods and topics to problems from number theory. Since Minkowski also created the basic notions for the field of classical convexity, he established the fundaments for the fruitful interplay between lattice point problems and parts of convexity. Without being exhaustive, we mention some basic books and surveys referring (not only) to this direction: Keller [53], Gioia [32, Chapter 9], Cassels [16], Gruber and Lekkerkerker [39], Erdős, Gruber, and Hammer [26], Gruber (see [36,37], and [38, § 21 to § 34]), Lagarias [59], Gritzmann and Wills [33], Olds, Lax, and Davidoff [78] as well as the problem books [20, Chapter E] and [12, Chapter 10]. More specified expository papers and books, discussing special aspects or notions emerging from lattice point problems, refer to Ehrhart polynomials (cf. [10,13,14,22,84] etc.), Minkowski's successive minima (see [45]), properties of lattice polytopes and also their connections to algebraic geometry (cf. [77, Chapter 2], [7,8,14,27, Chapter V] etc.), random and computational aspects ([9,52,88] etc.), applications regarding packing and covering ([19,28,34,102] etc.), lattice tilings (see the respective sections in [40,89]) and relations to crystallography [25].

In this paper we discuss and prove new geometric results related to the integer lattice. It is one of our main goals to demonstrate typical methods which successfully and elegantly work in this field. Most of the obtained results are not new, but we give new approaches, simplified proofs, or improved versions of them.

This paper is organized as follows.

Section 1: Introduction

Section 2: Lattice polygons and Pick's formula

Section 3: Special similarities and regular simplices

Section 4: Lattice angles and lattice polygons

Section 5: Lattice simplices

Section 6: Lattice points in a planar region

Section 7: Lattice points on quadratic curves

In Sect. 2, we recall some elementary and interesting results on lattice polygons (such as Scherrer's nonexistence-proof [85] and Pick's formula [80]), where a *lattice polygon* means a planar polygon whose vertices are all lattice points; see also Scott [93]. In Sect. 3, we present special similarities of $\mathbb{R}^n$ for $n = 2$ or $n = 4k$, (obtained by Maehara [61]), mapping a given line that passes through the origin and another lattice point to the first coordinate axis of $\mathbb{R}^n$, and mapping $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Such similarities are applied to present sufficient conditions for dimensions n such that $\mathbb{R}^n$ contains a regular n-simplex whose vertices are all lattice points.

An angle $\angle ABC$ determined by lattice points A, B, C in $\mathbb{R}^n$ is called a *lattice angle* in $\mathbb{R}^n$. In Sect. 4, we present Beeson's theorem [11] that characterizes lattice angles in $\mathbb{R}^n$. We also give a necessary and sufficient condition that a pair of lattice angles is a pair of interior angles of a lattice polygon. This result is applied to prove a theorem from Maehara [60] that describes when a polygon and a lattice polygon have the same cyclic sequence of interior angles.

In Sect. 5, we prove that a simplex σ is similar to a simplex whose vertices are all lattice points if and only if $\cos^2 \angle ABC$ is a rational number for every three vertices A, B, C of the simplex σ. We also consider the smallest dimension n such that a given lattice simplex in $\mathbb{R}^N$ for some N is similar to a lattice simplex in $\mathbb{R}^n$.

Further on, Sects. 6 and 7 concern mainly the two-dimensional lattice $\mathbb{Z}^2$. In Sect. 6, we introduce the notion of *lattice-generic curve* (see [63]), and prove that if a region bounded by a lattice-generic curve has area n, then it can be translated in the plane so that it contains exactly n lattice points. This solves a slightly generalized version of Steinhaus' circle-lattice-point problem. Applying this result, we also prove a similar result (but this time, besides translations, a rotation is needed) for beziergons, which are regions bounded by finite numbers of Bézier curves.

In Sect. 7, we introduce Schinzel's theorem (Schinzel [86], Maehara-Matsumoto [66]) that states that, for any n, there is a circle in the plane that passes through exactly n lattice points. Next, we define the $\mathbb{Z}^2$-spectrum of a curve C as the set of numbers n (including infinity ∞) such that there is a curve similar to C that passes through exactly n lattice points. Thus, the $\mathbb{Z}^2$-spectrum of a circle is a set of all positive integers by Schinzel's theorem. We show here several results, obtained in Kuwata-Maehara [58], on the $\mathbb{Z}^2$-spectra of quadratic curves.

2. Lattice polygons and Pick's formula

2.1. Lattice polygons

By a *polygon* in $\mathbb{R}^n, n \geq 2$, we mean a planar polygon lying on a 2-plane in $\mathbb{R}^n$. A *lattice polygon* in $\mathbb{R}^n$ is a polygon whose vertices are all lattice points in $\mathbb{Z}^n$. For example, the square in $\mathbb{R}^2$ with vertices $(0,0),(1,0),(1,1),(0,1)$ is a lattice square. For what $m > 0$ does a regular lattice m-gon (besides this simple example) exist in $\mathbb{R}^2$? Let us note here the following fundamental property of $\mathbb{Z}^n$.

> $(*)$ A translation of $\mathbb{R}^n$ that maps a lattice point to another lattice point maps the whole of $\mathbb{Z}^n$ to $\mathbb{Z}^n$.

Proposition 2.1. *No equilateral lattice triangle exists in the plane $\mathbb{R}^2$, and hence no regular lattice hexagon exists in $\mathbb{R}^2$.*

Proof. Suppose that there is an equilateral lattice triangle ABC in the plane. We may suppose that $B = (0,0)$, the origin. Put $A = (a,b)$, $C = (c,d)$. Since C is obtained by rotating A around B through the angle $60°$, we have

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then $c = \frac{1}{2}a - \frac{\sqrt{3}}{2}b$, $d = \frac{\sqrt{3}}{2}a + \frac{1}{2}b$, and since a, b are integers $\neq (0,0)$, one of c, d must be irrational, a contradiction. □

Proposition 2.2. *For $m \geq 5, m \neq 6$, no regular lattice m-gon exists in $\mathbb{R}^n$.*

Proof. There is a nice proof of this fact by Scherrer [85] (see also Hadwiger and Debrunner [43]): Let us consider the case $m = 5$. Suppose that there is a regular lattice pentagon in $\mathbb{R}^n$. Let $ABCDE$ be one with the *smallest edge length*. Translate the vertices A, B, C, D, E through the vectors $\overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EA}, \overrightarrow{AB}$, respectively, see Fig. 1. Then the translated points are lattice points in $\mathbb{R}^n$ by $(*)$, and they are the vertices of a regular lattice pentagon with *smaller* edge length, a contradiction. Similar arguments work for $m > 6$. (Note also that similar arguments fail for $m = 6$.) □

Thus, we have the following theorem.

Theorem 2.1. *In $\mathbb{R}^2$, there is no regular lattice polygon other than a square.*

In $\mathbb{R}^3$, there is an equilateral lattice triangle (see Fig. 2), and hence there is a regular lattice hexagon, too. Hence Proposition 2.2 implies the next theorem.

Theorem 2.2. (Schoenberg [87]) *Regular lattice m-gons exist in $\mathbb{R}^n$ $(n \geq 3)$ only for $m = 3, 4, 6$.*

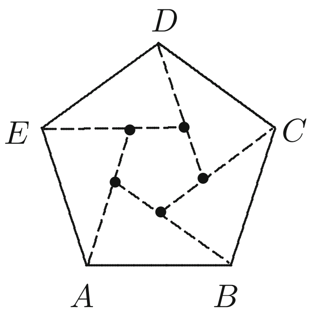

Figure 1. A small lattice pentagon inside a lattice pentagon

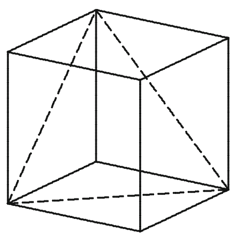

Figure 2. An equilateral lattice triangle in $\mathbb{R}^3$

Corollary 2.1. *A regular polyhedron in* $\mathbb{R}^3$ *whose vertices are all lattice points is either a regular tetrahedron, or a cube, or a regular octahedron.*

By a *Pythagorean triangle* we mean a right triangle whose edge lengths are all integers.

Corollary 2.2. *For an acute angle* θ *of a Pythagorean triangle,* θ/π *is an irrational number.*

Proof. Let $\cos\theta = a/c$, $\sin\theta = b/c$, where a, b, c are positive integers satisfying $a^2 + b^2 = c^2$. Suppose that $\theta/(2\pi) = l/m$ (irreducible fraction). Let ρ be the rotation of $\mathbb{R}^2$ around the origin through the angle θ, given by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a/c & -b/c \\ b/c & a/c \end{pmatrix}.$$

Let $P = (c^m, 0)$. Since l/m is an irreducible fraction, $P, \rho(P), \rho^2(P), \ldots, \rho^{m-1}(P)$ are distinct lattice points, and $\rho^m(P) = P$. Hence these m lattice points form a regular lattice m-gon. Then, by Theorem 2.1, we have $m = 4$. Since $\theta < \pi/2$, this is impossible. □

More generally, the following theorem holds; see Niven [75], Niven and Zuckerman [76].

Theorem 2.3. *If* $0 < \theta < \pi/2$ *and* $\cos\theta$ *is a rational number, then either* $\theta = \pi/3$ *or* θ/π *is an irrational number.*

2.2. Pick's formula

Note that a polygon is *simple* if its edges have no mutual intersections other than those of adjacent edges at the common vertices. By Pick's classical theorem the area of a simple lattice polygon is precisely determined in terms of the number of lattice points in its interior and that of lattice points in its boundary. There are many nice variants and generalizations of Pick's theorem; we briefly present now some of them, first mentioning the surveys [24,99]. In [41] Pick's theorem is extended to more general lattice polygons (allowing multiple intersections and even overlapping of their edges), and similar results were obtained in [42,91]. In [56] Pick's theorem is used to construct the (reciprocal of the) golden ratio as an irrational number, and in [82] a beautiful proof of Pick's theorem based on Minkowski's theorem on the volume of centrally symmetric convex bodies embedded in a lattice is presented. Analogues of Pick's theorem for hexagonal and triagonal lattices (with applications in computer graphics) are given in [23]. There are numerous generalizations of Pick's theorem to higher dimensions, staying with polygons or going up to polytopes; see, e.g., [44,55]. Interesting relations to algebraic geometry (with respect to toric varieties) also yield related higher dimensional theorems, see [73] and [29].

Lemma 2.1. *Let $ABCD$ be a lattice parallelogram in $\mathbb{R}^2$ that contains exactly four lattice points, namely as its four vertices. Then it has unit area.*

Proof. We may suppose that $A = (0,0)$, the origin. Put $B = (a,b), D = (c,d)$. Note that by translating $ABCD$ repeatedly, we can tessellate the whole plane without overlapping. Then all lattice points are obtained as the vertices of this tessellation. It follows that every lattice point is obtained as the integral combination of $(a,b),(c,d)$. Hence $(1,0)$, $(0,1)$ can be written as $(1,0) = p(a,b) + q(c,d)$, $(0,1) = r(a,b) + s(c,d)$, where $p,q,r,s \in \mathbb{Z}$. Thus

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since a,b,c,d,p,q,r,s are integers, we must have $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$, which implies that the area of $ABCD$ is 1. □

A lattice triangle in $\mathbb{R}^2$ that contains exactly three lattice points is called a *primitive lattice triangle*.

Corollary 2.3. *The area of a primitive lattice triangle in $\mathbb{R}^2$ is $1/2$.*

Proof. Let ABC be a primitive lattice triangle. By a rotation ρ of $\mathbb{R}^2$ around the midpoint of AC through $180°$, the lattice $\mathbb{Z}^2$ is mapped onto itself. Put $D = \rho(B)$, Then $ABCD$ is a lattice parallelogram that contains exactly four lattice points. Hence its area is 1 by Lemma 2.1, and hence the area of ABC is $1/2$. □

The following statement became famous as "Pick's theorem".

Theorem 2.4. (Pick [80]) *For a simple lattice polygon Γ in $\mathbb{R}^2$, let $i = i(\Gamma)$ denote the number of lattice points lying in the interior of Γ, and let $b = b(\Gamma)$ denote the number of lattice points lying on the boundary of Γ. Then the area of Γ is equal to $i + b/2 - 1$.*

Proof. Our proof is on the same lines as the one given by Funkenbusch [30]. It is possible to triangulate Γ into primitive lattice triangles. (Details of this part are omitted.) Let us regard the resulting triangulation as a connected planar graph. The number of vertices of this graph is $b + i$. Let t be the number of primitive lattice triangles. Then the number f of faces of this planar graph is $t + 1$. Let e be the number of edges of this graph. There are exactly b edges on the boundary of Γ. For each edge, put two pebbles on both sides of the edge. Then inside each primitive lattice triangle there are three pebbles, and outside Γ there are b pebbles. Thus, the total number of pebbles is $b + 3t$, which is equal to $2e$. Now, by Euler's formula, we have $v - e + f = 2$. Since $v-e+f = i+b-(b+3t)/2+t+1 = i+b/2-t/2+1$, we have $t/2 = i+b/2-1$. Hence the area of Γ is $t/2 = i + b/2 - 1$. □

Remark 2.1. Pick's theorem can also be proved in an elementary way, without using Euler's formula. See, e.g., Steinhaus [97, p. 96].

For $X \subset \mathbb{R}^2$ and a nonzero real λ, let $\lambda X = \{\lambda x \in \mathbb{R}^2 : x \in X\}$. If X is a lattice polygon and λ is a nonzero integer, then λX is also a lattice polygon.

Corollary 2.4. (Akopyan and Tagami [1]) *For every lattice polygon Γ in $\mathbb{R}^2$, 4Γ contains an odd number of lattice points in its interior.*

Proof. Let $i = i(\Gamma), b = b(\Gamma)$, and $i' = i(4\Gamma), b' = b(4\Gamma)$. It is clear that $b' = 4b$, and the area of 4Γ is equal to 16 times the area of Γ. By Pick's formula, $16(i+b/2-1) = (i'+b'/2-1) = (i'+2b-1)$. Hence $i' = 16i+8b-16-2b+1 = 16i + 6b - 15$, which is odd. □

Remark 2.2. Consider the tetrahedron in $\mathbb{R}^3$ having the four vertices

$$(0,0,0), (1,0,0), (0,1,0), (1,1,m),$$

where m is a positive integer. This tetrahedron is known as *Reeve tetrahedron* of height m. It contains exactly four lattice points for any $m > 0$, and its volume is $m/6$. This suggests that there is no simple analogue of Pick's formula for the volume of a lattice polyhedron in $\mathbb{R}^3$. Let Π be a lattice polyhedron in $\mathbb{R}^3$. Let I, B denote the number of lattice points interior to Π, and the number of lattice points on the boundary of Π, respectively. Similarly, for a positive integer m, let I_m, B_m be the number of lattice points interior to $m\Pi$, and the number of lattice points on the boundary of $m\Pi$, respectively. Let V be the volume of Π. Reeve [83] proved that

$$2m(m^2-1)V = 2(I_m - mI) + (B_m - mB)$$

holds for every positive integer m.

3. Special similarities and regular simplices

3.1. Special similarities of $\mathbb{R}^2$ and $\mathbb{R}^{4k}$

For square matrices $M_i, i = 1, 2, \ldots, k$, not necessarily of the same size, the block matrix with diagonals $M_1, \ldots, M_k$ is denoted by $M_1 \oplus M_2 \oplus \cdots \oplus M_k$.

Proposition 3.1. *For $X \subset \mathbb{Z}^{2k}$ and $(a,b) \in \mathbb{Z}^2 \backslash \{O\}$, the set $\sqrt{a^2+b^2}\,X$ is isometric to a subset of $\mathbb{Z}^{2k}$.*

Proof. Let

$$[[a,b]] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

and let M be the $2k \times 2k$-matrix defined by $M = \underbrace{[[a,b]] \oplus [[a,b]] \oplus \cdots \oplus [[a,b]]}_{k}$.

Then the linear transformation φ of $\mathbb{R}^{2k}$ induced by M is a similarity with ratio $\sqrt{a^2+b^2}$ and $\varphi(\mathbb{Z}^{2k}) \subset \mathbb{Z}^{2k}$. Hence $\varphi(X) \subset \mathbb{Z}^{2k}$ is isometric to $\sqrt{a^2+b^2}X$. □

Let us recall here Lagrange's four-square theorem for later use. For its proof, see Niven and Zuckerman [76].

Theorem 3.1. (Lagrange's four-square theorem) *Every positive integer can be represented as the sum of four squares of integers.*

Theorem 3.2. (Maehara [61]) *If $n = 2$ or $n = 4k$ $(k \geq 1)$, then for any $P \in \mathbb{Z}^n \backslash \{O\}$ there is a similarity transformation $f_P : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\begin{cases} f_P(O) = O, \ f_P(P) = (*, 0, \ldots, 0) \\ f_P(\mathbb{Z}^n) \subset \mathbb{Z}^n. \end{cases} \tag{3.1}$$

Proof. First, note that for any $1 \leq i < j \leq n$, there is an orthogonal transformation of $\mathbb{R}^n$ that switches the i-th coordinate and the j-th coordinate of every point of $\mathbb{R}^n$, and maps $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Such an orthogonal transformation is called a *coordinate-switching*.

(i) The case $n = 2$. For $P = (a,b) \in \mathbb{Z}^2 \backslash \{O\}$, let f_P be the linear transformation induced by $[[a,b]]$. Then f_P satisfies (3.1).

(ii) The case $n = 4$. For $P = (x,y,z,w) \in \mathbb{Z}^4 \backslash \{O\}$, put

$$[[x,y,z,w]] = \begin{pmatrix} x & -y & z & w \\ y & x & -w & z \\ z & -w & -x & -y \\ w & z & y & -x \end{pmatrix}.$$

This matrix induces a similarity f_P of $\mathbb{R}^4$ satisfying (3.1).

(iii) The case $n = 8$. Let $P = (\ldots, x, y, z, w) \in \mathbb{Z}^8$. By applying a suitable coordinate-switching if necessary, we may suppose that $(x,y,z,w) \neq (0,0,0,0)$. By the linear transformation defined by the matrix $[[x,y,z,w]] \oplus [[x,y,z,w]]$, P is sent to a point of the form $(p,q,r,s,m,0,0,0) \in \mathbb{Z}^8$, where $m = x^2+y^2+z^2+w^2$. By the matrix

$$\begin{pmatrix} [[p,q,r,s]] & -mI \\ mI & [[p,q,r,s]]^{tr} \end{pmatrix}$$

(where I is the identity matrix, and $(\ )^{tr}$ denotes the transpose of the matrix), this point is sent to a point of the form $(*, 0, \ldots, 0) \in \mathbb{Z}^8$. The product of these two matrices is a matrix of a similarity transformation of $\mathbb{R}^8$.

(vi) Now, let us show the theorem for $n = 4k$ by induction on k. By (ii), (iii), the assertion of the theorem is true for $k = 1, 2$. Suppose that the assertion of the theorem is true in the case $n = 4k$ for some $k \geq 2$, and consider the case $n = 4(k+1)$. Let $P \in \mathbb{Z}^{4(k+1)}, P \neq O$. By applying a suitable coordinate-switching if necessary, we may suppose that the last $4k$ coordinates of P are not all 0. Hence there is an integral $4k \times 4k$-matrix M with $MM^{tr} = \lambda I, \lambda > 0$, that sends the last $4k$ coordinates of P to $(*, 0, \ldots, 0)$. (Such matrix exists by the inductive assumption.) By Lagrange's four square theorem, there are integers a, b, c, d such that $\lambda = a^2 + b^2 + c^2 + d^2$. Then the $4(k+1) \times 4(k+1)$-matrix $[[a, b, c, d]] \oplus M$ sends P to a point Q whose last $4(k-1)$ coordinates are all 0. Let N be an integral $4k \times 4k$-matrix with $NN^{tr} = \mu I$ that sends the first $4k$ coordinates of this point to $(*, 0, \ldots, 0)$. There are integers a', b', c', d' such that $\mu = a'^2 + b'^2 + c'^2 + d'^2$. Then the transformation by $N \oplus [[a', b', c', d']]$ is a similarity transformation that sends Q to a point of the form $(*, 0, \ldots, 0)$. The product of these transformations is a similarity transformation of $\mathbb{R}^{4(k+1)}$ with integer entries only, and it satisfies (3.1). □

Remark 3.1. The assertion analogous to Theorem 3.2 does not hold for $2 < n \not\equiv 0 \pmod 4$. For the details regarding this fact see Maehara [61].

Theorem 3.3. (Maehara [61]) *If a subset $X \subset \mathbb{Z}^{4k} \subset \mathbb{R}^{4k}$ $(k \geq 1)$ lies on a hyperplane in $\mathbb{R}^{4k}$, then X is similar to a subset of $\mathbb{Z}^{4k-1} \subset \mathbb{R}^{4k-1}$.*

Proof. By translating if necessary, we may assume that X is contained in a hyperplane H that passes through the origin. There is a normal vector P of H having only integral coordinates. Each point of X is mapped by the similarity f_P of Theorem 3.2 to a point whose first coordinate is 0. Hence we can regard $f_P(X)$ as a subset of $\mathbb{Z}^{4k-1}$. □

3.2. Regular simplices

A k-dimensional simplex (simply a k-simplex) in $\mathbb{R}^n$ is called a *lattice k-simplex* if all its vertices are lattice points. Now, for what kind of n, does $\mathbb{R}^n$ contain a regular lattice n-simplex? Not for all n. For example, no equilateral lattice triangle exists in $\mathbb{R}^2$. This problem was completely solved by Schoenberg [87] by applying *Minkowski's theory of rational equivalence of quadratic forms.* (The exact statement of his solution will be given later in Remark 3.2). Let us show here some results on this problem in an elementary way, without invoking Minkowski's theory on quadratic forms.

For every n, the n points

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1) \in \mathbb{R}^n$$

span a regular $(n-1)$-simplex. Hence by Theorem 3.3 we have the following statement.

Corollary 3.1. (Schoenberg [87]) *If $n \equiv 3 \pmod 4$, then a regular lattice n-simplex exists in $\mathbb{R}^n$.*

Theorem 3.4. (1) *If $n \equiv 0 \pmod 4$ and a regular lattice n-simplex exists in $\mathbb{R}^n$, then a regular lattice $(n+1)$-simplex exists in $\mathbb{R}^{n+1}$.*

(2) *If $n+1$ is a perfect square, then a regular lattice n-simplex exists in $\mathbb{R}^n$.*

Proof. (1) Suppose that $n \equiv 0 \pmod 4$ and there is a regular lattice n-simplex in $\mathbb{R}^n$. Since the barycenter of this simplex is a rational point, by dilating and translating, we have a regular lattice n-simplex Δ^n in $\mathbb{R}^n$ whose barycenter is the origin O. Let us regard $\Delta^n \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Let $P = (0, \dots, 0, h) \in \mathbb{R}^{n+1}$, $h > 0$, be a point such that $\Delta^n \cup \{P\}$ span a regular $(n+1)$-simplex. For a vertex Q of Δ^n, $|OQ|^2 + h^2 = |PQ|^2$, which is equal to the square of the edge-length of Δ^n. Hence h^2 is an integer. Now, by Lagrange's four square theorem, there are $x, y, z, w \in \mathbb{Z}$ such that $h^2 = x^2 + y^2 + z^2 + w^2$. Let $[[x, y, z, w]]$ denote the 4×4-matrix which appeared in the proof of Theorem 3.2. Since $n \equiv 0 \pmod 4$, we can define the $n \times n$-matrix $[[x, y, z, w]] \oplus [[x, y, z, w]] \oplus \cdots \oplus [[x, y, z, w]]$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation induced by this matrix. Then f is a similarity transformation that magnifies each subset with ratio $\sqrt{x^2 + y^2 + z^2 + w^2} = h$. Then the vertices of $f(\Delta^n) \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and the point $(0, \dots, 0, h^2) \in \mathbb{R}^{n+1}$ are all lattice points of $\mathbb{R}^{n+1}$, and they span a regular $(n+1)$-simplex.

(2) Suppose that $n + 1 = k^2$. Put

$$A_0 = (1+k, 1+k, \dots, 1+k) \in \mathbb{R}^n \text{ and}$$
$$A_i = (0, \dots, 0, \overset{i}{n}, 0, \dots, 0) \in \mathbb{R}^n, \quad i = 1, \dots, n.$$

Then $|A_1A_2|^2 = 2n^2$ and

$$\begin{aligned} |A_0A_1|^2 &= (n-1-k)^2 + (n-1)(1+k)^2 = n^2 + n(1+k)^2 - 2n(1+k) \\ &= n^2 + n(1+k)(1+k-2) = n^2 + n(k^2-1) = 2n^2. \end{aligned}$$

Hence $A_0, A_1, \dots, A_n$ span a regular lattice n-simplex in $\mathbb{R}^n$. □

Since $(2k+1)^2 = 4k(k+1) + 1$, we have the following corollary.

Corollary 3.2. *If $n = 4k(k+1)$ or $n = 4k(k+1) \pm 1$, then $\mathbb{R}^n$ contains a regular lattice n-simplex.*

Theorem 3.5. *If there is a similarity transformation φ of $\mathbb{R}^n$ such that*

$$\begin{cases} \varphi(O) = O, \ \varphi(1, 1, \dots, 1) = (*, 0, \dots, 0) \\ \varphi(\mathbb{Z}^n) \subset \mathbb{Z}^n \end{cases} \tag{3.2}$$

then there is a regular lattice $(n-1)$-simplex in $\mathbb{R}^{n-1}$.

Proof. Let Δ denote the regular $(n-1)$-simplex in $\mathbb{R}^n$ spanned by the n points

$$(0, \dots, 0), (-1, 1, 0, \dots, 0), (-1, 0, 1, 0, \dots, 0), \dots, (-1, 0, \dots, 0, 1).$$

This simplex is contained in a hyperplane perpendicular to the vector $(1,1,\ldots,1)$. Then, $\varphi(\Delta) \subset \mathbb{Z}^n$. Since $\varphi(1,\ldots,1) = (*,0,\ldots,0)$, for every $p \in \Delta$, the first coordinate of $\varphi(p)$ is 0, that is, $\varphi(p)$ is a point of the form $(0,x_2,\ldots,x_n)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be an orthogonal projection such that $\pi(x_1,x_2,\ldots,x_n) = (x_2,\ldots,x_n)$. Then $\pi(\varphi(\Delta))$ is a regular lattice $(n-1)$-simplex in $\mathbb{R}^{n-1}$. □

Problem 3.1. *Find a simple condition for n to satisfy that there is a similarity transformation of $\mathbb{R}^n$ that satisfies* (3.2)*.*

Proposition 3.2. *If $n = 18$, then there is a similarity transformation φ of $\mathbb{R}^n$ that satisfies* (3.2)*.*

Proof. Let f be the similarity transformation of $\mathbb{R}^n$ with matrix $H_8 \oplus H_8 \oplus 2H_2$, where H_8, H_2 denote Hadamard matrices of order 8 and 2. (An Hadamard matrix H_n is a square matrix whose entries are either +1 or 1 and whose rows are mutually orthogonal.) Then f is a similarity and $f(1,1,\ldots,1) = (8,0,\ldots,0,8,0,\ldots,0,4,0)$. There is an orthogonal transformation g of $\mathbb{R}^n$ such that g maps $(8,0,\ldots,0,8,0,\ldots,0,4,0)$ to $(8,8,4,0,\ldots,0)$. Since $8^2+8^2+4^2 = 12^2$, the linear transformation h of $\mathbb{R}^n$ with matrix $[[8,8,4,0]] \oplus 12I$, where I denotes the 14×14-identity matrix, is a similarity transformation. Then the composite transformation $\varphi = h \circ g \circ f$ satisfies (3.2). □

Proposition 3.3. *If $n = 34$, then there is a similarity transformation φ of $\mathbb{R}^n$ that satisfies* (3.2)*.*

Proof. Let f be the linear transformation of $\mathbb{R}^n$ with matrix $H_{32} \oplus 4H_2$. Then f is a similarity and $f(1,\ldots,1) = (32,0\ldots,0,8,0)$. There is an orthogonal transformation g that sends $(32,0,\ldots,0,8,0)$ to $(32,8,0,\ldots,0)$. Let h be the linear transformation of $\mathbb{R}^n$ with matrix $[[32,8]] \oplus [[32,8] \oplus \cdots \oplus [[32,8]]$. Then the composite transformation $\varphi = h \circ g \circ f$ satisfies (3.2). □

From Corollaries 3.1, 3.2 and Propositions 3.2, 3.3 it follows that $\mathbb{R}^n$ ($n \leq 50$) contains a regular lattice n-simplex if n is equal to

$$1,3,7,8,9,11,15,17,19,23,24,25,27,31,33,35,39,43,47,48,49 \qquad (3.3)$$

Remark 3.2. Concerning the existence of a regular lattice n-simplex Δ^n in $\mathbb{R}^n$, Schoenberg [87] obtained the following complete results by applying Minkowski's theory of rational equivalence of quadratic forms.

- (i) If n is even, a regular lattice n-simplex exists in $\mathbb{R}^n$ if and only if $n+1 = k^2$ for some $k \in \mathbb{Z}$.
- (ii) If $n \equiv 3 \pmod 4$, then a regular lattice n-simplex always exists in $\mathbb{R}^n$.
- (iii) If $n \equiv 1 \pmod 4$, a regular lattice n-simplex exists in $\mathbb{R}^n$ if and only if $n+1$ is not divisible to an odd exponent by a prime number of the form $4k+3$.

Consulting this result, we can also confirm that the list (3.3) is the perfect list of numbers $n \le 50$ for which a regular lattice n-simplex exists in $\mathbb{R}^n$.

It is clear that embedding regular simplices into integral lattices is closely related to vertex embeddings of regular polytopes into regular polytopes or into integer lattices (particularly, to the respective simplex-cube pairing). It is well-known that the question, in which dimensions n a vertex embedding of the regular n-simplex into the vertex set of an n-cube exists, is equivalent to the question of the existence of Hadamard matrices of suitable orders. Regarding many results around this still unsettled problem (and its extension to other regular polytopes) we refer to [2,79], [67, § 4], and [68, § 2.4], related computational aspects are discussed in [35]; a recent contribution on (special types of) Hadamard matrices is [21].

4. Lattice angles and lattice polygons

4.1. Beeson's theorem

Recall that for lattice points A, B, C in $\mathbb{R}^n$, the angle $\angle ABC$ is called a *lattice angle* in $\mathbb{R}^n$. (Here we refer also to [51] for a related concept of lattice trigonometry.) Let Θ_n denote the set of (measured) lattice angles in $\mathbb{R}^n$. Thus

$$\Theta_n = \{\theta \mid \theta = \angle ABC,\ A, B, C \in \mathbb{Z}^n\}.$$

It is clear that $\Theta_n \subset \Theta_{n+1}$.

Theorem 4.1. (Beeson [11])

(1) $\theta \in \Theta_2 \Leftrightarrow \theta = \pi/2$ *or* $\tan\theta \in \mathbb{Q}$,
(2) $\theta \in \Theta_4 \Leftrightarrow \theta = \pi/2$ *or* $\tan^2\theta = (b^2 + b^2 + d^2)/a^2$ $(a, b, c, d \in \mathbb{Z})$,
(3) $\theta \in \Theta_5 \Leftrightarrow \cos^2\theta \in \mathbb{Q}$,
(4) $\Theta_2 \subsetneq \Theta_3 = \Theta_4 \subsetneq \Theta_5 = \Theta_6 = \dots$.

Preceding the proof of Beeson's theorem, we state a lemma.

Lemma 4.1. *For every $m \ge 0$, there are no integers a, b, c, d that satisfy*

$$a^2(8m + 7) = b^2 + c^2 + d^2,\ a \ne 0.$$

Proof. Suppose that some integers a, b, c, d satisfy this. We may assume that their greatest common divisor equals 1. Thus, we may assume that if a is even, then b is odd. Note that $n^2 \equiv 0, 1, 4 \pmod 8$ for any integer n. Hence, if a is odd, then $a^2(8m + 7) \equiv 7$ but $b^2 + c^2 + d^2 \not\equiv 7 \pmod 8$; if a is even, then $a^2(8m + 7) \equiv 0 \pmod 4$ but $b^2 + c^2 + d^2 \not\equiv 0 \pmod 4$. Thus, in either case we have a contradiction. □

Corollary 4.1. *An integer of the form $a^2(8m + 7)$ cannot be represented as a sum of three squares.*

Remark 4.1. Legendre's three-squares theorem (see, e.g., [3]) states that a positive integer can be represented as a sum of three squares if and only if it is not of the form $4^i(8m+7)$ $(i, m \geq 0)$.

Proof of Beeson's theorem. In (1) and (2), the implications "$\Leftarrow$" are obvious. So we show the implications "$\Rightarrow$" in (1) and (2).

(1) Suppose $\Theta_2 \ni \theta = \angle POQ$, where $P, O, Q \in \mathbb{Z}^2$ and O is the origin. Let f_P be the similarity of $\mathbb{R}^2$ in Theorem 3.2, and put $f_P(Q) = (a, b)$. Since $f_P(P)$ lies on the x-axis and $f_P(P) \neq (0,0)$, we have either $\theta = \pi/2$ or $\tan\theta = b/a \in \mathbb{Q}$.

(2) Suppose that $\pi/2 \neq \theta = \angle POQ$, $P, O, Q \in \mathbb{Z}^4$ and $O = (0,0,0,0)$. Let f_P be the similarity of Theorem 3.2. Then $f_P(P) = (*, 0, 0, 0)$. Put $f_P(Q) = (a, b, c, d) \in \mathbb{Z}^4$. Then $\tan^2(\theta) = (b^2 + c^2 + d^2)/a^2$.

(3) If $\theta \in \Theta_5$, then $\theta = \angle ABC$ for some $A, B, C \in \mathbb{Z}^5$. By the cosine law, we have $\cos\theta = (|AB|^2 + |BC|^2 - |AC|^2)/(2|AB||BC|)$. Hence $\cos^2\theta \in \mathbb{Q}$. Conversely, if $\theta \neq \pi/2$ and $\cos^2\theta \in \mathbb{Q}$, then $\tan^2\theta \in \mathbb{Q}$, and hence $\tan^2\theta = q/p = pq/p^2$ for some integers $p, q \neq 0$. By Lagrange's four-squares theorem, there are integers a, b, c, d such that $pq = a^2 + b^2 + c^2 + d^2$. Put $A = (p, 0, 0, 0, 0), B = (p, a, b, c, d)$. Then $\angle AOB = \theta \in \Theta_5$.

(4) Since $\tan\frac{\pi}{3} = \sqrt{3} \notin \mathbb{Q}$, we have $\pi/3 \notin \Theta_2$. Since $\pi/3 \in \Theta_3$, we have $\Theta_2 \subsetneq \Theta_3$. Since $\arctan\sqrt{7} \notin \Theta_4$ by Lemma 4.1 and (2) of Beeson' theorem, and since $\cos^2(\arctan\sqrt{7}) = 1/8$, we have $\arctan\sqrt{7} \in \Theta_5$. Hence $\Theta_4 \subsetneq \Theta_5$. By Corollary 3.1, we have $\Theta_3 = \Theta_4$. Finally, if $\theta \in \Theta_n$ for $n \geq 6$, then by the cosine law we have $\cos^2 \in \mathbb{Q}$, and hence $\theta \in \Theta_5$ by (3). Thus $\Theta_5 = \Theta_6 = \Theta_7 = \dots$. $\square$

Proposition 4.1. *If $\pi/2 \neq \theta \in \Theta_k$ for some $k \geq 2$, then $|\pi/2 - \theta| \in \Theta_k$.*

Proof. If $k = 2$ or 5, then the assertion is clear. Let us consider the case $k = 4$: If $\tan^2\theta = (b^2+c^2+d^2)/a^2$ for some $a, b, c, d \in \mathbb{Z}$, then $\tan^2|\pi/2-\theta| = a^2/(b^2+c^2+d^2) = \left((ab)^2 + (ac)^2 + (ad)^2\right)/(b^2+c^2+d^2)^2$. Hence $|\pi/2 - \theta| \in \Theta_4$. $\square$

If the interior angles of a polygon are all equal, then the polygon is called *equiangular*. It is known (see Scott [92]) that in $\mathbb{R}^2$, equiangular lattice m-gons exist only for $m = 4, 8$.

Theorem 4.2. *For every $n \geq 3$, an equiangular lattice m-gon exists in $\mathbb{R}^n$ if and only if $m \in \{3, 4, 6, 8, 12\}$.*

Proof. By Theorem 2.2, there are regular lattice m-gons in $\mathbb{R}^3$ for $m = 3, 4, 6$. If a regular lattice m-gon Γ exists, then an equiangular lattice $2m$-gon can be obtained from Γ by the "triple-and-truncate" method, see Fig. 3, which shows the case $m = 6$. Thus, for $m = 3, 4, 6, 8, 12$, equiangular lattice m-gons exist in $\mathbb{R}^3$.

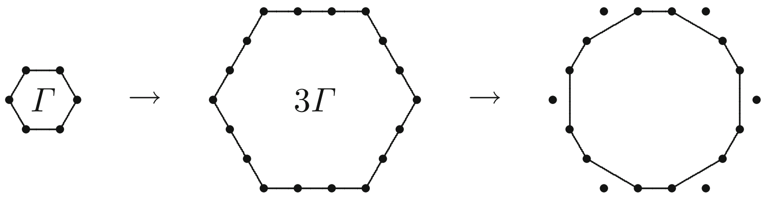

Figure 3. Triple-and-truncate

Now, let $\Gamma = A_1A_2\ldots A_m$ be an equiangular lattice m-gon in $\mathbb{R}^n, n \geq 3$. Its interior angle is equal to $\theta := \pi - 2\pi/m$. By the cosine law, we have

$$\cos\theta = \frac{|A_1A_2|^2 + |A_2A_3|^2 - |A_1A_3|^2}{2|A_1A_2||A_2A_3|}.$$

Since the squares of the edge-lengths of Γ are all integers, we have that $\cos^2\theta$ is rational, and since $\cos(4\pi/m) = \cos 2\theta = 2\cos^2\theta - 1$ and $\cos^2\theta \in \mathbb{Q}$, $\cos(4\pi/m)$ is rational. It follows now by Theorem 2.3 that the possible values of m are $3, 4, 6, 8, 12$. □

4.2. Coplanar lattice angles

For a polygon Γ in $\mathbb{R}^n$, the *cyclic* sequence of its interior angles is called the *angle-sequence* of Γ. For example, the angle-sequence of the (concave) hexagon in Fig. 4 is $\pi/2, 3\pi/4, \pi/4, 3\pi/2, \pi/3, 2\pi/3$. The angle-sequence of an equiangular m-gon is $(m-2)\pi/m, (m-2)\pi/m, \ldots, (m-2)\pi/m$.

By generalizing the relation between a regular m-gon and an equiangular m-gon in some way, we have the notion of *angle-equivalence*. Two polygons are called *angle-equivalent* if they have the same angle-sequence. Thus, an equiangular m-gon is angle-equivalent to a regular m-gon.

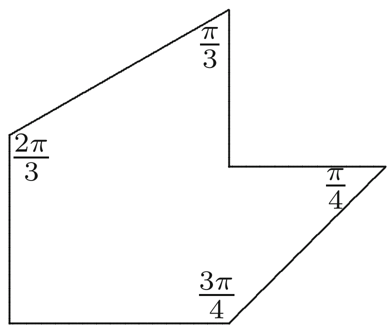

Figure 4. A hexagon

Now, it is natural to ask when a polygon is angle-equivalent to a lattice polygon. Is the polygon shown in Fig. 4 angle-equivalent to a lattice polygon? To answer these questions, let us first consider when a pair of lattice angles can be realized as lattice angles lying on the same plane. Such pairs of lattice angles are called *coplanar*. In other words, $\alpha, \beta \in \Theta_n$ are coplanar if there are coplanar lattice points A, B, C, X, Y, Z in $\mathbb{R}^n$ such that $\alpha = \angle ABC$ and $\beta = \angle XYZ$.

A *rational point* in $\mathbb{R}^n$ is a point whose coordinates are all rational numbers. If $\theta = \angle ABC$ for some three rational points $A, B, C \in \mathbb{R}^n$, then clearly $\theta \in \Theta_n$. A line in $\mathbb{R}^n$ determined by two rational points is called a *rational line*, and a plane determined by three non-collinear rational points in $\mathbb{R}^n$ is called a *rational plane*. Note that the set of rational points lying on a rational line is everywhere dense in the line, and the set of rational points on a rational plane is everywhere dense in the plane.

Lemma 4.2. (1) *If two rational lines intersect at a point, then the point is a rational point.* (2) *A line passing through a rational point and parallel to a rational line is itself a rational line.* (3) *The foot of the perpendicular dropped from a rational point to a rational line is a rational point.*

Proof. Let us show only (3). Let l be a rational line determined by two rational points A, B, and let P be a rational point not on l. Let F be the foot of the perpendicular dropped from P to l. Then we may put $F = (1-t)A + tB$. Since the lines PF and AB are perpendicular, we have

$$\begin{aligned} 0 = \overrightarrow{PF} \cdot \overrightarrow{BA} &= ((1-t)A + tB - P) \cdot (A - B) \\ &= (A - P) \cdot (A - B) - t(A - B) \cdot (A - B). \end{aligned}$$

Thus t is rational, and hence $F = (1-t)A + tB$ is a rational point. □

Lemma 4.3. *Let α, β be lattice angles in $\mathbb{R}^n$, and suppose that $\alpha \not\equiv 0 \pmod{\pi/2}$. Let $\alpha = \angle BAX$ for $A, B, X \in \mathbb{R}^n$, where A, B are rational points, and AX is a rational line. Let Y be a point on the plane determined by A, B, X such that $\beta = \angle ABY$, see Fig. 5. Then the line BY is a rational line if and only if $\cos^2(\alpha + \beta) \in \mathbb{Q}$.*

Proof. If the lines AX and BY are parallel, then BY is a rational line and $\cos^2(\alpha + \beta) \in \mathbb{Q}$. So we consider the case that $AX \nparallel BY$. Let C be the intersection of the lines AX and BY. Then, in the triangle ABC, $\angle A \equiv \pm\alpha, \angle B \equiv \pm\beta \pmod{\pi}$. Thus, $\cos^2(\angle A + \angle B)$ is equal to one of $\cos^2(\alpha + \beta)$, $\cos^2(\alpha - \beta)$. Now, if BY is a rational line, then the intersection point C is a rational point, and $\angle C = \angle ACB$ is a lattice angle. Thus, $\cos^2(\angle A + \angle B) = \cos^2(\pi - \angle C) \in \mathbb{Q}$. Since $\cos^2(\alpha - \beta) \in \mathbb{Q} \Leftrightarrow \cos^2(\alpha + \beta) \in \mathbb{Q}$, we have $\cos^2(\alpha + \beta) \in \mathbb{Q}$.

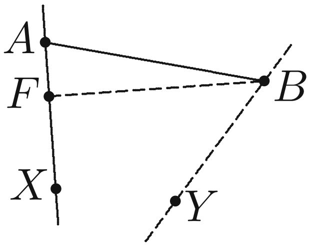

Figure 5. Lemma 4.3

Now suppose that $\cos^2(\alpha+\beta) \in \mathbb{Q}$. We show that C is a rational point. We may suppose that $A = O$, the origin. Let F be the foot of the perpendicular dropped from B to the line OX $(= AX)$. Since $\alpha \not\equiv 0 \pmod{\pi/2}$, $F \neq O$. Hence we can put $C = (1+y)F$. Then $|FC|^2 = y^2|OF|^2$. Since $|OB|/\sin\angle C = |OC|/\sin\angle B$ by the sine law, and since $\sin^2\angle B, \sin^2\angle C, |OB|^2$ are all rationals, we can deduce that $|OC|^2 = (1+y)^2|OF|^2 \in \mathbb{Q}$, and hence $(1+y)^2 \in \mathbb{Q}$. Since $\cot^2\angle C = |FC|^2/|BF|^2 = y^2|OF|^2/|BF|^2$, and $\cot^2\angle C, |OF|^2, |BF|^2$ are all rationals, y^2 is also rational. From $(1+y)^2 \in Q, y^2 \in \mathbb{Q}$, we have $y \in \mathbb{Q}$, and hence C is a rational point. Therefore the line BC (= the line BY) is a rational line. □

Remark 4.2. For two angles $\alpha, \beta \in \Theta_2$, we always have $\cos^2(\alpha+\beta) \in \mathbb{Q}$. However, for two angles $\alpha, \beta \in \Theta_3\backslash\Theta_2$, we do not necessarily have $\cos^2(\alpha+\beta) \in \mathbb{Q}$. For example, let $\alpha = \pi/3$ and $\beta = \arctan\sqrt{2}$. Then $\alpha, \beta \in \Theta_3\backslash\Theta_2$, but $\cos^2(\alpha+\beta) = 13/12 - 1/\sqrt{3}$ is not a rational number.

Remark 4.3. In Lemma 4.3, the restriction $\alpha \not\equiv 0 \pmod{\pi/2}$ is necessary. For example, if $\alpha = \pi/2$, $\beta = \pi/3$, then $\cos^2(\alpha+\beta) \in \mathbb{Q}$. Let $A = (0,0,0), B = (0,1,0)$, and $C = (2,0,0) \in \mathbb{R}^3$. The line BY on the plane of ABC is such that $\angle CBA = \beta$ intersects the line AC at the point $(\sqrt{3},0,0)$, and hence the line BY is not a rational line.

Corollary 4.2. *Let α, β be lattice angles and suppose that $\alpha \not\equiv 0 \pmod{\pi/2}$. If $\alpha \in \Theta_n, n < 5$, and $\beta \in \Theta_5\backslash\Theta_n$, then $\cos^2(\alpha+\beta)$ is an irrational number.*

Lemma 4.4. *Suppose that $\cos^2\alpha, \cos^2\beta, \cos^2\gamma, \cos^2(\alpha+\beta), \cos^2(\beta+\gamma)$ are all rationals. Then $\cos^2(\alpha+\gamma) \in \mathbb{Q}$ if and only if $\cos^2(\alpha+\beta+\gamma) \in \mathbb{Q}$.*

Proof. Since $\cos^2(\alpha+\beta) = \cos^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta - 2\cos\alpha\cos\beta\sin\alpha\sin\beta$, we have $\cos\alpha\cos\beta\sin\alpha\sin\beta \in \mathbb{Q}$. Similarly, $\cos\beta\cos\gamma\sin\beta\sin\gamma \in \mathbb{Q}$. Now, $\cos^2(\alpha+\beta+\gamma)$ is equal to $(\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\sin\gamma)^2$, and in the expansion of this, all the terms other than $-2\cos\alpha \cos\gamma\sin\alpha\sin\gamma\cos^2\beta$, $2\cos\alpha\cos\gamma\sin\alpha\sin\gamma\sin^2\beta$ are rational numbers. Hence $\cos^2(\alpha+\beta+\gamma) \in \mathbb{Q} \Leftrightarrow \cos\alpha\cos\gamma\sin\alpha\sin\gamma \in \mathbb{Q}$. Similarly, we have $\cos^2(\alpha+\gamma) \in \mathbb{Q} \Leftrightarrow \cos\alpha\cos\gamma\sin\alpha\sin\gamma \in \mathbb{Q}$. □

Lemma 4.5. *In a lattice pentagon $ABCDE$,* $\cos^2(\angle A + \angle C) \in \mathbb{Q}$.

Proof. Let $\alpha = \angle A, \beta = \angle B, \ldots, \varepsilon = \angle E$. If $AE \parallel BC$, then $\cos^2(\alpha + \beta) \in \mathbb{Q}$. If the lines AE and BC intersect at a point P, then, since P is a rational point, we have $\cos^2(\alpha + \beta) = \cos^2(\angle APB) \in \mathbb{Q}$. Similarly, it follows that $\cos^2(\beta + \gamma), \ldots, \cos^2(\delta + \varepsilon), \cos^2(\varepsilon + \alpha)$ are all rationals. Since $\cos^2(\alpha + \beta + \gamma) = \cos^2(3\pi - \delta - \varepsilon) = \cos^2(\delta + \varepsilon) \in \mathbb{Q}$, it follows from Lemma 4.4 that $\cos^2(\alpha + \gamma) \in \mathbb{Q}$. □

Remark 4.4. This lemma is true even if $ABCDE$ is a closed polygonal curve with self-intersections in which "interior angles" are defined suitably.

Theorem 4.3. *Two lattice angles α, β are coplanar if and only if* $\cos^2(\alpha + \beta)$ *is a rational number.*

Proof. Suppose that $\cos^2(\alpha + \beta)$ is rational. If $\alpha \equiv \beta \equiv 0 \pmod{\pi/2}$, then we can take lattice points A, B, C, X, Y, Z in $\mathbb{R}^2$ such that $\alpha = \angle ABC$ and $\beta = \angle XYZ$. So assume that $\alpha \not\equiv 0 \pmod{\pi/2}$. Let $\alpha = \angle XAB$, $X, A, B \in \mathbb{Z}^n$. Then, by Lemma 4.3, there is a lattice point Y in the plane XAB such that $\beta = \angle YBA$. Hence α, β are coplanar.

Conversely, suppose that $\alpha = \angle XAY$, and β is realized as the lattice angle $\angle PQR$ in the plane XAY. Let φ be the translation of the plane XAY along the vector $\overrightarrow{PY}$. Let $B = \varphi(Q), Z = \varphi(R)$. Then B, Z are also lattice points, and $\beta = \angle YBZ$. By adding the line segment XZ, we have a (possibly self-intersecting) lattice pentagon $AYBZX$. Now, by Lemma 4.5 (and Remark 4.4), we have $\cos^2(\angle A + \angle B) \in \mathbb{Q}$. Hence $\cos^2(\alpha + \beta) \in \mathbb{Q}$. □

4.3. Angle-equivalence to lattice polygons

Theorem 4.4. (Maehara [60]) *A polygon Γ is angle-equivalent to a lattice polygon in $\mathbb{R}^n$ if and only if all interior angles of Γ are lattice angles in $\mathbb{R}^n$, and* $\cos^2(\alpha + \beta) \in \mathbb{Q}$ *for every pair of interior angles α, β of Γ.*

For example, since $\cos^2(\pi/4 + \pi/3) \notin \mathbb{Q}$, the hexagon in Fig. 4 is not angle-equivalent to a lattice hexagon by Theorem 4.3.

Proof. First, suppose that Γ is angle-equivalent to a lattice polygon in $\mathbb{R}^n$. Then all interior angles of Γ are mutually coplanar. Hence $\cos^2(\alpha + \beta) \in \mathbb{Q}$ for every pair α, β of interior angles of Γ by Theorem 4.3.

Now, suppose that all interior angles of Γ are lattice angles in $\mathbb{R}^n$ and for every pair α, β of interior angles of Γ, $\cos^2(\alpha+\beta)$ is a rational number. If every angle of Γ is either $\pi/2$ or $3\pi/2$, then it is easy to see that there is a lattice polygon in $\mathbb{R}^2$ that is angle-equivalent to Γ. So, let us assume that there is an interior angle α of Γ such that $\alpha \not\equiv 0 \pmod{\pi/2}$. If Γ has corners where the angle is either $\pi/2$ or $3\pi/2$, then we modify Γ by cutting off, or attaching,

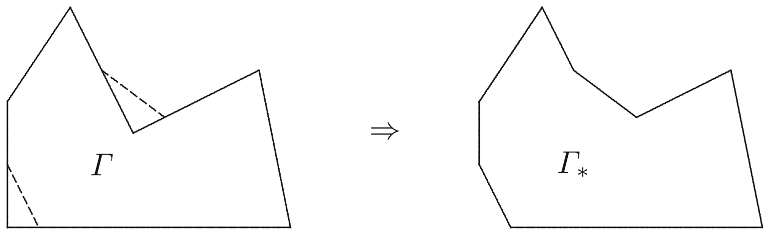

Figure 6. Elimination of right angles from Γ

small right triangles with one interior angle $\theta \equiv \alpha \pmod{\pi/2}$ at such corners, see Fig. 6.

Let $\Gamma_* = A_1A_2\ldots A_m \subset \mathbb{R}^n$ be the modified polygon, and put $\alpha_i = \angle A_i, i = 1, 2, \ldots, m$. Note that for every i and j, $\cos^2(\alpha_i + \alpha_j) \in \mathbb{Q}$ holds. It will be enough to show that there is a rational polygon in $\mathbb{R}^n$ that is angle-equivalent to the modified polygon $\Gamma_* = A_1A_2\ldots A_m$. Since α_1 is a lattice angle in $\mathbb{R}^n$, we may, by moving Γ_* in $\mathbb{R}^n$ if necessary, suppose that A_1 is a rational point, and the lines A_1A_m and A_1A_2 are rational lines in $\mathbb{R}^n$. Put $B_1 = A_1$. Since the set of rational points on a rational line is everywhere dense in the line, we can take a rational point B_2 on the line $B_1A_2\,(= A_1A_2)$ that is very close to A_2. Since $\cos^2(\alpha_1 + \alpha_2) \in \mathbb{Q}$, there is a rational line B_2X on the plane of Γ_* such that $\angle B_1B_2X = \angle A_2$ by Lemma 4.3. Note that the lines B_2X and A_2A_3 are parallel and very close. Take a rational point B_3 on the line B_2X that is near A_3. Repeating similarly, we can obtain a polygonal curve $B_1B_2B_3\ldots B_{m-1}$. Let $B_{m-1}Y$ be a rational line on the plane Γ_* such that $\angle B_{m-2}B_{m-1}Y = \alpha_{m-1}$. Let B_m be the intersection point of the lines B_1A_m and $B_{m-1}Y$. Then B_m is a rational point and $\angle B_1B_mB_{m-1} = \alpha_m$. By choosing B_2 sufficiently near to A_2, it is possible to choose subsequent $B_i, i = 3, 4, \ldots,$ which are also close to $A_i, i = 3, 4, \ldots,$ so that the resulting closed polygonal curve approximates the polygon Γ_*, and hence itself is also a polygon. Thus there is a rational polygon in $\mathbb{R}^n$ that is angle-equivalent to Γ_*. □

From Corollary 4.2 and Theorem 4.3, we have the following

Corollary 4.3. *Let ABC be a triangle whose interior angles are all lattice angles, and let $\angle A = \alpha \neq \pi/2$. Then the following statements hold.*

(1) *ABC is similar to a lattice triangle in $\mathbb{R}^5$.*
(2) *If $\alpha \in \Theta_4$, then ABC is similar to a lattice triangle in $\mathbb{R}^4$.*
(3) *If $\alpha \in \Theta_2$, then ABC is similar to a lattice triangle in $\mathbb{R}^2$.*

5. Lattice simplices

The beauty and depth of the geometry of lattice simplices might be demonstrated by the following more recent contributions: Kantor [50] showed that there are lattice n-simplices, having only integral vertices and no further integral points, whose width goes with n to infinity. Elegantly using barycentric coordinates and tools from the geometry of numbers, Averkov et al. (see [4,5]) established sharp upper bounds on the volume of lattice n-simplices with exactly one lattice interior point and on the volumes of their faces, thus confirming a known conjecture of Hensley from 1983; for the best known lower bound see [101]. And Nill [74] studied reflexive simplices, i.e., lattice simplices whose duals are also lattice simplices; these correspond to special toric Fano varieties, and a generalization of the Blaschke-Santaló inequality for reflexive simplices is also derived in [74].

5.1. Congruent embeddings

If a k-dimensional simplex σ^k is congruent to a lattice simplex in $\mathbb{Z}^n$ for some n, then we say that σ^k is *congruently embeddable* in $\mathbb{Z}^n$. Note that if a k-dimensional simplex σ^k is congruently embeddable in $\mathbb{Z}^n$ for some n, then σ^k clearly satisfies the following condition:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \in \mathbb{Z} \text{ for every three vertices } A, B, C. \tag{5.1}$$

In (5.1), A, B, C may not be different. If $B = C$, then $\overrightarrow{AB} \cdot \overrightarrow{AC} \in \mathbb{Z}$ implies $|AB|^2 \in \mathbb{Z}$.

For k linearly independent vectors $\vec{a}_1, \ldots, \vec{a}_k$ in $\mathbb{R}^n$, the set of integral combinations

$$\lambda_1\vec{a}_1 + \cdots + \lambda_k\vec{a}_k,\ \lambda_1, \ldots, \lambda_k \in \mathbb{Z}$$

is called a *k-dimensional lattice in $\mathbb{R}^n$ generated by $\vec{a}_i$*, $i = 1, \ldots, k$. Moreover, if the inner product $\vec{a}_i \cdot \vec{a}_j$ is an integer for every $1 \le i, j \le k$, then the lattice is called a k-dimensional *integral lattice*. It was proved by Mordell [72] and Ko [54] (see also Conway and Sloane [18]) that *if $k \le 5$, then every k-dimensional integral lattice is congruent to a subset of $\mathbb{Z}^{k+3}$, but there is a 6-dimensional integral lattice that is never congruent to a subset of $\mathbb{Z}^N$ for any N.*

If σ^k is a k-dimensional simplex in $\mathbb{R}^n$ satisfying (5.1), and one vertex of σ^k is the origin O, then the k vectors emanating from O to the other k vertices generate a k-dimensional integral lattice in $\mathbb{R}^n$. Hence we have the following theorem.

Theorem 5.1. (Mordell [72] + Ko [54], Conway and Sloane [18]) *If $k \le 5$, then a k-dimensional simplex that satisfies* (5.1) *is always congruently embeddable*

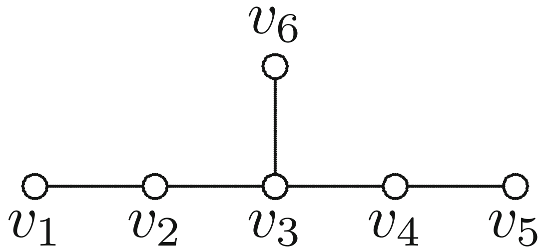

Figure 7. A Dynkin digram

in $\mathbb{Z}^{k+3}$*. On the other hand, there exists a* 6*-dimensional simplex that satisfies* (5.1)*, but is not congruently embeddable in* $\mathbb{Z}^n$ *for any* n*.*

A 6-dimensional simplex that satisfies (5.1) but is not congruently embeddable in any $\mathbb{Z}^n$ is generated by the six vectors $v_1, v_2, \ldots, v_6$ represented by the graph given in Fig. 7 (which is called a Dynkin diagram). In this graph, *each vertex represents a vector of length* $\sqrt{2}$*; the inner product of two vectors equals* -1 *if the corresponding vertices are adjacent, and it equals* 0 *otherwise.*

Thus the *Gram matrix* $A = (v_i \cdot v_j)$ of $v_1, v_2, \ldots, v_6$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

The fact that this simplex is not congruently embeddable in $\mathbb{Z}^n$ for any n is also easily checked by using the computer algebra system GAP in which the algorithm for solving the matrix equation $XX^{tr} = A$ over the integers (due to Plesken [81]) is implemented.

A simplex in $\mathbb{R}^n$ is called a *rational simplex* if all its vertices are rational points. If a simplex is congruent to a rational simplex in $\mathbb{R}^n$, then we say the simplex is *congruently embeddable* in $\mathbb{Q}^n$. If a simplex is congruently embeddable in some $\mathbb{Q}^n$, then it clearly satisfies the following condition:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \in \mathbb{Q} \text{ for every three vertices } A, B, C. \tag{5.2}$$

Theorem 5.2. (Kumada [57]) *If a* k*-dimensional simplex* σ^k *satisfies the condition* (5.2)*, then* σ^k *is congruently embeddable in* $\mathbb{Q}^{k+3}$*.*

It is generally impossible to reduce the dimension $k+3$ in this theorem. For example, the right triangles with edge-lengths $1, 7, \sqrt{50}$ satisfy the condition (5.2) as it is easily verified, but are not congruently embeddable in $\mathbb{Q}^{2+2}$, because $\arctan 7 \notin \Theta_4$, by Beeson's theorem.

Theorem 5.2 was proved by Kumada [57] using p-adic number theory. We show the theorem here invoking Mayer's theorem without proof.

Theorem 5.3. (Mayer [69]) *If a quadratic form of rank at least 5 with rational coefficients has a nontrivial zero over $\mathbb{R}$, then it has a nontrivial zero over $\mathbb{Q}$.*

For Mayer's theorem, see also J. Milnor and D. Husemoller [70], Ch. II §3, Cassel [17], and Serre [94]. As a direct consequence of this theorem, we have the following

Corollary 5.1. *For positive rationals a, b, c, d, e, the equation $ax^2 = by^2 + cz^2 + du^2 + dv^2$ has a solution $(x, y, z, u, v) \in \mathbb{Q}^5$ with $x \neq 0$.* □

Lemma 5.1. *Let $\sigma = A_1 \dots A_k$ be a $(k-1)$-dimensional rational simplex, and let F be a point on the affine hull of σ. If $|A_iF|^2 \in \mathbb{Q}$ $(i = 1 \dots, k)$, then F is a rational point.*

Proof. Let $\overrightarrow{A_1F} = x_2\overrightarrow{A_1A_2} + \cdots + x_k\overrightarrow{A_1A_k}$. Then we have

$$x_2\overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_i} + \cdots + x_k\overrightarrow{A_1A_k} \cdot \overrightarrow{A_1A_i} = \overrightarrow{A_1F} \cdot \overrightarrow{A_1A_i} \quad (i = 2, \dots, k). \tag{5.3}$$

Since σ satisfies (5.2), and $\overrightarrow{A_1F} \cdot \overrightarrow{A_1A_i} = (|A_1F|^2 + |A_1A_i|^2 - |FA_i|^2)/2 \in \mathbb{Q}$, (5.3) can be regarded as a simultaneous linear equation on $x_2, \dots, x_k$ whose coefficient-matrix is a non-singular matrix with rational entries. Therefore $x_2, \dots, x_k$ are all rationals, and hence F is a rational point. □

We denote the volume of a simplex σ by $|\sigma|$.

Lemma 5.2. *If a simplex σ satisfies the condition (5.2), then $|\sigma|^2 \in \mathbb{Q}$.*

Proof. Let $\sigma = A_0A_1 \dots A_k$, and for $1 \leq i, j \leq k$ let $a_{ij} = \overrightarrow{A_0A_i} \cdot \overrightarrow{A_0A_j}$. Then $|\sigma|^2 = \det(a_{ij})/(k!)^2$. If σ satisfies (5.2), then $a_{ij} \in \mathbb{Q}$, and hence $|\sigma|^2 \in \mathbb{Q}$. □

Proof of Theorem 5.2. The proof is by induction on the dimension k of the simplex. First, consider the case $k = 1$. Let $\sigma^1 = A_0A_1$. By using Lagrange's four-square theorem, represent $|A_0A_1|^2 \in \mathbb{Q}$ as $|A_0A_1|^2 = p^2 + q^2 + r^2 + s^2$ $(p, q, r, s \in \mathbb{Q})$, and put $A'_0 = (0, 0, 0, 0), A'_1 = (p, q, r, s)$. Clearly, $A'_0, A'_1 \in \mathbb{Q}^{1+3}$. Then $A'_0A'_1$ is congruent to A_0A_1.

Suppose that the theorem is true for $k = n-1$, and let $\sigma^n = A_0A_1 \dots A_n$ be an n-dimensional simplex that satisfies the condition (5.2). By the induction hypothesis, the facet $\tau := A_1 \dots A_n$ is congruently embeddable in $\mathbb{Q}^{n+2}$. Hence we may suppose that $\sigma^n \subset \mathbb{R}^{n+3}$ and $\{A_1, \dots, A_n\} \subset \mathbb{Q}^{n+2} \times \{0\} \subset \mathbb{Q}^{n+3}$. Let F be the foot of the perpendicular dropped from A_0 to the affine hull of the facet τ. Then $|\sigma^n| = |\tau| \times |FA_0|/k$. Hence $|A_0F|^2 \in \mathbb{Q}$, and therefore $|FA_i|^2 = |A_0A_i|^2 - |FA_0|^2 \in \mathbb{Q}$ for $i = 1, 2, \dots, n$. Thus, by Lemma 5.1, F is a rational point. By translating if necessary, we may suppose $F = O$, the origin. Then $\{A_1, \dots, A_n\}$ spans an $(n-1)$-dimensional subspace of the vector space $\mathbb{Q}^{n+3}$ over the rational field $\mathbb{Q}$. Hence there are four mutually orthogonal vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{Q}^{n+3}$, all orthogonal to the subspace spanned by $\{A_1, \ldots, A_n\}$. Let $a = |\vec{a}|^2, b = |\vec{b}|^2, c = |\vec{c}|^2, d = |\vec{d}|^2$, and $e = |A_0F|^2$. Since a, b, c, d, e are all positive rationals, the equation $ex^2 = ay^2 + bz^2 + cu^2 + dv^2$ has a solution $(x_0, y_0, z_0, u_0, v_0) \in \mathbb{Q}^5$ with $x_0 \neq 0$ by Corollary 5.1. Put $A_0' = F + (y_0/x_0)\vec{a} + (z_0/x_0)\vec{b} + (u_0/x_0)\vec{c} + (v_0/x_0)\vec{d}$. Then $A_0' \in \mathbb{Q}^{n+3}$, $|FA_0'| = e$, and $\overrightarrow{FA_0'}$ is orthogonal to the subspace spanned by $\{A_1, \ldots, A_n\}$. Hence the simplex $A_0'A_1 \ldots A_n$ is congruent to the simplex σ_n This completes the proof. □

5.2. Similar embeddings

As stated in the previous subsection, the 6-dimensional simplex generated by the vectors $v_1, \ldots, v_6$ described by the Dynkin diagram of Fig. 7 is not congruently embeddable in $\mathbb{Z}^n$ for any n. However, since this simplex satisfies the condition (5.2), it can be congruently embeddable in $\mathbb{Q}^{6+3}$ by Theorem 5.2, and hence, it is similar to a 6-dimensional simplex in $\mathbb{Z}^9$.

A simplex σ is said to be *similarly embeddable* in $\mathbb{Z}^n$ if there is a lattice simplex in $\mathbb{R}^n$ that is similar to σ. By Theorem 5.2, a k-dimensional simplex σ^k satisfying (5.2) is always similarly embeddable in $\mathbb{Z}^{k+3}$. Note that (5.2) implies that

$$\cos^2 \angle ABC \in \mathbb{Q} \text{ for every three vertices } A, B, C, \tag{5.4}$$

which is simply called the *angle condition*. For similar embeddings, we can relax the condition (5.2) to the angle condition (5.4).

Theorem 5.4. *A k-dimensional simplex σ is similarly embeddable in $\mathbb{Z}^{k+3}$ if and only if σ satisfies the angle condition* (5.4).

Proof. The "only if" part of the theorem is clear. So we show the "if" part by induction on the dimension m of the simplex σ. If $m \leq 1$, then the assertion is clearly true. Provided that the assertion is true for $m < k$, let us consider the case $m = k$. Let τ be a facet of σ. Since τ is a $(k-1)$-dimensional simplex that satisfies the angle condition (5.4), τ is similarly embeddable in $\mathbb{Z}^{k+2}$ by the induction hypothesis. Hence there is a k-dimensional simplex $\hat{\sigma}$ in $\mathbb{R}^{k+2}$ that is similar to σ and in which the facet $\hat{\tau}$ corresponding to the facet τ of σ is a lattice simplex. Note that $\hat{\sigma}$ also satisfies the angle condition (5.4). Let A be the vertex of $\hat{\sigma}$ opposite to $\hat{\tau}$, and let X, Y be any pair of vertices of $\hat{\tau}$. Then $|XY|^2 \in \mathbb{Q}$. Applying the sine law to the triangle AXY, we have $|XY|/\sin \angle A = |AX|/\sin \angle Y$. Since $\hat{\sigma}$ satisfies the angle condition (5.4), $\cos^2 \angle Y \in \mathbb{Q}$, and hence $\sin^2 \angle Y \in \mathbb{Q}$. Therefore $|AX|^2 \in \mathbb{Q}$. This implies that $\hat{\sigma}$ also satisfies the condition (5.2). Hence $\hat{\sigma}$ is similar to a rational simplex in $\mathbb{R}^{k+3}$, and hence similar to a lattice simplex in $\mathbb{R}^{k+3}$. □

Let $\delta(k)$ denote the minimum dimension n such that every k-dimensional simplex that satisfies the angle condition (5.4) is similarly embeddable in $\mathbb{Z}^n$.

By Theorem 5.3 we have $\delta(k) \leq k+3$. If $k \equiv 1 \pmod 4$, then by Theorem 3.3 we have $\delta(k) \leq k+2$. In the following, let us find $\delta(3)$.

A simplex of dimension ≥ 2 is called an *orthogonal simplex* if all edges emanating from one common vertex are mutually orthogonal. This vertex is called the *pivot* of the orthogonal simplex. For a given simplex $\sigma = A_0A_1 \dots A_k$ we have, applying the Schmidt orthogonalization method (without normalizing) to the vectors $\vec{v}_i := \overrightarrow{A_0A_i}$ $(i = 1, 2, \dots, k)$, mutually orthogonal vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$, where

$$\begin{aligned}
\vec{u}_1 &= \vec{v}_1 \,, \\
\vec{u}_2 &= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 \,, \\
\vec{u}_3 &= \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 \,, \\
&\dots .
\end{aligned}$$

Then, by putting $B_i = A_0 + \vec{u}_i$ $(i = 1, 2, \dots, n)$, we can get an orthogonal simplex $A_0B_1B_2 \dots B_k$, which is called an *orthogonal simplex with pivot* A_0 obtained from σ by Schmidt's method. Note that this orthogonal simplex depends on the choice of a vertex that becomes pivot, and on the order of other vertices of σ. Notice that $\{\vec{v}_1, \dots, \vec{v}_k\}$ and $\{\vec{u}_1, \dots, \vec{u}_k\}$ generate the same vector space over $\mathbb{Q}$. If σ is a rational simplex with A_0 at the origin, then the orthogonal simplex with pivot A_0 obtained from σ is contained in the vector space generated by $\vec{u}_1, \dots, \vec{u}_k$, and hence this orthogonal simplex is also a rational simplex. Thus, we have the following

Lemma 5.3. *A simplex σ of dimension ≥ 2 is similarly embeddable in $\mathbb{Z}^n$ if and only if an orthogonal simplex obtained from σ by Schmidt's method is similarly embeddable in $\mathbb{Z}^n$.* □

Lemma 5.4. *In a rational orthogonal tetrahedron, one of the three faces* (*right triangles*) *around the pivot is similarly embeddable in $\mathbb{Z}^4$.*

Proof. By (2) of Corollary 4.3, it is enough to show that one of the three right triangles around the pivot has an acute angle that belongs to Θ_4. Let ξ, η, ζ denote the lengths of the edges emanating from the pivot. Then $\xi^2, \eta^2, \zeta^2 \in \mathbb{Q}$. If $(\xi\eta)^2$ can be represented by the sum of three squares of integers, then since $(\xi/\eta)^2 = (\xi\eta)^2/\eta^4$, an acute angle of the right triangle with arms of lengths ξ, η belongs to Θ_4 by (2) from Beeson's theorem. Suppose that neither $(\xi\eta)^2$ nor $(\eta\zeta)^2$ can be represented by the sum of three squares. Then both $(\xi\eta)^2, (\eta\zeta)^2$ are numbers of the form $4^i(8m+7)$ by Legendre's three-squares theorem (see Remark 4.1). Since $7^2 \equiv 1 \pmod 8$, the number $(\xi\eta^2\zeta)^2$ is not of the form $4^i(8m+7)$; it can be represented as the sum of three squares. Therefore, $(\xi/\zeta)^2$ can be represented as $(b^2 + c^2 + d^2)/a^2$ $(a, b, c, d \in \mathbb{Z})$, and an acute angle of the right triangle with arms of lengths ξ, ζ belongs to Θ_4. □

Remark 5.1. Not every lattice tetrahedron has a planar angle that belongs to Θ_4. For example, a tetrahedron, that has a pair of opposite edges of length 2 and whose other four edges are of length $\sqrt{8}$, has no planar angle in Θ_4.

Theorem 5.5. *Every rational orthogonal tetrahedron is similarly embeddable in* $\mathbb{Z}^5$.

Applying Lemma 5.3, we have the following

Corollary 5.2. *Every tetrahedron that satisfies the angle condition* (5.4)*, is similarly embeddable in* $\mathbb{Z}^5$*. Thus,* $\delta(2) = \delta(3) = 5$.

Proof of Theorem 5.5. Let $OABC$ be a rational orthogonal tetrahedron with pivot O. By Lemma 5.4, one of the triangles OAB, OBC, OCA is similarly embeddable in $\mathbb{Z}^4$. So we may suppose that $O, A, B \in \mathbb{Z}^4$ and $O = (0, 0, 0, 0)$. Let $\eta = |OC|$. By multiplying A, B, C by an integer if necessary, we may suppose that η^2 is an integer. By the four-squares theorem, there are $x, y, z, w \in \mathbb{Z}$ such that $\eta^2 = x^2 + y^2 + z^2 + w^2$. Applying the similarity transformation induced by the matrix $[[x, y, z, w]]$ defined in the proof of Theorem 3.2, the triangle OAB is transformed to a triangle $OA'B'$ whose size is η times the size of OAB. Regarding $O, A', B' \in \mathbb{Z}^4$ as a subset of $\mathbb{Z}^4 \times \{0\} \subset \mathbb{Z}^5$, put $C' = (0, 0, 0, 0, \eta^2)$. Then the tetrahedron $OA'B'C'$ is similar to $OABC$ and is embedded in $\mathbb{Z}^5$. □

Problem 5.1. *Characterize those tetrahedra that are similarly embeddable in* $\mathbb{Z}^4$ *(and hence in* $\mathbb{Z}^3$*).*

Problem 5.2. *Find* $\delta(4)$.

6. Lattice points in a planar region

6.1. Steinhaus' lattice point problem

To avoid confusion, the reader is warned that there is a second "Steinhaus lattice problem" (see, e.g., [48]) different from the one discussed in the following.

In 1957, Steinhaus [96] posed the following problem in elementary mathematics. Is there a circle in $\mathbb{R}^2$ that encloses exactly m lattice points, for every $m > 0$?

He also proved the following theorem (see Honsberger [47]).

Theorem 6.1. (Steinhaus) *If a disk in* $\mathbb{R}^2$ *has area* m*, then it can be translated in* $\mathbb{R}^2$ *so that it contains exactly* m *lattice points in its interior.*

The author of [103] showed that Steinhaus' theorem below holds for Hilbert spaces; based on this, [49] extended the statement to certain classes of Banach spaces, e.g., to strictly convex norms. In [58] it is shown that the circular disk from Steinhaus' theorem can be replaced by any compact, convex figure in the plane. The polygonal version of this was extended in [65] to higher dimensions: any n-dimensional polyhedron (by which the author means a compact set bounded by an $(n-1)$-dimensional closed manifold that is contained in the union of finitely many hyperplanes) with volume $m+\alpha$, for some $|\alpha|<1$, has a congruent copy that contains exactly m points from the n-dimensional lattice.

Let us, in a more detailed way, generalize Theorem 6.1 to planar regions other than the disk. First, we recall the following theorem (in which area is meant in the Lebesgue sense).

Theorem 6.2. (Blichfeldt) *If a bounded region $X \subset \mathbb{R}^2$ has area $m+\alpha$ ($|\alpha|<1$), then it is possible to translate X to a position where X covers at least m lattice points, and it is also possible to translate X so that it covers at most m lattice points.*

We omit the proof and refer, for intuitive proofs of this result, to Honsberger [47] and Steinhaus [97].

For a set $X \subset \mathbb{R}^2$ and $p \in \mathbb{R}^2$, let $p+X$ denote the translate of X along $\vec{p}$. Further on, let X^* denote the set symmetric to X with respect to $O=(0,0)$, that is, $X^*=\{-x : x \in X\}$. Let us note here that

$$p \in w+X \Rightarrow p-w \in X \Rightarrow w-p \in X^* \Rightarrow w \in p+X^*. \tag{6.1}$$

A planar curve $C \in \mathbb{R}^2$ is called *lattice-generic* if $C \cap (p+C)$ is a finite set for every lattice point $p \neq (0,0)$. Note that if C is a lattice-generic curve, then $(p+C)\cap(q+C)$ is also a finite set for $p,q \in \mathbb{Z}^2, p \neq q$. Indeed, since $p \neq q$, $C \cap (q-p+C)$ is a finite set, and hence its translate $(p+C)\cap(q+C)$ is a finite set. If C is lattice-generic, then so is C^*, because $(p+C)^* = (-p)+C^*$. Since every circle is lattice-generic, Theorem 6.1 also follows from the next theorem.

Theorem 6.3. (Maehara [64]) *If $X \subset \mathbb{R}^2$ is a compact region of area $m+\alpha$ ($|\alpha|<1$) bounded by a lattice-generic curve, then X can be translated in $\mathbb{R}^2$ so that it covers exactly m lattice points.*

Proof. Let X° denote the interior of X, and C be the boundary curve of X. Then $area(X^\circ) = area(X) = m+\alpha$, and C is lattice-generic. By Blichfeldt's theorem, there are $u_0, v_0 \in \mathbb{R}^2$ such that

$$|(u_0+X)\cap \mathbb{Z}^2| \leq m, \quad |(v_0+X^\circ)\cap \mathbb{Z}^2| \geq m\,. \tag{6.2}$$

Let D be a disk that contains u_0, v_0 in its interior. The set S, defined by

$$S = \{p \in \mathbb{Z}^2 : (p+C^*)\cap D \neq \emptyset\}\,,$$

is a finite set. Since C^* is also lattice-generic, the set F defined by

$$F = \bigcup\{D \cap (p+C^*) \cap (q+C^*) : p,q \in S, p \neq q\}\,,$$

is also a finite set. Hence $D\backslash F$ is path-connected. Since X is compact, the minimum distance δ from a lattice point in the exterior of X to X is positive. Hence we have, for any point u in the $(\delta/2)$-neighborhood of u_0, $|(u+X)\cap\mathbb{Z}^2| \leq m$. Thus, replacing u_0 by a point close to it if necessary, we may suppose that $u_0 \in D\backslash F$. Similarly, we may assume that $v_0 \in D\backslash F$. Now, since $D\backslash F$ is path-connected, there is a simple curve J in $D\backslash F$ that connects u_0 and v_0. Note that if $p, q \in w + C$ for some $p, q \in \mathbb{Z}^2, p \neq q$, then $w \in (p + C^*) \cap (q + C^*)$ by (6.1), and hence $w \in F$. Therefore, if $w \in J$, then, since $J \cap F = \emptyset$, we have $|(w + C) \cap \mathbb{Z}^2| \leq 1$. Thus, when w moves from u_0 to v_0 along the curve J, $|(w+X)\cap\mathbb{Z}^2|$ changes one by one. Hence (6.2) implies that there is a point $w \in J$ such that $(w + X) \cap \mathbb{Z}^2| = m$. □

For a polynomial $f(x, y) \in \mathbb{R}[x, y]$, the set

$$V(f) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\} \subset \mathbb{R}^2$$

is called a plane (affine) *algebraic curve*. The equation $f(x, y)$ is called its *defining polynomial*. For example, quadratic curves are algebraic curves. If $C = V(f)$ is an algebraic curve, then so are $v + C$ and C^*. If the defining polynomial $f(x, y)$ is irreducible in $\mathbb{R}[x, y]$, then the algebraic curve $V(f)$ is called an *irreducible algebraic curve*. If C is an irreducible algebraic curve, then so are $v + C$ and C^*.

By Bézout's theorem (see, e.g., Silverman and Tate [95], Appendix A.4.), we have that if two irreducible algebraic curves have infinitely many points in common, then the two curves coincide completely.

Theorem 6.4. (Maehara [64]) *If an irreducible algebraic curve C is not a line, then C is lattice generic.*

Proof. If C is irreducible, then $p + C$ is also irreducible for $p \in \mathbb{Z}^2 - \{O\}$. Suppose that $C \cap (p + C)$ is an infinite set. Then C and $p + C$ coincide, from which it follows that $v \in C \Rightarrow v + kp \in C$ for $k = 1, 2, 3, \ldots$. Thus, C and the line represented by $(x, y) = v + tp$, $-\infty < t < \infty$, have infinitely many points in common, and hence C is a line. □

Since a line cannot bound a compact region, we have the following

Corollary 6.1. *If a compact region bounded by an irreducible algebraic curve has area m, then it is possible to translate X to a position where it covers exactly m lattice points.*

6.2. Bézier curves and beziergons

A Bézier curve is a parametric curve usually used in computer graphics. A cubic Bézier curve is determined by four control points $p_1, p_2, p_3, p_4 \in \mathbb{R}^2$, and it is presented by

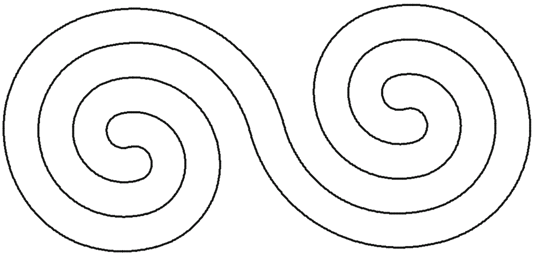

Figure 8. A beziergon

$$(x, y) = (1-t)^3 p_1 + 3(1-t)^2 t p_2 + 3(1-t)t^2 p_3 + t^3 p_4 \qquad (0 \le t \le 1).$$

This curve connects p_1 and p_4, and it is tangent to the line p_1p_2 at p_1, and tangent to the line p_3p_4 at p_4. We note here that if we put

$$p_2 = \tfrac{2}{3}p_1 + \tfrac{1}{3}p_4,\ p_3 = \tfrac{1}{3}p_1 + \tfrac{2}{3}p_4,$$

then we have $(x, y) = (1-t)p_1 + tp_4$, which is the line segment connecting p_1 and p_3. A translate of a Bézier curve is also a Bézier curve. A *beziergon* is a simple closed path composed of finitely many Bézier curves. Note that a polygon is a special case of the notion of beziergon. Figure 8 shows a beziergon composed by 28 Bézier curves.

Lemma 6.1. *A curve parametrized by using polynomials $\varphi(t), \psi(t)$ of t, such as*

$$(x, y) = (\varphi(t), \psi(t)) \ (a \le t \le b),$$

is a part of an irreducible algebraic curve. Hence, a Bézier curve is also a part of an irreducible algebraic curve.

Though this result is classical, let us present an outline of the proof.

Proof. By eliminating t from $\varphi(t)-x = 0$, $\psi(t)-y = 0$, we can get a polynomial equation $f(x, y) = 0$. (There are some algorithms to eliminate t by symbolic calculations, see, e.g., Buchberger [15] and Trott [98], pp. 26–29.) Hence $C \subset V(f)$. If $f = f_1^{e_1} f_2^{e_2} \dots f_k^{e_k}$ is a decomposition of f into irreducible polynomials, then $V(f) = V(f_1) \cup \cdots \cup V(f_k)$. Since $C \subset V(f)$, there is an i such that C and $V(f_i)$ have infinitely many points in common. In this case we have $f_i(\varphi(t), \psi(t)) = 0$ for infinitely many t. Therefore, $f_i(\varphi(t), \psi(t)) = 0$ holds identically, which implies $C \subset V(f_i)$. □

From this, we have the following

Corollary 6.2. *If two Bézier curves have infinitely many points in common, then they are parts of the same irreducible algebraic curve.*

Lemma 6.2. *Let B_1, B_2 be Bézier curves. If $p \neq O$, then the number of those $\rho \in SO(2)$ that satisfy $|\rho(B_1) \cap (p + \rho(B_2))| = \infty$ is at most two.*

Proof. Let V_1, V_2 be irreducible algebraic curves containing B_1 and B_2, respectively. If $|\rho(B_1) \cap (p + \rho(B_2))| = \infty$, then by Corollary 6.2 we have $\rho(V_1) = p + \rho(V_2)$, and hence $V_1 = \rho^{-1}(p) + V_2$. Suppose that two different ρ_1, ρ_2 satisfy the equation $V_1 = \rho^{-1}(p) + V_2$. If we put $v = \rho_2^{-1}(p) - \rho_1^{-1}(p)$, then $v \neq (0, 0)$ and $V_2 = v + V_2$. This implies that $u \in V_2 \Rightarrow u + kv \in V_2$ $(k = 1, 2, \ldots)$. Therefore, V_2 shares infinitely many points with a line, and hence V_2 itself is a line. Further on, V_1 is also a line. Now, $V_1 = \rho^{-1}(p) + V_2$ implies that for a fixed $x \in V_2$ we have $\rho^{-1}(p) \in (-x) + \in V_1$. Since $\{\rho^{-1}(p) : \rho \in SO(2)\}$ is a circle, it intersects the line $(-x) + V_1$ in at most two points. Hence there are at most two ρ that satisfy $\rho(V_1) = p + \rho(V_2)$. □

Lemma 6.3. *A beziergon can be rotated so that it becomes lattice generic.*

Proof. Let Ω be a beziergon consisting of m Bezier curves $B_1, \ldots, B_m$. and let D be a disk with center $O = (0, 0)$ that contains Ω. Let $K = \{p \in \mathbb{Z}^2 : D \cap (p + D) \neq \emptyset\}$. Then, for $\rho \in SO(2)$ and $q \in \mathbb{Z}^2 \backslash K$, we have $\rho(\Omega) \cap (q + \rho(\Omega)) = \emptyset$. If $|\rho(\Omega) \cap (p + \rho(\Omega)| = \infty$ for a $p \in K$, then there must be two B_i, B_j such that $|\rho(B_i) \cap (p + \rho(B_j))| = \infty$. Hence, for each $p \in K$, the number of $\rho \in SO(2)$ such that $|\rho(\Omega) \cap (p + \rho(\Omega)| = \infty$ is at most two by Lemma 6.3. Since K is a finite set, the set

$$\{\rho \in SO(2) : |\rho(\Omega) \cap (p + \rho(\Omega)| = \infty \text{ for some } p \in K\}$$

is also a finite set. Since $SO(2)$ has infinitely many elements, there must be an element $\rho_0 \in SO(2)$ such that $\rho_0(\Omega)$ is lattice generic. □

From Theorem 6.3, we have the following corollary.

Corollary 6.3. *If a region bounded by a beziergon has area m, then it can be rotated and translated so that it covers exactly m lattice points.*

As a special case, it follows that *every polygon in the plane having area m can be rotated and translated so that it contains exactly m lattice points.*

Similarly to the proof of Lemma 6.3, it can be proved that for a convex curve C there is a $\rho \in SO(2)$ such that $\rho(C)$ is lattice generic, see Maehara [64].

Problem 6.1. *Is there a compact planar region of area m for which it is impossible to move (rotate and translate) it so that the resulting set covers exactly m lattice points?*

7. Lattice points on quadratic curves

7.1. Schinzel's theorem

Lemma 7.1. *For an odd prime p that can be represented as the sum of two squares, the number of lattice points on the circle $x^2 + y^2 = p^k$ is $4(k + 1)$.*

Proof. Let us identify $(x, y) \in \mathbb{R}^2$ with the complex number $z = x + yi \in \mathbb{C}$. Then the lattice points on the circle $x^2 + y^2 = p^k$ correspond to the Gaussian integers $w \in \mathbb{Z}[i]$ satisfying $w\bar{w} = p^k$. Since p is written as $p = a^2+b^2$ $(a, b \in \mathbb{Z})$, we have the factorization $p = (a+bi)(a-bi)$, and $w\bar{w} = p^k = (a+bi)^k(a-bi)^k$. Since $|a + bi|^2 = |a - bi|^2 = p$, a prime, both $a + bi, a - bi$ are irreducible in $\mathbb{Z}[i]$. Moreover, since $\mathbb{Z}[i]$ is a unique factorization domain, w is one of

$$u(a + bi)^s(a - bi)^{k-s} \quad (s = 0, 1, 2, \dots, k,\ u = \pm 1, \pm i).$$

Hence the number of w's such that $w\bar{w} = p^k$ is equal to $4(k + 1)$. □

Remark 7.1. An odd prime p can be represented as the sum of two squares if and only if $p \equiv 1 \pmod 4$ (Fermat's two-squares theorem). Zagier [100] presents a very short proof of this fact.

Theorem 7.1. *For a prime p that satisfies $p \equiv 1 (\mathrm{mod}\ 4)$ and $p^k \equiv 1 \pmod 8$, the number of lattice points lying on the circle*

$$(4x - 1)^2 + (4y)^2 = p^k$$

is equal to $k + 1$.

Since $17 \equiv 1 \pmod 8$, it follows that $17^k \equiv 1 \pmod 8$ for every $k \geq 0$. Hence we may set $p = 17$ in this theorem.

Proof. By Lemma 7.1, the number of those $(X, Y) \in \mathbb{Z}^2$ that satisfy $X^2+Y^2 = p^k$ is equal to $4(k + 1)$. Since $(\text{an integer})^2 \equiv 0, 1, 4 \pmod 8$, $X^2 + Y^2 \equiv 1 \pmod 8$ implies that either $X^2 \equiv 1, Y^2 \equiv 0 \pmod 8$ or $X^2 \equiv 0, Y^2 \equiv 1 \pmod 8$. Therefore, $X^2 + Y^2 = p^k$ implies that $X \equiv \pm 1$, $Y \equiv 0 \pmod 4$, or $X \equiv 0$; $Y \equiv \pm 1 \pmod 4$. Thus the number of lattice points (x, y) satisfying $(4x - 1)^2 + (4y)^2 = p^k$ is equal to the number of $(X, Y) \in \mathbb{Z}^2$ satisfying

$$X^2 + Y^2 = p^k \ \text{ and } \ X \equiv -1 \pmod 4. \tag{7.1}$$

If $A^2 + B^2 = p^k$ $(B \equiv (0 \mod 4))$, then $(\pm A, B), (B, \pm A)$ are also solutions of $X^2 + Y^2 = p^k$. Among these four solutions, just one satisfies (7.1). Therefore, the number of lattice points on the circle $(4x - 1)^2 + (4y)^2 = p^k$ is equal to $4(k + 1)/4 = k + 1$. □

As a corollary, we have the following statement, which was obtained by Schinzel [86], see also Maehara-Matsumoto [66].

Corollary 7.1. (Schinzel's theorem) *For every positive integer m, there is a circle on which exactly m lattice points lie.* □

Remark 7.2. It was also shown by Maehara [62] that for every $m > k \geq 2$ there is a sphere in $\mathbb{R}^k$ that passes through exactly m lattice points in $\mathbb{Z}^k$, and that these m points span a k-dimensional polytope. Further results in this direction were obtained in [6].

7.2. $\mathbb{Z}^2$-spectra of quadratic curves

For a curve $C \subset \mathbb{R}^2$, define a subset $S(C)$ of $\mathbb{N} \cup \{\infty\}$ by

$$S(C) = \{|\mathbb{Z}^2 \cap \varphi(C)| : \varphi \text{ is a similarity of } \mathbb{R}^2 \text{ such that } \mathbb{Z}^2 \cap \varphi(C) \neq \emptyset\},$$

where $\mathbb{N}$ denotes the set of positive integers. Let us call the set $S(C)$ the $\mathbb{Z}^2$-*spectrum* of C. (This set $S(C)$ is called the *size-set* of C in [58].) For example, $S(\text{line segment}) = \mathbb{N}$, and $S(\text{line}) = \{1, \infty\}$. Since any circle passes through at most finitely many lattice points, we have $S(\text{circle}) = \mathbb{N}$ by Schinzel's theorem.

In this subsection, we show $\mathbb{Z}^2$-spectra for some quadratic curves. Almost all results presented in the following were obtained by Kuwata-Maehara [58].

Lemma 7.2. *An irreducible quadratic curve that passes through five lattice points can be represented by a quadratic equation with integral coefficients.*

Proof. Let $Ax^2+Bxy+Cy^2+Dx+Ey+F = 0$ be the equation of an irreducible quadratic curve that passes through five lattice points $(x_i, y_i), i = 1, \dots, 5$. We may suppose that F is an integer. By regarding A, B, C, D, E as unknown variables, we have simultaneous linear equations

$$x_i^2 A + x_i y_i B + y_i^2 C + x_i D + y_i E + F = 0, \quad i = 1, 2, \dots, 5\,, \tag{7.2}$$

for the variables A, B, C, D, E. Since (7.2) has a nontrivial solution in $\mathbb{R}^5$, it has a solution in $\mathbb{Q}^5$. Multiplying the rational solutions and F by a suitable nonzero integer, we have integers A', B', C', D', E', F'. Since the quadratic curves $A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0$ and $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ have five points in common, and since the latter is an irreducible quadratic curve, these two curves coincide. □

Recall that all parabolas in the plane are similar to each other.

Theorem 7.2. (Kuwata and Maehara [58]) $S(\textit{parabola}) = \{1, 2, 3, 4, \infty\}$.

Proof. First we show that if a parabola passes through five lattice points, then it passes through infinitely many lattice points.

Suppose that a parabola Γ passes through five lattice points $P_1, \dots, P_5$. By translating the parabola if necessary, we may suppose that P_1 is the origin. By Lemma 7.1, Γ is represented by a quadratic equation with only integral coefficients. Let P_3Q be the chord of Γ that is parallel to the chord P_1P_2.

Since the slope of P_1P_2 is rational, so is the slope of P_3Q, and we can deduce that Q is a rational point. Therefore the slope of the line passing through the midpoint of P_1P_2 and the midpoint of P_3Q is also rational, say $b/a\,(a,b\in\mathbb{Z})$. Since this line is parallel to the axis of the parabola Γ, the slope of the axis of Γ is also equal to the rational b/a. Now, by the similarity transformation φ of $\mathbb{R}^2$ defined by

$$\varphi:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}b & -a\\ a & b\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},$$

the point (a,b) goes to $(0,a^2+b^2)$. Hence $\varphi(\Gamma)$ is a parabola whose axis is parallel to the y-axis. Since $\varphi(\mathbb{Z}^2)\subset\mathbb{Z}^2$, $\varphi(\Gamma)$ also passes through (at least) five lattice points. Hence $\varphi(\Gamma)$ can be represented by an equation $Ay=Bx^2+Cx$ with integral coefficients. Then all the points

$$Q_m=(A(a^2+b^2)m, AB(a^2+b^2)^2m^2+C(a^2+b^2)m),\ m\in\mathbb{Z},$$

are lattice points on $\varphi(\Gamma)$. Since the matrix of the inverse transformation φ^{-1} of φ is given by

$$\begin{pmatrix}b/(a^2+b^2) & a/(a^2+b^2)\\ -a/(a^2+b^2) & b/(a^2+b^2)\end{pmatrix},$$

all the images $\varphi^{-1}(Q_n)$ are lattice points. They all lie on the parabola Γ. Hence Γ passes through infinitely many lattice points.

Now, the parabola $y=\sqrt{2}x^2$ passes through just one lattice point. The parabola $y=\sqrt{2}(x^2-1)$ passes through exactly two lattice points. The parabola $(x-\sqrt{2}y)^2=2(x+y)$ passes through exactly three lattice points. The quadratic curve

$$(6x+(-3+\sqrt{15})y)^2-72x+(-24+6\sqrt{15})y=0$$

is a parabola, and it passes through the four lattice points $(0,0),(0,1),(2,0)$ and $(1,3)$. Finally, the parabola $y=x^2$ passes through infinitely many lattice points. □

Ellipses and hyperbolas are classified in similarity classes by the *eccentricity* e. They are also classified regarding similarity by their *aspect ratio* $\lambda=$ (minor axis)/(major axis). The aspect ratio λ of an ellipse $x^2/a^2+y^2/b^2=1$ is $\lambda=\min\{b/a,\,a/b\}$. The aspect ratio of a hyperbola $x^2/a^2-y^2/b^2=1$ is $\lambda=a/b$. The eccentricity e and the aspect ratio λ are related by $e=\sqrt{1-\lambda^2}$ in the case of ellipses, and $\lambda=\sqrt{1+\lambda^2}$ for hyperbolas.

If d is a positive rational such that $\sqrt{d}\notin\mathbb{Q}$, then the set $\{a+b\sqrt{d}:a,b\in\mathbb{Q}\}$ is a *field*, which is called a *quadratic extension* of $\mathbb{Q}$. Let us denote the ellipse with aspect ratio λ by E_λ, and the hyperbola with aspect ratio λ by H_λ.

Lemma 7.3. *If E_λ or H_λ passes through five lattice points, then λ^2 belongs to a quadratic extension of $\mathbb{Q}$.*

Proof. By Lemma 7.2, the curve is represented by a quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with integral coefficients A, B, C, D, E, F. This equation changes to a form $ax^2 + by^2 + 2px + 2qy + r = 0$ (with no xy term) by a suitable orthogonal transformation of coordinates. In this case, a, b are eigenvalues of the matrix $\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$. Therefore a/b belongs to a quadratic extension of $\mathbb{Q}$, and since $\lambda^2 = |a/b|$, also λ^2 belongs to a quadratic extension of $\mathbb{Q}$. □

Theorem 7.3. *If λ^2 does not belong to any quadratic extension of $\mathbb{Q}$, then $S(E_\lambda) = S(H_\lambda) = \{1, 2, 3, 4\}$.*

Proof. By Lemma 7.3 we have $S(E_\lambda) \subset \{1, 2, 3, 4\}$ and $S(H_\lambda) \subset \{1, 2, 3, 4\}$. Put $\xi = \pm\lambda^2$ (+ for the ellipse case, and − for the hyperbola case), and let η be a transcendental number algebraically independent from ξ. The quadratic curve $\xi x^2 + y^2 + \eta x = 0$ passes through just one lattice point $(0, 0)$. The curve $\xi(x^2 - 1) + y^2 + \eta y = 0$ passes through just two lattice points $(\pm 1, 0)$. The curve $\xi x^2 + y^2 - 1 + (1 - \xi)x = 0$ passes through just three lattice points $(0, \pm 1), (1, 0)$. Finally, the curve $\xi x^2 + y^2 = \xi + 1$ passes through just four lattice points $(\pm 1, \pm 1)$. □

Theorem 7.4. (Kuwata and Maehara [58]) *If $\lambda \in \mathbb{Q}$, then $S(E_\lambda) = \mathbb{N}$.*

Proof. Let $\lambda = b/a \in \mathbb{Q}$ (an irreducible fraction). Since one of a, b is odd, let us suppose that a is odd. Let p be a prime such that $p \equiv 1 \pmod 8$. We show that the ellipse with equation

$$(4x/a - 1)^2 + (4y/b)^2 = p^k \tag{7.3}$$

passes through exactly $k + 1$ lattice points. For $(x_0, y_0) \in \mathbb{Z}^2$,

$$\begin{aligned} &(4x_0/a - 1)^2 + (4y_0/b)^2 = p^k \\ &\qquad \Rightarrow (4bx_0)^2 + (4ay_0)^2 = a^2b^2(p^k - 1) + 8ab^2x_0 \\ &\qquad \Rightarrow 8b^2 \mid 16a^2y_0^2 \Rightarrow b \mid y_0 \Rightarrow 4x_0/a \in \mathbb{Z} \Rightarrow a \mid x_0. \end{aligned}$$

Hence the number of lattice points on the ellipse (7.3) is equal to the number of integral solutions (X, Y) of (7.1), which is equal to $k + 1$. □

Proposition 7.1. *The equation $X^2 - 2Y^2 = 1$ has infinitely many integral solutions.*

Proof. Suppose (x_i, y_i) is a positive integral solution of the equation. Define $x_{i+1}, y_{i+1} \in \mathbb{Z}$ by

$$x_{i+1} + \sqrt{2}y_{i+1} = (x_i + \sqrt{2}y_i)^2, \tag{7.4}$$

namely, $x_{i+1} = x_i^2 + 2x_iy_i$, $y_{i+1} = 2x_iy_i$. Then $x_{i+1} - \sqrt{2}y_{i+1} = (x_i - \sqrt{2}y_i)^2$. Hence

$$\begin{aligned} x_{i+1}^2 - 2y_{i+1}^2 &= (x_{i+1} + \sqrt{2}y_{i+1})(x_{i+1} - \sqrt{2}y_{i+1}) \\ &= (x_i + \sqrt{2}y_i)^2(x_i - \sqrt{2}y_i)^2 = (x_i^2 - 2y_i^2)^2 = 1. \end{aligned}$$

Thus (x_{i+1}, y_{i+1}) is another integral solution of the equation, and since $x_i > 1$, we have $x_{i+1} > x_i$. Starting from an integral solution $(x_1, y_1) := (3, 2)$ of the equation, we can obtain infinitely many distinct integral solutions (x_i, y_i), $i = 1, 2, 3, \ldots$, of the equation by applying (7.4). □

Remark 7.3. For an integer $d > 0$ which is not a square, the equation $X^2 - dY^2 = 1$ is a so-called *Pell equation*. It is known that every Pell equation has infinitely many integral solutions.

The *rectangular hyperbola* is a hyperbola with aspect ratio 1.

Theorem 7.5. *The $\mathbb{Z}^2$-spectrum of the rectangular hyperbola is $\mathbb{N} \cup \{\infty\}$.*

Proof. Since the Pell equation $X^2 - 2Y^2 = 1$ has infinitely many integral solutions (see Proposition 7.1), the rectangular hyperbola $(x + y)^2 - 2y^2 = 1$ passes through infinitely many lattice points.

Let p be a prime such that $p \equiv 1 \pmod 6$, and consider the rectangular hyperbola $(3x+1)^2-(3y)^2 = p^{m-1}$. If (a, b) is a lattice point on this hyperbola, then $(3a + 1 + 3b)(3a + 1 - 3b) = p^{m-1}$. Hence there is a k, $0 \leq k \leq m - 1$, such that

$$\begin{cases} 3a + 1 + 3b = p^k \\ 3a + 1 - 3b = p^{m-k-1}. \end{cases}$$

Thus we have

$$\begin{cases} 6a + 2 &= p^k + p^{m-k-1} \\ 6b &= p^k - p^{m-k-1}. \end{cases}$$

Since $0 \leq k \leq m - 1$, the number of such lattice points (a, b) is at most m. Conversely, since $p \equiv 1 \pmod 6$, there is such a lattice point (a, b) for each k, $0 \leq k \leq m - 1$. Hence the number of lattice points on the hyperbola $(3x + 1)^2 - (3y)^2 = p^{m-1}$ is exactly m. Therefore, the $\mathbb{Z}^2$-spectrum of the rectangular hyperbola is $\mathbb{N} \cup \{\infty\}$. □

For further results on the $\mathbb{Z}^2$-spectra of ellipses and hyperbolas, see Kuwata and Maehara [58]. The $\mathbb{Z}^2$-spectra of ellipses and hyperbolas with general λ are not determined, yet. Thus, we finish with the following two problems.

Problem 7.1. *Determine completely the $\mathbb{Z}^2$-spectra of ellipses and hyperbolas.*

Problem 7.2. *Consider $\mathbb{Z}^2$-spectra of cubic curves.*

Acknowledgements

The authors are grateful to the referee for careful reading of the manuscript and helpful comments.

References

[1] Akopyan, A., Tagami, M.: On the number of points in a lattice polytope. Integers **11**(5), 619–622 (2011)
[2] Adams, J., Zvengrowski, P., Laird, P.: Vertex embeddings of regular polytopes. Expo. Math. **21**(4), 339–353 (2003)
[3] Ankeny, N.C.: Sums of three squares. Proc. Am. Math. Soc. **8**, 316–319 (1957)
[4] Averkov, G.: On the size of lattice simplices with a single interior lattice point. SIAM J. Discrete Math. **26**(2), 515–526 (2012)
[5] Averkov, G., Krümpelmann, J., Nill, B.: Largest integral simplices with one interior integral point: solution of Hensley's conjecture and related results. Adv. Math. **274**, 118–166 (2015)
[6] Bannai, E., Miezaki, T.: On a property of 2-dimensional integral Euclidean lattices. J. Number Theory **132**(3), 371–378 (2012)
[7] Bárány, I.: Extremal problems for convex lattice polytopes: a survey. In: Goodman, J.E., Pach, J., Pollack, R. (eds.) Surveys on Discrete and Computational Geometry, Contemporary Mathematics, vol. 453, pp. 87–103. American Mathematical Society, Providence (2008)
[8] Barvinok, A.: Integer Points in Polyhedra. Zürich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2008)
[9] Beck, J.: Randomness in lattice point problems. Comb. Graph Theory Algorithms Appl. Discrete Math. **229**(1–3), 29–55 (2001)
[10] Beck, M., Robins, S.: Computing the Continuous Discretely. Integer-Point Enumeration in Polyhedra. With illustrations by David Austin. Undergraduate Texts in Mathematics, vol. 2. Springer, New York (2015)
[11] Beeson, M.J.: Triangles with vertices on lattice points. Am. Math. Mon. **99**, 243–252 (1992)
[12] Brass, P., Moser, W., Pach, J.: Research Problems in Discrete Geometry. Springer, New York (2005)
[13] Braun, B.: Unimodality problems in Ehrhart theory. In: Recent Trends in Combinatorics, IMA, Mathematics with Applications, vol. 159, pp. 687–711. Springer, Cham (2016)
[14] Brion, M.: Polytopes convexes entiers. Gaz. Math. **67**, 21–42 (1996)
[15] Buchberger, B.: Gröbner basis: a short introduction for system theorists. In: Moreno-Diaz, R., et al. (eds.) EUROCAST 2001, Lecture Notes in Computer Science, vol. 2178, pp. 1–19. Springer, Berlin (2001)
[16] Cassels, J.W.S.: An Introduction to the Geometry of Numbers. Springer, New York (1971)
[17] Cassels, J.W.S.: Rational Quadratic Forms. Academic Press, New York (1978)
[18] Conway, J.H., Sloane, N.J.A.: Low dimensional lattices V: integral coordinates for integral lattices. Proc. R. Soc. Lond. Ser. A **426**, 211–232 (1989)
[19] Conway, J.H., Sloane, N.J.A.: Sphere packings, lattices and groups. With contributions by E. Bannai, J. Leech, S.P. Norton, A.M. Odlyzko, R.A. Parker, L. Queen, B.B. Venkov. Grundlehren der Mathematischen Wissenschaften 290. Springer, New York (1988)

[20] Croft, H.T., Falconer, K.J., Guy, R.K.: Unsolved Problems in Geometry, Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer, New York (1991)
[21] de Launey, W., Flannery, D.L., Horadam, K.J.: Cocyclic Hadamard matrices and difference sets. In: Coding, Cryptography and Computer Security (Lethbridge, AB, 1998), Discrete Applied Mathematics, vol. 102, no. 1–2, pp. 47–61 (2000)
[22] De Loera, J.: The many aspects of counting lattice points in polytopes. Math. Semesterber **52**(2), 175–195 (2005)
[23] Ding, R., Reay, J.R.: Areas of lattice polygons, applied to computer graphics. In: Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985). Zastos. Math. (1987), vol. 19, no. 3–4, pp. 547-556 (1988)
[24] Dubeau, F., Labbé, S.: Sur le théorème de Pick. Ann. Sci. Math. Québec **29**(2), 147–166 (2005)
[25] Engel, P.: Geometric crystallography. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry, vol. B, pp. 989–1041. North-Holland, Amsterdam (1993)
[26] Erdős, P., Gruber, P.M., Hammer, J.: Lattice Points. Longman, Essex (1989)
[27] Ewald, G.: Combinatorial Convexity and Algebraic Geometry. Graduate Texts in Mathematics, vol. 168. Springer, New York (1996)
[28] Fejes Tóth, G., Kuperberg, W.: Packing and covering with convex sets. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry, vol. B, pp. 799–860. North-Holland, Amsterdam (1993)
[29] Feldman, K.E.: Miraculous cancellation and Pick's theorem. In: Harada, M., Karshon, Y., Masuda, M., Panov, T. (eds.) Toric Topology, Contemporary Mathematics, vol. 460, pp. 71–86. American Mathematical Society, Providence (2008)
[30] Funkenbusch, W.W.: From Euler's formula to Pick's formula using an edge theorem. Am. Math. Mon. **81**, 647–648 (1974)
[31] Gauss, C.F.: Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber, Göttingische Gelehrte Anzeigen, 9.7.1831; Werke**2**, 188–196 (1836); J. reine angew. Math.**20**, 312–320 (1840)
[32] Gioia, A.A.: The Theory of Numbers: An Introduction. Markham Publishing Co., Chicago (1970)
[33] Gritzmann, P., Wills, J.M.: Lattice points. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry, vol. B, pp. 765–797. North-Holland, Amsterdam (1993)
[34] Gritzmann, P., Wills, J.M.: Finite packing and covering. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry, vol. B, pp. 861–897. North-Holland, Amsterdam (1993)
[35] Gritzmann, P., Klee, V.: On the complexity of some basic problems in computational convexity. I. Containment problems. Discrete Math. **136**(1–3), 129–174 (1994)
[36] Gruber, P.M.: Geometry of numbers. In: Tölke, J., Wills, J.M. (eds.) Contributions to Geometry, pp. 186–225. Birkhäuser, Basel (1979)
[37] Gruber, P.M.: Geometry of numbers. In: Handbook of Convex Geometry, vol. B, pp. 739–763 . North-Holland, Amsterdam (1993)
[38] Gruber, P.M.: Convex and Discrete Geometry. Springer, Berlin (2007)
[39] Gruber, P.M., Lekkerkerker, C.G.: Geometry of Numbers. North-Holland, Amsterdam (1987)
[40] Grünbaum, B., Shephard, G.C.: Tilings and Patterns. W. H. Freeman and Company, New York (1987)
[41] Grünbaum, B., Shephard, G.C.: Pick's theorem. Amer. Math. Mon. **100**(2), 150–161 (1993)
[42] Haase, C., Schicho, J.: Lattice polygons and the number $2i + 7$. Am. Math. Mon. **116**(2), 151–165 (2009)
[43] Hadwiger, H., Debrunner, H.: Combinatorial Geometry in the Plane (translated by Victor Klee, with a new chapter and other additional material supplied by the translator), Holt, Rinehart and Winston, New York (1964)

[44] Hadwiger, H., Wills, J.M.: Neuere Studien über Gitterpolygone. J. Reine Angew. Math. **280**, 61–69 (1976)
[45] Henk, M., Wills, J.M.: Minkowski's successive minima. In: Number Theory and Discrete Geometry. Ramanujan Mathematical Society, Lecture Notes Series, vol. 6, pp. 129–142. Ramanujan Mathematical Society, Mysore (2008)
[46] Hlawka, E.: 90 Jahre Geometrie der Zahlen (German). Jahrbuch Überbl. Math. 1980, pp. 9–41, Bibliographisches Inst., Mannheim (1980)
[47] Honsberger, R.: Mathematical Gems I. Mathematical Association of America, Toronto (1973)
[48] Jackson, S., Mauldin, R.D.: On a lattice problem of H. Steinhaus. J. Am. Math. Soc. **15**(4), 817–856 (2002)
[49] Kania, T., Kochanek, T.: Steinhaus' lattice-point problem for Banach spaces. J. Math. Anal. Appl. **446**(2), 1219–1229 (2017)
[50] Kantor, J.-M.: On the width of lattice-free simplices. Compos. Math. **118**(3), 235–241 (1999)
[51] Karpenkov, O.: Elementary notions of lattice trigonometry. Math. Scand. **102**(2), 161–205 (2008)
[52] Karpenkov, O.: Geometry of Continued Fractions, Algorithms and Computation in Mathematics, vol. 26. Springer, Heidelberg (2013)
[53] Keller, O.H.: Geometrie der Zahlen. Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, I 2, 27. Band I. Algebra und Zahlentheorie. 2. Teil, C. Reine Zahlentheorie. Heft 11, Teil III. B. G. Teubner Verlagsgesellschaft, Leipzig, p 84 (1954)
[54] Ko, C.: On the representation of a quadratic form as a sum of squares of linear forms. Quart. J. Math. Oxford **8**, 81–98 (1937)
[55] Kołodziejczyk, K.: Hadwiger–Wills-type higher-dimensional generalizations of Pick's theorem. The Branko Grünbaum birthday issue. Discrete Comput. Geom. **24**(2–3), 355–364 (2000)
[56] Krebs, M., Wright, T.: On Cantor's first uncountability proof, Pick's theorem, and the irrationality of the golden ratio. Am. Math. Mon. **117**(7), 633–637 (2010)
[57] Kumada, T.: Isometric embedding of metric $\mathbb{Q}$-vector space into $\mathbb{Q}^N$. Eur. J. Comb. **19**, 701–709 (1998)
[58] Kuwata, T., Maehara, H.: Lattice points on similar figures and conics. Graphs Comb. **27**(3), 441–450 (2011)
[59] Lagarias, J.C.: Point lattices. In: Handbook of Combinatorics, Vol. 1, 2, pp. 919–966. Elsevier Sci. B. V., Amsterdam (1995)
[60] Maehara, H.: Angles in a lattice polygon. Ryukyu Math. J. **6**, 9–19 (1993)
[61] Maehara, H.: Embedding a polytope in a lattice. Discrete Comput. Geom. **13**, 585–592 (1995)
[62] Maehara, H.: On a sphere that passes through n lattice points. Eur. J. Combin. **31**, 617–621 (2010)
[63] Maehara, H.: Euclidean embeddings of finite metric spaces. Discrete Math. **313**, 2848–2856 (2013)
[64] Maehara, H.: Circle lattice point problem, revisited. Discrete Math. **338**, 164–167 (2015)
[65] Maehara, H.: Steinhaus' lattice point problem for polyhedra. Colloq. Math. **146**(1), 123–128 (2017)
[66] Maehara, H., Matsumoto, M.: Is there a circle that passes through a given number of lattice points? Eur. J. Combin. **19**, 591–592 (1998)
[67] Martini, H.: A hierarchical classification of Euclidean polytopes with regularity properties. In: Polytopes: Abstract, Convex and Computational (Scarborough, ON, 1993), NATO Advanced Science Institute Series C: Mathematical and Physical Sciences, vol. 440, pp. 71–96.Kluwer Academic Publishers, Dordrecht (1994)

[68] Martini, H.: Reguläre Polytope und Verallgemeinerungen. In: Geometrie und ihre Anwendungen, pp. 247–281. Hanser-Verlag, München und Wien (1994)
[69] Mayer, A.: Mathematische Mittheilungen. Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich **29**, 209–222 (1984)
[70] Milnor, J., Husemoller, D.: Symmetric Bilinear Forms. Springer, Berlin (1973)
[71] Minkowski, H.: Geometrie der Zahlen, Teubner, Leipzig, 1896. Chelsea, New York. Johnson Reprint Corp., New York (1968)
[72] Mordell, L.J.: The representation of a quadratic form as a sum of two others. Ann. Math. **38**, 751–757 (1937)
[73] Morelli, R.: Pick's theorem and the Todd class of a toric variety. Adv. Math. **100**(2), 183–231 (1993)
[74] Nill, B.: Volume and lattice points of reflexive simplices. Discrete Comput. Geom. **37**(2), 301–320 (2007)
[75] Niven, I.: Numbers: Rational and Irrational. MAA, Washington (1961)
[76] Niven, I., Zuckerman, H.S.: An Introduction to the Theory of Numbers. Wiley, New York (1972)
[77] Oda, T.: Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15. Springer, Berlin (1988)
[78] Olds, C.D., Lax, A., Davidoff, G.: The Geometry of Numbers (Appendix I by P. D. Lax), Anneli Lax New Mathematical Library, vol. 41. Mathematical Association of America, Washington, D.C. (2000)
[79] Patruno, G.N.: The lattice polytope problem. Elem. Math. **38**(3), 69–71 (1983)
[80] Pick, G.: Geometrisches zur Zahlenlehre. Sitzungsberichte des deutschen naturwissenschaftlich-medicinischen Vereines für Böhmen "Lotos" in Prag (Neue Folge) **19**, 311–319 (1899)
[81] Plesken, W.: Solving $XX^{tr} = A$ over the integers. Linear Algebra Appl. **226**(228), 331–344 (1995)
[82] Ram Murty, M., Thain, N.: Pick's theorem via Minkowski's theorem. Am. Math. Mon. **114**(8), 732–736 (2007)
[83] Reeve, J.E.: On the volume of lattice polyhedra. Proc. Lond. Math. Soc. **7**, 378–395 (1957)
[84] Sam, S.V.: A bijective proof for a theorem of Ehrhart. Am. Math. Mon. **116**(8), 688–701 (2009)
[85] Scherrer, W.: Die Einlagerung eines regulären Vielecks in ein Gitter. Elem. Math. **1**, 97–98 (1946)
[86] Schinzel, A.: Sur l'existence d'un cercle passant par un nombre donne de points aux coordonnees entieres. Enseignement Math (2) **4**, 71–72 (1958)
[87] Schoenberg, I.J.: Regular simplices and quadratic forms. J. Lond. Math. Soc. **12**, 48–55 (1937)
[88] Schürmann, A.: Computational Geometry of Positive Definite Quadratic Forms. Polyhedral Reduction Theories, Algorithms, and Applications, University Lecture Series, vol. 48. American Mathematical Society, Providence (2009)
[89] Schulte, E.: Tilings. In: Gruber, P.M., Wills, J.M. (eds.) Handbook of Convex Geometry, vol. B, pp. 899–932. North-Holland, Amsterdam (1993)
[90] Schwermer, J.: Räumliche Anschauung und Minima positiv definiter quadratischer Formen. Zur Habilitation von Hermann Minkowski 1887 in Bonn (German). Jahresber. Deutsch. Math.-Verein, vol. 93, no. 2, pp. 49–105 (1991)
[91] Scott, P.R.: On convex lattice polygons. Bull. Aust. Math. Soc. **15**(3), 395–399 (1976)
[92] Scott, P.R.: Equiangular lattice polygons and semiregular lattice polygons. Coll. Math. J. **18**, 300–306 (1987)
[93] Scott, P.R.: The fascination of the elementary. Am. Math. Mon. **94**, 759–768 (1987)
[94] Serre, J.P.: A course in arithmetic, GTM, vol. 7. Springer, Berlin (1973)

[95] Silverman, J.H., Tate, J.: Rational Points on Elliptic Curves. Springer, New York (1992)
[96] Steinhaus, H.: One Hundred Problems in Elementary Mathematics. Dover Publications Inc., New York (1964)
[97] Steinhaus, H.: Mathematical Snapshots. Dover Publications Inc., New York (1983)
[98] Trott, M.: The Mathematica Guide Book for Symbolics. Springer, New York (2006)
[99] Varberg, D.E.: Pick's theorem revisited. Am. Math. Mon. **92**(8), 584–587 (1985)
[100] Zagier, D.: A one-sentence proof that every prime $p \equiv 1 \pmod 4$ is a sum of two squares. Am. Math. Mon. **97**, 144 (1990)
[101] Zaks, J., Perles, M.A., Wills, J.M.: On lattice polytopes having interior lattice points. Elem. Math. **37**(2), 44–46 (1982)
[102] Zong, C.: From deep holes to free planes. Bull. Am. Math. Soc. (N.S.) **39**(4), 533–555 (2002)
[103] Zwoleński, P.: Some generalization of Steinhaus' lattice points problem. Colloq. Math. **123**(1), 129–132 (2011)

Hiroshi Maehara
Ryukyu University
Nishihara Okinawa 903-0213
Japan
e-mail: hmaehara@edu.u-ryukyu.ac.jp

Horst Martini
Faculty of Mathematics
Chemnitz University of Technology
09107 Chemnitz
Germany
e-mail: martini@mathematik.tu-chemnitz.de

Received: November 3, 2017
Revised: March 19, 2018