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On the generalized Fréchet functional equation with constant coefficients and its stability

Janusz Brzdęk, Zbigniew Leśniak, and Renata Malejki

Abstract. We study a generalization of the Fréchet functional equation, stemming from a characterization of inner product spaces. We show, in particular, that under some weak additional assumptions each solution of such an equation is additive. We also obtain a theorem on the Ulam type stability of the equation. In its proof we use a fixed point result to show the existence of an exact solution of the equation that is close to a given approximate solution.

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1. Introduction

In this paper we study the following functional equation (with constant coefficients)

$$\begin{aligned} A_1F(x+y+z)+A_2F(x)+A_3F(y)+A_4F(z) &= A_5F(x+y) \\ +A_6F(x+z)+A_7F(y+z), \end{aligned} \tag{1}$$

where $A_1,\ldots,A_7\in\mathbb{K}$ and $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ ($\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively), in the class of functions $F:X\to Y$, where $(X,+)$ is a commutative monoid (i.e., a semigroup with a neutral element denoted by 0) and Y is a Banach space over the field $\mathbb{K}$.

This equation is a generalization of the Fréchet functional equation

$$F(x+y+z)+F(x)+F(y)+F(z)=F(x+y)+F(x+z)+F(y+z). \tag{2}$$

Namely, in case $A_i=A_j\neq 0$ for $i,j\in\{1,\ldots,7\}$, Eq. (1) can be easily reduced to Eq. (2). Therefore, we will be interested mainly in the case where $A_i\neq A_j$ for some i,j.

Fréchet [17] proved that a normed space $(X,\|\cdot\|)$ is an inner product space if and only if for all $x,y,z\in X$

$$\|x+y+z\|^2+\|x\|^2+\|y\|^2+\|z\|^2=\|x+y\|^2+\|x+z\|^2+\|y+z\|^2. \quad (3)$$

This means that a normed space X is an inner product space if and only if the function F, given by $F(x)\equiv\|x\|^2$, satisfies Eq. (2). For further similar results we refer to [1,4,5,7,13,15,20,27–29].

The first part of this paper contains some results on solutions of Eq. (1) in the case where $(X,+)$ is a monoid and Y is a Banach space. It is known that, if X is a group and Y is an abelian group divisible by 2, then the general solution of Eq. (2) is the sum of a quadratic and an additive function (see [23, pp. 249-250]). An analogous result is true for Eq. (1), but then each solution is a sum of a quadratic, an additive and a constant function (see [4, Proposition 7]). We will show that under the assumption that at least two coefficients A_i are not equal, every solution F of Eq. (1), with $F(0)=0$, is an additive function.

In the second part of this paper we will consider the problem of Ulam stability of Eq. (1). A stability result concerning Eq. (2) can be found in [5]. Analogous outcomes for Eq. (1) were proved in [26] (see also [4, Corollary 6]), under the assumptions that $(X,+)$ is a commutative group, $A_1\neq 0$ and $A_2+A_3+A_4=A_5+A_6+A_7$, which has motivated us to study a bit further the dependence of the stability of Eq. (1) on the values of the coefficients $A_1,\dots,A_7$. Moreover, we weaken the assumptions on X by assuming 'only' that it is a commutative monoid. In this way we also complement the recent outcomes in [4].

We use a fixed point approach, introduced in [10] (see also [4,5,13,26,30,34]). The main tool in the proof of our main stability result is the fixed point theorem in [11] (similar results can be found in, e.g., [2,12,14]).

Theorem 1. [11] *Let the following three hypotheses be valid.*

(H1) *S is a nonempty set, E is a Banach space, and functions $f_1,\dots,f_k: S\to S$ and $l_1,\dots,l_k: S\to\mathbb{R}_+$ are given.*

(H2) *$\mathcal{T}: E^S\to E^S$ is an operator satisfying the inequality*

$$\big\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\big\|\leq\sum_{i=1}^{k}l_i(x)\big\|\xi(f_i(x))-\mu(f_i(x))\big\|, \quad \xi,\mu\in E^S, x\in S.$$

(H3) *$\Lambda:{\mathbb{R}_+}^S\to{\mathbb{R}_+}^S$ is defined by*

$$\Lambda\delta(x):=\sum_{i=1}^{k}l_i(x)\delta(f_i(x)), \quad \delta\in{\mathbb{R}_+}^S, x\in S.$$

Assume that functions $\varepsilon: S\to\mathbb{R}_+$ and $\varphi: S\to E$ fulfil the following two conditions

$$\left\|\mathcal{T}\varphi(x) - \varphi(x)\right\| \leq \varepsilon(x), \quad x \in S, \tag{4}$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in S. \tag{5}$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in S.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \quad x \in S.$$

Using this theorem we prove that an operator defined in the space Y^X determines an exact solution of Eq. (1) as the limit of a sequence of its iterates on an approximate solution of this equation. Such a method has been used in, e.g., [4–6,10,26,30,33,34]. Moreover, the results that we provide correspond to the outcomes in [3,9,13,16,18,21,24,25,31,32] (for more details see, e.g., [8,19,22]) and complement [4, Corollary 6].

2. Additive solutions of the generalized Fréchet functional equation

In this section we present some auxiliary observations on solutions of Eq. (1). The main result of this part says that, under a natural additional assumption concerning the coefficients A_i, each solution F of Eq. (1), with $F(0) = 0$, is an additive function.

Throughout this section X is a monoid, Y is a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $A_1, \ldots, A_7 \in \mathbb{K}$. Let us recall that every solution of Eq. (2) satisfies the equality $F(0) = 0$. But this is not necessarily the case for Eq. (1). The following result gives a sufficient condition for the equality $F(0) = 0$.

Proposition 2. *If the condition*

$$A_1 + A_2 + A_3 + A_4 \neq A_5 + A_6 + A_7, \tag{6}$$

is fulfilled, then each solution $F : X \to Y$ of Eq. (1) satisfies the condition $F(0) = 0$.

Proof. Let F be a solution of Eq. (1). Putting $x = 0$, $y = 0$ and $z = 0$ into Eq. (1) we obtain

$$(A_1 + A_2 + A_3 + A_4)F(0) = (A_5 + A_6 + A_7)F(0). \tag{7}$$

Hence, if $F(0) \neq 0$, then

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7. \tag{8}$$

Consequently, by (6) we have $F(0) = 0$. □

From now on we are interested in solutions of Eq. (1) which satisfy the relation $F(0) = 0$.

Proposition 3. *If a nonzero function $F : X \to Y$, with $F(0) = 0$, satisfies Eq.* (1)*, then*

$$\begin{cases} A_2 = -A_1 + A_5 + A_6, \\ A_3 = -A_1 + A_5 + A_7, \\ A_4 = -A_1 + A_6 + A_7. \end{cases} \tag{9}$$

Proof. Putting $y = 0,\ z = 0$ in (1), we get

$$A_1F(x) + A_2F(x) = A_5F(x) + A_6F(x), \quad x \in X.$$

Thus

$$(A_1 + A_2 - A_5 - A_6)F(x) = 0, \quad x \in X,$$

whence

$$A_2 = -A_1 + A_5 + A_6,$$

since F is a nonzero function. In a similar way we obtain the other two relations in (9). More precisely, putting $x = 0,\ z = 0$ and $x = 0,\ y = 0$ we get $A_3 = -A_1 + A_5 + A_7$ and $A_4 = -A_1 + A_6 + A_7$, respectively. □

Proposition 4. *Assume that relations* (9) *are fulfilled. Then every additive function $a : X \to Y$ is a solution of Eq.* (1)*.*

Proof. Let $a : X \to Y$ be an additive function. Then

$$\begin{aligned} &A_5a(x + y) + A_6a(x + z) + A_7a(y + z) \\ &\quad = (A_5 + A_6)a(x) + (A_5 + A_7)a(y) + (A_6 + A_7)a(z). \end{aligned} \tag{10}$$

By (9) we obtain

$$\begin{aligned} &A_1a(x + y + z) + A_2a(x) + A_3a(y) + A_4a(z) = A_1a(x + y + z) \\ &\quad + (-A_1 + A_5 + A_6)a(x) + (-A_1 + A_5 + A_7)a(y) + (-A_1 + A_5 + A_7)a(z). \end{aligned}$$

Hence using the additivity of the function a once again we get

$$\begin{aligned} &A_1a(x + y + z) + A_2a(x) + A_3a(y) + A_4a(z) \\ &\quad = (A_5 + A_6)a(x) + (A_5 + A_7)a(y) + (A_6 + A_7)a(z). \end{aligned} \tag{11}$$

Thus by (10) and (11) we get

$$\begin{aligned} &A_1a(x + y + z) + A_2a(x) + A_3a(y) + A_4a(z) \\ &\quad = A_5a(x + y) + A_6a(x + z) + A_7a(y + z), \end{aligned}$$

i.e. a satisfies Eq. (1). □

Corollary 5. *A nonzero additive function $a : X \to Y$ satisfies Eq.* (1) *if and only if relations* (9) *hold.*

Proof. Let $a : X \to Y$ be a nonzero additive function. Assume that a satisfies Eq. (1). Then by Proposition 3 relations (9) hold, since $a(0) = 0$. The converse implication follows directly from Proposition 4. $\square$

Corollary 6. *If Eq.* (1) *has a nonzero solution $F : X \to Y$, with $F(0) = 0$, then each additive function $a : X \to Y$ satisfies Eq.* (1).

Proof. Assume that Eq. (1) has a nonzero solution $F : X \to Y$ such that $F(0) = 0$. Then by Proposition 3 relations (9) hold. Hence by Corollary 5 every nonzero additive function satisfies Eq. (1). Moreover, it is easily seen that the function $a : X \to Y$, given by $a(x) = 0$ for all $x \in X$, is a solution of Eq. (1). $\square$

Thus we have shown that the set of all nonzero solutions $F : X \to Y$ of Eq. (1) such that $F(0) = 0$, if non-empty, contains the set of all additive functions $a : X \to Y$. Under the assumption that relations (9) hold we obtain by Proposition 4 that the considered set of solutions of Eq. (1) is non-empty. The next result states that, under a suitable assumption on the coefficients A_i, the two sets coincide.

Theorem 7. *If $A_i \neq A_j$ for some $i, j \in \{1, \dots, 7\}$, then each solution $F : X \to Y$ of Eq.* (1), *with $F(0) = 0$, is an additive function.*

Proof. Clearly, if $F(x) \equiv 0$, then it is additive. So, assume that F is a nonzero solution of Eq. (1) such that $F(0) = 0$. We distinguish the cases $A_1 \neq A_5$ and $A_1 = A_5$. Let us note that by (9) we have

$$A_1 - A_5 = A_6 - A_2 = A_7 - A_3 := B. \tag{12}$$

First let us assume that $A_1 \neq A_5$, i.e., $B \neq 0$. Putting $z = 0$ into (1) we get

$$A_1F(x+y)+A_2F(x) + A_3F(y) = A_5F(x + y) + A_6F(x) + A_7F(y), \;\; x, y \in X.$$

Hence

$$(A_1 - A_5)F(x + y) = (A_6 - A_2)F(x) + (A_7 - A_3)F(y), \quad x, y \in X. \tag{13}$$

By (12) Eq. (13) can be written in the form

$$BF(x + y) = BF(x) + BF(y), \quad x, y \in X,$$

where $B \neq 0$. Consequently

$$F(x + y) = F(x) + F(y), \quad x, y \in X.$$

Now we proceed to the case where $A_1 = A_5$. Then by (12), $A_6 = A_2$ and $A_7 = A_3$ and Eq. (1) has the form

$$\begin{aligned} &A_1F(x + y + z) + A_2F(x) + A_3F(y) + A_4F(z) \\ &\quad = A_1F(x + y) + A_2F(x + z) + A_3F(y + z), \quad x, y, z \in X. \end{aligned}$$

Hence with $x = 0$ we obtain that

$$A_1F(y+z)+A_3F(y)+A_4F(z) = A_1F(y)+A_2F(z)+A_3F(y+z), \quad y,z \in X.$$

Thus

$$(A_1 - A_3)F(y+z) = (A_1 - A_3)F(y) + (A_2 - A_4)F(z), \quad y,z \in X. \tag{14}$$

Next, from (9) we get that

$$A_1 - A_7 = A_6 - A_4.$$

Hence

$$A_1 - A_3 = A_2 - A_4 := C, \tag{15}$$

since by (12) we have $A_7 = A_3$ and $A_6 = A_2$. Consequently, Eq. (14) can be written in the form

$$CF(y+z) = CF(y) + CF(z), \quad y,z \in X. \tag{16}$$

Now we consider the following two subcases: $A_1 \neq A_3$ and $A_1 = A_3$. If $A_1 \neq A_3$, then $C \neq 0$ and from (16) we obtain that F is additive. Now let us assume that $A_1 = A_3$. Then from (12) and (15) we obtain that $A_1 = A_3 = A_5 = A_7$ and $A_2 = A_4 = A_6$. Thus Eq. (1) has the form

$$\begin{aligned} &A_1F(x+y+z) + A_2F(x) + A_1F(y) + A_2F(z) \\ &\quad = A_1F(x+y) + A_2F(x+z) + A_1F(y+z), \quad x,y,z \in X. \end{aligned}$$

Hence putting $y = 0$ we get

$$A_1F(x+z)+A_2F(x)+A_2F(z) = A_1F(x)+A_2F(x+z)+A_1F(z), \quad x,z \in X.$$

Consequently

$$DF(x+z) = DF(x) + DF(z), \quad x,z \in X,$$

where $D =: A_1 - A_2$. Thus, if $A_1 \neq A_2$, then F is additive. □

Directly from Theorem 7 we obtain the following result.

Corollary 8. *If there exists a nonadditive nonzero solution $F : X \to Y$ of Eq. (1) such that $F(0) = 0$, then $A_i = A_j$ for all $i,j \in \{1,\ldots,7\}$.*

Moreover, from Theorem 7 and Proposition 2 we get the following description of the set of solutions of Eq. (1).

Corollary 9. *If $A_i \neq A_j$ for some $i,j \in \{1,\ldots,7\}$ and condition (6) holds, then each solution of Eq. (1) is an additive function.*

Now we proceed to the general case, without the assumption that $F(0) = 0$.

Corollary 10. *Assume that $A_i \neq A_j$ for some $i,j \in \{1,\ldots,7\}$. If $F : X \to Y$ is a solution of Eq. (1), then*

$$F(x) = a(x) + c, \quad x \in X, \tag{17}$$

where $a : X \to Y$ is an additive function and $c = F(0)$.

Proof. Let F satisfy Eq. (1). If $F(0) = 0$, then from Theorem 7 we obtain that (17) holds with $c = 0$. Now, let us assume that $F(0) \neq 0$. Define the function $F_0 : X \to Y$ by the formula

$$F_0(x) := F(x) - F(0), \quad x \in X.$$

Then F_0 is a solution of Eq. (1), since F satisfies Eq. (1) and consequently condition (7) holds. On account of Theorem 7 the function F_0 is additive. Thus F is a sum of an additive function and the constant $c := F(0)$. □

Let us note that the converse of Corollary 10, in general, is false. Consider the case where $A_1 = 2$ and $A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = 1$. Then by Proposition 2 for each solution F of Eq. (1) we have $F(0) = 0$, since condition (6) holds. Thus on account of Corollary 5 we obtain that the only solution of Eq. (1) is the zero function, since condition (9) is not fulfilled. In the case where $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = 1$ and $A_7 = 2$ the set of solutions of Eq. (1) consists of all constant functions, since none of conditions (6) and (9) holds.

Now consider the case where $A_1 = 3$, $A_2 = A_3 = A_4 = -1$ and $A_5 = A_6 = A_7 = 1$. Then by Proposition 2, $F(0) = 0$ for every solution F of Eq. (1). Thus from Proposition 4 we obtain that the set of solutions of Eq. (1) consists of all additive functions. In case $A_1 = 3$, $A_2 = A_3 = A_4 = 1$ and $A_5 = A_6 = A_7 = 2$ each function F of the form (17) is a solution of Eq. (1). In fact, we can state a more general result.

Corollary 11. *Assume that conditions* (8) *and* (9) *hold. If* $F : X \to Y$ *is of the form* (17), *where* $a : X \to Y$ *is an additive function and* $c \in Y$, *then* F *is a solution of Eq.* (1).

Proof. Since relations (9) hold, on account of Proposition 4 we have that each additive function satisfies Eq. (1). However, condition (8) implies that any constant function is a solution of Eq. (1). □

At the end of this section let us recall that in the only remaining case $A_1 = \cdots = A_7$ there exists a nonadditive solution F of Eq. (1) such that $F(0) = 0$. Namely, without loss of generality we can assume that $A_1 = \cdots = A_7 = 1$ (we exclude the trivial case $A_1 = \cdots = A_7 = 0$). Next, let $(X, \|\cdot\|)$ be an inner product space and $Y = \mathbb{R}$. Then the function $F : X \to \mathbb{R}$, given by $F(x) := \|x\|^2$, is a solution of Eq. (1), where the norm is derived from the inner product (cf. [17]). Moreover, in [26] it was proved that if $F : X \to \mathbb{R}$, $F(x) = \|x\|^2$ is a solution of (1), then $A_1 = \cdots = A_7$ (which corresponds to Corollary 8).

3. The stability result

In this section, as before, $(X, +)$ is a commutative group, $\widehat{X} := X^3 \setminus \{(0, 0, 0)\}$, Y is a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $A_1, \ldots, A_7 \in \mathbb{K}$.

We prove a stability result for the generalized Fréchet functional equation. As a corollary we obtain that near an approximate solution of Eq. (1) there exists an additive function. The proof of this fact is based on the results of the previous section.

Let us recall that the following theorem, concerning the stability of Eq. (1), was proved in [26].

Theorem 12. *Let* $A_1 \neq 0$ *and*

$$A_2 + A_3 + A_4 = A_5 + A_6 + A_7.$$

Assume that $f\colon X \to Y$, $c : \mathbb{Z}\backslash\{0\} \to [0,\infty)$ *and* $L\colon \widehat{X} \to [0,\infty)$ *satisfy the following three conditions:*

$$\mathcal{M} := \{m \in \mathbb{Z}\backslash\{0\} : |A_7|c(-2m) + |A_5 + A_6|c(m+1) + |A_3 + A_4|c(-m) + |A_2|c(2m+1) < |A_1|\} \neq \emptyset,$$

$$L(kx, ky, kz) \leq c(k)L(x,y,z), \qquad (x,y,z) \in \widehat{X}, m \in \mathcal{M}, \quad k \in \{-2m, m+1, -m, 2m+1\},$$

$$\|A_1 f(x+y+z) + A_2 f(x) + A_3 f(y) + A_4 f(z) - A_5 f(x+y) - A_6 f(x+z) - A_7 f(y+z)\| \leq L(x,y,z), \quad (x,y,z) \in X^3.$$

Then there is a unique function $F : X \to Y$ *satisfying Eq.* (1) *such that*

$$\|f(x) - F(x)\| \leq \rho_L(x), \quad x \in X\backslash\{0\},$$

where

$$\rho_L(x) := \inf_{m\in\mathcal{M}} \frac{L((2m+1)x, -mx, -mx)}{|A_1| - \beta_m},$$

$$\beta_m := |A_7|c(-2m) + |A_5 + A_6|c(m+1) + |A_3 + A_4|c(-m) + |A_2|c(2m+1).$$

The main theorem of this section corresponds to Theorem 12 and reads as follows.

Theorem 13. *Let* $A_2 + A_3 + A_4 \neq 0$,

$$\beta_0 := \left|\frac{A_5 + A_6 + A_7 - A_1}{A_2 + A_3 + A_4}\right| < 1,$$

and $L\colon X^3 \to [0,\infty)$ *satisfy the condition*

$$L(kx, ky, kz) \leq c_k L(x,y,z), \quad (x,y,z) \in \widehat{X}, k \in \{2,3\}, \tag{18}$$

with some $c_2, c_3 \in [0,\infty)$ *such that* $\beta := b_2c_2 + b_3c_3 < 1$, *where*

$$b_2 := \left|\frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4}\right|, \quad b_3 := \left|\frac{A_1}{A_2 + A_3 + A_4}\right|. \tag{19}$$

If $f\colon X \to Y$ fulfils the condition

$$\begin{aligned}\|A_1 f(x+y+z) + A_2 f(x) + A_3 f(y) + A_4 f(z) - A_5 f(x+y)\\ - A_6 f(x+z) - A_7 f(y+z)\| \le L(x,y,z), \quad (x,y,z) \in X^3, \qquad (20)\end{aligned}$$

then there exists a unique function $F : X \to Y$ satisfying (1) *such that $F(0) = 0$ and*

$$\|f(x) - F(x)\| \le \rho_L(x), \quad x \in X, \tag{21}$$

where

$$\rho_L(x) := \frac{L(x,x,x)}{|A_2 + A_3 + A_4|(1-\gamma(x))}, \quad x \in X, \tag{22}$$

with

$$\gamma(x) := \begin{cases} \beta & \text{if } x \neq 0; \\ \beta_0 & \text{if } x = 0. \end{cases}$$

Proof. In the first part of the proof we define an operator $\mathcal{T}$ and show that $\mathcal{T}$ satisfies the assumptions of Theorem 1. Taking $y = z = x$ in (20) we obtain

$$\|A_1 f(3x) + (A_2 + A_3 + A_4) f(x) - (A_5 + A_6 + A_7) f(2x)\| \le L(x,x,x), \quad x \in X.$$

Hence, for each $x \in X$,

$$\left\| f(x) - \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} f(2x) + \frac{A_1}{A_2 + A_3 + A_4} f(3x) \right\| \le \varepsilon(x), \tag{23}$$

where

$$\varepsilon(x) := \frac{L(x,x,x)}{|A_2 + A_3 + A_4|}.$$

Put

$$\mathcal{T}\xi(x) := \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4}\xi(2x) - \frac{A_1}{A_2 + A_3 + A_4}\xi(3x), \quad \xi \in Y^X,\ x \in X. \tag{24}$$

In particular, for $x = 0$ we have

$$\mathcal{T}\xi(0) = \frac{A_5 + A_6 + A_7 - A_1}{A_2 + A_3 + A_4}\xi(0), \quad \xi \in Y^X. \tag{25}$$

Let us note that the operator $\mathcal{T}$ is linear. From (23) and (24) we get directly that

$$\|f(x) - \mathcal{T} f(x)\| \le \varepsilon(x), \quad x \in X,$$

which means that condition (4) holds. In particular, on account of (23) and (25) with $x = 0$, we have

$$\|f(0) - \mathcal{T} f(0)\| = \left| 1 - \frac{A_5 + A_6 + A_7 - A_1}{A_2 + A_3 + A_4} \right| \|f(0)\| \le \varepsilon(0).$$

Now we will show that condition (H2) of Theorem 1 is satisfied with $k=3$, $S=X$, $E=Y$, $f_1(x)=2x$, $f_2(x)=3x$, $f_3(x)=x$ and

$$l_1(x) := \begin{cases} b_2 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$
$$l_2(x) := \begin{cases} b_3 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$
$$l_3(x) := \begin{cases} 0 & \text{if } x \neq 0; \\ \beta_0 & \text{if } x = 0, \end{cases}$$

i.e.,

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{3} l_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in Y^X, x \in X.$$

Let us fix $\xi, \mu \in Y^X$. Then for every $x \in X$ we have

$$\begin{aligned}\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| = \Bigg\| &\frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4}(\xi(2x) - \mu(2x)) \\ &- \frac{A_1}{A_2 + A_3 + A_4}(\xi(3x) - \mu(3x))\Bigg\|.\end{aligned}$$

Hence by the triangle inequality we obtain that

$$\begin{aligned}\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq &\left|\frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4}\right| \|\xi(2x) - \mu(2x)\| \\ &+ \left|\frac{A_1}{A_2 + A_3 + A_4}\right| \|\xi(3x) - \mu(3x)\|.\end{aligned}$$

Thus

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq b_2 \|\xi(2x) - \mu(2x)\| + b_3 \|\xi(3x) - \mu(3x)\|, \quad x \in X. \tag{26}$$

Actually, we use this relation for $x \in X\backslash\{0\}$. In case $x = 0$ we have a bit more:

$$\begin{aligned}\|\mathcal{T}\xi(0) - \mathcal{T}\mu(0)\| &= \left\|\frac{A_5 + A_6 + A_7 - A_1}{A_2 + A_3 + A_4}(\xi(0) - \mu(0))\right\| \\ &= \left|\frac{A_5 + A_6 + A_7 - A_1}{A_2 + A_3 + A_4}\right| \|\xi(0) - \mu(0)\|.\end{aligned}$$

Consequently

$$\|\mathcal{T}\xi(0) - \mathcal{T}\mu(0)\| = \beta_0 \|\xi(0) - \mu(0)\|, \tag{27}$$

which means that condition (H2) holds.

Define an operator $\Lambda : {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ as in (H3) by

$$\Lambda\eta(x) := \sum_{i=1}^{3} l_i(x)\eta(f_i(x)), \quad x \in X \tag{28}$$

for every $\eta \in {\mathbb{R}_+}^X$. Then for each $\eta \in {\mathbb{R}_+}^X$ we have

$$\Lambda\eta(x) := b_2\,\eta(2x) + b_3\,\eta(3x), \quad x \in X\backslash\{0\}$$

and

$$\Lambda\eta(0) := \beta_0\eta(0).$$

Let us note that the operator Λ is nondecreasing, i.e., $\Lambda\eta \le \Lambda\zeta$ for all $\eta, \zeta \in {\mathbb{R}_+}^X$ with $\eta \le \zeta$. Moreover, by (26) and (27)

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \Lambda(\|\xi - \mu\|)(x), \quad \xi, \mu \in Y^X, x \in X. \tag{29}$$

Now we will show that condition (5) is satisfied, i.e., the function series $\sum_{n=0}^{\infty} \Lambda^n\varepsilon(x)$ is convergent for each $x \in X$. Fix an $x \in X\backslash\{0\}$. In view of (28) and (18), we have

$$\begin{aligned}\Lambda\varepsilon(x) &= b_2\,\varepsilon(2x) + b_3\,\varepsilon(3x) = b_2\,\frac{L(2x,2x,2x)}{|A_2+A_3+A_4|} + b_3\,\frac{L(3x,3x,3x)}{|A_2+A_3+A_4|}\\ &\le b_2c_2\frac{L(x,x,x)}{|A_2+A_3+A_4|} + b_3c_3\frac{L(x,x,x)}{|A_2+A_3+A_4|}\\ &= (b_2c_2 + b_3c_3)\frac{L(x,x,x)}{|A_2+A_3+A_4|}.\end{aligned}$$

Thus

$$\Lambda\varepsilon(x) \le \beta\varepsilon(x). \tag{30}$$

By induction one can show that the monotonicity and linearity of Λ implies

$$\Lambda^n\varepsilon(x) \le \beta^n\varepsilon(x). \tag{31}$$

Consequently, for each $x \in X\backslash\{0\}$ we receive the following estimate:

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) \le \varepsilon(x)\left(1 + \sum_{n=1}^{\infty}\beta^n\right) = \frac{\varepsilon(x)}{1-\beta} = \frac{L(x,x,x)}{|A_2+A_3+A_4|(1-\beta)}.$$

In case $x = 0$ we have

$$\Lambda\varepsilon(0) = \beta_0\varepsilon(0). \tag{32}$$

Hence by induction we obtain

$$\Lambda^n\varepsilon(0) = \beta_0^n\varepsilon(0). \tag{33}$$

Therefore

$$\varepsilon^*(0) = \sum_{n=0}^{\infty} \Lambda^n\varepsilon(0) = \varepsilon(0)\left(1 + \sum_{n=1}^{\infty}\beta_0^n\right) = \frac{\varepsilon(0)}{1-\beta_0} = \frac{L(0,0,0)}{|A_2+A_3+A_4|(1-\beta_0)}.$$

Thus we have shown that

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) \le \frac{L(x,x,x)}{|A_2+A_3+A_4|(1-\gamma(x))} < \infty, \quad x \in X.$$

By Theorem 1 (with $S = X$ and $E = Y$), there exists a function $F\colon X \to Y$ satisfying the equation

$$F(x) = \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} F(2x) - \frac{A_1}{A_2 + A_3 + A_4} F(3x), \quad x \in X, \tag{34}$$

such that

$$\|f(x) - F(x)\| \le \varepsilon^*(x) \le \frac{L(x, x, x)}{|A_2 + A_3 + A_4|(1 - \gamma(x))}, \quad x \in X.$$

Moreover,

$$F(x) = \lim_{n\to\infty} \mathcal{T}^n f(x), \quad x \in X.$$

Let us note that condition (34) means that Eq. (1) is satisfied for $x = y = z$.

Next we will show that the function F satisfies Eq. (1) for all $x, y, z \in X$. To obtain this fact we will prove by induction that for all $(x, y, z) \in X^3$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned} &\|A_1 \mathcal{T}^n f(x+y+z) + A_2 \mathcal{T}^n f(x) + A_3 \mathcal{T}^n f(y) + A_4 \mathcal{T}^n f(z) \\ &\quad - A_5 \mathcal{T}^n f(x+y) - A_6 \mathcal{T}^n f(x+z) - A_7 \mathcal{T}^n f(y+z)\| \\ &\le \lambda^n\, L(x, y, z), \end{aligned} \tag{35}$$

where $\lambda := \max\{\beta, \beta_0\}$. For $n = 0$ condition (35) follows directly from (20). Assume that (35) holds for some $n \in \mathbb{N}_0$ and all $(x, y, z) \in X^3$. Then by (24) we have

$$\begin{aligned} &\big\|A_1 \mathcal{T}^{n+1} f(x+y+z) + A_2 \mathcal{T}^{n+1} f(x) + A_3 \mathcal{T}^{n+1} f(y) + A_4 \mathcal{T}^{n+1} f(z) \\ &\quad - A_5 \mathcal{T}^{n+1} f(x+y) - A_6 \mathcal{T}^{n+1} f(x+z) - A_7 \mathcal{T}^{n+1} f(y+z)\big\| \\ &= \Bigg\| \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_1 \mathcal{T}^n f(2(x+y+z)) \\ &\quad - \frac{A_1}{A_2 + A_3 + A_4} A_1 \mathcal{T}^n f(3(x+y+z)) \\ &\quad + \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_2 \mathcal{T}^n f(2x) - \frac{A_1}{A_2 + A_3 + A_4} A_2 \mathcal{T}^n f(3x) \\ &\quad + \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_3 \mathcal{T}^n f(2y) - \frac{A_1}{A_2 + A_3 + A_4} A_3 \mathcal{T}^n f(3y) \\ &\quad + \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_4 \mathcal{T}^n f(2z) - \frac{A_1}{A_2 + A_3 + A_4} A_4 \mathcal{T}^n f(3z) \\ &\quad - \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_5 \mathcal{T}^n f(2(x+y)) + \frac{A_1}{A_2 + A_3 + A_4} A_5 \mathcal{T}^n f(3(x+y)) \\ &\quad - \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_6 \mathcal{T}^n f(2(x+z)) + \frac{A_1}{A_2 + A_3 + A_4} A_6 \mathcal{T}^n f(3(x+z)) \\ &\quad - \frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4} A_7 \mathcal{T}^n f(2(y+z)) + \frac{A_1}{A_2 + A_3 + A_4} A_7 \mathcal{T}^n f(3(y+z)) \Bigg\| \end{aligned}$$

$$\leq \left|\frac{A_5+A_6+A_7}{A_2+A_3+A_4}\right| \lambda^n L(2x,2y,2z) + \left|\frac{A_1}{A_2+A_3+A_4}\right| \lambda^n L\big(3x,3y,3z\big)$$

for every $(x,y,z) \in \widehat{X}$. Hence by (18)

$$\begin{aligned} &\big\|A_1\mathcal{T}^{n+1}f(x+y+z) + A_2\mathcal{T}^{n+1}f(x) + A_3\mathcal{T}^{n+1}f(y) + A_4\mathcal{T}^{n+1}f(z) \\ &\quad -A_5\mathcal{T}^{n+1}f(x+y) - A_6\mathcal{T}^{n+1}f(x+z) - A_7\mathcal{T}^{n+1}f(y+z)\big\| \\ &\leq \lambda^n(b_2c_2+b_3c_3)L(x,y,z) \leq \lambda^{n+1}L(x,y,z) \end{aligned} \tag{36}$$

for $(x,y,z) \in \widehat{X}$. Finally, by (25),

$$\begin{aligned} &\big\|(A_1+A_2+A_3+A_4-A_5-A_6-A_7)\mathcal{T}^{n+1}f(0)\big\| \\ &= \left\|(A_1+A_2+A_3+A_4-A_5-A_6-A_7)\frac{A_5+A_6+A_7-A_1}{A_2+A_3+A_4}\mathcal{T}^n f(0)\right\| \\ &= \beta_0\big\|(A_1+A_2+A_3+A_4-A_5-A_6-A_7)\mathcal{T}^n f(0)\big\| \\ &\leq \beta_0\lambda^n L(0,0,0) \leq \lambda^{n+1}L(0,0,0), \end{aligned}$$

which ends the proof of (35). Letting $n \to \infty$ in (35), we obtain

$$\begin{aligned} &A_1F(x+y+z) + A_2F(x) + A_3F(y) + A_4F(z) \\ &\quad = A_5F(x+y) + A_6F(x+z) + A_7F(y+z), \quad (x,y,z) \in X^3. \end{aligned}$$

Thus, we have proved that, there exists a function $F : X \to Y$ satisfying Eq. (1) for which

$$\|f(x) - F(x)\| \leq \varepsilon^*(x) \leq \rho_L(x), \quad x \in X. \tag{37}$$

Now we will show that $F(0) = 0$. From (24) we get by induction that

$$\mathcal{T}^n\xi(0) = \left(\frac{A_5+A_6+A_7-A_1}{A_2+A_3+A_4}\right)^n \xi(0) = \beta_0^n\xi(0), \quad \xi \in Y^X, n \in \mathbb{N}.$$

Thus

$$\lim_{n\to\infty} \mathcal{T}^n\xi(0) = 0\ , \quad \xi \in Y^X, \tag{38}$$

since $\beta_0 < 1$. Consequently, we have $F(0) = \lim_{n\to\infty} \mathcal{T}^n f(0) = 0$.

It remains to prove the statement concerning the uniqueness of F. To get this fact we first show by induction that for all $\xi,\mu \in Y^X$, $n \in \mathbb{N}$

$$\|\mathcal{T}^n\xi(x) - \mathcal{T}^n\mu(x)\| \leq \Lambda^n(\|\xi-\mu\|)(x), \quad x \in X. \tag{39}$$

By (29) condition (39) holds for $n = 1$. Fix $\xi,\mu \in Y^X$ and assume that (39) holds for $n \in \mathbb{N}$. Then by (29)

$$\begin{aligned} \|\mathcal{T}^{n+1}\xi(x) - \mathcal{T}^{n+1}\mu(x)\| &= \|\mathcal{T}(\mathcal{T}^n\xi)(x) - \mathcal{T}(\mathcal{T}^n\mu)(x)\| \\ &\leq \Lambda(\|\mathcal{T}^n\xi - \mathcal{T}^n\mu\|)(x), \quad x \in X. \end{aligned}$$

Hence by the inductive hypothesis and the monotonicity of Λ we obtain

$$\|\mathcal{T}^{n+1}\xi(x) - \mathcal{T}^{n+1}\mu(x)\| \leq \Lambda(\Lambda^n(\|\xi-\mu\|))(x) = \Lambda^{n+1}(\|\xi-\mu\|)(x), \quad x \in X.$$

Let $G : X \to Y$ be also a solution of (1) such that $\|f(x) - G(x)\| \le \rho_L(x)$ for $x \in X$. Then

$$\|G(x) - F(x)\| \le 2\rho_L(x), \quad x \in X. \tag{40}$$

Hence by (39) we obtain

$$\|\mathcal{T}^n G(x) - \mathcal{T}^n F(x)\| \le 2\Lambda^n \rho_L(x) = \frac{2\Lambda^n \varepsilon(x)}{1 - \gamma(x)}, \quad x \in X,$$

since Λ is linear and monotone. Letting $n \to \infty$, by the convergence of the series $\sum_{n=0}^{\infty} \Lambda^n \varepsilon(x)$, we get

$$\lim_{n\to\infty} \|\mathcal{T}^n G(x) - \mathcal{T}^n F(x)\| = 0, \quad x \in X.$$

Hence $\|G(x) - F(x)\| = 0$ for $x \in X$, since G and F are fixed points of $\mathcal{T}$. Consequently $G(x) = F(x)$ for every $x \in X$. □

Now we proceed to some applications of the main result of this paper concerning additive functions. Let us note that the result presented below cannot be applied to Eq. (2), since its assumptions exclude the case where $A_1 = \cdots = A_7 = 1$.

Corollary 14. *Assume that* $A_i \neq A_j$ *for some* $i, j \in \{1, \ldots, 7\}$, $A_2 + A_3 + A_4 \neq 0$, $L\colon X^3 \to [0, \infty)$ *satisfy condition* (18)*, and* $\beta_0, \beta \in [0, 1)$*, where* β_0 *and* β *are defined as in Theorem* 13*. Let* $f\colon X \to Y$ *be a function satisfying condition* (20)*. Then there exists a unique additive solution* $a : X \to Y$ *of Eq.* (1) *such that*

$$\|f(x) - a(x)\| \le \rho_L(x), \quad x \in X,$$

where $\rho_L(x)$ *is given by* (22)*, i.e.,*

$$\rho_L(x) := \frac{L(x, x, x)}{|A_2 + A_3 + A_4|(1 - \gamma(x))}, \quad x \in X,$$

with

$$\gamma(x) := \begin{cases} \beta & \text{if } x \neq 0, \\ \beta_0 & \text{if } x = 0. \end{cases}$$

Proof. On account of Theorem 13 there exists a unique solution $F : X \to Y$ of Eq. (1) such that $F(0) = 0$ and

$$\|f(x) - F(x)\| \le \rho_L(x), \quad x \in X.$$

Then by Theorem 7 we obtain that F is additive. □

The assumption that $\beta_0 < 1$ can be omitted in Theorem 13, if we replace (18) by the subsequent somewhat stronger condition

$$L(kx, ky, kz) \le c_k L(x, y, z), \quad (x, y, z) \in X^3, k \in \{2, 3\}; \tag{41}$$

let us note that in the case $L(0, 0, 0) = 0$ these two conditions are equivalent. Namely, we have the following.

Proposition 15. *Let* $A_2 + A_3 + A_4 \neq 0$ *and* $L\colon X^3 \to [0,\infty)$ *satisfy condition* (41) *with some* $c_2, c_3 \in [0,\infty)$ *such that* $\beta := b_2c_2 + b_3c_3 < 1$*, where* b_2, b_3 *are given by* (19)*. If* $f\colon X \to Y$ *fulfils condition* (20)*, then there exists a unique function* $F : X \to Y$ *satisfying* (1) *for which*

$$\|f(x) - F(x)\| \leq \rho_L(x), \quad x \in X,$$

where

$$\rho_L(x) := \frac{L(x,x,x)}{|A_2 + A_3 + A_4|(1-\beta)}, \quad x \in X. \tag{42}$$

Proof. We give only an outline of the proof since the main steps are the same as in the proof of Theorem 13. Let us consider the operator $\mathcal{T}$ defined by (24). From (26) we obtain that condition (H2) is satisfied with $k = 2$, $S = X$, $E = Y$,

$$f_1(t) = 2t, \quad f_2(t) = 3t,$$
$$l_1(t) = b_2, \quad l_2(t) = b_3,$$

i.e.,

$$\|\mathcal{T}\xi(t) - \mathcal{T}\mu(t)\| \leq \sum_{i=1}^{2} l_i(t)\|\xi(f_i(t)) - \mu(f_i(t))\|, \quad \xi, \eta \in Y^X, t \in X.$$

Then the operator $\Lambda : {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ defined by

$$\Lambda\eta(t) := \sum_{i=1}^{2} l_i(t)\eta(f_i(t)), \quad \eta \in {\mathbb{R}_+}^X, t \in X$$

is of the form

$$\Lambda\eta(t) = b_2\eta(2t) + b_3\eta(3t), \quad \eta \in {\mathbb{R}_+}^X, t \in X.$$

From (41) we get that condition (30) holds for all $x \in X$ with

$$\varepsilon(x) := \frac{L(x,x,x)}{|A_2 + A_3 + A_4|}.$$

Hence we obtain that condition (31) also holds for all $x \in X$. Consequently, for each $x \in X$ we have

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \leq \varepsilon(x)\left(1 + \sum_{n=1}^{\infty} \beta^n\right) = \frac{\varepsilon(x)}{1-\beta} = \frac{L(x,x,x)}{|A_2 + A_3 + A_4|(1-\beta)}.$$

Let us yet note that (35) holds for $\lambda = \beta$, since we (36) holds also for $x = y = z = 0$ in view of (41).

The existence and uniqueness of F satisfying Eq. (1) can be obtained in the same way as in the proof of Theorem 13. $\square$

From the above result we can obtain the following counterpart of Corollary 14.

Corollary 16. *Let $A_i \neq A_j$ for some $i, j \in \{1, \dots, 7\}$, $A_2 + A_3 + A_4 \neq 0$ and relations (9) hold. Let a function $L\colon X^3 \to [0, \infty)$ satisfy condition (41) with $c_2, c_3 \in [0, \infty)$ such that $\beta := b_2c_2 + b_3c_3 < 1$, where b_2, b_3 are given by (19). If $f\colon X \to Y$ fulfils condition (20), then there exists a unique additive solution $a : X \to Y$ of Eq. (1) and a constant $c \in Y$ such that*

$$\|f(x) - a(x) - c\| \leq \rho_L(x), \quad x \in X,$$

where $\rho_L(x)$ is given by (42).

4. Final remarks

First, note that the condition

$$L(kx, ky, kz) \leq c_k L(x, y, z), \quad k \in \{2, 3\}, \tag{43}$$

for $(x, y, z) = (0, 0, 0)$, means that $L(0, 0, 0) = 0$ or $c_2, c_3 \in [1, \infty)$.

Let X be a normed space. Then the function $L : X^3 \to [0, \infty)$ given by

$$L(x, y, z) := (\alpha_1\|x\|^{p_1} + \alpha_2\|y\|^{p_2} + \alpha_3\|z\|^{p_3})^w, \quad (x, y, z) \in \widehat{X}, \tag{44}$$

with any fixed $p_i, w, \alpha_i \in \mathbb{R}$ such that $p_i > 0$ and $\alpha_i > 0$ for $i \in \{1, 2, 3\}$, satisfies condition (18) with

$$c_k = k^{pw}, \quad k \in \{2, 3\},$$

where

$$p = \begin{cases} \max\{p_1, p_2, p_3\} & \text{if } w > 0; \\ \min\{p_1, p_2, p_3\} & \text{if } w < 0. \end{cases}$$

By (22) we have

$$\rho_L(x) = \frac{(\alpha_1 + \alpha_2 + \alpha_3)^w\|x\|^{pw}}{|A_2 + A_3 + A_4|(1 - \beta)}, \quad x \in X\backslash\{0\}.$$

The value of L at $(x, y, z) = (0, 0, 0)$ can be taken arbitrarily. In particular, $L(0, 0, 0)$ can take the value of the left-hand side of condition (20) at $x = y = z = 0$. For $x = 0$ we have

$$\rho_L(0) = \frac{L(0, 0, 0)}{|A_2 + A_3 + A_4|(1 - \beta_0)}.$$

Thus an approximate solution f satisfying (20) need not satisfy the condition $f(0) = 0$.

Let $A_1 = -4$, $A_2 = A_3 = A_4 = 8$ and $A_5 = A_6 = A_7 = 2$. Then relations (9) hold and $b_2 = \frac{1}{4}$, $b_3 = \frac{1}{6}$. Consider the function $L : X^3 \to [0, \infty)$ given by

$$L(x, y, z) := \|x\|^p + \|y\|^p + \|z\|^p, \quad (x, y, z) \in X^3,$$

with some $p \in \mathbb{R}$ such that $p > 0$. It satisfies condition (41) with $c_k = |k|^p$ for $k \in \{2, 3\}$. Thus $\beta < 1$ if and only if $p \in (0, 1)$, since $2b_2 + 3b_3 = 1$. Therefore, in this case, Corollary 16 implies that, for a given function $f\colon X \to Y$ fulfilling condition (20), there exists an additive function F such that

$$\|f(x) - F(x)\| \le \rho_L(x), \quad x \in X,$$

where

$$\rho_L(x) = \frac{L(x,x,x)}{24(1-\beta)}, \quad x \in X.$$

Now let us consider the function $L : X^3 \to [0,\infty)$ given by

$$L(x,y,z) := (\alpha_1\|x\|^p + \alpha_2\|y\|^p + \alpha_3\|z\|^p)^w + \varepsilon, \quad (x,y,z) \in X,$$

with any fixed $p, w, \alpha_i \in \mathbb{R}$ such that $p > 0$, $w > 0$, $\alpha_i > 0$ for $i \in \{1,2,3\}$, and an $\varepsilon > 0$. Then condition (18) holds with $c_k = k^{pw}$, because $k^{pw} > 1$.

Let us note that under the assumptions of Theorem 13 and Corollary 14 we get that

$$-3A_1 + 2(A_5 + A_6 + A_7) \neq 0,$$

in the case where the solution of Eq. (1) occurring in each of these results is a nonzero function. Indeed, from Proposition 3 and Corollary 5, respectively, we obtain that relations (9) are satisfied. Hence we get that $A_2 + A_3 + A_4 = -3A_1 + 2(A_5 + A_6 + A_7)$.

Finally observe that also the function $L : X^3 \to [0,\infty)$, given by

$$L(x,y,z) := \alpha\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \varepsilon, \quad (x,y,z) \in \widehat{X},$$

with any fixed $p_1, p_2, p_3, \alpha, \varepsilon \in [0,\infty)$, fulfils (18) with

$$c_k = k^{p_1+p_2+p_3}, \quad k \in \{2,3\}.$$

References

[1] Alsina, C., Sikorska, J., Tomás, M.S.: Norm Derivatives and Characterizations of Inner Product Spaces. World Scientific Publishing Co., Singapore (2010)

[2] Badora, R., Brzdęk, J.: Fixed points of a mapping and Hyers–Ulam stability. J. Math. Anal. Appl. **413**, 450–457 (2014)

[3] Bahyrycz, A., Brzdęk, J., Leśniak, Z.: On approximate solutions of the generalized Volterra integral equation. Nonlinear Anal. Real World Appl. **20**, 59–66 (2014)

[4] Bahyrycz, A., Brzdęk, J., Jabłońska, E., Malejki, R.: Ulam's stability of a generalization of the Frechet functional equation. J. Math. Anal. Appl. **442**, 537–553 (2016)

[5] Bahyrycz, A., Brzdęk, J., Piszczek, M., Sikorska, J.: Hyperstability of the Fréchet equation and a characterization of inner product spaces. J. Funct. Spaces Appl. **2013,** 496361-1–496361-6 (2013)

[6] Bahyrycz, A., Ciepliński, K., Olko, J.: On Hyers–Ulam stability of two functional equations in non-Archimedean spaces. J. Fixed Point Theory Appl. **18**(2), 433–444 (2016)

[7] Bahyrycz, A., Olko, J.: Hyperstability of general linear functional equation. Aequ. Math. **90**(3), 527–540 (2016)

[8] Brillouët-Belluot, N., Brzdęk, J., Ciepliński, K.: On some recent developments in Ulam's type stability. Abstr. Appl. Anal. **2012,** 716936-1–716936-41 (2012)

[9] Brzdęk, J.: Remarks on hyperstability of the Cauchy functional equation. Aequ. Math. **86**, 255–267 (2013)
[10] Brzdęk, J.: Hyperstability of the Cauchy equation on restricted domains. Acta Math. Hung. **141**, 58–67 (2013)
[11] Brzdęk, J., Chudziak, J., Páles, Z.: A fixed point approach to stability of functional equations. Nonlinear Anal. **74**(17), 6728–6732 (2011)
[12] Brzdęk, J., Ciepliński, K.: A fixed point approach to the stability of functional equations in non-Archimedean metric spaces. Nonlinear Anal. **74**(18), 6861–6867 (2011)
[13] Brzdęk, J., Jabłońska, E., Moslehian, M.S., Pacho, P.: On stability of a functional equation of quadratic type. Acta Math. Hung. **149**, 160–169 (2016)
[14] Cădariu, L., Găvruţa, L., Găvruţa, P.: Fixed points and generalized Hyers–Ulam stability. Abstr. Appl. Anal. **2012**, 712743-1–712743-10 (2012)
[15] Dragomir, S.S.: Some characterizations of inner product spaces and applications. Stud. Univ. Babes-Bolyai Math. **34**(1), 50–55 (1989)
[16] Fechner, W.: On the Hyers–Ulam stability of functional equations connected with additive and quadratic mappings. J. Math. Anal. Appl. **322**, 774–786 (2006)
[17] Fréchet, M.: Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert. Ann. Math. (2) **36**(3), 705–718 (1935)
[18] Gselmann, E.: Hyperstability of a functional equation. Acta Math. Hung. **124**, 179–188 (2009)
[19] Hyers, D.H., Isac, G., Rassias, ThM: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
[20] Jordan, P., von Neumann, J.: On inner products in linear, metric spaces. Ann. Math. (2) **36**(3), 719–723 (1935)
[21] Jung, S.-M.: On the Hyers–Ulam stability of the functional equation that have the quadratic property. J. Math. Anal. Appl. **222**, 126–137 (1998)
[22] Jung, S.-M.: Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis, Springer Optimization and Its Applications, vol. 48. Springer, New York (2011)
[23] Kannappan, P.: Functional Equations and Inequalities with Applications. Springer Monographs in Mathematics. Springer, New York (2009)
[24] Lee, Y.-H.: On the Hyers–Ulam–Rassias stability of the generalized polynomial function of degree 2. J. Chuncheong Math. Soc. **22**(2), 201–209 (2009)
[25] Maksa, G., Páles, Z.: Hyperstability of a class of linear functional equations. Acta Math. Acad. Paedag. Nyìregyháziensis **17**, 107–112 (2001)
[26] Malejki, R.: Stability of a generalization of the Fréchet functional equation. Ann. Univ. Paedagog. Crac. Stud. Math. **14**, 69–79 (2015)
[27] Moslehian, M.S., Rassias, J.M.: A characterization of inner product spaces concerning an Euler–Lagrange identity. Commun. Math. Anal. **8**(2), 16–21 (2010)
[28] Nikodem, K., Pales, Z.: Characterizations of inner product spaces by strongly convex functions. Banach J. Math. Anal. **5**(1), 83–87 (2011)
[29] Rassias, Th M.: New characterizations of inner product spaces. Bull. Sci. Math. (2) **108**, 95–99 (1984)
[30] Piszczek, M.: Remark on hyperstability of the general linear equation. Aequ. Math. **88**, 163–168 (2014)
[31] Popa, D., Raşa, I.: The Fréchet functional equation with application to the stability of certain operators. J. Approx. Theory **164**, 138–144 (2012)
[32] Sikorska, J.: On a direct method for proving the Hyers–Ulam stability of functional equations. J. Math. Anal. Appl. **372**, 99–109 (2010)
[33] Zhang, D.: On Hyers–Ulam stability of generalized linear functional equation and its induced Hyers–Ulam programming problem. Aequ. Math. **90**, 559–568 (2016)
[34] Zhang, D.: On hyperstability of generalised linear functional equations in several variables. Bull. Aust. Math. Soc. **92**, 259–267 (2015)

Janusz Brzdęk, Zbigniew Leśniak and Renata Malejki
Department of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków
Poland
e-mail: zbigniew.lesniak@up.krakow.pl

Janusz Brzdęk
e-mail: janusz.brzdek@up.krakow.pl

Renata Malejki
e-mail: renata.malejki@up.krakow.pl

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