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**Aequationes Mathematicae**

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# The Cosine–Sine functional equation on a semigroup with an involutive automorphism

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**Abstract.** We determine the complex-valued solutions of the following extension of the Cosine–Sine functional equation

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in S,$$

where $S$ is a semigroup generated by its squares and $\sigma$ is an involutive automorphism of $S$. We express the solutions in terms of multiplicative and additive functions.

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## 1. Introduction

Let $S$ be a semigroup and let $\sigma$ be an involutive automorphism of $S$. That it is involutive means that $\sigma(\sigma(x)) = x$ for all $x \in S$.

The functional equation

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in S \tag{1.1}$$

includes a number of functional equations which have been treated by several authors in the literature. The case of the sine addition law

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in G, \tag{1.2}$$

has been treated on groups, semigroups and algebras. See for example [7, chapter 4] and [3]. Poulsen and Stetkær [5] derived the solution formulas for the functional equations

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y), \quad x, y \in G, \tag{1.3}$$
$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad x, y \in G,$$
$$g(x\sigma(y)) = g(x)g(y) + f(x)f(y), \quad x, y \in G,$$

on topological groups.

Chung et al. [2] solved the functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y)$$

on groups.

We refer also to [1], [6, Section 11.7] and [7] for further contextual and historical discussions.

Our main goal in this paper is to solve the functional equation (1.1) on semigroups generated by their squares. We notice here that (1.1) is a simple example of Levi-Civitá's functional equation, and there is a general theory about the general form of the structure of solutions of Levi-Civitá's functional equation on monoids, using matrix-coefficients of the right regular representation, see for example [7, Theorem 5.2]. But given a Levi-Civitá functional equation of a special form like (1.2) the application of the general theory is not the final word about its solutions. There will be a possible linear dependence between the functions on the right hand side of (1.1) to take into account as well as the internal structure of the monoid.

The remark above is illustrated by the treatment of the sine addition law (see for example [7, Corollary 4.4]). In this paper we take a more direct approach.

Replacing the semigroup $S$ in the functional equation (1.1) by a group we can provide a specialization of it. In particular, our results contain the solutions of the following functional equation

$$f(x-y) = f(x)g(y) + g(x)f(y) + h(x)h(y)$$

on abelian groups that are not in the literature.

Our main contributions to the knowledge about the Cosine–Sine functional equation (1.1) are the following

1) We extend the setting from groups to semigroups generated by their squares and with involutive automorphisms.
2) We relate the solutions of (1.1) to those of

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y), \quad x, y \in S$$

and

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in S.$$

3) We produce explicit solution formulas of Eq. (1.1).

It is intriguing to see that some methods of [2] carry over to the more general situation (1.1).

## 2. Notation and terminology

Throughout this paper $S$ denotes a semigroup (a set with an associative composition) generated by its squares. The map $\sigma : S \to S$ denotes an involutive automorphism. That $\sigma$ is involutive means that $\sigma(\sigma(x)) = x$ for all $x \in S$. Various examples of involutive automorphisms on semigroups can be found in [3].

Let $f : S \to \mathbb{C}$. We call $f_e := \frac{f+f\circ\sigma}{2}$ the even part of $f$ and $f_o := \frac{f-f\circ\sigma}{2}$ its odd part. We say that $f$ is even if $f = f\circ\sigma$ and that $f$ is odd if $f = -f\circ\sigma$. That is, $f$ is even or odd with respect to $\sigma$.

A function $f : S \to \mathbb{C}$ is said to be central if $f(xy) = f(yx)$ for all $x, y \in S$ and $f$ is said to be abelian if $f(x_1x_2\cdots x_n) = f(x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)})$ for all $x_1, x_2, \ldots, x_n \in S$, all permutations $\pi$ of $n$ elements and all $n = 1, 2, 3, \ldots$ [7, Definition B.3].

A multiplicative function on $S$ is a homomorphism $\chi : S \to (\mathbb{C}, \cdot)$. If $\chi \neq 0$, then $I_\chi := \{x \in S \mid \chi(x) = 0\}$ is either empty or a proper subset of $S$. $I_\chi$ is a two sided ideal in $S$ if not empty and $S\backslash I_\chi$ is a subsemigroup of $S$.

## 3. Basic results

The continuous solutions of the functional equation (1.3) were obtained on topological groups in [5] and on monoids generated by their squares in [3, Proposition 3.6]. We shall now extend these results to semigroups generated by their squares.

**Proposition 3.1.** *The solutions $f, g : S \to \mathbb{C}$ of the functional equation (1.3) can be listed as follows*

(a) $f = 0$ *and $g$ is arbitrary.*
(b) $f = \alpha(\chi_1 - \chi_2)$, $g = \frac{\chi_1+\chi_2}{2}$, *where $\alpha \in \mathbb{C}\backslash\{0\}$ is a constant and $\chi_1, \chi_2 : S \to \mathbb{C}$ are two multiplicative functions such that $\chi_1 \neq \chi_2$, $\chi_1\circ\sigma = \chi_1$ and $\chi_2\circ\sigma = \chi_2$.*
(c)

$$\begin{cases} f(x) = \chi(x)\, a(x), \quad g(x) = \chi(x) \quad \textit{for} \quad x \in S\backslash I_\chi \\ f(x) = 0, \quad g(x) = 0 \quad \textit{for} \quad x \in I_\chi, \end{cases}$$

*where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function and $a : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function such that $\chi\circ\sigma = \chi$ and $a\circ\sigma = a$.*

*Proof.* If $f = 0$ then $g$ is arbitrary. Assume that $f \neq 0$. Let $x, y, z \in S$. By interchanging $x$ and $y$ in Eq. (1.3) we get that $f(y\sigma(x)) = f(x\sigma(y))$, then $f\circ\sigma(xy) = f(\sigma(x)\sigma(y)) = f(yx)$ for all $x, y \in S$. Hence $f\circ\sigma(xyz) = f\circ\sigma(x(yz)) = f(yzx) = f\circ\sigma((zx)y) = f\circ\sigma(z(xy)) = f(xyz)$ for all $x, y, z \in S$. Since $S$ is generated by its squares, there exist $x_1, \ldots, x_n \in S$ such that $x = x_1^2\cdots x_n^2$. So we have $f\circ\sigma(x) = f\circ\sigma(x_1^2\cdots x_n^2)$. If $n = 1$ we obtain $f\circ\sigma(x) = f\circ\sigma(x_1^2) = f(x_1^2) = f(x)$. If $n \geq 2$ we have $f\circ\sigma(x) = f\circ\sigma(x_1x_1(x_2^2\cdot\cdot x_n^2)) = f(x_1x_1(x_2^2\cdots x_n^2)) = f(x)$. Hence $f$ is even with respect to $\sigma$ and central. By similar computations to the ones in the proof of [5, Theorem II.3] we get, for all $x, y \in S$,

$$\begin{aligned} f(x)g(y)+g(x)f(y) &= f(x\sigma(y)) = f\circ\sigma(x\sigma(y)) = f(\sigma(x)y) \\ &= f(\sigma(x))g(\sigma(y))+g(\sigma(x))f(\sigma(y)) \\ &= f(x)g(\sigma(y))+g(\sigma(x))f(y). \end{aligned}$$

We infer

$$f(x)[g(\sigma(y))-g(y)] = [g(x)-g(\sigma(x))]f(y), \quad x, y \in S.$$

Applying [7, Exercise 1.1(b)] to the last identity we get that $g\circ\sigma = g$, because $f \neq 0$. So the functional equation (1.3) implies the sine addition law

$$f(xy) = f(x)g(y)+g(x)f(y), \quad x, y \in S. \tag{3.1}$$

According to [7, Theorem 4.1] there exist two multiplicative functions $\chi_1$, $\chi_2$ : $S \to \mathbb{C}$ and a constant $c_1 \in \mathbb{C}$ such that $g = \frac{\chi_1+\chi_2}{2}$ and $2c_1 f = \chi_1 - \chi_2$. As $f\circ\sigma = f$, $g\circ\sigma = g$, we get that $\chi_1\circ\sigma = \chi_1$ and $\chi_2\circ\sigma = \chi_2$. We split the discussion into the cases $\chi_1 \neq \chi_2$ or $\chi_1 = \chi_2$.

Case 1: $\chi_1 \neq \chi_2$. Then $c_1 \neq 0$, and it follows that $f = \alpha(\chi_1 - \chi_2)$ with $\alpha := \frac{1}{2c_1} \in \mathbb{C}\backslash\{0\}$ a constant.

Case 2: $\chi_1 = \chi_2$. Putting $\chi = \chi_1 = \chi_2$ the functional equation (3.1) becomes

$$f(xy) = f(x)\chi(y)+\chi(x)f(y), \quad x, y \in S. \tag{3.2}$$

As $f \neq 0$ and $S$ is generated by its squares we get from (3.2) that $\chi \neq 0$. By similar computations to the ones in the proof of [3, Lemma 3.4] we deduce from (3.2) that there exists a nonzero additive function $a : S\backslash I_\chi \to \mathbb{C}$ such that $a\circ\sigma = a$, $f = \chi\, a$ on $S\backslash I_\chi$ and $f = 0$ on $I_\chi$.

Conversely we check by elementary computations that the pairs $(f, g)$ described in Proposition 3.1 are solutions of Eq. (1.3). This completes the proof. □

*Remark* 3.2. If $f, g : S \to \mathbb{C}$ satisfy the functional equation (1.3), the formulas in the cases $(a)$, $(b)$ and $(c)$ of Proposition 3.1 reveal that if $f \neq 0$ both $f$ and $g$ are abelian and even with respect to $\sigma$.

## 4. Main results

Chung et al. [2] solved the functional equation $f(xy) = f(x)g(y)+g(x)f(y)+h(x)h(y)$ where $f, g, h$ are unknown complex-valued functions defined on a group. In the present section we deal with the functional equation (1.1) on a semigroup generated by its squares. We start with the following properties.

### 4.1. Key properties of the solutions

**Lemma 4.1.** *Let $f, g, h : S \to \mathbb{C}$ be a solution of the functional equation (1.1).*

(a) $f(x\sigma(y)) = f(y\sigma(x))$ *for all* $x, y \in S$.
(b) $f \circ \sigma(xy) = f(yx)$ *for all* $x, y \in S$.
(c) $f \circ \sigma(xyz) = f(xyz)$ *for all* $x, y, z \in S$.
(d) $f$ *is even with respect to* $\sigma$ *and central.*
(e)

$$2f(x)g_o(y) + h(x)h(y) = -2f(y)g_o(x) + h(\sigma(x))h(\sigma(y)), \tag{4.1}$$

$$f(x)g_o(y) + h_e(x)h_o(y) = 0, \tag{4.2}$$

$$f(xy) - f(x\sigma(y)) = -2h_o(x)h_o(y), \tag{4.3}$$

*for all* $x, y \in S$.

*Proof.* (a) The right hand side of the functional equation (1.1) is invariant under the interchange of $x$ and $y$. So $f(x\sigma(y)) = f(y\sigma(x))$ for all $x, y \in S$.
(b) From (1) we get $f \circ \sigma(xy) = f(\sigma(x)\sigma(y)) = f(yx)$ for all $x, y \in S$.
(c) For all $x, y, z \in S$ we have, using the result (b), $f\circ\sigma(xyz) = f\circ\sigma(x(yz)) = f(yzx) = f \circ \sigma(zxy) = f(xyz)$.
(d) By the same computations used to prove that $f$ is even and central in the proof of Proposition 3.1.
(e) Using that $f$ is even we get $f(x\sigma(y)) = f \circ \sigma(\sigma(x)y) = f(\sigma(x)y)$. So, $f(x)g(y) + g(x)f(y) + h(x)h(y) = f(x)g(\sigma(y)) + g(\sigma(x))f(y) + h(\sigma(x))$ $h(\sigma(y))$, which implies Eq. (4.1).

Replacing $x$ by $\sigma(x)$ in Eq. (4.1) and taking into account that $f \circ \sigma = f$ and $g_o \circ \sigma = -g_o$ we obtain $2f(x)g_o(y) - 2f(y)g_o(x) + h(\sigma(x))h(y) - h(x)h(\sigma(y)) = 0$. When to this we add (4.1) we obtain $4f(x)g_o(y)+h(x)h(y)-h(\sigma(x))h(\sigma(y)) + h(\sigma(x))h(y) - h(x)h(\sigma(y)) = 0$. From this it follows that $4f(x)g_o(y)+h(x)[h(y)-h(\sigma(y))]+h(\sigma(x))[h(y)-h(\sigma(y))] = 0$, which implies that $4f(x)g_o(y) + [h(x) + h(\sigma(x))][h(y) - h(\sigma(y))] = 0$. This is (4.2).

On the other hand, by replacing $y$ by $\sigma(y)$ in Eq. (1.1) and taking into account that $f\circ\sigma = f$ we obtain $f(xy) = f(x)g(\sigma(y))+g(x)f(y)+h(x)h(\sigma(y))$, then $f(xy) - f(x\sigma(y)) = -f(x)(g(y) - g(\sigma(y))) - h(x)(h(y) - h(\sigma(y)))$. This implies $f(xy) - f(x\sigma(y)) = -2f(x)g_o(y) - 2h(x)h_o(y)$. From Eq. (4.2) we obtain

$$\begin{aligned} f(xy) - f(x\sigma(y)) &= 2h_e(x)h_o(y) - 2h(x)h_o(y) \\ &= -2(h(x) - h_e(x))h_o(y) = -2h_o(x)h_o(y). \end{aligned}$$

This proves (4.3) and completes the proof. □

To solve the functional equation (1.1) we will discuss two cases according to whether $f$ and $h$ are linearly independent or not.

### 4.2. The solutions of (1.1) when $f$ and $h$ are linearly dependent

**Theorem 4.2.** *The solutions $f, g, h : S \to \mathbb{C}$ of the functional equation (1.1) such that $f$ and $h$ are dependent can be listed as follows*

(a) $f = 0$, *$g$ arbitrary and* $h = 0$.

(b) $f = \alpha\,(\chi_1 - \chi_2)$, $g = \frac{\chi_1+\chi_2}{2} - \beta^2\,\frac{\chi_1-\chi_2}{2\alpha}$, $h = \beta\,(\chi_1 - \chi_2)$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$ *and* $\beta \in \mathbb{C}$ *are two constants,* $\chi_1, \chi_2 : S \to \mathbb{C}$ *are two multiplicative functions such that* $\chi_1 \neq \chi_2$, $\chi_1 \circ \sigma = \chi_1$, $\chi_2 \circ \sigma = \chi_2$.

(c)

$$\begin{cases} f = \chi\, a, \;\; g = \chi\left(1 - \frac{c^2}{2} a\right) \text{ and } h = c\,\chi\, a \text{ on } S\backslash I_\chi \\ f(x) = g(x) = h(x) = 0 \text{ for } x \in I_\chi \end{cases}$$

*where $c \in \mathbb{C}$ is a constant, $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function such that $\chi \circ \sigma = \chi$, $a : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function such that $a \circ \sigma = a$.*

*Proof.* Let $f, g, h : S \to \mathbb{C}$ be a solution of the functional equation (1.1) such that $f$ and $h$ are linearly dependent. If $f = 0$, then the functional equation (1.1) becomes $h(x)h(y) = 0$ for all $x, y \in S$, so $h = 0$ and $g$ is arbitrary. So during the rest of the proof we will assume that $f \neq 0$. Since $f$ and $h$ are assumed to be linearly dependent, there exists a constant $c \in \mathbb{C}$ such that $h = cf$. So Eq. (1.1) can be written as follows

$$\begin{aligned} f(x\sigma(y)) &= f(x)g(y) + g(x)f(y) + c^2 f(x)f(y) \\ &= f(x)\left[g(y) + \frac{c^2}{2}f(y)\right] + \left[g(x) + \frac{c^2}{2}f(x)\right]f(y), \end{aligned}$$

which becomes

$$f(x\sigma(y)) = f(x)k(y) + k(x)f(y), \quad x, y \in S, \tag{4.4}$$

where $k = g + \frac{c^2}{2}f$. According to Proposition 3.1 we have two cases:

Case 1: $f = \alpha\,(\chi_1 - \chi_2)$, $k = \frac{\chi_1+\chi_2}{2}$ where $\alpha \in \mathbb{C}\backslash\{0\}$ is a constant and $\chi_1, \chi_2 : S \to \mathbb{C}$ are two multiplicative functions such that $\chi_1 \neq \chi_2$, $\chi_1 \circ \sigma = \chi_1$ and $\chi_2 \circ \sigma = \chi_2$. As $h = c\,f$ we obtain $h = \beta\,(\chi_1 - \chi_2)$ where $\beta = \alpha c \in \mathbb{C}$ is a constant. On the other hand we have $g = k - \frac{c^2}{2}f = \frac{\chi_1+\chi_2}{2} - \frac{\alpha c^2}{2}(\chi_1 - \chi_2) = \frac{\chi_1+\chi_2}{2} - \beta^2\,\frac{\chi_1-\chi_2}{2\alpha}$.

Case 2:

$$\begin{cases} f(x) = \chi(x)\,a(x), \;\; k(x) = \chi(x) \;\; \text{for } \; x \in S\backslash I_\chi \\ f(x) = 0, \;\; k(x) = 0 \;\; \text{for } \; x \in I_\chi, \end{cases}$$

where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function and $a : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function such that $\chi \circ \sigma = \chi$ and $a \circ \sigma = a$.

Since $h = c\,f$ and $k = g + \frac{c^2}{2} f$, we find, on $S\backslash I_\chi$ where $f = \chi\, a$, that $g = \chi(1 - \frac{c^2}{2} a)$ and $h = c\,\chi\, a$. On the other hand we obtain $g = 0$ and $h = 0$ on $I_\chi$.

Conversely, if $f$, $g$ and $h$ are of the forms (a)-(c) in Theorem 4.2 we check by elementary computations that $f$, $g$ and $h$ satisfy the functional equation (1.1), and that $f$ and $h$ are linearly dependent. This completes the proof of Theorem 4.2. □

### 4.3. The solutions of (1.1) when $f$ and $h$ are linearly independent

Let $f, g, h : S \to \mathbb{C}$ satisfy the functional equation (1.1) so that $f$ and $h$ are linearly independent. According to Lemma 4.1(d) we have $f \circ \sigma = f$. Consequently Eq. (1.1) implies that

$$f(xy) = f(x)g(\sigma(y)) + g(x)f(y) + h(x)h(\sigma(y)), \quad x, y \in S.$$

Since $f \neq 0$, according to (4.2), $h_o = 0$ implies $g_o = 0$. So we will discuss the following possibilities: $h \circ \sigma = h$ and $h \circ \sigma \neq h$.

We notice here that if $h \circ \sigma = h$ Eq. (1.1) can be written as follows

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in S.$$

In this case we extend the results obtained in [2] on groups to semigroups generated by their squares.

#### 4.3.1. The case $h \circ \sigma = h$.

**Theorem 4.3.** *The solutions $f, g, h : S \to \mathbb{C}$ of the functional equation (1.1) with $f$ and $h$ linearly independent and $h \circ \sigma = h$ can be listed as follows*

(a)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \chi\, A_1 \\ \chi \\ \chi\, A \\ \chi\, A^2 \end{pmatrix} \quad on\ S\backslash I_\chi,$$

$f(x) = g(x) = h(x) = 0$ *for* $x \in I_\chi$
*where* $\chi : S \to \mathbb{C}$ *is a nonzero multiplicative function and* $A$, $A_1 : S\backslash I_\chi \to \mathbb{C}$ *are two additive functions with* $\chi \circ \sigma = \chi$, $A \circ \sigma = A$, $A_1 \circ \sigma = A_1$ *and* $A \neq 0$.

(b)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} c^2 & -c^2 & -c \\ 0 & 1 & 0 \\ c & -c & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \quad on\ S\backslash I_\chi,$$

$f(x) = c^2\, \mu(x)$, $g(x) = 0$ *and* $h(x) = c\, \mu(x)$ *for* $x \in I_\chi$, *where $c \in \mathbb{C}\backslash\{0\}$ is a constant, $\chi, \mu : S \to \mathbb{C}$ are two multiplicative functions where $\chi$ is nonzero, and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function such $\mu \neq \chi$, $\chi \circ \sigma = \chi$, $\mu \circ \sigma = \mu$ and $A \circ \sigma = A$.*

(c)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1 c_2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi A \end{pmatrix} \text{ on } S\backslash I_\chi,$$

*$f(x) = -c_1\,\mu(x)$, $g(x) = \frac{1}{2}\,\mu(x)$ and $h(x) = 0$ for $x \in I_\chi$, where $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ are two constants satisfying $1 + c_1 c_2^2 = 0$; $\mu, \chi : S \to \mathbb{C}$ are two multiplicative functions and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function such that $\chi \neq 0$, $\chi \neq \mu$, $\chi \circ \sigma = \chi$, $\mu \circ \sigma = \mu$ and $A \circ \sigma = A$.*

(d)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} c\,\beta & c\,(2-\beta) & -2\,c \\ \frac{1}{4}\beta & \frac{1}{4}\,(2-\beta) & \frac{1}{2} \\ \frac{1}{2\,\alpha} & -\frac{1}{2\,\alpha} & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix},$$

*where $\alpha, \beta, c \in \mathbb{C}\backslash\{0\}$ are three constants with $2\,c\,\alpha^2\,\beta\,(2-\beta) = 1$; $\chi_1, \chi_2, \chi_3 : S \to \mathbb{C}$ are three multiplicative functions such that $\chi_1 \neq \chi_2$, $\chi_1 \neq \chi_3$, $\chi_2 \neq \chi_3$, $\chi_1 \circ \sigma = \chi_1$, $\chi_2 \circ \sigma = \chi_2$ and $\chi_3 \circ \sigma = \chi_3$.*

(e)

$$\begin{cases} f = F, \\ g = -\frac{1}{2}\delta^2\,F + G + \delta\,H, \\ h = -\delta\,F + H, \end{cases}$$

*where $\delta \in \mathbb{C}$ is a constant and the functions $F, G, H : S \to \mathbb{C}$ are of the forms (a)-(d) with the same constraints.*

*Proof.* By using similar computations to the ones in the proof of [2, Section 3, Theorem]. □

**4.3.2. The case $h \circ \sigma \neq h$.** The following lemma (due to Stetkær) will be used later.

**Lemma 4.4.** *Let $A : S \to \mathbb{C}$ be an additive function and $\chi : S \to \mathbb{C}$ be a multiplicative function on a semigroup $S$.*

*If $\chi\,A = \Sigma_{j=1}^N c_j \chi_j$, where $c_j \in \mathbb{C}$ and $\chi_j : S \to \mathbb{C}$ is multiplicative for each $j = 1, 2, \ldots, N$, then $\chi\,A = 0$.*

*Proof.* It suffices to prove that $A = 0$ on the subsemigroup $\{x \in S \mid \chi(x) \neq 0\}$, so we may assume that $S = \{x \in S \mid \chi(x) \neq 0\}$. In that case we can divide by $\chi(x)$, so we may furthermore assume that $\chi = 1$. We can finally assume that $\{\chi_1, \chi_2, \ldots, \chi_N\}$ are different.

Let $y \in S$ be arbitrary. We shall show that $A(y) = 0$. The computation

$$A(y) = A(xy) - A(x) = \Sigma_{j=1}^{N} c_j[\chi_j(y) - 1]\chi_j(x) \textit{ for all } x \in S,$$

gives us that

$$-A(y) \cdot 1 + \Sigma_{j=1}^{N} c_j[\chi_j(y) - 1]\chi_j(x) = 0 \textit{ for all } x \in S.$$

If $\chi_j = 1$ for some $j$, then the corresponding term $c_j[\chi_j(y) - 1]\chi_j$ of the identity above vanishes, so the multiplicative function 1 does not occur in the sum. According to [7, Theorem 3.18] we obtain from the identity above that $A(y) = 0$. So, $y$ being arbitrary, we deduce that $\chi A = 0$. This completes the proof of Lemma 4.4. □

**Theorem 4.5.** *The solutions $f, g, h : S \to \mathbb{C}$ of the functional equation (1.1) with $f$ and $h$ linearly independent and $h \circ \sigma \neq h$ can be listed as follows:*

(a)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} \chi A_1 \\ \chi \\ \chi A \\ \chi A^2 \end{pmatrix} \textit{ on } S \backslash I_\chi,$$

$f(x) = g(x) = h(x) = 0 \quad$ *for* $\quad x \in I_\chi$,
*where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function; $A$, $A_1 : S \backslash I_\chi \to \mathbb{C}$ are two additive functions such that $\chi \circ \sigma = \chi$, $A \neq 0$, $A \circ \sigma = -A$ and $A_1 \circ \sigma = A_1$.*

(b)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -2\,\rho^2 & -2\,\rho^2 & 4\,\rho^2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \chi \circ \sigma \\ \mu \end{pmatrix},$$

*where $\rho \in \mathbb{C} \backslash \{0\}$ is a constant; $\chi, \mu : S \to \mathbb{C}$ are two multiplicative functions satisfying $\chi \circ \sigma \neq \chi$ and $\mu \circ \sigma = \mu$.*

(c)

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \rho^2 & -\frac{1}{2}\,\rho^2 & -\frac{1}{2}\,\rho^2 \\ 0 & 1 & 0 \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \mu \\ \mu \circ \sigma \end{pmatrix},$$

*where $\rho \in \mathbb{C} \backslash \{0\}$ is a constant; $\chi, \mu : S \to \mathbb{C}$ are two multiplicative functions such that $\chi \neq \mu$, $\chi \circ \sigma = \chi$ and $\mu \circ \sigma \neq \mu$.*

(d)

$$\begin{cases} f = F_0, \\ g = -\frac{1}{2}\delta^2\, F_0 + G_0 + \delta\, H_0, \\ h = -\delta\, F_0 + H_0, \end{cases}$$

*where $\delta \in \mathbb{C}$ is a constant and the functions $F_0, G_0, H_0 : S \to \mathbb{C}$ are of the forms (a)–(c) with the same constraints.*

*Proof.* Let $f, g, h : S \to \mathbb{C}$ satisfy the functional equation (1.1) so that $f$ and $h$ are linearly independent and $h \circ \sigma \neq h$. From the identity (4.2) and the fact that $h_o \neq 0$ we deduce that there exists a constant $\gamma \in \mathbb{C}$ such that

$$h_e = \gamma f, \tag{4.5}$$

hence $h_o = h - \gamma f$. As $f$ and $h$ are linearly independent we have $f \neq 0$, so we deduce from (4.2) that

$$g_o = -\gamma h_o. \tag{4.6}$$

We recall that $f \circ \sigma = f$ by Lemma 4.1(d). We split the discussion into the cases $\gamma = 0$ or $\gamma \neq 0$.

<u>Case A</u>: $\gamma = 0$. Then $h_e = 0$ and $g_o = 0$; hence $h \circ \sigma = -h$ and $g \circ \sigma = g$. So the functional equation (1.1) can be written as

$$f(xy) = f(x)g(y) + g(x)f(y) + k(x)k(y), \qquad x, y \in S, \tag{4.7}$$

where $k = i\,h$.

Using similar computations to the ones in the proof of [2, Section 3, Theorem] we have one of the following cases for the solutions $f$, $g$, $k$ of equation (4.7):

<u>Case A.1</u>:

$$\begin{pmatrix} f \\ g \\ k \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \chi A_1 \\ \chi \\ \chi\, A \\ \chi\, A^2 \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = g(x) = k(x) = 0 \quad \text{for} \quad x \in I_\chi$,

where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function and $A$, $A_1 : S \backslash I_\chi \to \mathbb{C}$ are two additive functions with $A \neq 0$. Then

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} \chi A_1 \\ \chi \\ \chi\, A \\ \chi\, A^2 \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = g(x) = h(x) = 0 \quad \text{for} \quad x \in I_\chi$.

Since $f \circ \sigma = f$, $g \circ \sigma = g$ and $h \circ \sigma = -h$, we get that $\chi \circ \sigma = \chi$, $(\chi\,(A_1 + A^2)) \circ \sigma = \chi\,(A_1 + A^2)$ and $(\chi\, A) \circ \sigma = -\chi\, A$, then $A \circ \sigma = -A$ and $A_1 \circ \sigma = A_1$. So we obtain a solution of the functional equation (1.1) of the form (a) in Theorem 4.5.

<u>Case A.2</u>:

$$\begin{pmatrix} f \\ g \\ k \end{pmatrix} = \begin{pmatrix} c^2 & -c^2 & -c \\ 0 & 1 & 0 \\ c & -c & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = c^2 \mu(x)$, $g(x) = 0$ and $k(x) = c\mu(x) \quad \text{for} \quad x \in I_\chi$, where $c \in \mathbb{C}\backslash\{0\}$ is a constant; $\chi \neq 0, \mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function. Then

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} c^2 & -c^2 & -c \\ 0 & 1 & 0 \\ -i\,c & i\,c & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\,A \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$f(x) = c^2\,\mu(x)$, $g(x) = 0$ and $h(x) = -i\,c\mu(x)$ for $x \in I_\chi$.
Notice that we can write $g(x) = \chi(x)$ and $h(x) = -i\,c\,(\mu(x) - \chi(x))$ for all $x \in S$. Since $g \circ \sigma = g$ and $h \circ \sigma = -h$, we get that $\chi \circ \sigma = \chi$ and $(\mu - \chi) \circ \sigma = -\mu + \chi$, then $\mu \circ \sigma + \mu = 2\chi$. According to [7, Corollary 3.19] either $\mu \circ \sigma = \chi$ or $\mu = \chi$, then $\mu = \chi$, hence $h = 0$. This contradicts the linear independence of $f$ and $h$. So the functional equation (1.1) has no solution in this case.
Case A.3:

$$\begin{pmatrix} f \\ g \\ k \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1c_2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\,A \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$f(x) = -c_1\,\mu(x)$, $g(x) = \frac{1}{2}\,\mu(x)$ and $k(x) = 0$ for $x \in I_\chi$,
where $c_1,\, c_2 \in \mathbb{C}\backslash\{0\}$ are two constants satisfying $1 + c_1c_2^2 = 0$; $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function. Then

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1c_2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}c_2 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\,A \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$f(x) = -c_1\,\mu(x)$, $g(x) = \frac{1}{2}\,\mu(x)$ and $h(x) = 0$ for $x \in I_\chi$.
We have $\sigma(S\backslash I_\chi) = S\backslash I_\chi$. Indeed, if there exists an element $x \in S\backslash I_\chi$ such that $\sigma(x) \in I_\chi$, then $h(x) = -i\,\chi(x)\,A(x)$ and $h(\sigma(x)) = 0$. Since $h(\sigma(x)) = -h(x)$ and $\chi(x) \neq 0$, we get that $A(x) = 0$. We infer from $f \circ \sigma = f$ and $g \circ \sigma = g$ that $-\mu(x) + \chi(x) = -\mu \circ \sigma(x)$ and $\mu(x) + \chi(x) = \mu \circ \sigma(x)$. So $\chi(x) = 0$, which contradicts that $x \in S\backslash I_\chi$. Hence $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$, then $\sigma^2(S\backslash I_\chi) \subseteq \sigma(S\backslash I_\chi)$. As $\sigma$ is involutive we get the converse inclusion. So $\sigma(S\backslash I_\chi) = S\backslash I_\chi$.
On the other hand, as $f \circ \sigma = f$, $g \circ \sigma = g$ and $h \circ \sigma = -h$, we obtain on $S\backslash I_\chi$:

$$-\mu \circ \sigma + \chi \circ \sigma - c_2\,\chi \circ \sigma\,A \circ \sigma = -\mu + \chi - c_2\,\chi\,A, \tag{4.8}$$

$$\mu \circ \sigma + \chi \circ \sigma - c_2\,\chi \circ \sigma\,A \circ \sigma = \mu + \chi - c_2\,\chi\,A, \tag{4.9}$$

$$\chi \circ \sigma\,A \circ \sigma = -\chi\,A. \tag{4.10}$$

Subtracting (4.9) and (4.8) we get that $\mu \circ \sigma = \mu$. Replacing $\mu \circ \sigma$ by $\mu$ in (4.9) and taking (4.10) into account we get that $\chi \circ \sigma - \chi = -2c_2\,\chi\,A$. Since $c_2 \neq 0$ and $\chi(x) \neq 0$ for all $x \in S\backslash I_\chi$, we get, according to Lemma 4.4, that $A = 0$, contradicting that $A$ is nonzero in Case A.3. So the functional equation (1.1) has no solution in this case.

Case A.4:

$$\begin{pmatrix} f \\ g \\ k \end{pmatrix} = \begin{pmatrix} c\,\beta & c\,(2-\beta) & -2\,c \\ \frac{1}{4}\beta & \frac{1}{4}\,(2-\beta) & \frac{1}{2} \\ \frac{1}{2\,\alpha_1} & \frac{-1}{2\,\alpha_1} & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \mu \end{pmatrix},$$

where $\alpha_1, \beta, c \in \mathbb{C}\backslash\{0\}$ are three constants with $2\,c\,\alpha_1^2\,\beta\,(2-\beta) = 1$; $\chi_1$, $\chi_2$, $\mu : S \to \mathbb{C}$ are three multiplicative functions satisfying $\chi_1 \neq \chi_2$, $\chi_1 \neq \mu$ and $\chi_2 \neq \mu$. Then

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} c\,\beta & c\,(2-\beta) & -2\,c \\ \frac{1}{4}\beta & \frac{1}{4}\,(2-\beta) & \frac{1}{2} \\ \frac{1}{2\,\alpha} & \frac{-1}{2\,\alpha} & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \mu \end{pmatrix},$$

where $\alpha := i\alpha_1, \beta, c \in \mathbb{C}\backslash\{0\}$ are three constants with $2\,c\,\alpha^2\,\beta\,(2-\beta) = -1$. Since $h \circ \sigma = -h$, we get that $\chi_1 \circ \sigma - \chi_2 \circ \sigma = -\chi_1 + \chi_2$, then $\chi_1 \circ \sigma + \chi_1 = \chi_2 \circ \sigma + \chi_2$. According to [7, Corollary 3.19] and taking into account that $\chi_1 \neq \chi_2$ we get that $\chi_1 \circ \sigma = \chi_2$. Since $f \circ \sigma = f$ and $g \circ \sigma = g$, we get that

$$\beta\,\chi_1 \circ \sigma + (2-\beta)\,\chi_2 \circ \sigma - 2\,\mu \circ \sigma = \beta\,\chi_1 + (2-\beta)\,\chi_2 - 2\,\mu$$
$$\beta\,\chi_1 \circ \sigma + (2-\beta)\,\chi_2 \circ \sigma + 2\,\mu \circ \sigma = \beta\,\chi_1 + (2-\beta)\,\chi_2 + 2\,\mu.$$

Subtracting these identities we get that $\mu \circ \sigma = \mu$. So $\beta\,\chi_2 + (2-\beta)\,\chi_1 = \beta\,\chi_1 + (2-\beta)\,\chi_2$, then $(1-\beta)\,\chi_1 = (1-\beta)\,\chi_2$. Since $\chi_1 \neq \chi_2$, we get that $\beta = 1$. So $2\,c\,\alpha^2 = -1$. By putting $\rho = \frac{1}{2\,\alpha}$ we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -2\,\rho^2 & -2\,\rho^2 & 4\,\rho^2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \chi \circ \sigma \\ \mu \end{pmatrix},$$

where $\rho \in \mathbb{C}\backslash\{0\}$ is a constant; $\chi$, $\mu : S \to \mathbb{C}$ are two multiplicative functions satisfying $\chi \circ \sigma \neq \chi$ and $\mu \circ \sigma = \mu$. As $f$ and $h$ are assumed to be linearly independent, we get that $\mu \neq 0$. So we obtain a solution of the functional equation (1.1) of the form (b) in Theorem 4.5.

Case A.5:

$$\begin{cases} f = F_1 \\ g = -\frac{1}{2}\delta^2\,F_1 + G_1 + \delta\,H_1 \\ k = -\delta\,F_1 + H_1 \end{cases}$$

where the functions $F_1, G_1, H_1 : S \to \mathbb{C}$ are of the forms in Cases A.1-A.4 and $\delta \in \mathbb{C}$ is a constant.

So

$$(I) \begin{cases} f = F_1 \\ g = -\frac{1}{2}\delta^2\,F_1 + G_1 + \delta\,H_1 \\ h = \delta i\,F_1 - iH_1. \end{cases}$$

The conditions $f \circ \sigma = f$, $g \circ \sigma = g$ and $h \circ \sigma = -h$ imply

$$(II)\begin{cases} F_1 \circ \sigma & = F_1 \\ (G_1 + \delta\, H_1) \circ \sigma = G_1 + \delta\, H_1 \\ H_1 \circ \sigma + H_1 & = 2\delta\, F_1. \end{cases}$$

Since $f$ and $h$ are linearly independent, so are $F_1$ and $H_1$. Then $H_1 \circ \sigma \neq H_1$. We have the following cases

Case A.5.1:

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \chi\, A_1 \\ \chi \\ \chi\, A \\ \chi\, A^2 \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$F_1(x) = G_1(x) = H_1(x) = 0, \quad \text{for} \quad x \in I_\chi,$
where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function and $A$, $A_1 : S\backslash I_\chi \to \mathbb{C}$ are two additive functions with $A \neq 0$.

We have $\sigma(S\backslash I_\chi) = S\backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$. Indeed, if there exists an element $x \in S\backslash I_\chi$ such that $\sigma(x) \in I_\chi$, then the first identity of $(II)$ implies $\chi(x)(A_1(x) + A^2(x)) = 0$. Since $\chi(x) \neq 0$, we get that $A^2(x) = -A_1(x)$. As $x^2 \in S\backslash I_\chi$ and $x^2 \in I_\chi$, we have similarly $A^2(x^2) = -A_1(x^2)$. Since the functions $A$, $A_1 : S\backslash I_\chi \to \mathbb{C}$ are additive, we get that $4\, A^2(x) = -2\, A_1(x) = 2\, A^2(x)$, which implies $A(x) = 0$, so $H_1(x) = 0$. As $\sigma(x) \in I_\chi$ we have $H_1(\sigma(x)) = 0$. From the second identity of $(II)$ we get that $G_1(x) = G_1(\sigma(x)) = 0$. Considering the formula of $G_1$ in the present case we get that $\chi(x) = 0$, which contradicts the assumption $x \in S\backslash I_\chi$. We deduce that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$. So $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

It follows that the second identity of $(II)$ becomes $(\chi\circ\sigma)^{-1}\,\chi\,(1+\delta\, A) = 1+\delta\,(A\circ\sigma)$. Then the function $m\,(1+a)-1$, defined from $S\backslash I_\chi$ into $\mathbb{C}$ is additive, where $m := (\chi \circ \sigma)^{-1}\,\chi : S\backslash I_\chi \to \mathbb{C}$ is multiplicative and $a := \delta\, A : S\backslash I_\chi \to \mathbb{C}$ is additive. Then $m(x^2)\,(1+a(x^2))-1 = 2\, m(x)\,(1+a(x))-2$ for all $x \in S\backslash I_\chi$, which implies $(m(x) - 1)\, m(x)\,(1 + 2\, a(x)) = m(x) - 1$ for all $x \in S\backslash I_\chi$. So $m(x) = 1$ for all $x \in S\backslash I_\chi$. Indeed, if not, there exists an element $x \in S\backslash I_\chi$ such that $m(x) \neq 1$ and $m(x^2) \neq 1$ because $S$ is generated by its squares. Then $m(x)\,(1+2\, a(x)) = 1$, which implies $2\, a(x) = (m(x))^{-1} - 1$. Similarly we have $2\, a(x^2) = (m(x^2))^{-1} - 1$. Using that $a$ is additive and $m$ is multiplicative we get that $4\, a(x) = (m(x))^{-2} - 1$. Then $(m(x))^{-2} - 1 = 2\,(m(x))^{-1} - 2$. It follows that $(m(x))^{-1} = 1$, which contradicts the assumption $m(x) \neq 1$. Hence $\chi \circ \sigma = \chi$. Since $G_1 = \chi$ on $S\backslash I_\chi$, we derive from the second identity of $(II)$ that $\delta\,(H_1 \circ \sigma - H_1) = 0$ on $S\backslash I_\chi$. As $H_1 = 0$ on $I_\chi$ and $\sigma(I_\chi) = I_\chi$ we get that $\delta\,(H_1 \circ \sigma - H_1) = 0$ on the semigroup $S$. Taking into account that $H_1$ and $F_1$ are linearly independent and satisfy the third identity of $(II)$ we get that $H_1 \circ \sigma \neq H_1$. So $\delta = 0$ and the solution $(f, g, h)$ of (1.1) is of the form in Case A.1 and fits into form (a) in Theorem 4.5. As in Case A.1 we derive immediately that $A\circ\sigma=-A$ and $A_1\circ\sigma=A_1$.

Case A.5.2:

$$\begin{pmatrix}F_1\\G_1\\H_1\end{pmatrix}=\begin{pmatrix}c^2 & -c^2 & -c\\0 & 1 & 0\\c & -c & 0\end{pmatrix}\begin{pmatrix}\mu\\\chi\\\chi A\end{pmatrix}\text{ on } S\backslash I_\chi,$$

$F_1(x)=c^2\,\mu(x)$, $G_1(x)=0$ and $H_1(x)=c\,\mu(x)$, for $x\in I_\chi$, where $c\in\mathbb{C}\backslash\{0\}$ is a constant, $\chi\neq 0$, $\mu:S\to\mathbb{C}$ are two multiplicative functions such that $\mu\neq\chi$ and $A:S\backslash I_\chi\to\mathbb{C}$ is a nonzero additive function.

We have $\sigma(S\backslash I_\chi)=S\backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S\backslash I_\chi)\subseteq S\backslash I_\chi$. Indeed, if there exists an element $x\in S\backslash I_\chi$ such that $\sigma(x)\in I_\chi$, the first identity in $(II)$ becomes $c^2\,\mu(x)-c^2\,\chi(x)-c\,\chi(x)\,A(x)=c^2\,\mu(\sigma(x))$, which implies $c\,(\mu(\sigma(x))-\mu(x))=-\chi(x)\,(c+A(x))$. The second identity in $(II)$ implies $\delta\,c\,\mu(\sigma(x))=\chi(x)+\delta\,c\,(\mu(x)-\chi(x))$, then $\delta\,c\,(\mu(\sigma(x))-\mu(x))=(1-\delta\,c)\,\chi(x)$. Hence $(1-\delta\,c)\,\chi(x)=-\delta\,\chi(x)\,(c+A(x))$, from which we get that $\delta\,A(x)=-1$. As $x^2\in S\backslash I_\chi$ and $x^2\in I_\chi$ we have, similarly, $\delta\,A(x^2)=-1$, then $2\,\delta\,A(x)=-1$, which is a contradiction. We deduce that $\sigma(S\backslash I_\chi)\subseteq S\backslash I_\chi$. So $\sigma(S\backslash I_\chi)=S\backslash I_\chi$ and $\sigma(I_\chi)=I_\chi$.

On the other hand, according to the result of Case A.2 we assume that $\delta\neq 0$. On $S\backslash I_\chi$ the second identity of $(II)$ implies $(\chi+\delta\,c\,(\mu-\chi))\circ\sigma=\chi+\delta\,c\,(\mu-\chi)$, from which we get that

$$(1-\delta\,c)\chi\circ\sigma-(1-\delta\,c)\,\chi-\delta\,c\,\mu+\delta\,c\,\mu\circ\sigma=0. \tag{4.11}$$

On $I_\chi$ we have $\chi\circ\sigma=\chi=0$ and since $F_1\circ\sigma=F_1$, we have $\mu\circ\sigma=\mu$, so the identity (4.11) is satisfied on the semigroup $S$. Since $\delta\,c\neq 0$, we derive, according to [7, Theorem 3.18], that the multiplicative functions $\chi$, $\mu$, $\chi\circ\sigma$ and $\mu\circ\sigma$ are not different. As $\chi\neq\mu$ and $\chi\circ\sigma\neq\mu\circ\sigma$ we have the following cases

Case A.5.2.1: $\chi\circ\sigma=\chi$. The identity (4.11) implies $\delta\,c\,(\mu\circ\sigma-\mu)=0$, then $\mu\circ\sigma=\mu$. So $H_1\circ\sigma=H_1$. Applying the third identity of $(II)$ we get that $H_1=\delta\,F_1$, which contradicts the linear independence of $F_1$ and $H_1$.

Case A.5.2.2: $\chi\circ\sigma=\mu$. Then $\mu\circ\sigma=\chi$. So $H_1\circ\sigma=-H_1$ on $S\backslash I_\chi$. From the third identity of $(II)$ we get that $F_1=0$ on $S\backslash I_\chi$. For all $x\in I_\chi$ we have $F_1(x)=c^2\,\mu(x)=c^2\,\chi(\sigma(x))=0$ because $\sigma(I_\chi)=I_\chi$. Hence $f(x)=F_1(x)=0$ for all $x\in S$, which contradicts the linear independence of $f$ and $h$.

Case A.5.2.3: $\mu\circ\sigma=\mu$. In this case the first identity of $(II)$ implies $c\,\chi\circ\sigma+\chi\circ\sigma\,A\circ\sigma=c\,\chi+\chi\,A$ on $S\backslash I_\chi$. Then $\chi\circ\sigma\,(c+A\circ\sigma)=\chi\,(c+A)$. So $1+b=m\,(1+a)$ where $m:=(\chi\circ\sigma)^{-1}\,\chi:S\backslash I_\chi\to\mathbb{C}$ is a multiplicative function, and $a:=c^{-1}\,A:S\backslash I_\chi\to\mathbb{C}$ and $b:=c^{-1}\,A\circ\sigma:S\backslash I_\chi\to\mathbb{C}$ are two additive functions. Proceeding exactly as in Case A.5.1 we derive that $\chi\circ\sigma=\chi$ on $S\backslash I_\chi$. As $\chi\circ\sigma=\chi=0$ on $I_\chi$ we get that $\chi\circ\sigma=\chi$ on the semigroup $S$. So we go back to Case A.5.2.1.

We conclude that the functional equation (1.1) has no solution in Case A.5.2.

Case A.5.3:

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1 c_2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$F_1(x) = -c_1\, \mu(x)$, $G_1(x) = \frac{1}{2}\, \mu(x)$ and $H_1(x) = 0$ for $\quad x \in I_\chi$,
where $c_1$, $c_2 \in \mathbb{C}\backslash\{0\}$ are two constants such that $1 + c_1 c_2^2 = 0$, $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function.

We split the discussion into the cases $\delta \neq 0$ and $\delta = 0$.

Case A.5.3.1: $\delta = 0$. In this case we get

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1 c_2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} c_2 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = -c_1\, \mu(x)$, $g(x) = \frac{1}{2}\, \mu(x)$ and $h(x) = 0$ for $\quad x \in I_\chi$,
where $c_1$, $c_2 \in \mathbb{C}\backslash\{0\}$, $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions and $A : S\backslash I_\chi \to \mathbb{C}$ is an additive function satisfying the same assumptions as the ones above. This is Case A.3. As seen earlier this case has no solution.

Case A.5.3.2: $\delta \neq 0$. In this case $\sigma(S\backslash I_\chi) = S\backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$. Indeed, if there exists an element $x \in S\backslash I_\chi$ such that $\sigma(x) \in I_\chi$ then we get that $H_1(x) = \chi(x)\, A(x)$, $H_1(\sigma(x)) = 0$ and $F_1(\sigma(x)) = -c_1\, \mu(\sigma(x))$. Using the first and the third identities of $(II)$ we obtain

$$\chi(x)\, A(x) = -2\, \delta\, c_1\, \mu(\sigma(x)).$$

Since $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$, we have similarly $\chi(x^2)\, A(x^2) = -2\, \delta\, c_1$ $\mu(\sigma(x^2))$. Then $2\, (\chi(x))^2\, A(x) = -2\, \delta\, c_1 (\mu(\sigma(x)))^2$. Hence

$$\delta\, c_1\, (\mu(\sigma(x)))^2 = 2\, \delta\, c_1\, \chi(x)\, \mu(\sigma(x)),$$

which implies

$$\delta\, c_1\, \mu(\sigma(x))\, [\mu(\sigma(x)) - 2\, \chi(x)] = 0.$$

Notice that $c_1 \neq 0$ and $\delta \neq 0$.

If $\mu(\sigma(x)) = 0$ then $F_1(\sigma(x)) = -c_1\, \mu(\sigma(x)) = 0$ because $\sigma(x) \in I_\chi$. Using the first identity in $(II)$ we get that $-c_1\, \mu(x) + c_1\, \chi(x) - c_1\, c_2\, \chi(x)\, A(x) = 0$. As $c_1 \neq 0$ and $\chi(x)\, A(x) = -2\, \delta\, c_1\, \mu(\sigma(x)) = 0$ we find that $\chi(x) = \mu(x)$. Since $x \in S\backslash I_\chi$, we have $G_1(x) = \frac{1}{2}\, [\mu(x) + \chi(x) - c_2\, \chi(x)\, A(x)] = \mu(x) = \chi(x)$. Moreover from $H_1(x) = 0$, $H_1(\sigma(x)) = 0$ and $(G_1 + \delta\, H_1) \circ \sigma = G_1 + \delta\, H_1$ we find that $G_1(\sigma(x)) = G_1(x)$. As $\sigma(x) \in I_\chi$ we have $G_1(\sigma(x)) = \frac{1}{2}\, \mu(\sigma(x))$. Hence $\chi(x) = \mu(x) = \frac{1}{2}\, \mu(\sigma(x)) = 0$, which contradicts the assumption $x \in S\backslash I_\chi$.

Hence $\mu(\sigma(x)) \neq 0$. Here we get that $\mu(\sigma(x)) = 2\, \chi(x)$. As $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$, and $\mu(\sigma(x^2)) \neq 0$ because $\mu(\sigma(x^2)) = (\mu(\sigma(x)))^2$, we obtain similarly $(\mu(\sigma(x)))^2 = 2\,(\chi(x))^2$. Then $4\,(\chi(x))^2 = 2\,(\chi(x))^2$, so $\chi(x) = 0$, which contradicts the assumption $x \in S\backslash I_\chi$. We conclude that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$, so $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

On the other hand, from the first and the second identities of $(II)$ we have, respectively, on $S\backslash I_\chi$ the two identities below

$$\mu\circ\sigma - \mu = \chi\circ\sigma - \chi + c_2\,\chi\,A - c_2\,\chi\circ\sigma\,A\circ\sigma.$$
$$\mu - \mu\circ\sigma = 2\,\delta\,[\chi\circ\sigma\,A\circ\sigma - \chi\,A] + \chi\circ\sigma - \chi + c_2\,\chi\,A - c_2\,\chi\circ\sigma\,A\circ\sigma.$$

It follows that

$$\mu - \mu\circ\,\sigma = \delta\,[\chi\circ\sigma\,A\circ\sigma - \chi\,A]. \tag{4.12}$$

When we substitute this back into the first identity above we get that

$$\delta\,(\mu - \mu\circ\sigma + \chi\circ\sigma - \chi) = c_2\,(\mu - \mu\circ\sigma),$$

hence

$$(c_2 - \delta)\,\mu\circ\sigma + (\delta - c_2)\,\mu + \delta\,\chi\circ\sigma - \delta\,\chi = 0. \tag{4.13}$$

Moreover on $I_\chi$ we have $\chi\circ\sigma = \chi = 0$, and $\mu\circ\sigma = \mu$ because $F_1\circ\sigma = F_1$, so the identity (4.13) is satisfied on the semigroup $S$. As $\delta \neq 0$ in the present case we conclude, by [7, Theorem 3.18], that the multiplicative functions $\mu\circ\sigma$, $\mu$, $\chi\circ\sigma$ and $\chi$ are not different. Since $F_1$ and $H_1$ are linearly independent, we infer from the third identity of $(II)$ that $H_1\circ\sigma \neq H_1$. Then $\chi\circ\sigma\,A\circ\sigma \neq \chi\,A$. So from (4.12) we get that $\mu\circ\sigma \neq \mu$. Since $\mu \neq \chi$, we have the following cases:

Case A.5.3.2.1: $\mu\circ\sigma = \chi$. Here the identity (4.13) implies $(c_2 - 2\,\delta)\,(\chi - \mu) = 0$. Since $\chi \neq \mu$, we get that $c_2 = 2\,\delta$. So the third identity of $(II)$ becomes, on $S\backslash I_\chi$,

$$\chi\circ\sigma\,A\circ\sigma + \chi\,A = c_1\,c_2(-\mu + \chi - c_2\,\chi\,A),$$

which implies

$$\chi\circ\sigma\,A\circ\sigma + (1 + c_1\,c_2^2)\chi\,A = c_1\,c_2(\chi - \mu).$$

As $1 + c_1\,c_2^2 = 0$ we get that $A\circ\sigma = \dfrac{1}{c_2}\,(\chi\circ\sigma)^{-1}\,\mu - \dfrac{1}{c_2}\,(\chi\circ\sigma)^{-1}\chi$. By applying Lemma 4.4 on the subsemigroup $S\backslash I_\chi$ and taking into account that $\sigma : S \rightarrow S$ is an involution and that $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ we get $A = 0$, which contradicts that $A$ is nonzero in Case A.5.3.

Case A.5.3.2.2: $\chi\circ\sigma = \chi$. Then the identity (4.12) implies $\chi^{-1}\,\mu - \chi^{-1}\,\mu\circ\sigma = \delta\,(A\circ\sigma - A)$. Since $A\circ\sigma - A : S\backslash I_\chi \rightarrow \mathbb{C}$ is an additive function, we proceed like in Case A.5.3.2.1 above and get $A\circ\sigma = A$. So $H_1\circ\sigma = H_1$ on $S\backslash I_\chi$. Since $H_1\circ\sigma = H_1 = 0$ on $I_\chi$, we get that $H_1\circ\sigma = H_1$. Then the third identity of $(II)$ implies $H_1 = \delta\,F_1$, which contradicts the linear independence of $H_1$ and $F_1$.

We conclude that the functional equation (1.1) has no solution if $F_1$, $G_1$ and $H_1$ are of the form in Case A.5.3.

Case A.5.4:

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} c\,\beta & c\,(2-\beta) & -2\,c \\ \frac{1}{4}\beta & \frac{1}{4}\,(2-\beta) & \frac{1}{2} \\ \frac{1}{2\,\alpha} & -\frac{1}{2\,\alpha} & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix},$$

where $\alpha,\ \beta,\ c \in \mathbb{C}\backslash\{0\}$ are three constants with $2\,c\,\alpha^2\,\beta\,(2-\beta) = 1$; $\chi_1,\ \chi_2,\ \chi_3 : S \to \mathbb{C}$ are three multiplicative functions such that $\chi_1 \neq \chi_2$, $\chi_1 \neq \chi_3$ and $\chi_2 \neq \chi_3$.

As in Case A.5.3 we split the discussion into the cases $\delta \neq 0$ and $\delta = 0$.
Case A.5.4.1: $\delta = 0$. Here $h \circ \sigma = -h$ and $g \circ \sigma = g$. So we go back to Case A.4 and the solution occurs in (b) of the list of Theorem 4.5.
Case A.5.4.2: $\delta \neq 0$. The third and the first identities of $(II)$ imply, respectively,

$$\chi_1 \circ \sigma - \chi_2 \circ \sigma + (1 - 4\,c\,\alpha\,\beta\,\delta)\chi_1 - (1 + 4\,c\,\alpha\,(2-\beta)\,\delta)\,\chi_2 + 8\,c\,\alpha\,\delta\,\chi_3 = 0 \quad (4.14)$$

and

$$\beta\,\chi_1 \circ \sigma + (2-\beta)\,\chi_2 \circ \sigma - 2\,\chi_3 \circ \sigma - \beta\,\chi_1 - (2-\beta)\,\chi_2 + 2\,\chi_3 = 0. \quad (4.15)$$

According to [7, Theorem 3.18] we derive from (4.15) that the multiplicative functions $\chi_1$, $\chi_2$, $\chi_3$, $\chi_1 \circ \sigma$, $\chi_2 \circ \sigma$ and $\chi_3 \circ \sigma$ are not different. As $\chi_1$, $\chi_2$ and $\chi_3$ are different, so are $\chi_1 \circ \sigma$, $\chi_2 \circ \sigma$ and $\chi_3 \circ \sigma$. Since $\delta \neq 0$ and $F_1 \neq 0$, we derive from the third identity of $(II)$ that $H_1 \circ \sigma \neq -H_1$, hence $\chi_1 \circ \sigma \neq \chi_2$. Moreover $\chi_3 \circ \sigma \neq \chi_3$. Indeed, if $\chi_3 \circ \sigma = \chi_3$ the identity (4.15) implies

$$\beta\,\chi_1 \circ \sigma + (2-\beta)\,\chi_2 \circ \sigma - \beta\,\chi_1 - (2-\beta)\,\chi_2 = 0, \quad (4.16)$$

and as $\chi_1 \neq \chi_2$ and $\chi_1 \circ \sigma \neq \chi_2$ we deduce from the identity (4.16), according to [7, Theorem 3.18], that $\chi_1 \circ \sigma = \chi_1$ and $\chi_2 \circ \sigma = \chi_2$ because $\beta\,(2-\beta) \neq 0$. Considering the formula for $H_1$ of the present case we obtain $H_1 \circ \sigma = H_1$. Using the third equality of $(II)$ we get $H_1 = \delta\,F_1$, which contradicts the linear independence of $H_1$ and $F_1$. Then we have the following possibilities:
Case A.5.4.2.1: $\chi_1 \circ \sigma = \chi_1$. In this case the identity (4.15) becomes

$$(2-\beta)\,\chi_2 \circ \sigma - 2\,\chi_3 \circ \sigma - (2-\beta)\,\chi_2 + 2\,\chi_3 = 0. \quad (4.17)$$

On the other hand since $F_1$ and $H_1$ are linearly independent, we get from the third identity of $(II)$ that $H_1 \circ \sigma \neq H_1$, then $\chi_1 \circ \sigma - \chi_2 \circ \sigma \neq \chi_1 - \chi_2$. Since $\chi_1 \circ \sigma = \chi_1$, we get that $\chi_2 \circ \sigma \neq \chi_2$. As $\beta \neq 2$ we derive from (4.17) that $\chi_2 \circ \sigma = \chi_3$. So (4.17) becomes $(4-\beta)\,\chi_3 - \chi_2 = 0$. As $\chi_3 \neq \chi_2$ we deduce that $\beta = 4$. Since $2\,c\,\alpha^2\,\beta\,(2-\beta) = 1$, we get that $4\,\alpha^2 = -\frac{1}{4\,c}$. With $\beta = 4$ the identity (4.14) implies

$$2\,(1 - 8\,c\,\alpha\,\delta)\,\chi_1 - (1 - 8\,c\,\alpha\,\delta)\,\chi_2 - (1 - 8\,c\,\alpha\,\delta)\,\chi_3 = 0.$$

Since the multiplicative functions $\chi_1$, $\chi_2$ and $\chi_3$ are different, we get, according to [7, Theorem 3.18], that $1 - 8\,c\,\alpha\,\delta = 0$. So $\delta = -2\,\alpha$, which implies $\delta^2 =$

$4\,\alpha^2=-\frac{1}{4\,c}$. Using $(I)$ and the expressions of $F_1$, $G_1$ and $H_1$ in term of $\chi_1$, $\chi_2$ and $\chi_3$ we get that

$$\begin{cases} f=4\,c\,\chi_1-2\,c\,\chi_2-2\,c\,\chi_3\\ g=\frac{1}{2}\,\chi_1+\frac{1}{4}\,\chi_2+\frac{1}{4}\,\chi_3\\ h=\frac{i}{4\,\alpha}\,\chi_2-\frac{i}{4\,\alpha}\,\chi_3. \end{cases}$$

Putting $\chi=\chi_2$, $\mu=\chi_1$ and $\rho=\frac{i}{4\,\alpha}$ we obtain

$$\begin{pmatrix} f\\ g\\ h \end{pmatrix}=\begin{pmatrix} -2\rho^2 & -2\rho^2 & 4\rho^2\\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\\ \rho & -\rho & 0 \end{pmatrix}\begin{pmatrix} \chi\\ \chi\circ\sigma\\ \mu \end{pmatrix},$$

which is a solution of the form (b) in Theorem 4.5.

Case A.5.4.2.2: $\chi_1\circ\sigma=\chi_3$. So $\chi_3\circ\sigma=\chi_1$ and the identity (4.15) becomes

$$(2+\beta)\,\chi_1+(2-\beta)\,\chi_2-(2+\beta)\,\chi_3-(2-\beta)\,\chi_2\circ\sigma=0. \tag{4.18}$$

The coefficients $2+\beta$ and $2-\beta$ can not be zero at the same time, and we conclude, by [7, Theorem 3.18], that the multiplicative functions $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_2\circ\sigma$ are not different. As $\chi_1$, $\chi_2$ and $\chi_3$ are different, $\chi_2\circ\sigma\neq\chi_1$ and $\chi_2\circ\sigma\neq\chi_3$ we derive that $\chi_2\circ\sigma=\chi_2$. Interchanging $\chi_1$ and $\chi_2$, and replacing $\beta$ by $2-\beta$, and $\alpha$ by $-\alpha$ we go back to Case A.5.4.2.1 and the solution occurs in (b) of the list of Theorem 4.5.

Case A.5.4.2.3: $\chi_2\circ\sigma=\chi_2$. Interchanging $\chi_1$ and $\chi_2$, and replacing $\beta$ by $2-\beta$, and $\alpha$ by $-\alpha$ we go back to Case A.5.4.2.1 and the solution occurs in (b) of the list of Theorem 4.5.

Case A.5.4.2.4: $\chi_2\circ\sigma=\chi_3$. So $\chi_3\circ\sigma=\chi_2$ and we get from (4.15) that

$$\beta\,(\chi_1\circ\sigma)-\beta\,\chi_1+(\beta-4)\,\chi_2-(\beta-4)\,\chi_3=0. \tag{4.19}$$

Since $\beta\neq 0$, we deduce from (4.19), according to [7, Theorem 3.18], that the multiplicative functions $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_1\circ\sigma$ are not different. As $\chi_1$, $\chi_2$, $\chi_3$ are different, $\chi_1\circ\sigma\neq\chi_2$ and $\chi_1\circ\sigma\neq\chi_3$ we get that $\chi_1\circ\sigma=\chi$. So we go back to Case A.5.4.2.1 and the solution occurs in (b) of the list of Theorem 4.5.

Case B: $\gamma\neq 0$. Here the functional equation (4.3) becomes

$$\begin{aligned} f(xy)&=f(x)g(y)+g(x)f(y)+h(x)h(y)-2(h(x)-\gamma f(x))(h(y)-\gamma f(y))\\ &=f(x)g(y)+g(x)f(y)+h(x)h(y)\\ &\quad -2[h(x)h(y)-\gamma h(x)f(y)-\gamma f(x)h(y)+\gamma^2 f(x)f(y)]\\ &=f(x)[g(y)-\gamma^2 f(y)+2\gamma h(y)]+f(y)[g(x)-\gamma^2 f(x)+2\gamma h(x)]\\ &\quad -h(x)h(y), \end{aligned}$$

for all $x,y\in S$, which implies

$$f(xy)=f(x)G(y)+G(x)f(y)+H(x)H(y),\qquad x,y\in S, \tag{4.20}$$

where

$$G := g - \gamma^2 f + 2\gamma h, \tag{4.21}$$
$$H := i h. \tag{4.22}$$

Since $f \circ \sigma = f$, according to Lemma 4.1(d), we get from (4.21) that

$$G \circ \sigma = g \circ \sigma - \gamma^2 f + 2\gamma h \circ \sigma,$$

then

$$\begin{aligned} G - G \circ \sigma &= g - g \circ \sigma + 2\gamma (h - h \circ \sigma) \\ &= g - g \circ \sigma - 2(g - g \circ \sigma) = -(g - g \circ \sigma). \end{aligned}$$

Using (4.6) we get that

$$(g + G) \circ \sigma = g + G. \tag{4.23}$$

On the other hand, by similar computations to the ones in the proof of [2, Section 3, Theorem] we find that we have one of the following cases for the solutions $f, G, H$ of the functional equation (4.20):
Case B.1:

$$\begin{pmatrix} f \\ G \\ H \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \chi A_1 \\ \chi \\ \chi A \\ \chi A^2 \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = G(x) = H(x) = 0 \quad \text{for} \quad x \in I_\chi,$
where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function and $A$, $A_1 : S \backslash I_\chi \to \mathbb{C}$ are two additive functions with $A \neq 0$. Using (4.21) and (4.22) we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2}\gamma^2 & 1 & 2\gamma i & \frac{1}{2}\gamma^2 \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} \chi A_1 \\ \chi \\ \chi A \\ \chi A^2 \end{pmatrix} \text{ on } S \backslash I_\chi.$$

$f(x) = g(x) = h(x) = 0 \quad \text{for} \quad x \in I_\chi.$
We have $\sigma(S \backslash I_\chi) = S \backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S \backslash I_\chi) \subseteq S \backslash I_\chi$. Indeed, if there exists an element $x \in S \backslash I_\chi$ such that $\sigma(x) \in I_\chi$; we obtain, according to (4.5), $g(x) + G(x) = g(\sigma(x)) + G(\sigma(x))$, hence $g(x) + \chi(x) = 0$. This implies $\chi(x) \left[\frac{1}{2}\gamma^2 A_1(x) + 1 + 2\gamma i A(x) + \frac{1}{2}\gamma^2 A^2(x)\right] = -\chi(x)$. As $\chi(x) \neq 0$ we get that

$$\frac{1}{2}\gamma^2 A_1(x) + 2\gamma i A(x) + \frac{1}{2}\gamma^2 A^2(x) = -2.$$

Since $x^2 \in S \backslash I_\chi$ and $\sigma(x^2) \in I_\chi$, we also have

$$\frac{1}{2}\gamma^2 A_1(x^2) + 2\gamma i A(x^2) + \frac{1}{2}\gamma^2 A^2(x^2) = -2,$$

then

$$\frac{1}{2}\gamma^2 A_1(x) + 2\gamma i\, A(x) + \gamma^2 A^2(x) = -1.$$

It follows that $\gamma^2 A^2(x) = 2$. As $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$ we have the same result for $x^2$, i.e $\gamma^2 A^2(x^2) = 2$. Using the additivity of $A$ we obtain $4\gamma^2 A^2(x) = 2$, which contradicts the fact that $\gamma^2 A^2(x) = 2$. We conclude that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$, so $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

On the other hand, since $f$ is even with respect to $\sigma$, we get that

$$\chi\, A_1 + \chi\, A^2 = \chi\circ\sigma\, A_1\circ\sigma + \chi\circ\sigma\,(A\circ\sigma)^2. \tag{4.24}$$

Using (4.23) we deduce

$$\frac{1}{2}\gamma^2(\chi\, A_1 + \chi\, A^2) + 2(\chi + \gamma i\,\chi\, A) = \frac{1}{2}\gamma^2(\chi\circ\sigma\, A_1\circ\sigma + \chi\circ\sigma(A\circ\sigma)^2) + 2(\chi\circ\sigma + \gamma i\,\chi\circ\sigma\, A\circ\sigma).$$

Taking (4.24) into account we obtain

$$\gamma i\,[\chi\, A - \chi\circ\sigma\, A\circ\sigma] = \chi\circ\sigma - \chi. \tag{4.25}$$

Furthermore (4.5) means that $h + h\circ\sigma = 2\gamma\, f$, so

$$-\gamma i\,[\chi\, A + \chi\circ\sigma\, A\circ\sigma] = \gamma^2\,[\chi\, A_1 + \chi\, A^2],$$

which we reformulate to

$$-\frac{i}{\gamma}\,[A(x) + \chi_1(x)\, A(\sigma(x))] = A_1(x) + A(x)^2 \quad \text{for} \quad x \in S\backslash I_\chi,$$

where $\chi_1 := \chi\circ\sigma/\chi$ on $S\backslash I_\chi$. Replacing $x$ by $x^n$, where $n = 1, 2, \ldots$, in the identity above, and dividing by $n^2$ we get

$$\frac{A(x)}{n} + \frac{\chi_1(x)^n}{n}\, A(\sigma(x)) = -\frac{\gamma}{i}\left[\frac{A_1(x)}{n} + A(x)^2\right]. \tag{4.26}$$

We derive a contradiction from (4.26) using the elementary fact from the theory of orders of growth that

$$|\frac{z^n}{n}| \to \begin{cases} \infty & \text{for} \quad |z| > 1 \\ 0 & \text{for} \quad |z| \leq 1. \end{cases}$$

Let $x = x_0 \in S\backslash I_\chi$ be arbitrary, but fixed in (4.26).

If $|\chi_1(x_0)| > 1$, then we let $n \to \infty$ and we get $|\frac{z^n}{n}| \to \infty$ by the above mentioned elementary fact. So the sequence $\frac{\chi_1(x)^n}{n}$ is unbounded, while the other terms of (4.26) are bounded. We deduce that $A(\sigma(x_0)) = 0$ in this case, which reduces the identity (4.26) to

$$\frac{A(x_0)}{n} = -\frac{\gamma}{i}\left[\frac{A_1(x_0)}{n} + A(x_0)^2\right].$$

Letting $n \to \infty$ we obtain that $A(x_0) = 0$.

If $|\chi_1(x_0)| \leq 1$, then we let $n \to \infty$ in (4.26) and we get $A(x_0) = 0$ by the above mentioned elementary fact.

Thus $A(x_0) = 0$ in both cases. So, $x_0$ being arbitrary, we deduce that $A = 0$, contradicting that $A$ is nonzero in Case B.1

We conclude that the functional equation (1.1) has no solution in Case B.1.

Case B.2:

$$\begin{pmatrix} f \\ G \\ H \end{pmatrix} = \begin{pmatrix} c^2 & -c^2 & -c \\ 0 & 1 & 0 \\ c & -c & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = c^2\, \mu(x)$, $G(x) = 0$ and $H(x) = c\, \mu(x)$, for $x \in I_\chi$,
where $c \in \mathbb{C}\backslash\{0\}$ is a constant, $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function. Using (4.21) and (4.22) we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} c^2 & -c^2 & -c \\ \gamma\, c\, (\gamma\, c + 2i) & (\gamma\, ci - 1)^2 & -\gamma^2\, c \\ -ic & ic & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$f(x) = c^2\, \mu(x)$, $g(x) = \gamma\, c\, (\gamma\, c + 2i)\, \mu(x)$ and $h(x) = -ic\, \mu(x)$ for $x \in I_\chi$.
We have $\sigma(S\backslash I_\chi) = S\backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$. Indeed, if there exists $x \in S\backslash I_\chi$ such that $\sigma(x) \in I_\chi$, then $c\, \mu(x) - c\, \chi(x) - \chi(x)\, A(x) = c\, \mu(\sigma(x))$ because $f \circ \sigma = f$. Hence

$$\chi(x)\, A(x) = c\, [\mu(x) - \chi(x) - \mu(\sigma(x))]. \tag{4.27}$$

By using (4.23) we get that

$$\gamma c(\gamma c + 2i)\, \mu(\sigma(x)) = \gamma c(\gamma c + 2i)\, \mu(x) + [(\gamma ci - 1)^2 + 1]\, \chi(x) - \gamma^2 c\, \chi(x)\, A(x).$$

Taking (4.27) into account we derive

$$\begin{aligned} \gamma^2 c\, \chi(x)\, A(x) &= -\gamma c(\gamma c + 2i)\, \mu(\sigma(x)) + \gamma c(\gamma c + 2i)\, \mu(x) \\ &\quad + (-\gamma^2 c^2 - 2\gamma ci + 2)\, \chi(x) \\ &= \gamma c(\gamma c + 2i)\, [\mu(x) - \chi(x) - \mu(\sigma(x))] + 2\, \chi(x) \\ &= (\gamma^2 c + 2\gamma i)\, \chi(x)\, A(x) + 2\chi(x). \end{aligned}$$

Since $\gamma \neq 0$ and $\chi(x) \neq 0$, we obtain $\gamma\, A(x) = i$. As $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$ we have similarly $\gamma\, A(x^2) = i$. The function $A : S\backslash I_\chi \to \mathbb{C}$ is additive, so $2i = i$, which is a contradiction. We conclude that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$, so $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

On the other hand, since the functions $f$ and $g + G$ are even with respect to $\sigma$, we have on $S\backslash I_\chi$

$$c\, \mu \circ \sigma - c\, \chi \circ \sigma - \chi \circ \sigma\, A \circ \sigma = c\, \mu - c\, \chi - \chi\, A, \tag{4.28}$$

and

$$\gamma c(\gamma c+2i)\,\mu\circ\sigma+[(\gamma ci-1)^2+1]\,\chi\circ\sigma-\gamma^2c\,\chi\circ\sigma\,A\circ\sigma$$
$$=\gamma c(\gamma c+2i)\,\mu+[(\gamma ci-1)^2+1]\,\chi-\gamma^2c\chi\,A,$$

which implies

$$(\gamma^2c^2+2\gamma ci)\,\mu\circ\sigma+[-\gamma^2c^2-2\gamma ci+2]\,\chi\circ\sigma-\gamma^2c\,\chi\circ\sigma\,A\circ\sigma$$
$$=(\gamma^2c^2+2\gamma ci)\,\mu+[-\gamma^2c^2-2\gamma ci+2]\,\chi-\gamma^2c\,\chi\,A,$$

so that

$$\gamma^2c\,[c\,\mu\circ\sigma-c\,\chi\circ\sigma-\chi\circ\sigma\,A\circ\sigma]+2\gamma ci\,\mu\circ\sigma-2i(\gamma c+i)\,\chi\circ\sigma$$
$$=\gamma^2c\,[c\,\mu-c\,\chi-\chi\,A]+2\gamma ci\,\mu-2i(\gamma c+i)\,\chi. \tag{4.29}$$

It follows from (4.28) and (4.29) that

$$\gamma c\,\mu\circ\sigma(x)-\gamma c\,\mu(x)-(\gamma c+i)\,\chi\circ\sigma(x)+(\gamma c+i)\,\chi(x)=0,$$

for all $x\in S\backslash I_\chi$. As $f\circ\sigma=f$ and $\sigma(I_\chi)=I_\chi$ we get $\mu\circ\sigma(x)=\mu(x)$ and $\chi\circ\sigma(x)=\chi(x)=0$ for all $x\in I_\chi$. Then

$$\gamma c\,\mu\circ\sigma-\gamma c\,\mu-(\gamma c+i)\,\chi\circ\sigma+(\gamma c+i)\,\chi=0. \tag{4.30}$$

Since $\gamma c\neq 0$, we get, according to [7, Theorem 3.18], that the multiplicative functions $\mu\circ\sigma$, $\mu$, $\chi\circ\sigma$ and $\chi$ are not different. So we have two cases according to whether $\mu\circ\sigma=\mu$ or $\mu\circ\sigma=\chi$. Notice that if $\chi\circ\sigma=\chi$ then (4.30) gives $\mu\circ\sigma=\mu$.

<u>Case B.2.1</u>: $\mu\circ\sigma=\mu$. Here (4.30) becomes $(\gamma c+i)\,(\chi\circ\sigma-\chi)=0$. If $\gamma c\neq -i$ then $\chi\circ\sigma=\chi$, hence $h\circ\sigma=h$, which contradicts the assumption on $h$. If $\gamma c=-i$, then (4.5) implies $-ic\,(2\,\mu-\chi-\chi\circ\sigma)=2\gamma\,f=-2\frac{i}{c}\,f$, then $c^2\,(2\,\mu-\chi-\chi\circ\sigma)=2(c^2\,\mu-c^2\chi-c\chi\,A)$ on $S\backslash I_\chi$. Hence $\frac{2}{c}\,A=\chi^{-1}\,(\chi\circ\sigma)-1$. Using the same computations as the ones in Case B.1 we obtain $\chi^{-1}\,(\chi\circ\sigma)=1$, then $\chi\circ\sigma=\chi$; so $h\circ\sigma=h$, which contradicts the assumption on $h$.

<u>Case B.2.2</u>: $\mu\circ\sigma=\chi$. In this case $\mu(x)=\chi(x)=0$ for all $x\in I_\chi$, then $f(x)=0$ for all $x\in I_\chi$. Let $x\in S\backslash I_\chi$, then applying (4.5) we obtain $2\gamma\,f(x)=h(x)+h\circ\sigma(x)=-ic\,(\mu(x)-\chi(x))-ic\,[\mu\circ\sigma(x)-\chi\circ\sigma(x)]=0$. So $f(x)=0$ for all $x\in S$, contradicting that $f$ and $h$ are linearly independent.

We conclude that the functional equation (1.1) has no solution in Case B.2.

<u>Case B.3</u>:

$$\begin{pmatrix} f\\ G\\ H\end{pmatrix}=\begin{pmatrix} -c_1 & c_1 & -c_1c_2\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}c_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mu\\ \chi\\ \chi\,A\end{pmatrix}\ \text{on } S\backslash I_\chi,$$

$f(x)=-c_1\,\mu(x)$, $G(x)=\frac{1}{2}\,\mu(x)$ and $H(x)=0$ for $\quad x\in I_\chi$, where $c_1,\, c_2 \in \mathbb{C}\backslash\{0\}$ are two constants such that $1 + c_1 c_2^2 = 0$, $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function. Using (4.21) and (4.22) we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1 c_2 \\ -\gamma^2 c_1 + \frac{1}{2} & \gamma^2 c_1 + \frac{1}{2} & -c_2\left(\gamma^2 c_1 + \frac{1}{2}\right) + 2\gamma i \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi A \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$f(x) = -c_1\, \mu(x)$, $g(x) = (-\gamma^2 c_1 + \frac{1}{2})\, \mu(x)$ and $h(x) = 0$ for $x \in I_\chi$.
We have $\sigma(S\backslash I_\chi) = S\backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$. Indeed, if there exists $x \in S\backslash I_\chi$ such that $\sigma(x) \in I_\chi$ then $h(x) = -i\, \chi(x)\, A(x)$, $h(\sigma(x)) = 0$ and $f(x) = f(\sigma(x)) = -c_1\, \mu(\sigma(x))$ because $f \circ \sigma = f$. So (4.5) implies

$$i\, \chi(x)\, A(x) = 2\gamma c_1\, \mu(\sigma(x)). \tag{4.31}$$

Since $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$, we have similarly $2i\, (\chi(x))^2\, A(x) = 2\gamma c_1 (\mu(x))^2$. Then $4\gamma c_1\, \chi(x)\, \mu(\sigma(x)) = 2\gamma c_1\, (\mu(\sigma(x)))^2$. As $\gamma \neq 0$ and $c_1 \neq 0$ we get

$$\mu(\sigma(x))\, [\mu(\sigma(x)) - 2\, \chi(x)] = 0. \tag{4.32}$$

If $\mu(\sigma(x)) = 0$, then $\mu(x) = \chi(x)$ because $f \circ \sigma = f$, hence

$$g(x) = \left(-\gamma^2 c_1 + \frac{1}{2}\right) \chi(x) + \left(\gamma^2 c_1 + \frac{1}{2}\right) \chi(x) = \mu(x).$$

Moreover $g(\sigma(x)) = (-\gamma^2 c_1 + \frac{1}{2})\, \mu(\sigma(x)) = 0$. Taking (4.6) into account we get $\chi(x) = \gamma i\, \chi(x)\, A(x) = 0$.

If $\mu(\sigma(x)) \neq 0$, then (4.32) implies $\mu(\sigma(x)) = 2\, \chi(x)$. As $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$ we have similarly $\mu(\sigma(x^2)) = 2\, \chi(x^2)$, and it follows that $4\, (\chi(x))^2 = 2\, (\chi(x))^2$; hence $\chi(x) = 0$.

Thus $\chi(x) = 0$ in both cases, which contradicts the assumption on $x$.

We conclude that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$, so $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

Let $x \in I_\chi$, then it follows from $f \circ \sigma = f$ and (4.5), that $2\gamma\, f(x) = 0$ so that $\mu(x) = 0$. Hence $f(x) = g(x) = h(x) = 0$.

On the other hand, since $f \circ \sigma = f$, we have

$$-\mu + \chi - c_2\, \chi\, A = -\mu \circ \sigma + \chi \circ \sigma - c_2\, \chi \circ \sigma\, A \circ \sigma, \tag{4.33}$$

on the subsemigroup $S\backslash I_\chi$; then

$$-\mu \circ \sigma + \mu + \chi \circ \sigma - \chi = c_2\, [\chi \circ \sigma\, A \circ \sigma - \chi\, A]. \tag{4.34}$$

Moreover, taking (4.6) into account we get

$$\left(-\gamma^2 c_1+\frac{1}{2}\right)\mu+\left(\gamma^2 c_1+\frac{1}{2}\right)\chi+\left(-\gamma^2 c_1 c_2+2\gamma i-\frac{1}{2}c_2\right)\chi A$$
$$-\left(-\gamma^2 c_1+\frac{1}{2}\right)\mu\circ\sigma-\left(\gamma^2 c_1+\frac{1}{2}\right)\chi\circ\sigma$$
$$-\left(-\gamma^2 c_1 c_2+2\gamma i-\frac{1}{2}c_2\right)\chi\circ\sigma\, A\circ\sigma=\gamma i\,[\chi\, A-\chi\circ\sigma\, A\circ\sigma],$$

and it follows that

$$\left(\gamma^2 c_1+\frac{1}{2}\right)[-\mu+\chi-c_2\,\chi\, A]+\mu+2\gamma i\,\chi\, A-\left(\gamma^2 c_1+\frac{1}{2}\right)$$
$$[-\mu\circ\sigma+\chi\circ\sigma-c_2\,\chi\circ\sigma\, A\circ\sigma]-\mu\circ\sigma-2\gamma i\,\chi\circ\sigma\, A\circ\sigma$$
$$=\gamma i\,[\chi\, A-\chi\circ\sigma\, A\circ\sigma].$$

Taking (4.33) into account we get that

$$\mu-\mu\circ\sigma=\gamma i\,[\chi\circ\sigma\, A\circ\sigma-\chi\, A]. \tag{4.35}$$

(4.34) and (4.35) imply $c_2\,(\mu-\mu\circ\sigma)=\gamma i\,(-\mu\circ\sigma+\mu+\chi\circ\sigma-\chi)$. Hence

$$(\gamma i-c_2)\,\mu\circ\sigma-(\gamma i-c_2)\,\mu-\gamma i\,\chi\circ\sigma+\gamma i\,\chi=0. \tag{4.36}$$

As $\gamma i\neq 0$ we get, according to [7, Theorem 3.18], that the multiplicative functions $\mu\circ\sigma$, $\mu$, $\chi\circ\sigma$ and $\chi$ are not different. Since $h\circ\sigma\neq h$, we have $\chi\circ\sigma\, A\circ\sigma\neq\chi\, A$ and we get, from (4.35), that $\mu\circ\sigma\neq\mu$. As $\mu\neq\chi$ we have two cases according to whether $\mu\circ\sigma=\chi$ or $\chi\circ\sigma=\chi$.

Case B.3.1: $\mu\circ\sigma=\chi$. Then (4.36) becomes $(2\gamma i-c_2)\,(\chi-\mu)=0$. Since $\mu\neq\chi$, we get that $2\gamma i-c_2=0$. Using (4.5) and taking into account that $1+c_1c_2^2=0$ we find by elementary computations that $c_2\, A\circ\sigma=(\chi\circ\sigma)^{-1}\,\mu-(\chi\circ\sigma)^{-1}\chi$. Proceeding exactly like in Case A.5.3.2.1 we get that $A=0$, which contradicts that $A$ is nonzero in Case B.3.

Case B.3.2: $\chi\circ\sigma=\chi$. In this case (4.35) becomes $\mu\circ\sigma-\mu=c_2\,(\chi\, A-\chi\, A\circ\sigma)$. Then $\chi^{-1}\,\mu\circ\sigma-\chi^{-1}\,\mu=c_2\,(A-A\circ\sigma)$. As $A-A\circ\sigma$ is an additive function on the subsemigroup $S\backslash I_\chi$ we get, by Lemma 4.4, that $A\circ\sigma=A$. So $\chi\, A=\chi\circ\sigma\, A\circ\sigma$. Hence $h(\sigma(x))=h(x)$ for all $x\in S\backslash I_\chi$. Moreover $h(\sigma(x))=h(x)=0$ for all $x\in I_\chi$, so $h\circ\sigma=h$, which contradicts the assumption on $h$.

We conclude that the functional equation (1.1) has no solution in Case B.3.

Case B.4:

$$\begin{pmatrix} f\\ G\\ H\end{pmatrix}=\begin{pmatrix} c\beta & c(2-\beta) & -2c\\ \frac{1}{4}\beta & \frac{1}{4}(2-\beta) & \frac{1}{2}\\ \frac{1}{2\alpha} & -\frac{1}{2\alpha} & 0\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix},$$

where $\alpha$, $\beta$, $c \in \mathbb{C}\backslash\{0\}$ are three constants with $2c\alpha^2\beta(2-\beta) = 1$; $\chi_1$, $\chi_2$, $\chi_3$ : $S \to \mathbb{C}$ are three multiplicative functions. Using (4.21) and (4.22) we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} c\beta & c(2-\beta) & -2c \\ c\beta\gamma^2 + \frac{1}{4}\beta + \frac{\gamma}{\alpha}i & -\left(c\beta\gamma^2 + \frac{1}{4}\beta + \frac{\gamma}{\alpha}i\right) + 2\gamma^2 c + \frac{1}{2} & -2\gamma^2 c + \frac{1}{2} \\ -\frac{1}{2\alpha}i & \frac{1}{2\alpha}i & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}.$$

From (4.5) we deduce $\chi_1 - \chi_2 + \chi_1 \circ \sigma - \chi_2 \circ \sigma = 4\gamma\alpha i\, f$. Since $\gamma, \alpha \in \mathbb{C}\backslash\{0\}$ and $f \neq 0$, we get that $\chi_1 \circ \sigma \neq \chi_2$. Since $f$ and $h$ are linearly independent, so are $f$ and $H$. Then we get that $\chi_2 \neq \chi_3$ and $\chi_1 \neq \chi_3$.

On the other hand, since $f \circ \sigma = f$, we get that

$$\beta\,\chi_1 + (2-\beta)\,\chi_2 - 2\,\chi_3 - \beta\,\chi_1 \circ \sigma - (2-\beta)\,\chi_2 \circ \sigma + 2\,\chi_3 \circ \sigma = 0. \quad (4.37)$$

Since $\beta \neq 0$, we deduce from (4.37), according to [7, Theorem 3.18], that the multiplicative functions $\chi_1$, $\chi_2$, $\chi_3$, $\chi_1 \circ \sigma$, $\chi_2 \circ \sigma$ and $\chi_3 \circ \sigma$ are not different. Since $\chi_1 \neq \chi_2$, $\chi_1 \neq \chi_3$, $\chi_2 \neq \chi_3$ and $\chi_1 \neq \chi_2 \circ \sigma$ we have two cases according to whether $\chi_1 \circ \sigma = \chi_1$ or $\chi_1 \circ \sigma = \chi_3$.

Using [7, Theorem 3.18] we check that the remaining cases $\chi_2 = \chi_2 \circ \sigma$, $\chi_2 = \chi_3 \circ \sigma$ and $\chi_3 = \chi_3 \circ \sigma$ can be subsumed within the first two ones $\chi_1 \circ \sigma = \chi_1$ and $\chi_1 \circ \sigma = \chi_3$.

Case B.4.1: $\chi_1 \circ \sigma = \chi_1$. In this case $\chi_3 = \chi_2 \circ \sigma$ and $\beta = 4$. Putting $\rho = -\frac{1}{2\alpha}i \in \mathbb{C}\backslash\{0\}$ we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \rho^2 & -\frac{1}{2}\rho^2 & -\frac{1}{2}\rho^2 \\ (\rho\gamma - 1)^2 & \frac{1}{2}(-\rho^2\gamma^2 + 4\rho\gamma - 1) & \frac{1}{2}(-\rho^2\gamma^2 + 1) \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_2 \circ \sigma \end{pmatrix}.$$

Using (4.23) we find, by elementary computations, that $\rho\gamma = 1$. Hence

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \rho^2 & -\frac{1}{2}\rho^2 & -\frac{1}{2}\rho^2 \\ 0 & 1 & 0 \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_2 \circ \sigma \end{pmatrix}.$$

So we obtain a solution of the form (c) in Theorem 4.5.

Case B.4.2: $\chi_1 \circ \sigma = \chi_3$. In this case $\chi_3 = \chi_1 \circ \sigma$, $\chi_2 = \chi_2 \circ \sigma$ and $\beta = -2$. Putting $\lambda = \frac{1}{2\alpha}i \in \mathbb{C}\backslash\{0\}$ we obtain

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\lambda^2 & \lambda^2 & -\frac{1}{2}\lambda^2 \\ \frac{1}{2}(-\lambda^2\gamma^2 + 4\lambda\gamma - 1) & (\lambda\gamma - 1)^2 & \frac{1}{2}(-\lambda^2\gamma^2 + 1) \\ -\lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_1 \circ \sigma \end{pmatrix}.$$

Using (4.23) we find, by elementary computations, that $\lambda\gamma = 1$. Hence

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\lambda^2 & \lambda^2 & -\frac{1}{2}\lambda^2 \\ 1 & 0 & 0 \\ -\lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_1 \circ \sigma \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \lambda^2 & -\frac{1}{2}\lambda^2 & -\frac{1}{2}\lambda^2 \\ 0 & 1 & 0 \\ \lambda & -\lambda & 0 \end{pmatrix} \begin{pmatrix} \chi_2 \\ \chi_1 \\ \chi_1 \circ \sigma \end{pmatrix}.$$

So we obtain a solution of the form (c) in Theorem 4.5.

Case B.5:

$$\begin{cases} f = F_1 \\ G = -\frac{1}{2}\delta^2 \, F_1 + G_1 + \delta \, H_1 \\ H = -\delta \, F_1 + H_1 \end{cases}$$

where the functions $F_1, G_1, H_1 : S \to \mathbb{C}$ are of the forms in cases B.1-B.4 and $\delta \in \mathbb{C}$ is a constant.

From (4.21) and (4.22) we derive

$$(III) \begin{cases} f = F_1 \\ g = \left[\frac{1}{2}\delta^2 - \eta^2\right] \, F_1 + G_1 + (\eta + \gamma i) \, H_1 \\ h = \delta i \, F_1 - i H_1 \end{cases} ,$$

where $\eta = \delta + \gamma i$.

On the other hand the properties $f \circ \sigma = f$, (4.5), (4.6) and (4.23) imply

$$(IV) \begin{cases} F_1 \circ \sigma & = F_1 \\ (G_1 + \eta \, H_1) \circ \sigma = G_1 + \eta \, H_1 \\ H_1 \circ \sigma + H_1 & = 2\eta \, F_1. \end{cases}$$

We have the following cases

Case B.5.1:

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \chi \, A_1 \\ \chi \\ \chi \, A \\ \chi \, A^2 \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$F_1(x) = G_1(x) = H_1(x) = 0 \quad \text{for} \quad x \in I_\chi$,

where $\chi : S \to \mathbb{C}$ is a nonzero multiplicative function and $A$, $A_1 : S \backslash I_\chi \to \mathbb{C}$ are two additive functions with $A \neq 0$.

By the same computations as used in Case B.1 we prove that $\sigma(S \backslash I_\chi) = S \backslash I_\chi$, $\sigma(I_\chi) = I_\chi$ and $\chi \circ \sigma = \chi$. Then $G_1 \circ \sigma = G_1$. We split the discussion into the cases $\eta \neq 0$ and $\eta = 0$.

Case B.5.1.1: $\eta \neq 0$. As $(G_1 + \eta \, H_1) \circ \sigma = G_1 + \eta \, H_1$ we get that $\eta \, H_1 \circ \sigma = \eta \, H_1$, so $H_1 \circ \sigma = H_1$. Taking into account that $H_1 \circ \sigma + H_1 = 2\eta \, F_1$ we get that $H_1 = \eta \, F_1$. Then $h = (\delta - \eta) i f$, contradicting that $f$ and $h$ are linearly independent.

Case B.5.1.2: $\eta = 0$. Then $\delta = -\gamma i$ and the conditions $(IV)$ become

$$\begin{cases} F_1 \circ \sigma = F_1 \\ G_1 \circ \sigma = G_1 \\ H_1 \circ \sigma = -H_1. \end{cases}$$

By the same method as the one in Case A.1 we get that $A \circ \sigma = -A$ and $A_1 \circ \sigma = A_1$. Moreover by putting $F_0 = F_1$, $G_0 = G_1$ and $H_0 = -iH_1$ and writing $\delta i$ instead of $\delta$ in $(III)$ we obtain that

$$\begin{cases} f = F_0 \\ g = -\frac{1}{2}\delta^2 F_0 + G_0 + \delta\, H_0 \\ h = -\delta\, F_0 + H_0 \end{cases},$$

where

$$\begin{pmatrix} F_0 \\ G_0 \\ H_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} \chi\, A_1 \\ \chi \\ \chi\, A \\ \chi\, A^2 \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$F_0(x) = G_0(x) = H_0(x) = 0 \quad \text{for} \quad x \in I_\chi,$
is a solution of the form (a) in Theorem 4.5. The solutions occur in (d) of the list of Theorem 4.5.

Case B.5.2:

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} c^2 & -c^2 & -c \\ 0 & 1 & 0 \\ c & -c & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S\backslash I_\chi,$$

$F_1(x) = c^2\,\mu(x)$, $G_1(x) = 0$ and $H_1(x) = c\,\mu(x)$, for $x \in I_\chi$,
where $c \in \mathbb{C}\backslash\{0\}$ is a constant, $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S\backslash I_\chi \to \mathbb{C}$ is a nonzero additive function.

We have $\sigma(S\backslash I_\chi) = S\backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S\backslash I_\chi) \subseteq S\backslash I_\chi$. Indeed, if there exists an element $x \in S\backslash I_\chi$ such that $\sigma(x) \in I_\chi$, then the first and the second identities of $(IV)$ imply $c\,\mu(\sigma(x)) = c\,(\mu(x) - c\,\chi(x)) - \chi(x)\,A(x)$ and $\chi(x) + \eta\,c(\mu(x) - \chi(x)) = \eta\,c\,\mu(\sigma(x))$. Hence $\eta\,\chi(x)\,A(x) = -\chi(x)$. Since $\chi(x) \neq 0$, we get that $\eta\,A(x) = -1$. As $x^2 \in S\backslash I_\chi$ and $\sigma(x^2) \in I_\chi$ we get similarly that $\eta\,A(x^2) = -1$. Using the additivity of $A$ we obtain $2\,\eta\,A(x^2) = -1$, which is a contradiction. We deduce that $\sigma(S\backslash I_\chi) = S\backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

On the other hand, using the third identity of $(IV)$, we obtain, on $S\backslash I_\chi$,

$$c\,\mu \circ \sigma - c\,\chi \circ \sigma + c\,\mu - c\,\chi = 2\,\eta\,c(c\,\mu - c\,\chi - \chi\,A),$$

so that

$$2\,\eta\,\chi\,A = (1 - 2\,\eta\,c)\,\chi - (1 - 2\,\eta\,c)\,\mu + \chi \circ \sigma - \mu \circ \sigma.$$

Applying Lemma 4.4 on the last identity, on the subsemigroup $S\backslash I_\chi$, we get that $2\,\eta\,\chi\,A = 0$.

If $\eta \neq 0$, then $A = 0$, which contradicts that $A$ is nonzero in Case B.5.2.

If $\eta = 0$, then the identity above gives $\chi \circ \sigma + \chi = \mu \circ \sigma + \mu$ while the second identity of $(IV)$ becomes $\chi \circ \sigma = \chi$, so $2\,\chi = \mu \circ \sigma + \mu$ on the subsemigroup $S \backslash I_\chi$. According to [7, Corollary 3.19] we get that $\chi = \mu$ on the subsemigroup $S \backslash I_\chi$, then $H_1(x) = 0$ for all $x \in S \backslash I_\chi$. Moreover, for all $x \in I_\chi$, the first and the third identities of $(IV)$ imply $\mu(\sigma(x)) = \mu(x)$ and $\mu(\sigma(x)) = -\mu(x)$, so $\mu(x) = 0$. Hence $H_1(x) = 0$ for all $x \in S \backslash I_\chi$, so $H_1 = 0$. Considering the first and the third identities of $(III)$ we get that $h = \delta i\, f$, which contradicts the linear independence of $f$ and $h$.

We conclude that the functional equation (1.1) has no solution in Case B.5.2.

Case B.5.3:

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 & -c_1 c_2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \chi \\ \chi\, A \end{pmatrix} \text{ on } S \backslash I_\chi,$$

$F_1(x) = -c_1\, \mu(x)$, $G_1(x) = \frac{1}{2}\, \mu(x)$ and $H_1(x) = 0$ for $\quad x \in I_\chi$,
where $c_1,\ c_2 \in \mathbb{C} \backslash \{0\}$ are two constants such that $1 + c_1 c_2^2 = 0$, $\chi \neq 0$, $\mu : S \to \mathbb{C}$ are two multiplicative functions such that $\mu \neq \chi$ and $A : S \backslash I_\chi \to \mathbb{C}$ is a nonzero additive function.

We have $\sigma(S \backslash I_\chi) = S \backslash I_\chi$. Like in Case A.3 it suffices to check that $\sigma(S \backslash I_\chi) \subseteq S \backslash I_\chi$. Indeed, if there exists an element $x \in S \backslash I_\chi$ such that $\sigma(x) \in I_\chi$, then the first and the second identities of $(IV)$ imply

$$-\mu(x) + \chi(x) - c_2\, \chi(x)\, A(x) = -\mu(\sigma(x))$$

and

$$\mu(x) + \chi(x) - (c_2 - 2\,\eta)\, \chi(x)\, A(x) = \mu(\sigma(x)).$$

Adding the two last identities we obtain $2\,\chi(x) - 2\,(c_2 - \eta)\, \chi(x)\, A(x) = 0$. Since $\chi(x) \neq 0$, we get that $(c_2 - \eta)\, A(x) = 1$. As $x^2 \in S \backslash I_\chi$ and $\sigma(x^2) \in I_\chi$ we get similarly that $(c_2 - \eta)\, A(x^2) = 1$. Since $A$ is additive, we get $2\,(c_2 - \eta)\, A(x) = 1$, which is a contradiction. We deduce that $\sigma(S \backslash I_\chi) = S \backslash I_\chi$ and $\sigma(I_\chi) = I_\chi$.

On the other hand the first and the second identities of $(IV)$ imply, on $S \backslash I_\chi$,

$$-\,\mu \circ \sigma + \chi \circ \sigma - c_2\, \chi \circ \sigma\, A \circ \sigma = -\mu + \chi - c_2\, \chi\, A \tag{4.38}$$

and

$$\mu \circ \sigma + \chi \circ \sigma - (c_2 - 2\,\eta)\, \chi \circ \sigma\, A \circ \sigma = \mu + \chi - (c_2 - 2\,\eta)\, \chi\, A, \tag{4.39}$$

and subtracting (4.38) from (4.39) we derive

$$\eta\, \chi\, A - \eta\, \chi \circ \sigma\, A \circ \sigma = \mu \circ \sigma - \mu. \tag{4.40}$$

Moreover, using the third identity of $(IV)$ we get

$$\chi \circ \sigma\, A \circ \sigma + (1 + 2\,\eta\, c_1\, c_2)\, \chi\, A = 2\,\eta\, c_1\, (\chi - \mu). \tag{4.41}$$

Multiplying both sides of (4.41) by $\eta$ and adding the identity obtained to (4.40) we get

$$2\,\eta\,(1+\eta\,c_1\,c_2)\,\chi\,A = 2\,\eta^2\,c_1\,\chi - (1+2\,\eta^2\,c_1)\,\mu + \mu\circ\sigma. \tag{4.42}$$

Applying Lemma 4.4 on the subsemigroup $S\backslash I_\chi$ and taking into account that $\chi(x)\neq 0$ for all $x\in S\backslash I_\chi$ we get that

$$\eta\,(1+\eta\,c_1\,c_2)\,A = 0. \tag{4.43}$$

If $\eta=0$, then we get, from (4.40) and (4.41) respectively, that $\mu\circ\sigma=\mu$ and $\chi\circ\sigma\,A\circ\sigma=-\chi\,A$. Taking (4.39) into account it follows that $\chi\circ\sigma-\chi=-2c_2\,\chi\,A$. Since $c_2\neq 0$ and $\chi(x)\neq 0$ for all $x\in S\backslash I_\chi$, we get, according to Lemma 4.4, that $A=0$, contradicting that $A$ is nonzero in Case B.5.3.

If $\eta\neq 0$ and $1+\eta\,c_1\,c_2\neq 0$, then we get from (4.43) that $A=0$, which contradicts that $A$ is nonzero in Case B.5.3.

If $1+\eta\,c_1\,c_2=0$, then $\eta=c_2$ because $1+c_1\,c_2^2=0$. Replacing $\eta$ by $c_2$ in (4.42) we get that $2\,\chi=\mu+\mu\circ\sigma$ on the subsemigroup $S\backslash I_\chi$. According to [7, Corollary 3.19] we get that $\chi=\mu=\mu\circ\sigma$ on the subsemigroup $S\backslash I_\chi$. Using (4.38) we get that $\chi\circ\sigma\,A\circ\sigma=\chi\,A$, so $H_1\circ\sigma=H_1$ on the subsemigroup $S\backslash I_\chi$. As $H_1(x)=0$ for all $x\in I_\chi$ and $\sigma(I_\chi)=I_\chi$ we get that $H_1\circ\sigma=H_1$. It follows, from the third identity of $(IV)$, that $H_1=\eta\,F_1=c_2\,F_1$. Using the first and the third identities of $(III)$ we get that $h=(\delta-c_2)if$, contradicting that $f$ and $h$ are linearly independent.

We conclude that the functional equation (1.1) has no solution in Case B.5.3.

<u>Case B.5.4</u>:

$$\begin{pmatrix}F_1\\ G_1\\ H_1\end{pmatrix}=\begin{pmatrix}c\,\beta & c\,(2-\beta) & -2\,c\\ \frac{1}{4}\beta & \frac{1}{4}\,(2-\beta) & \frac{1}{2}\\ \frac{1}{2\,\alpha} & -\frac{1}{2\,\alpha} & 0\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix},$$

where $\alpha,\ \beta,\ c\in\mathbb{C}\backslash\{0\}$ are three constants with $2c\alpha^2\beta(2-\beta)=1$; $\chi_1,\ \chi_2,\ \chi_3: S\to\mathbb{C}$ are three multiplicative functions such that $\chi_1\neq\chi_2$, $\chi_1\neq\chi_3$ and $\chi_2\neq\chi_3$.

We split the discussion into the cases $\eta\neq 0$ and $\eta=0$.

<u>Case B.5.4.1</u>: $\eta=0$. Then $\delta=-\gamma i$ and the conditions $(IV)$ become

$$\begin{cases} F_1\circ\sigma = F_1\\ G_1\circ\sigma = G_1\\ H_1\circ\sigma = -H_1.\end{cases}$$

So, writing $\alpha$ instead of $\alpha i$, $F_0:=F_1$, $G_0:=G_1$ and $H_0:=-iH_1$ is a solution of the functional equation (1.1) of the form

$$\begin{pmatrix}F_0\\ G_0\\ H_0\end{pmatrix}=\begin{pmatrix}c\,\beta & c\,(2-\beta) & -2\,c\\ \frac{1}{4}\beta & \frac{1}{4}\,(2-\beta) & \frac{1}{2}\\ \frac{1}{2\,\alpha} & -\frac{1}{2\,\alpha} & 0\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix},$$

where $\alpha$, $\beta$, $c \in \mathbb{C}\backslash\{0\}$ are three constants with $2\,c\,\alpha^2\,\beta\,(2-\beta) = -1$; $\chi_1$, $\chi_2$, $\chi_3 : S \rightarrow \mathbb{C}$ are three different multiplicative functions. Indeed, for all $x, y \in S$ we have

$$F_1(xy) = F_1(x)G_1(y) + G_1(x)F_1(y) + H_1(x)H_1(y),$$

so that

$$\begin{aligned} F_1(x\sigma(y)) &= F_1(x)G_1(\sigma(y)) + G_1(x)F_1(\sigma(y)) + H_1(x)H_1(\sigma(y)) \\ &= F_1(x)G_1(y) + G_1(x)F_1(y) - H_1(x)H_1(y) \\ &= F_1(x)G_1(y) + G_1(x)F_1(y) + (-iH_1(x))(-iH_1(y)), \end{aligned}$$

hence $F_0(x\sigma(y)) = F_0(x)G_0(y) + G_0(x)F_0(y) + H_0(x)H_0(y)$ for all $x, y \in S$. On the other hand $H_0 \circ \sigma = -H_0$. Since $f$ and $h$ are linearly independent, so are $F_0$ and $H_0$. It follows, according to the result of Case A.4, that $F_0$, $G_0$ and $H_0$ are of the form (b) in Theorem 4.5. Moreover taking into account that $F_0 = F_1$, $G_0 = G_1$, $H_0 = -iH_1$, $\eta = 0$ and $\delta = -\gamma i$ we obtain, by writing $-\delta$ instead of $\delta i$ in $(III)$,

$$\begin{cases} f = F_0 \\ g = -\frac{1}{2}\delta^2 F_0 + G_0 + \delta\, H_0\,. \\ h = -\delta\, F_0 + H_0 \end{cases}$$

The solutions occur in (d) of the list of Theorem 4.5.

Case B.5.4.2: $\eta \neq 0$. By similar computations to the ones in Case B.4 now applied to $(F_1, G_1, H_1)$ we prove that

$$\begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} -\rho^2 & \frac{1}{2}\rho^2 & \frac{1}{2}\rho^2 \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \mu \\ \mu\circ\sigma \end{pmatrix},$$

where $\rho \in \mathbb{C}\backslash\{0\}$ is a constant such that $\rho\eta = -1$ and $\chi, \mu : S \rightarrow \mathbb{C}$ are two multiplicative functions such that $\chi \circ \sigma = \chi$ and $\chi \neq \mu$. Using $(III)$ we deduce

$$\begin{aligned} g &= \left[\frac{1}{2}\delta^2 - \eta^2\right] F_1 + G_1 + (\eta + \gamma i)H_1 \\ &= \frac{1}{2}\delta^2 F_1 - \eta^2 F_1 + G_1 + (\eta + \gamma i)H_1 \\ &= \frac{1}{2}\delta^2 F_1 + (\rho\eta)^2\chi - \frac{1}{2}(\rho\eta)^2\mu \\ &\quad -\frac{1}{2}(\rho\eta)^2\,\mu\circ\sigma + \chi - \frac{1}{2}\mu + \frac{1}{2}\,\mu\circ\sigma + (\eta + \gamma i)H_1 \\ &= \frac{1}{2}\delta^2 F_1 + 2\,(\chi - \mu) + \mu + (\eta + \gamma i)H_1. \end{aligned}$$

Since $H_1 = \rho\,(\chi - \mu)$ and $\rho\eta = -1$, we get $\chi - \mu = -\eta H_1$. Hence

$$g = \frac{1}{2}\delta^2 F_1 + \mu + (\gamma i - \eta)H_1 = \frac{1}{2}\delta^2 F_1 + \mu - \delta H_1.$$

By putting $F_0 = F_1$, $G_0 = \mu$ and $H_0 = -iH_1$ and writing $-\delta$ instead of $\delta i$ in $(III)$ we obtain

$$\begin{cases} f = F_0 \\ g = -\frac{1}{2}\delta^2 F_0 + G_0 + \delta\, H_0 \\ h = -\delta\, F_0 + H_0, \end{cases}$$

with

$$\begin{pmatrix} F_0 \\ G_0 \\ H_0 \end{pmatrix} = \begin{pmatrix} -\rho^2 & \frac{1}{2}\rho^2 & \frac{1}{2}\rho^2 \\ 0 & 1 & 0 \\ -\rho i & \rho i & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \mu \\ \mu\circ\sigma \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} F_0 \\ G_0 \\ H_0 \end{pmatrix} = \begin{pmatrix} \rho^2 & -\frac{1}{2}\rho^2 & -\frac{1}{2}\rho^2 \\ 0 & 1 & 0 \\ \rho & -\rho & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \mu \\ \mu\circ\sigma \end{pmatrix},$$

where $\rho \in \mathbb{C}\backslash\{0\}$, so $F_0$, $G_0$ and $H_0$ are of the form (c) in Theorem 4.5 and the solutions occur in (d) of the list of Theorem 4.5.

Conversely if $f, g$ and $h$ are of the forms (a)-(d) in Theorem 4.5 we check by elementary computations that $f, g$ and $h$ satisfy the functional equation (1.1), and $f$ and $h$ are linearly independent and that $h \circ \sigma \neq h$. This completes the proof of Theorem 4.5. □

*Remark* 4.6. Let $S$ be a semigroup generated by its squares and $\sigma : S \to S$ be an involutive automorphism. We consider the variant

$$f(\sigma(y)x) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in S \tag{4.44}$$

of the functional equation (1.1).

Let $f, g, h : S \to \mathbb{C}$ satisfy the functional equation (4.44). The right hand side of (4.44) is invariant under the interchange of $x$ and $y$. So $f(\sigma(y)x) = f(\sigma(x)y)$ for all $x, y \in S$. By the same computations as the ones in the proof of Proposition 3.1 we derive that $f$ is central. Then $f, g$ and $h$ satisfy the functional equation (1.1). Conversely we check similarly that if $f, g, h : S \to \mathbb{C}$ satisfy the functional equation (1.1) then they satisfy (4.44). Thus the functional equation (1.1) and its variant (4.44) have the same solutions.

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