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On certain generalizations of the Levi-Civita and Wilson functional equations

J. M. Almira and E. V. Shulman

Abstract. We study the functional equation

$$\sum_{i=1}^{m} f_i(b_i x + c_i y) = \sum_{k=1}^{n} u_k(y) v_k(x)$$

with $x, y \in \mathbb{R}^d$ and $b_i, c_i \in GL(d, \mathbb{R})$, both in the classical context of continuous complex-valued functions and in the framework of complex-valued Schwartz distributions, where these equations are properly introduced in two different ways. The solution sets are, typically, exponential polynomials and, in some particular cases, related to the so called characterization problem of the normal distribution in Probability Theory, they reduce to ordinary polynomials.

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1. Introduction

The Levi-Civita functional equation (see Levi-Cività [21]) has the form

$$f(x+y) = \sum_{k=1}^{n} u_k(y) v_k(x), \tag{1}$$

where f, u_k, v_k, $(1 \leq k \leq n)$, are complex-valued functions defined on a semigroup $(G, +)$. This equation can be restated by claiming that $\tau_y(f) \in W$ for all $y \in G$, where $W = \text{span}\{v_k\}_{k=1}^n$ is a finite-dimensional space of functions defined on G and $\tau_y(f)(x) = f(x+y)$.

In this paper we deal with the case $G = \mathbb{R}^d$ and study a more general functional equation

$$\sum_{i=1}^{m} f_i(b_i x + c_i y) = \sum_{k=1}^{n} u_k(y) v_k(x) \tag{2}$$

for all $x, y \in \mathbb{R}^d$, where f_i, v_k, u_k, for $1 \leq i \leq m$, $1 \leq k \leq n$, are functions defined on $\mathbb{R}^d$, and $b_i, c_i \in GL(d, \mathbb{R})$. Our main result is that all continuous solutions of (2) are exponential polynomials. Moreover, using a result from [26] we extend this statement to equations of the form

$$\sum_{i=1}^{m} f_i(b_i x + c_i y) = \sum_{k=1}^{n} u_k(y) v_k(x) + \sum_{s=1}^{N} P_s(x) w_s(y) \exp \langle x, \varphi_s(y) \rangle, \tag{3}$$

where P_s are polynomials and the functions w_s, φ_s are arbitrary.

In the one-dimensional case, addition theorems with such a left-hand side were studied by Wilson [33] a hundred years ago. Applying an elimination method, which now is a classical one, he showed that all continuous solutions of the equation

$$\sum_{i=1}^{m} f_i(\alpha_i x + \beta_i y) = f(x) + g(y)$$

are polynomials of degree not greater than m.

Eq. (2) includes the equations in iterated differences:

$$\sum_{j=1}^{m} \lambda_j \Delta_y^j f = \sum_{k=1}^{n} u_k(y) v_k \ , \ y \in \mathbb{R}^d,$$

where Δ_y is the difference operator $\Delta_y(f)(x) = f(x+y) - f(x)$. Indeed

$$\Delta_y^m f(x) = 1 \cdot f(x) + \sum_{p=1}^{m} \binom{m}{p} (-1)^{m-p} f(I_d x + (pI_d) y),$$

where I_d denotes the identity matrix of size d. A simplest example is Frechet's equation $\Delta_y^m f = 0$.

Eq. (2) also extends the functional equation

$$\frac{1}{N} \sum_{k=0}^{N-1} f(z + w^k h) = 0, \text{ for all } z, h \in \mathbb{C},$$

where w is any primitive N-th root of 1. This equation, characterizing harmonic polynomials, was introduced by Kakutani and Nagumo [19], Walsh [32] in the 1930's, and intensively studied by Haruki [15–17] in the 1970's and 1980's.

Another special case of (2) is the equation

$$\sum_{i=1}^{m} f_i(b_i x + c_i y) = \sum_{i=1}^{m} f_i(b_i x) + \sum_{i=1}^{m} f_i(c_i y), \tag{4}$$

which is a linearized (by taking logarithms) form of the Skitovich–Darmois functional equation:

$$\prod_{i=1}^{m} \widehat{\mu_i}(b_i x + c_i y) = \prod_{i=1}^{m} \widehat{\mu_i}(b_i x) \prod_{i=1}^{m} \widehat{\mu_i}(c_i y).$$

Here, $\widehat{\mu_i}$ represents the characteristic function of a probability distribution μ_i. This equation is connected to the characterization problem of normal distributions. Concretely, its study leads to a proof of the following result (Linnik [22], Ghurye–Olkin [14,18]):

Assume that X_i, $i = 1, \ldots, m$ are independent d-dimensional random vectors such that the linear forms $L_1 = b_1^t X_1 + \cdots + b_m^t X_m$ and $L_2 = c_1^t X_1 + \cdots + c_m^t X_m$ are independent, with $b_i, c_i \in GL(d, \mathbb{R})$ for $i = 1, \ldots, m$. Then X_i is Gaussian for all i.

The equation (4) (with changing the matrices b_i, c_i by automorphisms) and its applications to probability distributions, has been studied in great detail by Feld'man [13], for functions defined on locally compact commutative groups.

A more general specialization of (2) was considered by Ghurye and Olkin in [14]:

$$\sum_{i=1}^{m} f_i(x + c_i y) = A(x, y) + B(y, x), \tag{5}$$

where f_i map $\mathbb{R}^d$ to $\mathbb{C}$, and the functions A, B are such that, for each $y \in \mathbb{R}^d$, $A(x, y)$ and $B(x, y)$ are polynomials in the variable x with degrees not greater than r and s, respectively (here r, s do not depend on y). This equation has also proven to be a useful tool in the study of probability distributions (see, for example, [24, Chapter 7]).

The equation (1) can be formulated also for distributions, since the shift operator $\tau_y : f(x) \mapsto f(x + y)$ and dilation operator $\sigma_b : f(x) \mapsto f(bx)$ can be extended to the space $\mathcal{D}(\mathbb{R}^d)'$ of Schwartz complex-valued distributions (as the adjoint of the corresponding operators on $\mathcal{D}(\mathbb{R}^d)$). Our results in this setting extend the Anselone–Korevaar [9] theorem on finite-dimensional shift-invariant subspaces of $\mathcal{D}(\mathbb{R}^d)'$ and the results of [20,23] (see also [1–8]). They show in particular that the continuity restrictions on solutions and coefficients of (1) can be weakened at least to local integrability.

It should be underlined that in all settings if the functions or distributions u_k, v_k in (2) are linearly independent (which can always be assumed), then they are linear combinations of (shifted) functions f_i. Therefore, proving that f_i are exponential polynomials we simultaneously prove the same for u_k and v_k.

2. Solution of Eq. (2)

Our aim here is to establish the following result:

Theorem 1. *If the functions v_k are continuous, the matrices b_i, c_i and $b_i^{-1}c_i - b_j^{-1}c_j$ (for $i \neq j$) are invertible, then all solutions $f_i \in C(\mathbb{R}^d)$ of (2) are exponential polynomials.*

The condition that the matrices $b_i^{-1}c_i - b_j^{-1}c_j$ (for $i \neq j$) are invertible already existed in the literature. In particular, a condition of this type is explicitly stated in [28, Theorem 3.9].

Note that the substitution $\widetilde{f}_i(x) = f_i(b_i x)$ reduces Eq. (2) to the case that $b_i = I_d$, the identity matrix, that is to the equation

$$\sum_{i=0}^{m} f_i(x + c_i y) = \sum_{k=1}^{n} u_k(y) v_k(x). \tag{6}$$

So in what follows we mostly consider this case.

For $y \in \mathbb{R}^d$, let τ_y denote the shift operator on $C(\mathbb{R}^d)$: $\tau_y(f)(x) = f(x+y)$. Then denoting by W the subspace generated by $v_1, \ldots, v_n$, we may reformulate (6) saying that all functions $\sum_{i=1}^{m} \tau_{c_i y}(f_i)$ belong to W.

In the proof of Theorem 1 we will use the following result from [26]:

Proposition 2. *Let π be a continuous representation of a topologically finitely generated semigroup G on a topological linear space X. Suppose that a vector $x \in X$ and a finite-dimensional subspace $L \subset X$ have the property that for any $g \in G$, there is a finite-dimensional π-invariant subspace $R(g) \subset X$ with $\pi(g)(x) \in L + R(g)$. Then x belongs to a finite-dimensional π-invariant subspace of X.*

In this section Proposition 2 will be applied to the representation $y \mapsto \tau_y$ of the group $\mathbb{R}^d$ by shifts on the space $C(\mathbb{R}^d)$.

By the above, Theorem 1 is equivalent to the following:

Theorem 3. *Assume that $\{f_i\}_{i=1}^{m} \subset C(\mathbb{R}^d)$ and, for all $y \in \mathbb{R}^d$,*

$$\sum_{i=1}^{m} \tau_{c_i y}(f_i) \in W \textit{ for all } y \in \mathbb{R}^d \tag{7}$$

where $W \subset C(\mathbb{R}^d)$ is a finite-dimensional subspace. If all matrices c_i and $c_i - c_j$ (for $i \neq j$) are invertible, then all f_i are exponential polynomials.

Proof. We use induction on m. The case $m = 1$ is known. Indeed, in this case (6) is the Levi-Civita equation, its solutions on abelian groups are described, for example, in [10, Theorem 1] (see also [29–31] and references there).

Suppose that the statement is true whenever the number of summands appearing in the left-hand side of the equation is strictly smaller than m. Take arbitrary $h \in \mathbb{R}^d$ and substitute in (7) $y - c_1^{-1}h$ for y. Then applying the operator τ_h we will obtain

$$\sum_{i=1}^{m} \tau_{h + c_i(y - c_1^{-1}h)}(f_i) \in \tau_h(W) \text{ for all } y \in \mathbb{R}^d. \tag{8}$$

Comparing the left-hand sides of (7) and (8), and setting $W^* := \tau_h(W) + W$ we obtain

$$\sum_{i=2}^{m}(\tau_{c_iy+(I_d-c_ic_1^{-1})h}(f_i) - \tau_{c_iy}(f_i)) \in W^*, \text{ for all } y \in \mathbb{R}^d. \quad (9)$$

Clearly $\dim W^* < \infty$. Setting $d_i = I_d - c_ic_1^{-1}$, define, for a fixed h, the functions g_i by $g_i(x) = f_i(x + d_ih) - f_i(x)$. Then (9) will have the form

$$\sum_{i=2}^{m}\tau_{c_iy}(g_i) \in W^* \text{ for all } y \in \mathbb{R}^d. \quad (10)$$

By the induction hypothesis, we obtain that all functions g_i are continuous exponential polynomials.

The matrices $d_i = I_d - c_ic_1^{-1}$ are invertible, since

$$\ker d_i = c_1 \ker(c_i - c_1) = \{0\}.$$

Thus, the condition "$f_i(x+d_ih) - f_i(x)$ is a continuous exponential polynomial for all h" can be written as "$f_i(x + y) - f_i(x)$ is a continuous exponential polynomial for all y". Since any exponential polynomial is contained in an invariant finite-dimensional subspace, we see that each function $\tau_y f_i$ belongs to the sum of the one-dimensional subspace $\mathbb{C}f_i$ and some invariant finite-dimensional subspace. By Proposition 2, f_i is contained in an invariant finite-dimensional subspace, so f_i is an exponential polynomial. Here $i = 2, \ldots, m$, but clearly the same is true for f_1 by symmetry. □

3. A more general class of equations

Here we consider Eq. (3). Since the second term in the right-hand side of (3) is an exponential polynomial in x for each y, the study of this equation reduces to the following extension of Theorem 3.

Theorem 4. *Let $\{f_k\}_{k=1}^m \subset C(\mathbb{R}^d)$, W be a finite-dimensional subspace of $C(\mathbb{R}^d)$, $c_i \in GL(d, \mathbb{R})$. Suppose that, for each $y \in \mathbb{R}^d$, there is a finite-dimensional translation invariant space $R(y) \subset C(\mathbb{R}^d)$ with*

$$\sum_{i=1}^{m}\tau_{c_iy}(f_i) \in W + R(y). \quad (11)$$

If all matrices $c_i - c_j$ (for $i \neq j$) are invertible, then all f_i are exponential polynomials.

Proof. We proceed by induction on m. The case $m = 1$ is solved by Proposition 2, since c_1 is invertible. Take $m > 1$ and let $y \in \mathbb{R}^d$, so (11) holds for a finite-dimensional invariant subspace $R(y)$ of $C(\mathbb{R}^d)$. Choosing $h \in \mathbb{R}^d$, we apply the assumption to $y - c_1^{-1}h$:

$$\sum_{i=1}^{m} \tau_{c_i y}(f_i) \in W + R(y - c_1^{-1}h). \tag{12}$$

Applying the operator τ_h to both sides of (12) one obtains

$$\sum_{i=1}^{m} \tau_{h+c_i(y-c_1^{-1}h)}(f_i) \in \tau_h(W) + R(y - c_1^{-1}h). \tag{13}$$

Subtracting (12) from (13) we get

$$\sum_{i=2}^{m} (\tau_{c_i y+(I_d - c_i c_1^{-1})h}(f_i) - \tau_{c_i y}(f_i)) \in W_1 + R_1(y), \tag{14}$$

where $W_1 = \tau_h(W) + W$ and $R_1(y) = R(y) + R(y - c_1^{-1}h)$. Clearly W_1 and $R_1(y)$ are finite-dimensional and $R_1(y)$ is translation invariant.

Setting $d_i = I_d - c_i c_1^{-1}$ define the functions g_i by $g_i(x) = f_i(x + d_i h) - f_i(x)$. Then (14) will have the form

$$\sum_{i=2}^{m} \tau_{c_i y}(g_i) \in W_1 + R_1(y), \text{ for all } y \in \mathbb{R}^d.$$

Thus, the induction step confirms that all functions g_i, for $i = 2, \ldots, m$, are exponential polynomials. Since h is arbitrary, the proof can be finished in the same way as the proof of Theorem 3. □

Corollary 5. *If the functions v_k are continuous, the matrices b_i, c_i and $b_i^{-1}c_i - b_j^{-1}c_j$ (for $i \neq j$) are invertible, then all continuous solutions f_i of (3) are exponential polynomials.*

4. Distributions

The functions which are the coefficients of the equation (2) (as well as its solutions) could a priori belong to a more general class than $C(\mathbb{R}^d)$. To handle a more wide variety of situations we will now study it in the distributional setting.

We will distinguish two variants of the distributional view of Eq. (2). If u_k are usual (arbitrary!) functions while v_k are distributions then the equation means that the sum in the left-hand side for every y belongs to the linear span of the distributions v_k. So we come to the following setting:

Theorem 6. *Assume that $\{f_k\}_{k=1}^m \subset \mathcal{D}(\mathbb{R}^d)'$ and, for all $y \in \mathbb{R}^d$,*

$$\sum_{i=1}^{m} \tau_{c_i y}(f_i) \in W \text{ for all } y \in \mathbb{R}^d$$

for an n-dimensional subspace W of $\mathcal{D}(\mathbb{R}^d)'$. If all matrices c_i and $c_i - c_j$ (for $i \neq j$) are invertible, then all f_k are continuous exponential polynomials.

Proof. The proof is similar to the proof of Theorem 3 and uses, in a fundamental form, the distributional part of the Anselone–Korevaar theorem [9]. In particular, the case $m = 1$ follows from that result. The induction step goes as above with the only distinction that we apply Proposition 2 to the representation of $\mathbb{R}^d$ by shifts on the space $\mathcal{D}(\mathbb{R}^d)'$. □

Corollary 7. *The statements of Theorem 1 and Corollary 5 extend to the case that v_k and f_i are locally summable.*

On the other hand one can consider (2) in the case that both u_k and v_k are distributions. In this setting we should regard both sides of the equation as elements of $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)'$.

Theorem 8. *Assume that*

$$\sum_{i=1}^{m} f_i(x + c_i y) = \sum_{k=1}^{n} u_k(y) v_k(x), \tag{15}$$

where $f_i, u_k, v_k \in \mathcal{D}(\mathbb{R}^d)'$, $c_i \in GL(d, \mathbb{R})$ and both sides of (15) are considered as elements of $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)'$. If all matrices $c_i - c_j$ (for $i \neq j$) are invertible, then f_k is a continuous exponential polynomial for $k = 1, \ldots, m$.

Proof. Let us denote by $\Delta_{(h,k)}$ the general difference operator in $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)'$ given by:

$$\langle \Delta_{(h,k)} F(x, y), \phi(x, y) \rangle := \langle F(x, y), \phi(x - h, y - k) - \phi(x, y) \rangle,$$

where $\phi \in \mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$ denotes an arbitrary test function. We will use the fact that the equality

$$\Delta_{(h,k)}(f(x + cy)) = (\Delta_{h+ck}(f))(x + cy) \tag{16}$$

holds for all $f \in \mathcal{D}(\mathbb{R}^d)'$, $h, k \in \mathbb{R}^d$, and $c \in GL(d, \mathbb{R})$; its validity can be checked by direct calculation.

As above we proceed by induction on m, the number of summands in the left-hand side of the equation (15). As we have already noticed, the case $m = 1$ of this equation is known (see, e.g., [11,12] for $d = 1$ and [25,27] for domains of $\mathbb{R}^d$). Assume the result holds true whenever we have less than m summands. Let f_i, u_k, v_k satisfy (15).

Let us apply the operator $\Delta_{(h, -c_1^{-1} h)}$ to both sides of the Eq. (this is equivalent to substituting x by $x + h$ and y by $y - c_1^{-1} h$ in the equation). Then (16) implies that

$$\Delta_{(h,-c_1^{-1}h)}\left[\sum_{i=1}^m f_i(x+c_iy)\right] = \sum_{i=1}^m \Delta_{(h,-c_1^{-1}h)} f_i(x+c_iy)$$
$$= \sum_{i=1}^m (\Delta_{h-c_ic_1^{-1}h}(f_i))(x+c_iy)$$
$$= \sum_{i=2}^m (\Delta_{(I_d-c_ic_1^{-1})h}(f_i))(x+c_iy)$$
$$= \sum_{i=2}^m g_i(x+c_iy),$$

with $g_i = \Delta_{(I_d-c_ic_1^{-1})h}(f_i) \in \mathcal{D}(\mathbb{R}^d)'$ for $i = 2, \ldots, m$. Hence, after applying the operator $\Delta_{(h,-c_1^{-1}h)}$ to the left-hand side of the equation, we reduce by 1 the number of summands in the equation. On the other hand, in the right-hand side of the equation we get

$$\Delta_{(h,-c_1^{-1}h)}\left(\sum_{k=1}^n u_k(y)v_k(x)\right) = \sum_{k=1}^n \tau_{-c_1^{-1}h}(u_k)(y)\tau_h(v_k)(x) - \sum_{k=1}^n u_k(y)v_k(x),$$

which is an expression of the form

$$\sum_{k=1}^{2n} U_k(y)V_k(x)$$

with $U_k, V_k \in \mathcal{D}(\mathbb{R}^d)'$ for $k = 1, \ldots, 2n$. Hence we can use the induction hypothesis to conclude that $g_i = \Delta_{(I_d-c_ic_1^{-1})h}(f_i) \in \mathcal{D}(\mathbb{R})'$ is a continuous exponential polynomial for $i = 2, \ldots, m$. As in the proof of Theorem 6 we conclude, using Proposition 2, that all f_i are continuous exponential polynomials. □

Corollary 9. *Assume that*

$$\sum_{i=1}^m f_i(b_ix+c_iy) = \sum_{k=1}^n u_k(y)v_k(x),$$

where $f_i, u_k, v_k \in \mathcal{D}(\mathbb{R}^d)'$ *and* $b_i, c_i \in GL(d, \mathbb{R})$. *If all matrices* $b_i^{-1}c_i - b_j^{-1}c_j$ *(for* $i \neq j$*) are invertible, then* f_k *is a continuous exponential polynomial for* $k = 1, \ldots, m$.

As a consequence, the results of Kakutani-Nagumo, Walsh, Ghurie-Olkin and others mentioned in the Introduction, extend to distributions. For example, we have

Theorem 10. *Assume that* $f_i, a_\alpha, b_\beta \in \mathcal{D}(\mathbb{R}^d)'$ *for* $1 \leq i \leq m$, $0 \leq |\alpha| \leq r$ *and* $0 \leq |\beta| \leq s$, *and equation* (5) *is satisfied with* $A(x,y) = \sum_{|\alpha|\leq r} x^\alpha \cdot a_\alpha(y)$ *and* $B(y,x) = \sum_{|\beta|\leq s} b_\beta(x) \cdot y^\beta$. *Assume, furthermore, that all matrices* c_i *(for all* *i) and* $c_i - c_j$ *(for* $i \neq j$*) are invertible. Then all* f_i *are (in the distributional sense) ordinary polynomials.*

Proof. It follows from Theorem 6 that f_i are continuous exponential polynomials, which implies immediately the same about $b_\beta(x)$. Therefore one can apply results of [14] where the statement was proved for continuous functions. □

Compliance with ethical standards

Conflict of interest The authors declare that there are no conflicts of interest.

References

[1] Aksoy, A., Almira, J.M.: On Montel and Montel–Popoviciu Theorems in several variables. Aequat. Math. **89**(5), 1335–1357 (2015)
[2] Almira, J.M.: Montel's theorem and subspaces of distributions which are Δ^m-invariant. Numer. Funct. Anal. Optim. **35**(4), 389–403 (2014)
[3] Almira, J.M.: On Popoviciu–Ionescu functional equation. Ann. Math. Sil. **30**, 5–15 (2016)
[4] Almira, J.M.: On Loewner's characterization of polynomials. Jaén J. Approx. **8**(2), 175–181 (2016)
[5] Almira, J.M., Abu-Helaiel, K.F.: On Montel's theorem in several variables. Carpath. J. Math. **31**, 1–10 (2015)
[6] Almira, J.M., Abu-Helaiel, K.F.: A note on invariant subspaces and the solution of some classical functional equations. Ann. Tiberiu Popoviciu Semin. Funct. Eq. Approx. Convexity **11**, 3–17 (2013)
[7] Almira, J.M., Székelyhidi, L.: Local polynomials and the Montel theorem. Aequat. Math. **89**, 329–338 (2015)
[8] Almira, J.M., Székelyhidi, L.: Montel-type theorems for exponential polynomials. Demonstr. Math. **49**(2), 197–212 (2016)
[9] Anselone, P.M., Korevaar, J.: Translation invariant subspaces of finite dimension. Proc. Am. Math. Soc. **15**, 747–752 (1964)
[10] Engert, M.: Finite-dimensional translation invariant subspaces. Pacific J. Maths. **32**(2), 333–343 (1970)
[11] Fenyö, I.: Über eine Lösungsmethode gewisser Funktionalgleichungen. Acta Math. Acad. Sci. Hungar. **7**, 383–396 (1956)
[12] Fenyö, I.: Sur les équations distributionnelles. In: Forte, B. (ed.) Functional Equations and Inequalities. C.I.M.E. Summer Schools, vol. 54, pp. 45–109. Springer, Berlin (1971) (Reprinted in 2010)
[13] Feldman, G.: Functional Equations and Characterization Problems on Locally Compact Abelian Groups. In: EMS Tracts in Mathematics, vol. 5, European Mathematical Society (EMS), Zurich (2008)
[14] Ghurye, S.G., Olkin, I.: A characterization of the multivariate normal distribution. Ann. Math. Stat. **33**(2), 533–541 (1962)
[15] Haruki, S.: On the mean value property of harmonic and complex polynomials. Proc. Japan Acad. Ser. A **57**, 216–218 (1981)
[16] Haruki, S.: On the theorem of S. Kakutani-M. Nagumo and J.L. Walsh for the mean value property of harmonic and complex polynomials. Pacific J. Math. **94**(1), 113–123 (1981)
[17] Haruki, S.: On two functional equations connected with a mean-value property of polynomials. Aequat. Math. **6**, 275–277 (1971)

[18] Kagan, A.M., Linnik, YuV, Rao, C.R.: Characterization Problems in Mathematical Statistics. Wiley, Hoboken (1973)
[19] Kakutani, S., Nagumo, M.: About the functional equation $\sum_{v=0}^{n-1} f(z+e^{(\frac{2v\pi}{2})i}) = nf(z)$. Zenkoku Shijô Danwakai **66**, 10–12 (1935). (in Japanese)
[20] Leland, K.O.: Finite-dimensional translation invariant spaces. Am. Math. Monthly **75**, 757–758 (1968)
[21] Levi-Cività, T.: Sulle funzioni che ammettono una formula d'addizione del tipo $f(x+y) = \sum_{i=1}^{n} X_i(x)Y_i(y)$. Atti Accad. Naz. Lincei Rend. **22**(5), 181–183 (1913)
[22] Linnik, YuV: A remark on the Cramer's theorem on the decomposition of the normal law. Theory Probab. Appl. **1**(4), 435–436 (1956)
[23] Loewner, C.: On some transformation semigroups invariant under Euclidean and non-Euclidean isometries. J. Math. Mech. **8**, 393–409 (1959)
[24] Mathai, A.M., Pederzoli, G.: Characterizations of the Normal Probability Law. Wiley Eastern Limited New Delhi (1977)
[25] Shulman, E.: Addition theorems for distributions on domains in $\mathbb{R}^n$ (to appear)
[26] Shulman, E.: Decomposable functions and representations of topological semigroups. Aequat. Math. **79**, 13–21 (2010)
[27] Shulman, E.: Functional equations of homological type. Ph.D. Thesis, Moscow State Institute of Elektronics and mathematics (1994)
[28] Székelyhidi, L.: On a class of linear functional equations. Publ. Math. (Debrecen) 29:19–28 (1982)
[29] Székelyhidi, L.: *On the Levi-Civita functional equation*, Berichte der Mathematisch-Statistischen Sektion in der Forschungsgesellschaft Joanneum, 301, Forschungszentrum Graz, Mathematisch-Statistische Sektion, Graz (1988)
[30] Székelyhidi, L.: Convolution type Functional Equations on Topological Abelian Groups. World Scientific, Singapore (1991)
[31] Stetkær, H.: Functional Equations on Groups. World Scientific, Singapore (2013)
[32] Walsh, J.L.: A mean value theorem for polynomials and harmonic polynomials. Bull. Am. Math. Soc. **42**, 923–930 (1936)
[33] Wilson, W.H.: On a certain general class of functional equations. Am. J. Math. **40**(3), 263–282 (1918)

J. M. Almira
Departamento de Matemáticas, E.P.S. Linares
Universidad de Jaén
Campus Científico Tecnológico de Linares
23700 Linares
Spain

and

Departamento de Ingeniería y Tecnología de Computadores, Facultad de Informática
Universidad de Murcia
Campus de Espinardo
30100 Murcia
Spain

e-mail: jmalmira@ujaen.es;
jmalmiraujaen@gmail.com

E. V. Shulman
Department of Mathematics
Vologda State University
S. Orlova 6
Vologda
Russia 160035
e-mail: shulmanka@gmail.com;
ekaterina.shulman@us.edu.pl

and

Instytut of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland

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