Aequat. Math. 91 (2017), 419–428
© Springer International Publishing 2017
0001-9054/17/030419-10
published online April 4, 2017
DOI 10.1007/s00010-017-0476-9

Aequationes Mathematicae

Extensions of Jacobson's lemma for Drazin inverses

Dijana Mosić

Abstract. We study the generalization of Jacobson's lemma for the group inverse, Drazin inverse, generalized Drazin inverse and pseudo Drazin inverse of $1 - bd$ (or $1 - ac$) in a ring when $1-ac$ (or $1-bd$) has a corresponding inverse, $acd = dbd$ and $bdb = bac$ (or $dba = aca$). Thus, we recover some recent results.

Mathematics Subject Classification. 16B99, 15A09, 16U99.

Keywords. Jacobson's lemma, Group inverse, Drazin inverse, Generalized Drazin inverse, Pseudo Drazin inverse, Ring.

1. Introduction

Let $\mathcal{R}$ be a ring with the unit 1. We use $\mathcal{R}^{-1}$ and $\mathcal{R}^{nil}$ to denote the set of all invertible and nilpotent elements of $\mathcal{R}$, respectively.

Recall that an element $a \in \mathcal{R}$ has a Drazin inverse [3] if there exists $x \in \mathcal{R}$ such that

$$xax = x, \quad ax = xa \quad \text{and} \quad a^k = a^{k+1}x,$$

for some $k \geq 0$. The smallest such integer k is called the Drazin index of a, denoted $\operatorname{ind}(a)$. The element x above is unique if it exists and is denoted by a^D. The notation a^π means $1 - aa^D$ for any Drazin invertible element $a \in \mathcal{R}$. Observe that by the definition of the Drazin inverse, $aa^\pi \in \mathcal{R}^{nil}$ and the nilpotency index of aa^π is the Drazin index of a. If $\operatorname{ind}(a) = 1$, then a is group invertible and the group inverse of a is denoted by $a^\#$. Thus, $a^\#$ satisfies $a^\# a a^\# = a^\#$, $a^\# a = aa^\#$ and $aa^\# a = a$. The subsets of $\mathcal{R}$ composed of Drazin invertible and group invertible elements will be denoted by $\mathcal{R}^D$ and $\mathcal{R}^\#$, respectively.

Dijana Mosić: The author is supported by the Ministry of Education and Science, Republic of Serbia, Grant No. 174007.

Birkhäuser

Lemma 1.1. [5] *Let* $a \in \mathcal{R}$. *Then* $a \in \mathcal{R}^{\#}$ *if and only if* $a \in a^2\mathcal{R} \cap \mathcal{R}a^2$. *Moreover, if* $a = xa^2 = a^2y$ *for some* $x, y \in \mathcal{R}$, *then* $a^{\#} = xay = x^2a = ay^2$.

Lemma 1.2. *Let* $a \in \mathcal{R}$. *Then* a *is Drazin invertible if and only if* a^k *is group invertible for some* $k \geq 1$. *In addition,* $a^D = a^{k-1}(a^k)^{\#} = (a^k)^{\#}a^{k-1}$.

For any element $a \in \mathcal{R}$ the commutant and the double commutant of a, respectively, are defined by

$$\mathrm{comm}(a) = \{x \in \mathcal{R} : ax = xa\},$$

$$\mathrm{comm}^2(a) = \{x \in \mathcal{R} : xy = yx \text{ for all } y \in \mathrm{comm}(a)\}.$$

If $a \in \mathcal{R}^D$, then $a^D \in \mathrm{comm}^2(a)$ [6].

In [4], quasinilpotent elements of a ring $\mathcal{R}$ are introduced as follows: $q \in \mathcal{R}$ is quasinilpotent, if $1 + xq \in \mathcal{R}^{-1}$ for all $x \in \mathrm{comm}(q)$. We use $\mathcal{R}^{qnil}$ to denote the set of all quasinilpotent elements of $\mathcal{R}$.

The generalized Drazin inverse of $a \in \mathcal{R}$ is defined in [6] as the element $a^d = x$ satisfying:

$$x \in \mathrm{comm}^2(a), \quad xax = x \quad \text{and} \quad a(1 - ax) \in \mathcal{R}^{qnil}.$$

If a^d exists, then it is unique [6]. In Banach algebras it is enough to assume $x \in \mathrm{comm}(a)$ instead of $x \in \mathrm{comm}^2(a)$. We use $\mathcal{R}^d$ to denote the set of all generalized Drazin invertible elements of $\mathcal{R}$.

Lemma 1.3. [6, Theorem 4.2] *Let* $a \in \mathcal{R}$. *Then* $a \in \mathcal{R}^d$ *if and only if there exists* $p = p^2 \in \mathcal{R}$ *such that*

$$p \in \mathrm{comm}^2(a), \qquad a + p \in \mathcal{R}^{-1} \qquad \text{and} \qquad ap \in \mathcal{R}^{qnil}.$$

In this case, $p = 1 - aa^d$ *is a spectral idempotent of* a *and will be denoted by* a^{π}.

Wang and Chen [7] introduced the pseudo Drazin inverse in associative rings as an intermedium between the Drazin inverse and generalized Drazin inverse. An element $a \in \mathcal{R}$ is pseudo Drazin invertible if there exists $x \in \mathcal{R}$ such that

$$x \in \mathrm{comm}^2(a), \quad xax = x \quad \text{and} \quad a^k - a^{k+1}x \in J(\mathcal{R}),$$

for some $k \geq 0$, where $J(\mathcal{R})$ is the Jacobson radical of $\mathcal{R}$. Any element $x \in \mathcal{R}$ satisfying the above equations is called a pseudo Drazin inverse of a, which is unique if it exists, and is denoted by a^{pD}. The set of all pseudo Drazin invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{pD}$. Also, $a^{\pi} = 1 - aa^{pD}$.

Jacobson's lemma states that if $1 - ab$ is invertible, then so is $1 - ba$, i.e. the following holds:

Lemma 1.4. *Let* $a, b \in \mathcal{R}$. *If* $1 - ab \in \mathcal{R}^{-1}$, *then* $1 - ba \in \mathcal{R}^{-1}$ *and* $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$.

In recent years, it has been proved that Jacobson's lemma has suitable analogues for the group, Drazin and generalized Drazin inverses [1,9].

Corach et al. [2] generalized Jacobson's lemma to the case that $aba = aca$. Precisely, they showed that if $1 - ab$ is invertible and $aba = aca$, then $1 - ba$ is invertible too and $(1 - ba)^{-1} = 1 + b(1 - ac)^{-1}a$.

In [8], a new extension of Jacobson's lemma for bounded linear operators between Banach spaces, was studied whenever $acd = dbd$ and $dba = aca$. Evidently, for $d = a$, $aba = aca$.

Notice that, when $acd = dbd$, $bdb = bac$ and $1-ac$ is invertible, then $1-bd$ is invertible too and

$$(1 - bd)^{-1} = 1 + b(1 - ac)^{-1}d.$$

If $acd = dbd$, $dba = aca$ and $1 - bd$ is invertible, we observe that $1 - ac$ is invertible and

$$(1 - ac)^{-1} = 1 + [1 + d(1 - bd)^{-1}b]ac.$$

In this paper, we investigate the generalization of Jacobson's lemma in a ring when $acd = dbd$ and ($bdb = bac$ or $dba = aca$). In the case that $acd = dbd$ and $bdb = bac$, we prove that if $1 - ac$ is group invertible, Drazin invertible, generalized Drazin invertible or pseudo Drazin invertible, then so is $1-bd$ and give expressions for the group, Drazin, generalized Drazin and pseudo Drazin inverses of $1 - bd$ in terms of the corresponding inverse of $1 - ac$. Also, we study the group and Drazin invertibility of $1 - ac$ when $acd = dbd$, $dba = aca$ and $1 - bd$ is group or Drazin invertible. As a consequence of these results, we get some results in [1,9]. In the end, we state as a conjecture the generalized Drazin and pseudo Drazin invertibility of $1 - ac$ when $acd = dbd$, $dba = aca$ and $1 - bd$ is generalized Drazin or pseudo Drazin invertible.

2. Extensions of Jacobson's lemma

In the first theorem of this section, if $1 - ac$ is group invertible, we prove that $1 - bd$ is group invertible under the conditions $acd = dbd$ and $bdb = bac$.

Theorem 2.1. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $bdb = bac$. If $1-ac \in \mathcal{R}^{\#}$, then $1 - bd \in \mathcal{R}^{\#}$ and*

$$(1 - bd)^{\#} = 1 + b[(1 - ac)^{\#} - (1 - ac)^{\pi}]d. \tag{1}$$

Proof. Denote by y the right hand side of (1). Then

$$\begin{aligned}(1 - bd)y &= 1 - bd + b(1 - ac)[(1 - ac)^{\#} - (1 - ac)^{\pi}]d \\ &= 1 - b(1 - ac)^{\pi}d.\end{aligned}$$

In the same way, we get $y(1-bd) = 1-b(1-ac)^\pi d$. Thus, $(1-bd)y = y(1-bd)$. Further, we have

$$\begin{aligned}(1-bd)y(1-bd) &= [1-b(1-ac)^\pi d](1-bd)\\ &= 1-bd-b(1-ac)^\pi(1-ac)d\\ &= 1-bd.\end{aligned}$$

Since db commutes with $1-ac$, we deduce that db commutes with $(1-ac)^\#$ and $(1-ac)^\pi$. Now, as

$$\begin{aligned}y(1-bd)y &= y[1-b(1-ac)^\pi d]\\ &= y-b(1-ac)^\pi d - b[(1-ac)^\# - (1-ac)^\pi](1-ac)^\pi dbd\\ &= y-b(1-ac)^\pi d + b(1-ac)^\pi acd\\ &= y-b(1-ac)^\pi(1-ac)d\\ &= y,\end{aligned}$$

we conclude that $1-bd \in \mathcal{R}^\#$ and $(1-bd)^\# = y$. □

If $c=b$ and $d=a$ in Theorem 2.1, we obtain [1, Theorem 3.5]:

Corollary 2.1. *Let $a, b \in \mathcal{R}$. If $1-ab \in \mathcal{R}^\#$, then $1-ba \in \mathcal{R}^\#$ and*

$$(1-ba)^\# = 1+b[(1-ab)^\# - (1-ab)^\pi]a.$$

In a ring $\mathcal{R}$ with involution (which is any map $* : \mathcal{R} \to \mathcal{R}$ satisfying $(b^*)^* = b$, $(by)^* = y^*b^*$, $(b+y)^* = b^*+y^*$, for any $b, y \in \mathcal{R}$), an element $a \in \mathcal{R}$ is Moore–Penrose invertible if there exists a unique element $x = a^\dagger \in \mathcal{R}$ such that $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$. Recall that an element a is EP if a is Moore-Penrose invertible and $aa^\dagger = a^\dagger a$ which is equivalent to that a is group invertible and $(a^\pi)^* = a^\pi$ [6]. If $1-ac$ is EP in a ring with involution, the necessary and sufficient conditions for $1-bd$ to be EP, are given now applying Theorem 2.1.

Corollary 2.2. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $bdb = bac$. If $1-ac$ is EP, then $1-bd$ is EP if and only if $b(1-ac)^\pi d = d^*(1-ac)^\pi b^*$. In addition, $(1-bd)^\dagger$ is represented by (1).*

To prove in a ring that $1-ac \in \mathcal{R}^\#$ in the case that $1-bd \in \mathcal{R}^\#$, we replace the condition $bdb = bac$ of Theorem 2.1 with $dba = aca$ and obtain an expression for $(1-ac)^\#$.

Theorem 2.2. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. If $1-bd \in \mathcal{R}^\#$, then $1-ac \in \mathcal{R}^\#$ and*

$$(1-ac)^\# = 1+ac+d[(1-bd)^\# - 2(1-bd)^\pi]bac.$$

Proof. If we denote $y = 1+ac+d[(1-bd)^\# - (1-bd)^\pi]bac$, then we can check, as in the proof of Theorem 2.1, that $(1-ac)y = y(1-ac)$ and $(1-ac)y(1-ac) = 1-ac$. Using Lemma 1.1, we obtain that $1-ac \in \mathcal{R}^\#$ and $(1-ac)^\# = y(1-ac)y = y-d(1-bd)^\pi bac$. □

Exchanging the roles of a and b, and c and d in Theorem 2.2, we get the next result with a hypothesis and conclusion which are different from Theorem 2.1.

Corollary 2.3. *Let $a, b, c, d \in \mathcal{R}$ satisfy $bdc = cac$ and $cab = bdb$. If $1-ac \in \mathcal{R}^{\#}$, then $1 - bd \in \mathcal{R}^{\#}$ and*

$$(1 - bd)^{\#} = 1 + bd + c[(1 - ac)^{\#} - 2(1 - ac)^{\pi}]abd.$$

In the case that $d = a$ in Theorem 2.2, we get the following result as a consequence.

Corollary 2.4. *Let $a, b, c \in \mathcal{R}$ satisfy $aba = aca$. If $1 - ba \in \mathcal{R}^{\#}$, then $1 - ac \in \mathcal{R}^{\#}$ and*

$$(1 - ac)^{\#} = 1 + ac + a[(1 - ba)^{\#} - 2(1 - ba)^{\pi}]bac.$$

Remark. Let $a, b \in \mathcal{R}$ and $1 - ba \in \mathcal{R}^{\#}$. If we suppose that $c = b$ in Corollary 2.4, then $1 - ab \in \mathcal{R}^{\#}$ and

$$(1 - ab)^{\#} = 1 + ab + a[(1 - ba)^{\#} - 2(1 - ba)^{\pi}]bab := X_1.$$

Exchanging the roles of a and b in Corollary 2.1, $1 - ab \in \mathcal{R}^{\#}$ and

$$(1 - ab)^{\#} = 1 + a[(1 - ba)^{\#} - (1 - ba)^{\pi}]b := X_2.$$

Notice that these two expressions X_1 and X_2 for $(1 - ab)^{\#}$ are equal, since

$$\begin{aligned} X_1 &= 1 + ab + a(1 - ba)^{\#}b - a(1 - ba)^{\#}(1 - ba)b - a(1 - ba)^{\pi}bab \\ &\quad + a(1 - ba)^{\pi}(1 - ba)b - a(1 - ba)^{\pi}b \\ &= 1 + ab + a(1 - ba)^{\#}b - a(1 - ba)^{\#}(1 - ba)b - abab \\ &\quad + a(1 - ba)^{\#}(1 - ba)bab - a(1 - ba)^{\pi}b \\ &= 1 + a(1 - ba)b + a(1 - ba)^{\#}b - a(1 - ba)^{\#}(1 - ba)^2b - a(1 - ba)^{\pi}b \\ &= X_2. \end{aligned}$$

Using Theorem 2.1, we verify the Drazin invertibility of $1-bd$, when $1-ac$ is Drazin invertible. Throughout this section, if the lower limit of a sum is greater than its upper limit, we always define the sum to be 0. For example, the sum $\sum_{k=0}^{-1} * = 0$ and so the following theorem recovers the cases $1 - ac \in \mathcal{R}^{-1}$ (for $k = 0$) and $1 - ac \in \mathcal{R}^{\#}$ (for $k = 1$).

Theorem 2.3. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $bdb = bac$. If $1-ac \in \mathcal{R}^{D}$, then $1 - bd \in \mathcal{R}^{D}$ and*

$$(1 - bd)^{D} = 1 + b[(1 - ac)^{D} - (1 - ac)^{\pi}r]d,$$

where $r = \sum_{j=0}^{k-1}(1 - ac)^{j}$ and $\mathrm{ind}(1 - ac) = k$.

Proof. Suppose that $k \geq 2$, $s = \sum_{j=0}^{k-1}(1-db)^j$. Since $1-rac = (1-ac)^k \in \mathcal{R}^{\#}$,

$$racsd = racd\sum_{j=0}^{k-1}(1-bd)^j = rdbd\sum_{j=0}^{k-1}(1-bd)^j = \sum_{j=0}^{k-1} d(1-bd)^j bsd = sdbsd$$

and

$$brac = \sum_{j=0}^{k-1}(1-bd)^j bac = \sum_{j=0}^{k-1}(1-bd)^j bdb = bsdb,$$

by Theorem 2.1, $1 - bsd \in \mathcal{R}^{\#}$ and

$$\begin{aligned}(1-bsd)^{\#} &= 1 + b[(1-rac)^{\#} - (1-rac)^{\pi}]sd \\ &= 1 + b[((1-ac)^D)^k - (1-ac)^{\pi}]sd.\end{aligned}$$

From $1-bsd = (1-bd)^k$ and Lemma 1.2, $1-bd \in \mathcal{R}^D$ and, for $s' = \sum_{j=0}^{k-2}(1-db)^j$,

$$\begin{aligned}(1-bd)^D &= [(1-bd)^k]^{\#}(1-bd)^{k-1} \\ &= (1-bd)^{k-1} + b[((1-ac)^D)^k - (1-ac)^{\pi}](1-ac)^{k-1}sd \\ &= 1 - bs'd + b(1-ac)^D(1+(1-ac)s')d - b(1-ac)^{\pi}(1-ac)^{k-1}d \\ &= 1 - b[(1-ac)^D - (1-ac)^{\pi}s' - (1-ac)^{\pi}(1-ac)^{k-1}]d \\ &= 1 - b[(1-ac)^D - (1-ac)^{\pi}s]d \\ &= 1 - b[(1-ac)^D - (1-ac)^{\pi}r]d.\end{aligned}$$

□

For $c = b$ and $d = a$ in Theorem 2.3, we have [1, Theorem 3.6].

Corollary 2.5. *Let $a, b \in \mathcal{R}$. If $1-ab \in \mathcal{R}^D$, then $1-ba \in \mathcal{R}^D$ and*

$$(1-ba)^D = 1 + b[(1-ab)^D - (1-ab)^{\pi}r_1]a,$$

where $r_1 = \sum_{j=0}^{k-1}(1-ab)^j$ and $\operatorname{ind}(1-ab) = k$.

Like Theorem 2.3, we prove the following result.

Theorem 2.4. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. If $1-bd \in \mathcal{R}^D$, then $1-ac \in \mathcal{R}^D$ and*

$$\begin{aligned}(1-ac)^D = {} & (1+sac)(1-ac)^{k-1} + s^2d(1-bd)^D bac \\ & -2d(1-bd)^{\pi}(1-bd)^{k-1}bac,\end{aligned}$$

where $s = \sum_{j=0}^{k-1}(1-db)^j$ and $\operatorname{ind}(1-bd) = k \geq 1$.

If $d = a$ in Theorem 2.4, we get the next expression for $(1-ac)^D$ in terms of $(1-ba)^D$.

Corollary 2.6. *Let $a, b, c \in \mathcal{R}$ satisfy $aba = aca$. If $1 - ba \in \mathcal{R}^D$, then $1 - ac \in \mathcal{R}^D$ and*

$$(1-ac)^D = (1+r_1ac)(1-ac)^{k-1} + r_1^2 a(1-ba)^D bac \\ -2d(1-ba)^\pi(1-ba)^{k-1}bac,$$

where $r_1 = \sum_{j=0}^{k-1}(1-ab)^j$ and $\operatorname{ind}(1-ba) = k \geq 1$.

Under the assumptions $acd = dbd$ and $bdb = bac$, we prove that the generalized Drazin invertibility of $1-ac$ implies the generalized Drazin invertibility of $1 - bd$ in a ring.

Theorem 2.5. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $bdb = bac$. If $1-ac \in \mathcal{R}^d$, then $1 - bd \in \mathcal{R}^d$ and*

$$(1-bd)^d = 1 + b[(1-ac)^d - (1-ac)^\pi(1-(1-ac)^\pi(1-ac))^{-1}]d. \qquad (2)$$

Proof. Let y be the right hand side of (2), $\alpha = 1 - ac$ and $\beta = 1 - bd$. Then, by Lemma 1.3, $1 - \alpha^\pi\alpha \in \mathcal{R}^{-1}$ and

$$\begin{aligned} y(1-bd) &= 1 - bd + b[\alpha^d - \alpha^\pi(1-\alpha^\pi\alpha)^{-1}]\alpha d \\ &= 1 - b\alpha^\pi d - b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}\alpha d \\ &= 1 - b\alpha^\pi[1 + (1-\alpha^\pi\alpha)^{-1}\alpha\alpha^\pi]d \\ &= 1 - b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d. \end{aligned}$$

Since db commutes with α, we deduce that db commutes with α^d, α^π and $(1-\alpha^\pi\alpha)^{-1}$. Hence,

$$\begin{aligned} y(1-bd)y &= y - b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d + bdb\alpha^\pi(1-\alpha^\pi\alpha)^{-2}d \\ &= y - b\alpha^\pi(1-\alpha^\pi\alpha)^{-2}(1-\alpha^\pi\alpha - ac)d \\ &= y - b\alpha^\pi(1-\alpha^\pi\alpha)^{-2}(\alpha^\pi\alpha - \alpha^\pi\alpha)d \\ &= y. \end{aligned}$$

To prove that

$$(1-bd) - (1-bd)y(1-bd) = b\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \in \mathcal{R}^{qnil},$$

assume that $z \in \mathcal{R}$ satisfies $b\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dz = zb\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d$. Then $db\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dzb = dzb\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}db$ which gives, since db commutes with α, $\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}acdzb = dzbac\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}$. Now, from $ac\alpha^\pi = \alpha^\pi(1-\alpha^\pi\alpha)$, we get $\alpha\alpha^\pi dzb = dzb\alpha\alpha^\pi$. Because $\alpha\alpha^\pi \in \mathcal{R}^{qnil}$ and $\alpha\alpha^\pi$ commutes with $(1-\alpha^\pi\alpha)^{-1}dzb$, we have that $1 + (1-\alpha^\pi\alpha)^{-1}dzb\alpha\alpha^\pi \in \mathcal{R}^{-1}$. Using Lemma 1.4, we have $1 + b\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dz \in \mathcal{R}^{-1}$ which yields that $b\alpha\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \in \mathcal{R}^{qnil}$.

In order to show that $y \in \operatorname{comm}^2(1-bd)$, suppose that, for $z \in \mathcal{R}$, $z(1-bd) = (1-bd)z$. So, $zbd = bdz$ and $dzb\alpha = dz\beta b = d\beta zb = \alpha dzb$. Because dzb

commutes with α, notice that dzb commutes with α^d, α^π and $(1-\alpha^\pi\alpha)^{-1}$. From

$$\begin{aligned} zb\alpha^\pi d &= zb\alpha^\pi(1-\alpha^\pi\alpha)(1-\alpha^\pi\alpha)^{-1}d = zbac\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \\ &= zbdb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d = bdzb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \\ &= b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dzbd = b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dbdz \\ &= bac\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dz = b\alpha^\pi(1-\alpha^\pi\alpha)(1-\alpha^\pi\alpha)^{-1}dz \\ &= b\alpha^\pi dz, \end{aligned}$$

we have $zb\alpha\alpha^d d = b\alpha\alpha^d dz$, that is

$$zb\alpha^d d - zbdb\alpha^d d = b\alpha^d dz - b\alpha^d dbdz.$$

Since $zbdb\alpha^d d = bdzb\alpha^d d = b\alpha^d dzbd = b\alpha^d dbdz$, we obtain

$$zb\alpha^d d = b\alpha^d dz. \tag{3}$$

The equalities

$$\begin{aligned} b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dzb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d &= bdzb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \\ &= zbac\alpha^\pi(1-\alpha^\pi\alpha)^{-2}d \\ &= zb\alpha^\pi(1-\alpha^\pi\alpha)(1-\alpha^\pi\alpha)^{-2}d \\ &= zb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \end{aligned}$$

and

$$\begin{aligned} b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}dzb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d &= b(1-\alpha^\pi\alpha)^{-2}\alpha^\pi dzbd \\ &= b(1-\alpha^\pi\alpha)^{-2}\alpha^\pi acdz \\ &= b(1-\alpha^\pi\alpha)^{-1}\alpha^\pi dz \end{aligned}$$

imply

$$zb\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d = b(1-\alpha^\pi\alpha)^{-1}\alpha^\pi dz. \tag{4}$$

Using (3) and (4), we conclude that $zy = yz$. Thus, $y \in \mathrm{comm}^2(1-bd)$ and, by the definition of the generalized Drazin inverse, $1-bd \in \mathcal{R}^d$ and $(1-bd)^d = y$. □

In the case that $c = b$ and $d = a$ in Theorem 2.5, we recover [9, Theorem 2.3].

Corollary 2.7. *Let $a, b \in \mathcal{R}$. If $1-ab \in \mathcal{R}^d$, then $1-ba \in \mathcal{R}^d$ and*

$$(1-ba)^d = 1 + b[(1-ab)^d - (1-ab)^\pi(1-(1-ab)^\pi(1-ab))^{-1}]a.$$

We consider the pseudo Drazin invertibility of $1-bd$ in the next result.

Theorem 2.6. *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $bdb = bac$. If $1-ac \in \mathcal{R}^{pD}$, then $1-bd \in \mathcal{R}^{pD}$ and*

$$(1-bd)^{pD} = 1 + b[(1-ac)^{pD} - (1-ac)^\pi(1-(1-ac)^\pi(1-ac))^{-1}]d. \tag{5}$$

Proof. If y is equal to the right hand side of (5), then we verify that $y(1-bd)y = y$ and $y \in \mathrm{comm}^2(1-bd)$ as in the proof of Theorem 2.5. For $\alpha = 1-ac$, from $\alpha^k\alpha^\pi \in J(\mathcal{R})$, we have that

$$(1-bd)^k b\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d = b\alpha^k\alpha^\pi(1-\alpha^\pi\alpha)^{-1}d \in J(\mathcal{R}).$$

Hence, $1-bd \in \mathcal{R}^{pD}$ and $(1-bd)^{pD} = y$. □

Corollary 2.8. *Let $a, b \in \mathcal{R}$. If $1-ab \in \mathcal{R}^{pD}$, then $1-ba \in \mathcal{R}^{pD}$ and*

$$(1-ba)^{pD} = 1 + b[(1-ab)^{pD} - (1-ab)^\pi(1-(1-ab)^\pi(1-ab))^{-1}]a.$$

At the end of this section, the question is how to express the generalized Drazin and pseudo Drazin inverses of $1-ac$ when $acd = dbd$, $dba = aca$ and $1-bd$ is generalized Drazin or pseudo Drazin invertible. So, we state it as a conjecture.

Conjecture. Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$.

(i) If $1-bd \in \mathcal{R}^d$, then $1-ac \in \mathcal{R}^d$.
(ii) If $1-bd \in \mathcal{R}^{pD}$, then $1-ac \in \mathcal{R}^{pD}$.

Acknowledgements

The author is grateful to the referee for constructive comments and careful reading of the paper.

References

[1] Castro-González, N., Mendes Araújo, C., Patrício, P.: Generalized inverses of a sum in rings. Bull. Aust. Math. Soc. **82**, 156–164 (2010)
[2] Corach, G., Duggal, B., Harte, R.E.: Extensions of Jacobson's lemma. Commun. Algebra **41**, 520–531 (2013)
[3] Drazin, M.P.: Pseudoinverse in associative rings and semigroups. Am. Math. Mon. **65**, 506–514 (1958)
[4] Harte, R.E.: On quasinilpotents in rings. Panamer. Math. J. **1**, 10–16 (1991)
[5] Hartwig, R.E., Luh, J.: A note on the group structure of unit regular ring elements. Pac. J. Math. **71**(2), 449–461 (1977)
[6] Koliha, J.J., Patrício, P.: Elements of rings with equal spectral idempotents. J. Aust. Math. Soc. **72**, 137–152 (2002)
[7] Wang, Z., Chen, J.: Pseudo Drazin inverses in associative rings and Banach algebras. Linear Algebra Appl. **437**, 1332–1345 (2012)
[8] Yan, K., Fang, X.C.: Common properties of the operator products in local spectral theory. Acta Math. Sin. (Engl. Ser.) **31**, 1715–1724 (2015)
[9] Zhuang, G., Chen, J., Cui, J.: Jacobsons lemma for the generalized Drazin inverse. Linear Algebra Appl. **436**(3), 742–746 (2012)

Dijana Mosić
Faculty of Sciences and Mathematics
University of Niš
P. O. Box 224, Nis 18000
Serbia
e-mail: dijana@pmf.ni.ac.rs

Received: August 1, 2016
Revised: January 20, 2017