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Best constant in stability of some positive linear operators

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Abstract. We prove that the kernels of Bernstein, Stancu and Kantorovich operators are proximinal sets, therefore the infimum of Hyers–Ulam constants is also a Hyers–Ulam constant for the above mentioned operators. Moreover, we investigate what happens when the supremum norm is replaced by the L_1-norm.

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1. Introduction

Hyers–Ulam stability is one of the main topics in functional equation theory. Ulam formulated a problem concerning the stability of the equation of homomorphism of a metric group in 1940 and a year later D.H. Hyers gave a first answer to Ulam's problem for the Cauchy functional equation in Banach spaces. Recall the result of Hyers [9]: Let X, Y be two real Banach spaces and $\varepsilon > 0$. Then for every mapping $f : X \to Y$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in X, \tag{1.1}$$

there exists a unique additive mapping $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq \varepsilon, \quad x \in X. \tag{1.2}$$

This is the reason why this type of stability is called after their names [9,23]. The result of Hyers was extended later by Aoki [2] and Rassias [20] by replacing ε in (1.1) with a function depending on x and y. Generally, we say that an equation is stable in the Hyers–Ulam sense if for every solution of a perturbation of the equation (approximate solution) there exists a solution of the equation (exact solution) near it. Hyers–Ulam stability was considered by numerous mathematicians, especially during the last 50 years, due to its connections with many branches of mathematics: differential equations, operator theory, dynamical systems theory, functional analysis; see for more details, results and approaches [3–6, 10–12, 16, 18]. The Hyers–Ulam stability of operators was considered for the first time in the papers by Hatori, Hirasawa, Miura et al. who obtained a characterization of the stability of linear operators and a representation for their best constants [7, 8, 13, 22]. Recall also the results obtained by Moslehian and Sadeghi on the stability of linear operators and their best constants [14, 15]. The authors of the present paper studied the stability of some classical operators from approximation theory obtaining also an explicit formula for the infimum of Hyers–Ulam constants for Bernstein, Stancu and Kantorovich operators. They proved also that in a class of generalized positive linear operators the infimum of Hyers–Ulam constants for Bernstein operators has a minimality property [17, 19]. Generally, the infimum of Hyers–Ulam constants of an operator is not a Hyers–Ulam constant of that operator. An example can be found in [7]. The goal of this paper is to give a positive answer to this problem for Bernstein, Stancu and Kantorovich operators by showing that their kernels are proximinal sets in the space $C[0,1]$ endowed with the supremum norm. The case where the supremum norm is replaced by the L_1-norm $\|\|_1$ is studied in Section 3 where we show that the Bernstein operator B_n, considered on the space $(C[0,1], \|\cdot\|_1)$, is HU-stable with an arbitrary small constant $K > 0$. We obtain a similar result for the Szász–Mirakjan operator L_n on the space $C_b[0,+\infty)$ with the generalized norm $\|\cdot\|_1$; remark that on $C_b[0,+\infty)$ with the supremum norm, L_n is not HU-stable.

2. The HUS-constant for Bernstein, Stancu and Kantorovich operators

Let A, B be normed spaces and $T : A \to B$ an operator. The following definition can be found in [7].

Definition 2.1. We say that T has the Hyers–Ulam stability property (briefly, T is HU-stable) if there exists a constant $K > 0$ such that for every $g \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\|Tf - g\| \le \varepsilon$, there exists $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \le K\varepsilon$. The number K is called a Hyers–Ulam constant of T (briefly, HUS-constant) and the infimum of all HUS-constants of T is denoted by K_T. If K_T is a HUS-constant of T then it is called the best HUS-constant of T.

Remark 2.2. If $T : A \to B$ is a linear operator then T is HU-stable with HUS-constant $K > 0$ if and only if for any $f \in A$ with $\|Tf\| \le 1$ there exists $f_0 \in A$, $Tf_0 = 0$, such that $\|f - f_0\| \le K$. See [22, p. 587] and [19, Remark 2.3] for more details.

The following result can be found in [7, p. 390, Corollary].

Theorem 2.3. *Suppose that T is an HU-stable linear operator and its kernel $N(T)$ is a proximinal set. Then K_T is the best HUS-constant of T.*

Recall that a subset M of the normed space $(A, \|\cdot\|)$ is called a **proximinal** set if for every $f \in A$ there exists $g \in M$ such that

$$\mathrm{dist}(f, M) = \|f - g\|,$$

where $\mathrm{dist}(f, M) = \inf\{\|f - h\| : \ h \in M\}$.

Let $C[0,1]$ be the space of all continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the supremum norm denoted by $\|\cdot\|_\infty$, and $a, b \in \mathbb{R}$, $0 \le a \le b$. Let $\prod_n \subset C[0,1]$ be the space of all polynomial functions of degree $\le n$.

The Stancu operator [21] $S_n : C[0,1] \to \Pi_n$ is defined by

$$S_n f(x) = \sum_{k=0}^{n} f\left(\frac{k+a}{n+b}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad f \in C[0,1].$$

We have

$$N(S_n) = \left\{ f \in C[0,1] : \ f\left(\frac{k+a}{n+b}\right) = 0, \quad 0 \le k \le n \right\}.$$

For $a = b = 0$ the Stancu operator reduces to the classical Bernstein operator B_n.

Now let $X = \{f : [0,1] \to \mathbb{R} \mid \ f$ is bounded and Riemann integrable$\}$ and suppose that X is endowed also with the supremum norm denoted by $\|\cdot\|_\infty$. The Kantorovich operator [1] is defined by

$$K_n f(x) = (n+1)\sum_{k=0}^{n}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\right)\binom{n}{k} x^k (1-x)^{n-k},$$

for every $f \in X$ and $x \in [0,1]$. The kernel of K_n is given by

$$N(K_n) = \left\{ f \in X : \ \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt = 0, \ 0 \le k \le n \right\}.$$

In [17] it is proved that S_n, B_n, K_n are HU-stable operators and, moreover, the following result holds.

Theorem 2.4.

$$K_{S_n} = K_{B_n} = K_{K_n} = \binom{2n}{2\left[\frac{n}{2}\right]} \Big/ \binom{n}{\left[\frac{n}{2}\right]}.$$

We prove in what follows that $K_{S_n}, K_{B_n}, K_{K_n}$ are HUS-constants for the corresponding operators. Our proof is based on Theorem 2.3. Classical results on the proximinality of a subset of a normed space are given for reflexive spaces; $C[0,1]$ is not reflexive, therefore we will give a direct proof for the proximinality of the kernels of S_n, B_n, K_n.

Lemma 2.5. *Let* $0 \leq x_0 < x_1 < \ldots < x_n \leq 1$ *and*

$$N := \{g \in C[0,1] : \; g(x_i) = 0, \quad i = 0, 1, \ldots, n\}.$$

Then N *is proximinal in* $(C[0,1], \|\cdot\|_\infty)$.

Proof. Let $f \in C[0,1]$. We have to find a function $g \in N$ such that

$$dist(f, N) = \|f - g\|_\infty . \tag{2.1}$$

Let $m := max\{|f(x_i)| : i = 0, 1, \ldots, n\}$. If $m = 0$, then $f \in N$ and (2.1) is satisfied with $g = f$.

It remains to consider the case when $m > 0$. First, it is easy to verify that for each $h \in N$ one has $\|f - h\|_\infty \geq m$, hence

$$m \leq dist(f, N). \tag{2.2}$$

On the other hand, let $\lambda \in C[0,1]$ be the piecewise affine function such that

$$\lambda(0) = \lambda(x_0) = \frac{m - f(x_0)}{2m}, \; \lambda(x_n) = \lambda(1) = \frac{m - f(x_n)}{2m},$$
$$\lambda(x_i) = \frac{m - f(x_i)}{2m}, \quad i = 1, \ldots, n-1.$$

Since $0 \leq \lambda(x_i) \leq 1$, $i = 0, 1, \cdots, n$, and λ is affine on each subinterval $[0, x_0]$, $[x_0, x_1]$,$\cdots$, $[x_{n-1}, x_n]$, $[x_n, 1]$, we deduce that $0 \leq \lambda(x) \leq 1$, $x \in [0,1]$. Therefore, $\|2\lambda - 1\|_\infty \leq 1$.

Consider the function $g \in C[0,1]$, $g(x) := f(x) + m(2\lambda(x) - 1)$, $x \in [0,1]$. It is easy to verify that $g(x_i) = 0$, $i = 0, 1, \cdots, n$, hence $g \in N$. This entails

$$dist(f, N) \leq \|f - g\|_\infty = m \|2\lambda - 1\|_\infty \leq m.$$

Considering (2.2), we get

$$dist(f, N) = \|f - g\|_\infty = m.$$

So the function $g \in N$ satisfies (2.1), and this concludes the proof. □

Lemma 2.6. $N(K_n)$ *is proximinal in* $(X, \|\cdot\|_\infty)$.

Proof. Let $I_k := \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right)$, $k = 0, 1, \ldots, n-1$, and $I_n := \left[\frac{n}{n+1}, 1\right]$. Let $\|h\|_k$ be the supremum norm of a bounded function h defined on I_k, $k = 0, \ldots, n$.

Let $f \in X$ and $h \in N(K_n)$. Then $|f(t) - h(t)| \leq \|f - h\|_k$, $t \in I_k$, and this entails

$$(n+1) \left| \int_{I_k} f(t)dt \right| \leq \|f - h\|_k, \quad k = 0, \ldots, n. \tag{2.3}$$

Set $m := \max\left\{(n+1)\left|\int_{I_k} f(t)dt\right| : \; k = 0, \ldots, n\right\}$. From (2.3) we get $m \leq \|f - h\|_\infty$, for all $h \in N(K_n)$ and so

$$m \leq \operatorname{dist}(f, N(K_n)). \tag{2.4}$$

Now let $g \in X$ be the function defined by

$$g(t) = f(t) - (n+1) \int_{I_k} f(s)ds, \ t \in I_k, \quad k = 0, \ldots, n.$$

Then

$$\int_{I_k} g(t)dt = \int_{I_k} f(t)dt - \int_{I_k} f(s)ds = 0, \quad i = 0, 1, \cdots, n,$$

hence $g \in N(K_n)$. Moreover,

$$\begin{aligned} \|f-g\|_\infty &= max\left\{\|f-g\|_k : k = 0, 1, \cdots, n\right\} \\ &= max\left\{(n+1)\left|\int_{I_k} f(s)ds\right| : k = 0, 1, \cdots, n\right\} = m. \end{aligned}$$

Therefore $g \in N(K_n)$ and $\|f-g\|_\infty = m$; using also (2.4) we get

$$dist(f, N(K_n)) \leq \|f-g\|_\infty = m \leq dist(f, N(K_n)).$$

This yields $dist(f, N(K_n)) = \|f-g\|_\infty$, and the lemma is proved. □

The results proved in Lemma 2.5, Lemma 2.6, Theorem 2.3 and Theorem 2.4 lead to the following conclusion.

Corollary 2.7. *The best HUS-constant for Bernstein, Stancu and Kantorovich n-th operators is*

$$\binom{2n}{2\left[\frac{n}{2}\right]} \Big/ \binom{n}{\left[\frac{n}{2}\right]}.$$

3. HU-stability of operators with respect to different norms

For $f \in C[0,1]$ let $\|f\|_\infty$ be the supremum norm and $\|f\|_1 = \int_0^1 |f(x)|dx$. Let $n \geq 1$ be given, and

$$N := \left\{g \in C[0,1] : \ g\left(\frac{k}{n}\right) = 0, \quad k = 0, 1, \ldots, n\right\}.$$

Theorem 3.1. *N is dense in $(C[0,1], \|\cdot\|_1)$.*

Proof. Let $f \in C[0,1]$, $f \neq 0$, and $\varepsilon > 0$. We have to find a function $g \in N$ such that $\|f-g\|_1 \leq \varepsilon$. Set

$$\delta := \min\left\{\frac{1}{3n}, \frac{\varepsilon}{4n\|f\|_\infty}\right\}. \tag{3.1}$$

Consider the intervals $I_0 = [0, \delta]$, $I_n = [1-\delta, 1]$, $I_k = \left[\frac{k}{n} - \delta, \frac{k}{n} + \delta\right]$, $k = 1, 2, \ldots, n-1$. Due to (3.1), they are pairwise disjoint.

Let $g \in C[0,1]$ be the function defined by:

- $g\left(\frac{k}{n}\right) = 0, \ k = 0, 1, \ldots, n;$

- $g(x) = f(x),\ x \in [0,1] \setminus \bigcup_{k=0}^{n} I_k$;
- g is affine on each of the intervals $I_0,\ I_n,\ \left[\frac{k}{n} - \delta, \frac{k}{n}\right],\ \left[\frac{k}{n}, \frac{k}{n} + \delta\right],\ k = 1, \ldots, n-1$.

Then $g \in N$ and $\|g\|_\infty \le \|f\|_\infty$, hence $|f(x) - g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty \le 2\|f\|_\infty,\ x \in [0,1]$. Let $I := \bigcup_{k=0}^{n} I_k$. Then the Lebesgue measure of I is $2n\delta$, and

$$\int_0^1 |f(x) - g(x)|dx = \int_I |f(x) - g(x)|dx \le 2\|f\|_\infty meas(I) = 4\|f\|_\infty n\delta \le \varepsilon.$$

Therefore $\|f - g\|_1 \le \varepsilon$, and this concludes the proof. □

Theorem 3.2. *Let $K > 0$. The Bernstein operator $B_n : (C[0,1], \|\cdot\|_1) \to (C[0,1], \|\cdot\|_1)$ is HU-stable with HUS-constant K.*

Proof. Let $f \in C[0,1]$ with $\|B_n f\|_1 \le 1$. According to Theorem 3.1, there exists $g \in N = \ker B_n$ such that $\|f - g\|_1 \le K$. This means that K is a HUS-constant for B_n in view of Remark 2.2.
(Let us remark that the assumption $\|B_n f\|_1 \le 1$ is not used in the proof!). □

Remark 3.3. Theorem 3.2 provides an example of an operator for which the infimum of the HUS-constants is 0, and this infimum is not an HUS-constant. Another example of an operator T for which the infimum K_T is not an HUS-constant can be found in [7].

In what follows, for $f \in C_b[0,+\infty)$ let $\|f\|_\infty$ be the supremum norm; consider also the generalized norm $\|f\|_1 := \int_0^\infty |f(x)|dx$.

Let $n \ge 1$ be given, and

$$M := \left\{ g \in C_b[0,+\infty) :\ g\left(\frac{k}{n}\right) = 0,\ k = 0, 1, \ldots \right\}.$$

Lemma 3.4. *M is dense in $(C_b[0,+\infty), \|\cdot\|_1)$, i.e., for each $\varepsilon > 0$ and for each $f \in C_b[0,+\infty)$ there exists $g \in M$ with $\|f - g\|_1 \le \varepsilon$.*

Proof. Let $f \in C_b[0,+\infty),\ f \ne 0$, and $\varepsilon > 0$.

Let

$$0 < \delta_k < \frac{1}{2n}, k \ge 0, \tag{3.2}$$

such that

$$\delta_0 + 2\sum_{k=1}^{\infty} \delta_k \le \frac{\varepsilon}{2\|f\|_\infty}. \tag{3.3}$$

Consider the intervals $I_0 = [0, \delta_0],\ I_k = \left[\frac{k}{n} - \delta_k, \frac{k}{n} + \delta_k\right],\ k \ge 1$. Due to (3.2) they are pairwise disjoint. Let $g \in C_b[0,+\infty)$ be the function defined by

- $g\left(\frac{k}{n}\right) = 0,\ k \ge 0$;
- $g(x) = f(x),\ x \in [0,+\infty) \setminus \bigcup_{k=0}^{\infty} I_k$;
- g is affine on each of the intervals $I_0,\ \left[\frac{k}{n} - \delta_k, \frac{k}{n}\right],\ \left[\frac{k}{n}, \frac{k}{n} + \delta_k\right],\ k \ge 1$.

Then $g \in M$ and $\|g\|_\infty \le \|f\|_\infty$, hence $|f(x) - g(x)| \le 2\|f\|_\infty$, $x \in [0,1]$. Let $I := \bigcup_{k=0}^{\infty} I_k$. Then, using (3.3) we get

$$\int_0^\infty |f(x) - g(x)|dx = \int_I |f(x) - g(x)|dx \le 2\|f\|_\infty meas(I)$$
$$= 2\|f\|_\infty \left(\delta_0 + 2\sum_{k=1}^{\infty} \delta_k\right) \le \varepsilon.$$

Therefore $\|f - g\|_1 \le \varepsilon$, which shows that M is dense in $(C_b[0, +\infty), \|\cdot\|_1)$. □

Theorem 3.5. *Let* $K > 0$.

The Szász–Mirakjan operator $L_n : (C_b([0, +\infty), \|\cdot\|_1) \to (C_b[0, +\infty), \|\cdot\|_1)$ *defined by*

$$L_n f(x) := e^{-nx} \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) \frac{n^i}{i!} x^i, \;\; f \in C_b[0, +\infty), \;\; x \ge 0,$$

is HU-stable with HUS-constant K.

Proof. Similar to that of Theorem 3.2. □

Remark 3.6. $L_n : (C_b[0, +\infty), \|\cdot\|_\infty) \to (C_b[0, +\infty), \|\cdot\|_\infty)$ is not HU-stable; see [19].

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