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Some identities of the r-Whitney numbers

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Abstract. In this paper we establish some algebraic properties involving r-Whitney numbers and other special numbers, which generalize various known identities. These formulas are deduced from Riordan arrays. Additionally, we introduce a generalization of the Eulerian numbers, called r-Whitney–Eulerian numbers and we show how to reduce some infinite summation to a finite one.

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1. Introduction

The r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and the second kind $W_{m,r}(n,k)$ were defined by Mező [19] as the connecting coefficients between some special polynomials. We note that these numbers, under a different name, appear in the work of Corcino et al. in [12]. Specifically, for non-negative integers n, k and r with $n \geq k \geq 0$ and for any integer $m > 0$

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}}, \tag{1}$$

and

$$m^n x^{\underline{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k, \tag{2}$$

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where

$$x^{\underline{n}} = \begin{cases} x(x-1)\cdots(x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

The r-Whitney numbers of the first kind and the second kind satisfy the following recurrence, respectively [19]

$$w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (m - nm - r)w_{m,r}(n-1,k), \tag{3}$$

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km + r)W_{m,r}(n-1,k). \tag{4}$$

Moreover, these numbers have the following rational generating function [8]:

$$\sum_{k=0}^{n} w_{m,r}(n,n-k)x^k = \prod_{k=0}^{n-1}(1-(r+mk)x), \tag{5}$$

$$\sum_{k=0}^{n} W_{m,r}(n,k)x^n = \frac{x^k}{(1-rx)(1-(r+m)x)\cdots(1-(r+mk)x)}. \tag{6}$$

Note that if $(m,r) = (1,0)$ we obtain the Stirling numbers [14], if $(m,r) = (1,r)$ we have the r-Stirling (or noncentral Stirling) numbers [7], and if $(m,r) = (m,0)$ we have the Whitney numbers [4,5]. See [3,8,21] for combinatorial interpretations of the r-Whitney numbers, [16–18] for their connections to elementary symmetric functions, [11] for asymptotic expansions of $W_{m,r}(n,k)$ and [20] for their connections to matrix theory.

In this paper we extend the work of Cheon et al. [9]. We use the fundamental theorem of Riordan arrays to establish some combinatorial sums which involve the r-Whitney numbers and other special numbers. Additionally, we introduce a generalization of the Eulerian numbers, which are called r-Whitney–Eulerian numbers, and we obtain some combinatorial properties of them.

2. Preliminary definitions and basic identities

A Riordan array $L = [l_{n,k}]_{n,k\in\mathbb{N}}$ is defined by a pair of generating functions $g(z) = 1 + g_1z + g_2z^2 + \cdots$ and $f(z) = f_1z + f_2z^2 + \cdots$, where $f_1 \neq 0$, so that the k-th column satisfies

$$\sum_{n\geqslant 0} l_{n,k}z^n = g(z)\left(f(z)\right)^k,$$

the first column being indexed by 0. It is clear that $l_{n,k} = [z^n]\, g(z)\left(f(z)\right)^k$, where $[z^n]$ is the coefficient operator. The matrix corresponding to the pair $f(z), g(z)$ is denoted by $\mathcal{R}(g(z), f(z))$ or $(g(z), f(z))$. The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by:

$$(g(z), f(z)) * (h(z), l(z)) = \left(g(z)h\left(f(z)\right), l\left(f(z)\right)\right).$$

The set of all Riordan matrices is a group under the operator $*$, [23]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is $(g(z), f(z))^{-1} = \left(1/\left(g \circ \overline{f}\right)(z), \overline{f}(z)\right)$, where $\overline{f}(z)$ is the compositional inverse of $f(z)$.

Example 1. The Pascal matrix is given by the following Riordan array.

$$\left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & \\ 1 & 3 & 3 & 1 & 0 & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

The following theorem is known as the fundamental theorem of Riordan arrays or summation property.

Theorem 2. [24] *If $[l_{n,k}]_{n,k\in\mathbb{N}} = (g(z), f(z))$ is a Riordan array, then for any sequence $\{h_k\}_{k\in\mathbb{N}}$*

$$\sum_{k=0}^{n} l_{n,k} h_k = [z^n]\, g(z) h(f(z)),$$

where $h(z)$ is the generating function of the sequence $\{h_k\}_{k\in\mathbb{N}}$.

From the fundamental theorem of Riordan arrays we can obtain the following identities:

Proposition 3. *For any integers $n, k \geq 0$,*

1. $\displaystyle\sum_{j=0}^{i} \binom{i}{j} w_{m,r}(n, n-j) = \sum_{j=0}^{i} \binom{n+i-j}{n} w_{m,r+1}(n, n-j).$
2. $\displaystyle\sum_{j=k}^{i} \binom{i}{j} W_{m,r}(j, k) = W_{m,r+1}(i, k).$

Proof. 1. Let $h(x) = \prod_{k=0}^{n-1}(1-(r+mk)x)$. From Eq. (5) and by applying Theorem 2 to the Riordan array $P = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ with the generating function $h(x)$, we obtain

$$\begin{aligned} \sum_{j=0}^{i} \binom{i}{j} w_{m,r}(n, n-j) &= \left[x^i\right] \frac{1}{1-x} h\left(\frac{x}{1-x}\right) \\ &= \left[x^i\right] \frac{1}{1-x} \prod_{k=0}^{n-1} \left(1-(r+mk)\frac{x}{1-x}\right) \end{aligned}$$

$$= \left[x^i\right] \frac{1}{(1-x)^{n+1}} \prod_{k=0}^{n-1} (1-(1+r+mk)x)$$

$$= \sum_{j=0}^{i} \binom{n+i-j}{n} w_{m,r+1}(n, n-j).$$

2. Let $h(x) = \frac{x^k}{(1-rx)(1-(r+m)x)\cdots(1-(r+mk)x)}$. From Eq. (6) and by applying Theorem 2 to the Riordan array $P = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ with the generating function $h(x)$, we obtain

$$\sum_{j=0}^{i} \binom{i}{j} W_{m,r}(j,k) = \left[x^i\right] \frac{1}{1-x} h\left(\frac{x}{1-x}\right)$$

$$= \left[x^i\right] \frac{1}{1-x} \cdot \frac{\left(\frac{x}{1-x}\right)^k}{\left(1-r\left(\frac{x}{1-x}\right)\right)\left(1-(r+m)\left(\frac{x}{1-x}\right)\right)\cdots\left(1-(r+mk)\left(\frac{x}{1-x}\right)\right)}$$

$$= \left[x^i\right] \frac{x^k}{(1-(1+r)x)(1-(1+r+m)x)\cdots(1-(1+r+mk)x)}$$

$$= W_{m,r+1}(i,k).$$

□

3. Main theorem

Theorem 4. *Let $g(t)$ be the generating function of a sequence $(g_k)_{k\in\mathbb{N}}$ for which the below series is convergent. Then we have*

$$\sum_{j=0}^{n} W_{m,r}(n,j)(mt)^j g^{(j)}(t) = \sum_{k=0}^{\infty} (mk+r)^n g_k t^k, \quad \textit{for} \quad n = 0, 1, 2, \ldots,$$

where $W_{m,r}(n,j)$ is an r-Whitney number of the second kind and $g^{(j)}(t)$ is the j-th derivative of the function $g(t)$ with respect to t.

Proof. We proceed by induction on n. If $n = 1$,

$$\sum_{k=0}^{\infty} (mk+r) g_k t^k = m \sum_{k=0}^{\infty} k + r \sum_{k=0}^{\infty} g_k t^k = mtg'(t) + rg(t)$$

$$= W_{m,r}(1,1) mtg'(t) + W_{m,r}(1,0) g(t),$$

so the statement holds. Now, supposing that the result is true for all $j < n+1$, we prove it for $n+1$. From recurrence (4) we obtain:

$$
\begin{aligned}
\sum_{k=0}^{\infty}(mk+r)^{n+1}g_k t^k &= mt\frac{d}{dt}\sum_{j=0}^{n}W_{m,r}(n,j)(mt)^j g^{(j)}(t)\\
&\quad + r\sum_{j=0}^{n}W_{m,r}(n,j)(mt)^j g^{(j)}(t)\\
&= mt\left[\sum_{j=0}^{n}jW_{m,r}(n,j)(mt)^{j-1}mg^{(j)}(t)+\sum_{j=0}^{n}W_{m,r}(n,j)(mt)^j g^{(j+1)}(t)\right]\\
&\quad + r\sum_{j=0}^{n}W_{m,r}(n,j)(mt)^j g^{(j)}(t)\\
&= \sum_{j=0}^{n}(mj+r)W_{m,r}(n,j)(mt)^j g^{(j)}(t)+\sum_{j=0}^{n}W_{m,r}(n,j)(mt)^{j+1}g^{(j+1)}(t)\\
&= rW_{m,r}(n,0)g(t)+(m+r)W_{m,r}(n,1)(mt)g'(t)\\
&\quad + \sum_{j=2}^{n}(mj+r)W_{m,r}(n,j)(mt)^j g^{(j)}(t)+W_{m,r}(n,0)(mt)g'(t)\\
&\quad + \sum_{j=1}^{n-1}W_{m,r}(n,j)(mt)^{j+1}g^{(j+1)}(t)+W_{m,r}(n,n)(mt)^{n+1}g^{n+1}(t)\\
&= W_{m,r}(n+1,0)g(t)+((m+r)W_{m,r}(n,1)+W_{m,r}(1,0))(mt)g'(t)\\
&\quad + \sum_{j=1}^{n-1}\Big[(m(j+1)+r)W_{m,r}(n,j+1)(mt)^{j+1}g^{(j+1)}(t)\\
&\quad + W_{m,r}(n,j)(mt)^{j+1}g^{(j+1)}(t)\Big]\\
&\quad + W_{m,r}(n+1,n+1)(mt)^{n+1}g^{n+1}(t)\\
&= W_{m,r}(n+1,0)g(t)+W_{m,r}(n+1,1)(mt)g'(t)\\
&\quad + \sum_{j=1}^{n-1}\left[(m(j+1)+r)W_{m,r}(n,j+1)+W_{m,r}(n,j)\right](mt)^{j+1}g^{(j+1)}(t)\\
&\quad + W_{m,r}(n+1,n+1)(mt)^{n+1}g^{n+1}(t)\\
&= W_{m,r}(n+1,0)g(t)+W_{m,r}(n+1,1)(mt)g'(t)\\
&\quad + \sum_{j=1}^{n-1}W_{m,r}(n+1,j+1)(mt)^{j+1}g^{(j+1)}(t)\\
&\quad + W_{m,r}(n+1,n+1)(mt)^{n+1}g^{n+1}(t)\\
&= W_{m,r}(n+1,0)g(t)+W_{m,r}(n+1,1)(mt)g'(t)
\end{aligned}
$$

$$+\sum_{j=1}^{n} W_{m,r}(n+1, j+1)(mt)^{j+1}g^{(j+1)}(t)$$
$$=\sum_{j=0}^{n+1} W_{m,r}(n+1, j)(mt)^{j}g^{(j)}(t).$$

□

Corollary 5. *For any integer $n \geq 0$, we have*

$$\sum_{j=0}^{n} S_r(n,j)t^j g^{(j)}(t) = \sum_{k=0}^{\infty}(k+r)^n g_k t^k,$$

where $S_r(n,j)$ is an r-Stirling number of the second kind.

Corollary 6. [9] *For any integer $n \geq 0$, we have*

$$\sum_{j=0}^{n} S(n,j)t^j g^{(j)}(t) = \sum_{k=0}^{\infty} k^n g_k t^k,$$

where $S(n,j)$ is a Stirling number of the second kind.

4. Some identities

From Theorem 4 and the fundamental theorem of Riordan arrays we get the following identities.

Theorem 7. *Let x be a nonzero real number. For any integers $n, h \geq 0$, we have*

1. $\displaystyle\sum_{j=0}^{i}\binom{i}{j}(mj+r)^n x^{i-j} = \sum_{j=0}^{n} W_{m,r}(n,j)m^j j!\binom{i}{j}(1+x)^{i-j}.$
2. $\displaystyle\sum_{j=0}^{i}\binom{i}{j}\binom{h}{j}(mj+r)^n x^{i-j} = \sum_{j=0}^{n}\sum_{l=0}^{i-j} W_{m,r}(n,j)m^j j!\binom{h}{j}\binom{h+l}{h}\binom{h-j}{i-j-l}$ $x^l(1-x)^{i-j-l}.$

Proof. 1. Let $g(t) = \frac{1}{1-t}$, and let $f(t) = \sum_{j=0}^{n} W_{m,r}(n,j)(mt)^j g^{(j)}(t)$. Then $g_l = 1$ for all $l \geq 0$, and $f_k = (mk+r)^n$ for all $k, n \geq 0$. By applying Theorem 2 to the Riordan array $P[x] = \left(\frac{1}{1-xt}, \frac{t}{1-xt}\right)$ with the generating function $f(t)$, we have

$$\begin{aligned}\sum_{j=0}^{i} p_{ij} f_j &= \sum_{j=0}^{i} \binom{i}{j} x^{i-j}(mj+r)^n = \left[t^i\right] \frac{1}{1-xt} f\left(\frac{t}{1-xt}\right) \\ &= \left[t^i\right] \frac{1}{1-xt} \sum_{j=0}^{n} W_{m,r}(n,j) \left(\frac{mt}{1-xt}\right)^j \frac{j!}{\left(\frac{1-xt-t}{1-xt}\right)^{j+1}} \\ &= \sum_{j=0}^{n} W_{m,r}(n,j) m^j j! \left[t^{i-j}\right] \left(\frac{1}{1-(1+x)t}\right)^{j+1} \\ &= \sum_{j=0}^{n} W_{m,r}(n,j) m^j j! \binom{i}{j} (1+x)^{i-j}.\end{aligned}$$

2. Let $g(t) = (1+t)^h$ for $h \geq 0$, then $g_l = \binom{h}{l}$ for all $l \leq 0$. Therefore $f_k = (mk+r)^n \binom{h}{k}$ for all $n \geq 0$. By applying Theorem 2 to the Riordan array $P[x] = \left(\frac{1}{1-xt}, \frac{1}{1-xt}\right)$ with the generating function $f(t)$, we obtain

$$\begin{aligned}\sum_{j=0}^{i} p_{ij} f_j &= \sum_{j=0}^{i} \binom{i}{j}\binom{h}{j} x^{i-j}(mj+r)^n \\ &= \left[t^i\right] \frac{1}{1-xt} f\left(\frac{t}{1-xt}\right) \\ &= \left[t^i\right] \frac{1}{1-xt} \sum_{j=0}^{n} W_{m,r}(n,j) \left(\frac{mt}{1-xt}\right)^j j! \binom{h}{j} \left(1+\frac{t}{1-xt}\right)^{h-j} \\ &= \sum_{j=0}^{n} W_{m,r}(n,j) m^j j! \binom{h}{j} \left[t^{i-j}\right] \left(\frac{1}{1-(1+x)t}\right)^{h+1} (1+(1-x)t)^{h-j} \\ &= \sum_{j=0}^{n} W_{m,r}(n,j) m^j j! \binom{h}{j} \sum_{l=0}^{i-j} \left[t^l\right] \left(\frac{1}{1-xt}\right)^{h+1} \left[t^{i-j-l}\right] (1+(1-x)t)^{h-j} \\ &= \sum_{j=0}^{n}\sum_{l=0}^{i-j} W_{m,r}(n,j) m^j j! \binom{h}{j}\binom{h+l}{h}\binom{h-j}{i-j-l} x^l (1-x)^{i-j-l}.\end{aligned}$$

□

From the above theorem we obtain the following identities.

Corollary 8. *For any integers $n, h \geq 0$, we have*

1. $\displaystyle\sum_{j=0}^{i} \binom{i}{j} (mj+r)^n = \sum_{j=0}^{n} W_{m,r}(n,j) m^j j! \binom{i}{j} 2^{i-j}.$
2. $\displaystyle W_{m,r}(n,i) = \frac{1}{m^i i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} (mj+r)^n.$

3. $\sum_{j=0}^{i}\binom{i}{j}\binom{h}{j}x^{i-j}=\sum_{l=0}^{i}\binom{h+l}{h}\binom{h}{i-l}x^{l}(1-x)^{i-l}$.
4. $\sum_{j=0}^{i}\binom{i}{j}\binom{h}{j}(mj+r)=r\binom{h+i}{h}+mh\binom{h+i-1}{h}$.

Proof. (1) Taking $n=0$ in Theorem 7-(1). (2) Taking $x=1$ in Theorem 7-(1). (3) Taking $n=0$ in Theorem 7-(2). (4) Taking $n=1$ and $x=1$ in Theorem 7-(2). □

Note that (3) and (4) are two generalizations of the classical Chu–Vandermonde identity

$$\sum_{j=0}^{i}\binom{i}{j}\binom{h}{j}=\binom{h+i}{h}.$$

5. A generalization of the Eulerian numbers

In the combinatorics of permutations the $A(n,k)$ Eulerian numbers play an important role [6]. They can be defined by the $S(n,k)$ Stirling numbers as

$$A(n,k)=\sum_{j=0}^{k}S(n,j)j!\binom{n-j}{k-j}(-1)^{k-j}\quad (n,k\geq 1). \tag{7}$$

Moreover, the Eulerian polynomials are defined as

$$A_n(x)=\sum_{k=0}^{n}A(n,k)x^k.$$

It is a nice fact that these polynomials help to calculate special infinite sums, because

$$\frac{A_n(x)}{(1-x)^{n+1}}=\sum_{k=0}^{\infty}k^n x^k$$

holds for any non-negative integer n [10, p.244].

Our master formula in Theorem 4 can be used to generalize the Eulerian numbers and polynomials. Recall that

$$\sum_{j=0}^{n}W_{m,r}(n,j)(mt)^j g^{(j)}(t)=\sum_{k=0}^{\infty}(mk+r)^n g_k x^k$$

for any generating function g keeping convergence. Choosing simply $g(x) = \frac{1}{1-x}$ (and so $g_k \equiv 1$) we have that

$$\sum_{j=0}^{n} W_{m,r}(n,j)(mx)^j \frac{j!}{(1-x)^{j+1}} = \sum_{k=0}^{\infty}(mk+r)^n x^k.$$

Carrying out $\frac{1}{(1-x)^{n+1}}$ we get a polynomial on the left whose coefficient can be found by the binomial theorem:

$$\begin{aligned}
&\sum_{j=0}^{n} W_{m,r}(n,j)(mx)^j \frac{j!}{(1-x)^{j+1}}\\
&\quad= \frac{1}{(1-x)^{n+1}} \sum_{j=0}^{n} W_{m,r}(n,j)(mt)^j j!(1-x)^{n-j}\\
&\quad= \frac{1}{(1-x)^{n+1}} \sum_{j=0}^{n} W_{m,r}(n,j)m^j j! \sum_{i=0}^{n-j} \binom{n-j}{i}(-1)^i x^{i+j}.
\end{aligned}$$

Introducing the new index $k = i + j$ (and changing the index in the infinite sum from k to i to avoid index conflicts) we have that

$$\frac{1}{(1-x)^{n+1}} \sum_{k=0}^{n} x^k \left(\sum_{j=0}^{n} W_{m,r}(n,j)m^j j! \binom{n-j}{k-j}(-1)^{k-j} \right) = \sum_{i=0}^{\infty}(mi+r)^n x^i.$$

We have thus obtained the following theorem.

Theorem 9. *For non-negative integers r, n and positive m we have that*

$$\sum_{i=0}^{\infty}(mi+r)^n x^i = \frac{A_{n,m,r}(x)}{(1-x)^{n+1}},$$

and the polynomial $A_{n,m,r}(x)$ equals

$$A_{n,m,r}(x) = \sum_{k=0}^{n} x^k \left(\sum_{j=0}^{n} W_{m,r}(n,j)m^j j! \binom{n-j}{k-j}(-1)^{k-j} \right).$$

This result shows that one can define a generalized version of the Eulerian numbers—which we may call r-Whitney–Eulerian numbers—as

$$A_{m,r}(n,k) = \sum_{j=0}^{n} W_{m,r}(n,j)m^j j! \binom{n-j}{k-j}(-1)^{k-j}.$$

Note that

$$A_{1,0}(n,k) = \sum_{j=0}^{n} W_{1,0}(n,j)j!\binom{n-j}{k-j}(-1)^{k-j}$$
$$= \sum_{j=0}^{n} S(n,j)j!\binom{n-j}{k-j}(-1)^{k-j} = A(n,k)$$

which is just (7) again. The polynomials $A_{n,m,r}(x)$ are then called r-Whitney–Eulerian polynomials. Their exponential generating function is easy to determine.

Theorem 10. *We have that*

$$\sum_{n=0}^{\infty} A_{n,m,r}(x)\frac{y^n}{n!} = \frac{(1-x)\exp(ry(1-x))}{1-x\exp(my(1-x))}.$$

Proof. By our previous theorem

$$A_{n,m,r}(x) = (1-x)^{n+1}\sum_{i=0}^{\infty}(mi+r)^n x^i,$$

so

$$\sum_{n=0}^{\infty} A_{n,m,r}(x)\frac{y^n}{n!} = (1-x)\sum_{i=0}^{\infty} x^i \sum_{n=0}^{\infty}\frac{[y(mi+r)(1-x)]^n}{n!}$$
$$= (1-x)\sum_{i=0}^{\infty} x^i \exp(y(mi+r)(1-x))$$
$$= (1-x)\exp(ry(1-x))\sum_{i=0}^{\infty} x^i \exp(my(1-x)i)$$
$$= \frac{(1-x)\exp(ry(1-x))}{1-x\exp(my(1-x))}.$$

□

For a similar class of Eulerian numbers connected to the Whitney numbers see the paper of Rahmani [22].

6. Some additional special cases of the main theorem

The simplest case of Theorem 4 is discussed in the previous section, when $g_k = 1$ for all k. We discuss other special cases now.

First, let $g_k = \frac{1}{k!k}$. For this sequence let

$$\sum_{k=1}^{\infty}\frac{1}{k!k}x^k = g(x).$$

It is not hard to see that the derivatives of g at $x = 1$ are as follows

$$g^{(j)}(1) = (-1)^{j-1}\left(eD_{j-1} - (j-1)!\right) \quad (j \geq 1).$$

Here

$$D_j = j!\sum_{i=0}^{j}\frac{(-1)^i}{i!}$$

is the j-th derangement number [6]. Combinatorially, in a permutation on j elements there are exactly D_j permutations without fixed points.

It is well known that

$$g(1) = \mathrm{Ei}(1) - \gamma,$$

where

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{dt}{te^t}$$

is the exponential integral function [2, p.143]. Hence we readily get the following summation formula for all $n \geq 1$

$$\sum_{k=1}^{\infty}\frac{(mk+r)^n}{k!k} = W_{m,r}(n,0)(\mathrm{Ei}(1)-\gamma) - \sum_{j=1}^{n} m^j W_{m,r}(n,j)(-1)^j\left(eD_{j-1} - (j-1)!\right).$$

Especially, when $m = 1$, $r = 0$,

$$\sum_{k=1}^{\infty}\frac{k^n}{k!k} = e\sum_{j=1}^{n} S(n,j)(-1)^{j-1}D_{j-1}.$$

Now we take $g(x) = \cos(2\pi x)$ and $g(x) = \sin(2\pi x)$. These special cases result that

$$\sum_{k=0}^{\infty}(2mk+r)^n\frac{(-1)^k(2\pi)^{2k}}{(2k)!} = \sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j(2\pi m)^{2j}W_{m,r}(n,2j),$$

$$\sum_{k=0}^{\infty}(m(2k+1)+r)^n\frac{(-1)^k(2\pi)^{2k}}{(2k+1)!} = m\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j(2\pi m)^{2j}W_{m,r}(n,2j+1).$$

For the Stirling numbers this specializes to

$$\sum_{k=0}^{\infty}(2k)^n\frac{(-1)^k(2\pi)^{2k}}{(2k)!} = \sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j(2\pi)^{2j}S(n,2j),$$

$$\sum_{k=0}^{\infty}(2k+1)^n\frac{(-1)^k(2\pi)^{2k}}{(2k+1)!} = \sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j(2\pi)^{2j}S(n,2j+1).$$

Another interesting example comes if we take $g(x) = x\psi(x)$, where ψ is the Digamma function [2, p.15]. The derivatives of $g(x)$ are

$$g^{(n)}(x) = n\psi^{(n-1)}(x) + x\psi^{(n)}(x), \text{ if } n \geq 1.$$

It is also known that

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1}n!(2^{n+1}-1)\zeta(n+1) \quad (n \geq 2),$$

where ζ is the Riemann zeta function. Moreover, $g(1/2) = -\gamma/2 - \log(2)$ and $g'(1/2) = \pi^2/4 - \gamma - 2\log(2)$. From these one can easily prove that for any $n, m \geq 1$ and $r \geq 0$

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{-1}{2}\right)^k \zeta(k)(mk+r)^n \\ &= r^n + \frac{\gamma}{2}(m+r)^n - \left(\frac{\gamma}{2} + \log(2)\right) W_{m,r}(n,0) \\ &\quad + \left(\frac{\pi^2}{8} - \log(2) - \frac{\gamma}{2}\right) W_{m,r}(n,1) \\ &\quad + \sum_{j=2}^{n} W_{m,r}(n,j)\left(-\frac{1}{2}\right)^j j!\left[(2^j-1)\zeta(j) - \frac{2^{j+1}-1}{2}\zeta(j+1)\right],\end{aligned}$$

with the special case

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{-1}{2}\right)^k \zeta(k)k^n \\ &= \frac{\pi^2}{8} - \log(2) + \sum_{j=2}^{n} S(n,j)\left(-\frac{1}{2}\right)^j j!\left[(2^j-1)\zeta(j) - \frac{2^{j+1}-1}{2}\zeta(j+1)\right],\end{aligned}$$

which also holds for any $n \geq 1$ assuming that the empty sum equals zero.

We call the reader to prove that the sum of the even and odd indexed second kind Stirling numbers can be expressed as

$$\sum_{j=0}^{\lfloor n/2 \rfloor} S(n,2j) = \cosh(1)\sum_{k=0}^{\infty}\frac{(2k)^n}{(2k)!} - \sinh(1)\sum_{k=0}^{\infty}\frac{(2k+1)^n}{(2k+1)!},$$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} S(n,2j+1) = \cosh(1)\sum_{k=0}^{\infty}\frac{(2k+1)^n}{(2k+1)!} - \sinh(1)\sum_{k=0}^{\infty}\frac{(2k)^n}{(2k)!}.$$

(These identities appear in an American Mathematical Monthly problem of A. Fekete [13]. See [1] for a solution, and see the book of Comtet [10, p.225–226] for a more general setting.)

Finally, let C_n be the n-th Catalan number. It is well known that

$$C(x) := \sum_{k=0}^{\infty} C_k x^k = \frac{1-\sqrt{1-4x}}{2x}.$$

In [15], the author showed that

$$C^{(n)}(x) = \frac{n!}{(x(1-4x))^n}\left(a_{n-1}(x) + b_n(x)C(x)\right),$$

where

$$a_n(x) = \sum_{k=0}^{n} C_k x^k (4x-1)^{n-k}, \text{ and } b_n(x) = -2\sum_{k=0}^{n} C_{k-1} x^k (4x-1)^{n-k},$$

with $C_{-1} = -1/2$. Therefore,

$$\sum_{k=0}^{\infty}(mk+r)^n C_k t^k = \sum_{j=0}^{n} W_{m,r}(n,j)(mt)^j \frac{j!}{(t(1-4t))^j}\left(a_{j-1}(t) + b_j(t)C(t)\right).$$

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